Fourier Multipliers and Maximal Regularity for Integro-differential Equations in Banach spaces

Verónica Poblete Oviedo[1]

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Chapter 1

Introduction

The aim of this thesis is to study the existence and qualitative properties of solutions for some integro-differential equations with delay by using methods of maximal regularity in spaces of vector valued functions. The study of maximal regularity is very useful for treating semilinear and quasilinear problems and results in this direction have been studied extensively in recent years (see for example, [1], [21], [42], and the recent survey by W. Arendt [6] and the bibliography therein). One of the most important tools to prove maximal regularity is the theory of Fourier multipliers. They play an important role in the analysis of parabolic problems. In recent years it has become apparent that one needs not only the classical theorems but also vector-valued extensions with operator-valued multiplier functions or symbols. These extensions allow to treat certain problems for evolution equations with partial differential operators in an elegant and efficient manner in analogy to ordinary differential equations. For some recent papers on the subjet, we refer to Weis [52, 53], Kalton-Lancien [32], Denk-Hieber-Prüss [25], Schweiker [46], Arendt-Bu [9, 11], Amann [1, 2], Arendt-Batty-Bu [8] . Operator valued Fourier multiplier theorems for Besov spaces, have been obtained and studied by Amann [2], Weis [54], Girardi-Weis [27, 28], Arendt-Bu [10].

We characterize well-posedness of some linear integro-differential equations in L^p spaces, Besov and Hölder spaces. In the case of $L^p_{2\pi}(\mathbb{R}; X)$ (periodic boundary conditions), our results involve UMD spaces, the concept of $R-$boundedness and a condition on the resolvent operator. We remark that many of the most powerful modern theorems are valid in UMD spaces, i.e., Banach spaces in which martingale are unconditional differences. The probabilistic definition of UMD spaces turns out to be equivalent to the L^p-boundedness of the Hilbert transform, a transformation which is, in a sense, the typical representative example of a multiplier operator. On the other hand the notion of $R-$boundedness has played an important role in the functional analytic approach to partial differential equations. It was shown in [52] (see also [15, 29]) that $R-$boundedness provides a proper setting for boundedness theorems for operator-valued Fourier multipliers. Workable criteria for $R-$boundedness have been established recently by Girardi-Weiss in [29].

In the case of, Besov and Hölder spaces, our results involve only boundedness of the

resolvent and are therefore, more suitable for the applications.
We study equations containing some hereditary characteristics. In the modelling of many evolution phenomena arising in physics, biology, engineering, etc., some time delay can appear. Typical examples can be found in the researches on materials with thermal memory, biochemical reactions, population models (See for instance, Wu [56] and references cited therein).
We will consider the following three problems, the first and second one with periodic boundary conditions, the third one on the real line.
First problem. Denote by $B^s_{pq}(\mathbb{T};X)$ the periodic Besov spaces and let $f \in B^s_{pq}(\mathbb{T};X)$. We consider the following integro-differential equation with infinite delay

$$\begin{cases} u''(t) + \alpha u'(t) = Au(t) + \displaystyle\int_{-\infty}^{t} c(t-s)Au(s)ds + f(t) \quad 0 \le t \le 2\pi \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi)\,, \end{cases} \tag{1.1}$$

where A is a closed linear operator defined on the Banach space X, $c \in L^1(\mathbb{R}_+)$ is an scalar-valued kernel.
We say that equation (1.1) is $B^s_{p,q}-$ well-posed or that there exists a classical solution with maximal regularity, if for each $f \in B^s_{pq}(\mathbb{T};X)$ there exists a unique solution $u \in B^s_{pq}(\mathbb{T};X) \cap B^{s+2}_{pq}(\mathbb{T};[D(A)])$.
We will obtain maximal regularity results for (1.1) inspired by a recent paper by Keyantuo-Lizama [34] where the second order problem without integral term is studied. Note that the results presented here corresponding to equation (1.1) are the subject of the paper [43].

Second problem. We achieve in this work is the perturbed equation

$$\begin{cases} u(t) &= \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)ds + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t) \\ u(0) &= u(2\pi)\,, \end{cases} \tag{1.2}$$

where A and B are closed linear operators defined on a UMD space X, such that $D(A) \subset D(B)$ and $a(\cdot)\,,\, b(\cdot) \in L^1(\mathbb{R}_+)$ are scalar-valued kernels.
By $L^p_{2\pi}(\mathbb{R};X)$ we denote all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0,2\pi]$ is p-integrable. We say that the problem (1.2) is L^p well-posed or that there exists a classical solution with maximal regularity if, for each $f \in L^p_{2\pi}(\mathbb{R};X)$ there exists a unique solution $u \in L^p_{2\pi}(\mathbb{R};[D(A)])$.
Equations of the form (1.2) has been studied by Pugliese [45] (see also Prüss [44]). Maximal regularity for integro-differential equations similar to (1.2) using operator-valued Fourier multiplier theorems have been studied recently in [33] and [35]. Our case is more difficult to handle in opposition to those cases treated, for example to [9], [33], [35], because the presence of the perturbing operator B.

In the problems above, we shall assume that ik , $k \in \mathbb{Z}$, is contained in the resolvent set of A. We will study existence of solutions for (1.1) and (1.2) if some ik does not belong to the resolvent set. We also give a representation formula for all the solutions. We call this problem: resonance case. We remark that a similar case was studied by Da Prato and Lunardi in [24] when A generates an analytic semigroup. Our results extends and improve those in [24].
The study of equation (1.2) was done in joint work with C. Lizama [39].

Third problem. We consider

$$u'(t) = Au(t) + Fu_t + f(t), \qquad t \in \mathbb{R}, \tag{1.3}$$

where $(A, D(A))$ is a (unbounded) linear operator on a Banach space X, $u_t(\cdot) = u(t+\cdot)$ on $[-r, 0], r > 0$, here the delay operator F belong to $\mathcal{B}(C([-r, 0], X), X)$ and $f \in C^\alpha(\mathbb{R}, X)$.
Some of the earlier investigation on equation (1.3) were done by back to J. Hale [30] and G. Webb [51]. More recently a general and systematic study of linear delay equations with emphasis on the qualitative behavior and asymptotic properties can be found in the recent monograph by Bátkai and Piazzera [13]. (See also [56]). The problem to obtaining conditions for all solutions of (1.3) to be in the same space as f arises naturally from new studies on maximal regularity and their application to nonlinear problems in the theory of evolution equations. See the monograph by Denk-Hieber-Prüss [25].
A significant progress has been made in finding sufficient conditions for operator valued functions to be C^α- Fourier multipliers; see [8]. In particular, in [12] the theory of operator-valued Fourier multipliers is applied to obtain results on the hyperbolicity of delay equations and in [37] to obtain stability of linear control systems in Banach spaces. Also, existence and uniqueness of periodic solutions for equation (5.1) via L^p-Fourier multiplier theorems has been obtained in [41].
Our goal is to prove that problem (1.3) is C^α well-posed. We characterize the maximal regularity of solutions on the real line by operator-valued Fourier multipliers methods. (See [8]). We remark that the study of equation (1.3) was done in joint work with C. Lizama [40].

A few words about the organization of this work. It is divided in four chapters. In the first chapter, we collect basic definitions and notations which we use throughout, for example, $R-$boundedness of operator families, Fourier multipliers, UMD and Besov spaces. Moreover, we present previous results on maximal regularity via periodic multipliers; see [9], [10], [33]. In the remaining chapters we study the three problems described above.

Chapter 2

Preliminaries

In this chapter we introduce some of the concepts to be used there after. We also review the classical results that provide material for a better understanding of the thesis. We study the notion of $R-$boundedness, giving a review of its basic properties of $R-$bounds. We present the basic theory of $UMD-$spaces and establish their basic properties. In the Section 4, we present the notion of multipliers. Fourier multiplier theorems are of crucial importance in the study of maximal regularity of evolution equations. In Section 5, we establish a connection between sequences that satisfy Marcinkiewicz estimates of order k ($k = 1, 2, 3$) and $k-$regular sequences. Furthermore, we establish new properties of $k-$regular sequences.
Let X, Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ be the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$.

2.1 R-bounded Families

The notion of $R-$boundedness has proved to be a significant tool in the study of abstract multiplier operators. Preliminary concepts for the definition and properties of $R-$boundedness that we will use may be found in [9], [31], [25].
For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0, 1]$, i.e. $r_j(t) = sgn(\sin(2^j \pi t))$. For $x \in X$ we write $r_j x$ for the vector valued function $t \to r_j(t)x$. We use the notation $L^p(a, b; X)$ for the L^p-space of all functions X-valued integrable on $[a, b]$. The definition of $R-$boundedness is given as follows.

Definition 2.1 *A family $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded if there exists $c_p \geq 0$ so that*

$$|| \sum_{j=1}^{n} r_j T_j x_j ||_{L^p(0,1;Y)} \leq c_p || \sum_{j=1}^{n} r_j x_j ||_{L^p(0,1;X)} \tag{2.1}$$

for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$ and $n \in \mathbb{N}$, where $1 \leq p < \infty$. We denote by $R_p(\mathcal{T})$ the smallest constant c_p such that (2.1) holds.

Theorem 2.2 *(Khintchine-Kahane inequality) For* $0 < p,q < \infty$ *, there exist finite constants* $K_{p,q}$ *such that*

$$|| \sum_{j=1}^{n} r_j x_j ||_{L^q(0,1;X)} \leq K_{p,q} || \sum_{j=1}^{n} r_j x_j ||_{L^p(0,1;X)} \tag{2.2}$$

for all $x_j \in X$ *,* $j = 1 \ldots n$ *.*

Proof. See [31, Corollary 3.12]. ■

Remark 2.3

(a) By the Khintchine-Kahane inequality, the definition of $R-$boundedness is independent of the value of p in the sense that any $\mathcal{T} \subset \mathcal{B}(X,Y)$ either satisfies the condition for all $p \in [1,\infty)$ or for none of them. However, note that the $R-$bound $R_p(\mathcal{T})$ may depend on p. In fact, the Khintchine-Kahane inequality shows that we could take different exponents $p,q \in [1,\infty)$ on the two sides of the inequality defining $R-$boundedness, and the resulting inequality either holds for all pairs (p,q) or for none of them.

(b) From the definition it is clear that any R-bounded family is bounded.
In fact, if $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded then it is uniformly bounded, with

$$\sup_{T \in \mathcal{T}} ||T||_{\mathcal{B}(X,Y)} \leq \inf_{p \in [0,\infty)} R_p(\mathcal{T}).$$

The converse of this assertion holds only for spaces which are isomorphic to Hilbert spaces. For more details, we refer to Arendt and Bu [9] .

Example 2.4 *Let* $X = Y = L^p(a,b;\mathbb{C}) = L^p(a,b)$ *for some* $a,b \in \mathbb{R}$ *with* $a < b$ *. Then* $\mathcal{T} \subset \mathcal{B}(X,Y)$ *is R-bounded if and only in there is a constant* $M > 0$ *such that the following square function estimate holds*

$$\left\| \left(\sum_{j=1}^{n} |T_j f_j|^2 \right)^{1/2} \right\|_{L^p(a,b)} \leq M \left\| \left(\sum_{j=1}^{n} |f_j|^2 \right)^{1/2} \right\|_{L^p(a,b)} \tag{2.3}$$

for all $n \in \mathbb{N}$ *,* $f_j \in L^p(a,b)$ *, and* $T_j \in \mathcal{T}$ *.*

This is a consequence of the Khintchine-Kahane inequality: For each $p \in [1,\infty)$ there is a constant $K_p > 0$ such that

$$K_p^{-1} \left\| \sum_{j=1}^{n} r_j a_j \right\|_{L^p(0,1;\mathbb{C})} \leq \left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \leq K_p \left\| \sum_{j=1}^{n} r_j a_j \right\|_{L^p(0,1;\mathbb{C})} \tag{2.4}$$

for all $n \in \mathbb{N}$, $a_j \in \mathbb{C}$, and for all Rademacher functions r_j, $j = 1 \dots n$.
If (2.3) holds, we have by (2.4)

$$\begin{aligned}
\left\| \sum_{j=1}^{n} r_j T_j f_j \right\|_{L^p(0,1;L^p(a,b))} &= \left\| \sum_{j=1}^{n} r_j T_j f_j \right\|_{L^p(a,b;L^p(0,1))} \\
\leq K_p \left\| \left(\sum_{j=1}^{n} |T_j f_j|^2 \right)^{1/2} \right\|_{L^p(a,b)} &\leq K_p M \left\| \left(\sum_{j=1}^{n} |f_j|^2 \right)^{1/2} \right\|_{L^p(a,b)} \\
\leq K_p^2 M \left\| \sum_{j=1}^{n} r_j f_j \right\|_{L^p(a,b;L^p(0,1))} &= K_p^2 M \left\| \sum_{j=1}^{n} r_j f_j \right\|_{L^p(0,1;L^p(a,b))} .
\end{aligned}$$

The proof of the converse is similar.
In what follows, we give a survey of some simple properties of $R-$boundedness and provide further examples of $R-$bounded sets of operators. More details and proofs, can be founded in the recent monograph of Denk-Hieber-Prüss [25].

Proposition 2.5 *(a) Any finite family $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded.*
(b) A subset of an R-bounded set is also R-bounded.
(c) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X,Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$.
(d) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \leq R_p(\mathcal{S}) R_p(\mathcal{T})$.

Lemma 2.6 *([31, Lemma 4.6]) To check the $R-$boundedness of a family $\mathcal{T} \subset \mathcal{B}(X,Y)$, it is sufficient to verify the inequality (2.1) for all sequences of distint elements $T_k \in \mathcal{T}$. The best constants are the same.*

Corollary 2.7 *([31, Corollary 4.7])If $\mathcal{T} = \{T^k\}_{k=1}^{\infty} \subset \mathcal{B}(X,Y)$ is a countable sequence of operators, then it is sufficient to verify the inequality (2.1) for all truncated sequences $\{T^k\}_{k=1}^{n}$ of the first n members of the sequence.*

It is clear that the $R-$boundedness of the countable set $\mathcal{T}$ is independent of the order in which we enumerate its element. Thus it is interesting that, given any enumeration, the subset of n first members of the sequence are fully representative of all finite subsets of $\mathcal{T}$ in view of $R-$boundedness, this is what the assertion above states.
A very useful tool in connection with R-boundedness is the contraction principle of Kahane, which we state as a lemma. A proof can be found in [25, Lemma 3.5] and [38].

Lemma 2.8 *Let X be a Banach space, $n \in \mathbb{N}$, $x_j \in X$, and $\alpha_j \in \mathbb{C}$ for each $j = 1, ..., n$. Then*

$$||\sum_{j=1}^{n} \alpha_j \, r_j \, x_j||_p \leq 2 \max_{j=1,\dots,n} |\alpha_j| \; ||\sum_{j=1}^{n} r_j \, x_j||_p \, .$$

The constant 2 can be omitted in case where α_j is real.

2.2 UMD Spaces

The definition of a Banach space with the unconditional martingale difference property or UMD was introduced by D.L. Burkholder in [18, Section 9], and is given as follows

Definition 2.9 *A Banach space X is said to have the unconditional martingale difference property (UMD) if for each $p \in (1, \infty)$ there is a constant C_p such that for any martingale $\{f_n\}_{n \geq 0} \subset L^p(\Omega, \Sigma, \mu; X)$ and any choice of signs $\{\varepsilon_n\}_{n \geq 0} \subset \{-1, 1\}^{\mathbb{N}}$ and any $N \in \mathbb{N}$ the following estimate holds.*

$$||f_0 + \sum_{n=1}^{N} \varepsilon_n \, (f_n - f_{n-1})||_{L^p(\Omega,\Sigma,\mu;X)} \leq C_p \, ||f_N||_{L^p(\Omega,\Sigma,\mu;X)}$$

We recall that those Banach spaces X for which the Hilbert transform defined by

$$(Hf)(t) = \lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \frac{1}{\pi} \int_{\epsilon \leq |s| \leq R} \frac{f(t-s)}{s} ds$$

is bounded on $L^p(\mathbb{R}, X)$ for some $p \in (1, \infty)$ are called $\mathcal{HT}$ spaces. The limit in the above formula is to be understood in the L^p sense.
For more information and details on the Hilbert transform and the UMD Banach spaces we refer to [5, Section III.4.3-III.4.5]. The UMD property turns out to be equivalent to several important properties of certain Banach spaces. Burkholder and McConnell proved that a UMD space is a $\mathcal{HT}$ space (see [18, Section 9]) and Bourgain proved the converse in [17].

The following are examples the UMD spaces. For its proof see [31] and [25].

Example 2.10 *The UMD spaces include Hilbert spaces, Sobolev spaces $W^s_p(\Omega)$, $1 < p < \infty$ (see [3]), Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when X is a UMD space and the Schatten-von Neumann classes $C_p(H)$, $1 < p < \infty$ of operators on Hilbert spaces.*

Example 2.11 *Every closed subspace of a UMD space is a UMD space.*

Example 2.12 *Every UMD space is reflexive.*

Example 2.13 *A Banach space X is UMD if and only if its dual X^* is UMD.*

For more information on $R-$boundedness and $UMD-$spaces we refer to the recent thesis of Hytönen [31].

2.3 Periodic Besov Spaces

Besov spaces form a class of function spaces which are of special interest. The relatively complicated definition is rewarded by useful applications to differential equations (see Amann [1] for a concrete model). We briefly recall the definition of periodic Besov spaces in the vector-valued setting as case introduced in [10]. For the scalar case, see Triebel [49, Chapter 9] and Schmeisser-Triebel [47] . An approach to periodic Besov spaces based on semigroup theory and abstract interpolation is presented in [19, Chapter 4].
Let X be a Banach space and let $\mathbb{T} = [0, 2\pi]$ where the points 0 and 2π are identified. Let $\mathcal{D}(\mathbb{T})$ be the space of all complex-valued infinitely differentiable functions on $\mathbb{T}$. The usual locally convex topology in $\mathcal{D}(\mathbb{T})$ is generated by the semi-norms $||f||_n = \sup_{t\in\mathbb{T}} ||f^{(n)}(t)||$, where $n \in \mathbb{N} \cup \{0\}$. We let $\mathcal{D}'(\mathbb{T}; X) := \mathcal{B}(\mathcal{D}(\mathbb{T}); X)$. Elements in $\mathcal{D}'(\mathbb{T}; X)$ are called $X-$valued distributions on $\mathbb{T}$.

For $f \in \mathcal{D}'(\mathbb{T}; X)$, denote by $\hat{f}(k)$, for $k \in \mathbb{Z}$, the k-th Fourier coefficient of f as

$$\langle \hat{f}, l \rangle = \langle f, \hat{l} \rangle, \quad l \in \mathcal{D}(\mathbb{T}).$$

In what follows we identify $\hat{l}$ with $\hat{f}$ which is standard in the theory of Besov and Triebel spaces (see [50], pages 49-50).
Let $\mathcal{S}$ be the Schwartz space on $\mathbb{R}$ and $\Phi(\mathbb{R})$ be the set of all systems $\phi = \{\phi_j\}_{j\geq 0} \subset \mathcal{S}$ satisfying

$$\operatorname{supp}(\phi_0) \subset [-2, 2],$$

$$\operatorname{supp}(\phi_j) \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{k+1}], \quad j \geq 1,$$

$$\sum_{j\geq 0} \phi_j(t) = 1, \quad t \in \mathbb{R}$$

and for $n \in \mathbb{N} \cup \{0\}$, there exists $C_n > 0$ such that

$$\sup_{j\geq 0,\, x\in\mathbb{R}} 2^{nj} ||\phi_j^{(n)}(x)|| \leq C_n . \tag{2.5}$$

Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $\phi = (\phi_j)_{j\geq 0} \in \Phi(\mathbb{R})$. Let $e_k(t) = e^{ikt}$. For $x \in X$ we write $e_k x$ the vector valued function $t \to e_k(t)x$. The $X-$valued periodic Besov spaces are defined by

$$B^{s,\phi}_{p,q}(\mathbb{T}; X) = \{ f \in \mathcal{D}'(\mathbb{T}; X) \ : \ ||f||_{B^{s,\phi}_{p,q}} = \left(\sum_{j\geq 0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} e_k \phi_j(k) \hat{f}(k) \right\|_p^q \right)^{1/q} < \infty \},$$

where for $x \in X$.
We make the usual modification if $q = \infty$. Note also, that the space $B^s_{\infty,\infty}$ is the familiar space of all Hölder continuous functions of index s if $s \in (0, 1)$.

We remark that the spaces $B^{s,\phi}_{p,q}$ are independent of $\phi \in \Phi(\mathbb{R})$, and the norms $||\cdot||_{B^{s,\phi}_{p,q}}$ are equivalent. We will simply denote $||\cdot||_{B^{s,\phi}_{p,q}}$ by $||\cdot||_{B^{s}_{p,q}}$ for some $\phi \in \Phi(\mathbb{R})$.

Remark 2.14

We summarize some useful properties of $B^s_{p,q}(\mathbb{T};X)$.

(i) $B^s_{p,q}(\mathbb{T};X)$ is a Banach space.

(ii) The natural injection from $B^s_{p,q}(\mathbb{T};X)$ into $L^p(\mathbb{T};X)$ is a continuous linear operator for $s>0$.

(iii) The natural injection from $B^{s+\varepsilon}_{p,q}(\mathbb{T};X)$ in $B^s_{p,q}(\mathbb{T};X)$ is a continuous linear operator for $\varepsilon > 0$.

(iv) Lifting Property: Let $f \in \mathcal{D}'(\mathbb{T};X)$ and $\eta \in \mathbb{R}$. Then $f \in B^s_{p,q}(\mathbb{T};X)$ if and only if $\sum_{k\neq 0} e_k \otimes k^\eta \hat{f}(k) \in B^{s-\eta}_{p,q}(\mathbb{T};X)$.

(v) Let $s>0$. Then $f \in B^{1+s}_{p,q}(\mathbb{T};X)$ if and only if f is differentiable a.e. and $f' \in B^s_{p,q}(\mathbb{T};X)$.

For a proof see [10, Theorem 2.3]

2.4 Multipliers

In the classical context, the notion of multipliers emerges in Fourier analysis. It turns out that certain important bounded linear transformations of L^p to L^q, $1 \le p,q < \infty$ have a multiplier structure when viewed in the Fourier domain.
We fix some notation. We identify the spaces of (vector or operator-valued) functions defined on $[0,2\pi]$ to their periodic extensions to $\mathbb{R}$.
For a function $f \in L^1_{2\pi}(\mathbb{R};X)$, denote by $\hat{f}(k)$, for $k \in \mathbb{Z}$, the k-th Fourier coefficient of f, that is,

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)dt,$$

with $t \in \mathbb{R}$. By Fejér's theorem

$$\sigma_n(f) := \frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e_k \hat{f}(k) = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right) e_k \hat{f}(k)$$

converges to f as $n \to \infty$.
We begin with some preliminaries about operator-valued Fourier multipliers. More information may be found in Arendt-Bu [9] for the periodic case and Amann [5], Weis [52] for the non-periodic case.

Definition 2.15 *For* $1 \le p \le \infty$, *we say that a sequence* $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ *is an* $L^p_{X,Y}$*-multiplier (resp.* $B^s_{pq}-$*multiplier), if for each* $f \in L^p_{2\pi}(\mathbb{R};X)$ *(resp.* $f \in B^s_{p,q}(\mathbb{T};X)$ *) there exists* $u \in L^p_{2\pi}(\mathbb{R};Y)$ *(resp.* $u \in B^s_{p,q}(\mathbb{T};Y)$ *) such that*

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

If $\{M_k\}_{k\in\mathbb{Z}}$ is a $L^p_{X,Y}$-multiplier then the uniqueness theorem and the closed graph theorem show that the mapping

$$M : L^p_{2\pi}(\mathbb{R};X) \to L^p_{2\pi}(\mathbb{R};Y)$$

is linear and continuous. We call M the operator associated with $\{M_k\}_{k\in\mathbb{Z}}$. One has

$$Mf = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k M_k \hat{f}(k)$$

in $L^p_{2\pi}(\mathbb{R};Y)$ for all $f \in L^p_{2\pi}(\mathbb{R};X)$, (analogously, if $f \in B^s_{p,q}(\mathbb{T};X)$).

Example 2.16 *On a Hilbert space* X *each bounded sequence* $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$ *is an* L^2*-multiplier. This follows from the fact that the Fourier transform given by*

$$f \in L^p_{2\pi}(\mathbb{R};X) \to \{\hat{f}(k)\}_{k\in\mathbb{Z}} \in \ell^2(X)$$

is an isometric isomorphism if X *is a Hilbert space.*

Remark 2.17

Let X, Y and Z be Banach spaces. If $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ and $\{N_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(Y,Z)$ are $B^s_{pq}-$multipliers then $\{N_k M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{pq}-$multiplier. This follows directly from the definition.

Remark 2.18

Let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ be an $L^p_{X,Y}-$ multiplier, where $1 \le p < \infty$. An inspection of the proof of [9, Proposition 1.11] shows that the set $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded.
The following condition on sequences $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ appears in [9] to study Fourier multipliers in the L^p-context. It is also used in the study of multipliers of Besov spaces.

Definition 2.19 *A sequence* $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ *satisfies a Marcinkiewicz estimate of order 1 if*

$$\sup_{k\in\mathbb{Z}} ||M_k|| < \infty\,, \quad \sup_{k\in\mathbb{Z}} ||k\,(M_{k+1} - M_k)|| < \infty\;, \tag{2.6}$$

If in addition we have that

$$\sup_{k\in\mathbb{Z}} ||k^2\,(M_{k+1} - 2M_k + M_{k-1})|| < \infty\,. \tag{2.7}$$

then we say that $\{M_k\}_{k\in\mathbb{Z}}$ *satisfies a Marcinkiewicz estimate of order 2.*
Finally, if in addition to (2.6) and (2.7) we have that

$$\sup_{k\in\mathbb{Z}} ||k^3\,(M_{k+1} - 3M_k + 3M_{k-1} - M_{k-2})|| < \infty\,. \tag{2.8}$$

then we say that $\{M_k\}_{k\in\mathbb{Z}}$ *satisfies a Marcinkiewicz estimate of order 3.*

The following theorem establishes regularity properties of sequences that satisfies Marcinkiewicz estimates.

Theorem 2.20 *If* $\{M_k\}_{k\in\mathbb{Z}}$ *and* $\{N_k\}_{k\in\mathbb{Z}}$ *satisfy Marcinkiewicz estimate of order* k $(k = 1, 2, 3)$ *then* $\{M_k \pm N_k\}_{k\in\mathbb{Z}}$ *satisfy Marcinkiewicz estimates of the same order.*

The proof is obvious and we omit it. In the scalar case, we have

Theorem 2.21 *If* $\{a_k\}_{k\in\mathbb{Z}}$ *and* $\{b_k\}_{k\in\mathbb{Z}}$ *are sequences that satisfy Marcinkiewicz estimate of order* k $(k = 1, 2, 3)$ *then* $\{a_k\ b_k\}_{k\in\mathbb{Z}}$ *satisfy Marcinkiewicz estimate the same.*

Proof. By the hypotheses, is clear that $\sup_{k\in\mathbb{Z}} |a_k\ b_k| < \infty$. To verify Marcinkiewicz estimates of order $k\,(k = 1, 2, 3)$, we have the following identities,

(i) For Marcinkiewicz estimates of order 1

$$k(a_{k+1}b_{k+1} - a_kb_k) = k\,(a_{k+1} - a_k)\,b_{k+1} + k\,(b_{k+1} - b_k)\,a_k\,.$$

(ii) For Marcinkiewicz estimates of order 2

$$k^2\,(a_{k+1}b_{k+1} - 2a_kb_k + a_{k-1}b_{k-1})$$

$$= k^2\,(a_{k+1} - 2a_k + a_{k-1})\,b_{k+1} + k^2\,(b_{k+1} - 2b_k + b_{k-1})a_k + k\,(a_k - a_{k-1})\,k\,(b_{k+1} - b_{k-1})\,.$$

(iii) For Marcinkiewicz estimates of order 3

$$k^3\,(a_{k+1}b_{k+1} - 3a_kb_k + 3a_{k-1}b_{k-1} - a_{k-2}b_{k-2})$$

$$\begin{aligned} &= k^3(a_{k+1} - 3a_k + 3a_{k-1} - a_{k-2})\,b_{k+1} + k^2\,(a_k - 2a_{k-1} + a_{k-2})\,k\,(b_{k+1} - b_{k-2}) \\ &+ k^3\,(b_{k+1} - 3b_k + 3b_{k-1} - b_{k-2})a_k + 2k^2\,(b_k - 2b_{k-1} + b_{k-2})\,k(a_k - a_{k-1}) \\ &+ k\,(a_k - a_{k-1})\,k^2\,(b_{k+1} - 2b_k + b_{k-1})\,. \end{aligned}$$

Since $\{a_k\}$ and $\{b_k\}$ satisfy Marcinkiewicz estimates of order $k\,(k = 1, 2, 3)$ we obtain that $\{a_k\,b_k\}$ satisfies (2.6), (2.7) and (2.8).

∎

The following general multiplier theorem is due to Arendt and Bu [10, Theorem 4.5] and plays an important role in our investigations.

Theorem 2.22 *Let X and Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ satisfy Marcinkiewicz estimates of order 2 . Then for $1 \le p,q \le \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{pq}-$multiplier.*

For the Hölder spaces, Arendt, Batty and Bu in [8, Theorem 3.4], proved the following theorem.

Theorem 2.23 *If $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ satisfies a Marcinkiewicz estimate of order 2 then $\{M_k\}_{k\in\mathbb{Z}}$ is a $C^\alpha_{2\pi}$-multiplier.*

The following theorem due to Weis [52] is the discrete analog of the operator- valued version of Mikhlin's theorem, and will be of fundamental importance, see also [22] .

Theorem 2.24 *(Marcinkiewicz operator-valued multiplier theorem) Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If the families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an $L^p_{X,Y}$-multiplier for $1 < p < \infty$.*

We remark that Witvliet's thesis [55] contains an extensive treatment of modern multiplier theorems and applications. Some of the results have also appeared in Clément et. al. [20].

2.5 $k-$regular Sequences

The notion of $1-$regular and $2-$regular scalar sequences were introduced by Keyantuo and Lizama in [33] to study maximal regularity on periodic Besov spaces. This concept is the discrete analogue for the notion of $k-$regularity related to Volterra integral equations (see [44, Chapter I, Section 3.2]). Subsequently, Bu and Fang in [16] introduced the notion of 3-regular scalar sequence to study maximal regularity on Triebel-Lizorkin spaces.

Definition 2.25 *A sequence $\{a_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}\setminus\{0\}$ is called*
(a) $1-$regular if the sequence $\{k\,\dfrac{(a_{k+1}-a_k)}{a_k}\}_{k\in\mathbb{Z}}$ is bounded;
(b) $2-$regular if it is $1-$regular and the sequence $\{k^2\,\dfrac{(a_{k+1}-2a_k+a_{k-1})}{a_k}\}_{k\in\mathbb{Z}}$ is bounded;
(c) $3-$regular if it is $2-$regular and the sequence $\{k^3\,\dfrac{(a_{k+1}-3a_k+3a_{k-1}-a_{k-2})}{a_k}\}_{k\in\mathbb{Z}}$ is bounded.

Example 2.26 *It is not difficult to see that the sequence $a_k = b/(ik+c)$, where $b \in \mathbb{R}$ and $c \neq 0$ is 3-regular.*

In the next Lemma we give some useful properties of $k-$regular $(k = 1,2,3)$ sequences.

Lemma 2.27 *(i) If $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are $k-$regular sequences such that* $\sup_k \left|\dfrac{a_k}{a_k+b_k}\right| < \infty$, *then the sequence $\{a_k+b_k\}_{k\in\mathbb{Z}}$ is $k-$regular.*

(ii) If the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are $k-$regular, then the sequence $\{a_k\, b_k\}_{k\in\mathbb{Z}}$ is $k-$regular.

(iii) The sequence $\{a_k\}_{k\in\mathbb{Z}}$ is $k-$regular if and only if the sequence $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}}$ is $k-$regular.

(iv) If the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are $k-$regular, then the sequence $\{a_k/b_k\}_{k\in\mathbb{Z}}$ is $k-$regular.

Proof. We first prove (i). Observe that for 1-regularity observe that

$$\begin{aligned}
&k\frac{a_{k+1}\ +\ b_{k+1}\ -\ (a_k+b_k)}{a_k+b_k}\\
&\quad= k\frac{a_{k+1}-a_k\ +\ b_{k+1}-b_k}{a_k+b_k}\\
&\quad= k\frac{a_{k+1}-a_k}{a_k}\,\frac{a_k}{a_k+b_k}\ +\ k\,\frac{b_{k+1}-b_k}{b_k}\,\frac{b_k}{a_k+b_k}\\
&\quad= k\frac{a_{k+1}-a_k}{a_k}\,\frac{a_k}{a_k+b_k}\ +\ k\,\frac{b_{k+1}-b_k}{b_k} - k\,\frac{b_{k+1}-b_k}{b_k}\,\frac{a_k}{a_k+b_k}\,.
\end{aligned}$$

To verify 2-regularity, we have

$$\begin{aligned}
&k^2\frac{a_{k+1}\ +\ b_{k+1}\ -\ 2(a_k+b_k)+a_{k-1}+b_{k-1}}{a_k+b_k}\\
&\quad= k^2\frac{a_{k+1}-2a_k\ +\ a_{k-1}}{a_k}\,\frac{a_k}{a_k+b_k}\ +\ k^2\frac{b_{k+1}-2b_k\ +\ b_{k-1}}{b_k}\,\frac{b_k}{a_k+b_k}\\
&\quad= k^2\frac{a_{k+1}-2a_k\ +\ a_{k-1}}{a_k}\,\frac{a_k}{a_k+b_k}\ +\ k^2\frac{b_{k+1}-2b_k\ +\ b_{k-1}}{b_k} - k^2\frac{b_{k+1}-2b_k\ +\ b_{k-1}}{b_k}\,\frac{a_k}{a_k+b_k}\,.
\end{aligned}$$

Finally, to verify the 3-regularity, note that

$$\begin{aligned}
&k^3\frac{a_{k+1}\ +\ b_{k+1}\ -\ 3(a_k+b_k)+3(a_{k-1}+b_{k-1})-(a_{k-2}+b_{k-2})}{a_k+b_k}\\
&\quad= k^3\frac{a_{k+1}-3a_k+3a_{k-1}-a_{k-2}}{a_k}\,\frac{a_k}{a_k+b_k}\ +\ k^3\frac{b_{k+1}-3b_k+3b_{k-1}-b_{k-2}}{b_k}\,\frac{b_k}{a_k+b_k}\\
&\quad= k^3\frac{a_{k+1}-3a_k+3a_{k-1}-a_{k-2}}{a_k}\,\frac{a_k}{a_k+b_k}\ +\ k^3\frac{b_{k+1}-3b_k+3b_{k-1}-b_{k-2}}{b_k}\\
&\quad -k^3\frac{b_{k+1}-3b_k+3b_{k-1}-b_{k-2}}{b_k}\,\frac{a_k}{a_k+b_k}\,.
\end{aligned}$$

Hence we obtain (i).

To prove (ii), note that

$$k\frac{a_{k+1}b_{k+1} - a_kb_k}{a_kb_k} = k\frac{a_{k+1}b_{k+1} - a_kb_{k+1} + a_kb_{k+1} - a_kb_k}{a_kb_k}$$

$$= k\frac{a_{k+1} - a_k}{a_k}\,\frac{b_{k+1}}{b_k} + k\,\frac{b_{k+1} - b_k}{b_k}\,.$$

Since $\{a_k\}$ and $\{b_k\}$ are 1-regular sequences, it follows that $\{a_k\, b_k\}$ is 1-regular. In order to prove that $\{a_k\, b_k\}$ is 2-regular, we have the following identity

$$k^2\,\frac{a_{k+1}b_{k+1} - 2a_kb_k + a_{k-1}b_{k-1}}{a_kb_k} = k^2\,\frac{a_{k+1} - 2a_k + a_{k-1}}{a_k}\,\frac{b_{k+1}}{b_k}$$

$$+\ k^2\,\frac{b_{k+1} - 2b_k + b_{k-1}}{b_k} + k\,\frac{a_k - a_{k-1}}{a_k}\,k\,\frac{b_{k+1} - b_{k-1}}{b_k}\,.$$

Since $\{a_k\}$ and $\{b_k\}$ are 2-regular sequences, it follows that $\{a_k\, b_k\}$ is 2-regular. Finally, to prove that $\{a_k\, b_k\}$ is 3-regular, we have

$$k^3\,\frac{a_{k+1}b_{k+1} - 3a_kb_k + 3a_{k-1}b_{k-1} - a_{k-2}b_{k-2}}{a_kb_k}$$

$$= \quad k^3\frac{a_{k+1} - 3a_k + 3a_{k-1} - a_{k-2}}{a_k}\,\frac{b_{k+1}}{b_k} + k^2\,\frac{a_k - 2a_{k-1} + a_{k-2}}{a_{k-1}}\,k\,\frac{b_{k+1} - b_{k-2}}{b_k}\,\frac{a_{k-1}}{a_k}$$

$$+\quad k^3\,\frac{b_{k+1} - 3b_k + 3b_{k-1} - b_{k-2}}{b_k} + 2k^2\,\frac{b_k - 2b_{k-1} + b_{k-2}}{b_{k-1}}\,k\frac{a_k - a_{k-1}}{a_k}\,\frac{b_{k-1}}{b_k}$$

$$+\quad k\,\frac{a_k - a_{k-1}}{a_k}\,k^2\,\frac{b_{k+1} - 2b_k + b_{k-1}}{b_k}\,,$$

and hence (ii) follows.
To verify (iii) observe that

$$k\frac{1/a_{k+1} - 1/a_k}{1/a_k} = -k\frac{a_{k+1} - a_k}{a_k}\,\frac{a_k}{a_{k+1}}\,.$$

Since $\{a_k\}$ is a 1-regular sequence, it follows that $|\frac{a_{k+1}}{a_k} - 1| \le M/|k|\,, k \neq 0$, for some $M > 0$, and hence $a_k/a_{k+1} \to 1$, from which it follows that $\{1/a_k\}$ is 1-regular.

To verify 2-regularity, we write

$$k^2\frac{1/a_{k+1} - 2/a_k + 1/a_{k-1}}{1/a_k} = k^2\,\frac{a_{k-1}a_k - 2a_{k-1}a_{k+1} + a_ka_{k+1}}{a_{k-1}\ a_k\ a_{k+1}}\ a_k$$

$$= k^2 \, \frac{a_{k-1}(a_k - a_{k+1}) + a_{k+1}(a_k - a_{k-1})}{a_{k-1}\; a_{k+1}}$$

$$= k^2 \, \frac{a_{k-1}(a_k - a_{k+1}) - a_{k+1}(a_k - a_{k+1}) + a_{k+1}(a_k - a_{k+1}) + a_{k+1}(a_k - a_{k-1})}{a_{k-1}\; a_{k+1}}$$

$$= k^2 \, \frac{(a_{k-1} - a_{k+1})\,(a_k - a_{k+1}) - a_{k+1}(a_{k+1} - 2a_k + a_{k-1})}{a_{k-1}\; a_{k+1}}$$

$$= k\, \frac{a_{k-1} - a_{k+1}}{a_{k-1}}\; k\, \frac{a_k - a_{k+1}}{a_{k+1}} \;-\; k^2 \frac{a_{k+1} - 2a_k + a_{k-1}}{a_{k-1}}\,.$$

Finally, to verify 3-regularity, we write

$$k^3 \frac{1/a_{k+1} \,-\, 3/a_k \,+\, 3/a_{k-1} \,-\, 1/a_{k-2}}{1/a_k}$$

$$= \frac{a_{k-1}}{a_{k-2}}\; \frac{a_k}{a_{k-1}}\; \frac{a_k}{a_{k+1}}\; k^3 \frac{-a_{k+1} + 3a_k - 3a_{k-1} + a_{k-2}}{a_k}$$

$$-\;\; 3\; \frac{a_{k-1}}{a_{k-2}}\; \frac{a_k}{a_{k+1}}\; k \frac{a_{k-1} - a_k}{a_{k-1}}\;\; k^2 \frac{a_{k-1} - 2a_k + a_{k+1}}{a_k}$$

$$+\;\; 3\; \frac{a_k}{a_{k+1}}\;\; k \frac{a_{k-1} - a_k}{a_{k-1}}\;\; k \frac{a_{k-1} - a_{k-2}}{a_{k-2}}\;\; k \frac{a_{k+1} - a_k}{a_k}$$

$$+\;\; 3\; \frac{a_{k-2}}{a_{k+1}}\;\; k \frac{a_{k-1} - a_k}{a_{k-1}}\;\; k \frac{a_{k-1} - a_{k-2}}{a_{k-2}}\;\; k \frac{a_{k-1} - a_{k-2}}{a_{k-2}}$$

$$-\;\; 3\; \frac{a_{k-1}}{a_{k+1}}\;\; k \frac{a_{k-1} - a_k}{a_{k-1}}\;\; k^2 \frac{a_{k-2} - 2a_{k-1} + a_k}{a_{k-1}}\,,$$

and hence the result follows.
Note that (iv) follows from (ii) and (iii). This completes the proof of the Lemma.

■

Remark 2.28

Note that (i) hold substituting the condition $\sup_k \left| \dfrac{a_k}{a_k + b_k} \right| < \infty$ by $\sup_k \left| \dfrac{b_k}{a_k + b_k} \right| < \infty$.

Proposition 2.29 *If $\{a_k\}_{k\in\mathbb{Z}}$ is a bounded and $k-$regular sequence, then it satisfies a Marcinkiewicz estimate of order k for $k = 1, 2, 3$.*

Remark 2.30

The converse of the above proposition is false. In fact, the sequence $a_k = e^{-k^2}$ satisfies Marcinkiewicz of order 3 and not 3-regular.

Theorem 2.31 *If $\{a_k\}_{k\in\mathbb{Z}}$ satisfies a Marcinkiewicz estimate of order k and $\{\frac{1}{a_k}\}$ is bounded then $\{a_k\}_{k\in\mathbb{Z}}$ is a $k-$regular sequence $(k = 1, 2, 3)$.*

Proof. It follows directly from the definition of $k-$regular sequence. ∎

Corollary 2.32 *If $\{a_k\}_{k\in\mathbb{Z}}$ satisfies a Marcinkiewicz estimate of order k and $\{\frac{1}{a_k}\}$ is bounded then $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}}$ is a $k-$regular sequence $(k = 1, 2, 3)$.*

Proof. By Theorem 2.31 we have that $\{a_k\}_{k\in\mathbb{Z}}$ is $k-$regular sequence. From Lemma 2.27 (iii), the result follows.

∎

2.6 Maximal Regularity via Periodic Multipliers

In this section, we review some recent work where maximal regularity of integro-differential problems is studied via periodic Fourier multipliers.
For a linear operator A on X, we denote its domain by $D(A)$ and its resolvent set by $\rho(A)$, and for $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1}$.

2.6.1 Strong solutions of periodic problems on Lebesgue spaces

Given a closed linear operator A defined on a UMD space, Arendt and Bu (see [9]), characterize maximal regularity of the following non-homogeneous problem with periodic boundary conditions

$$\begin{aligned} u'(t) &= Au(t) + f(t)\,, \quad t \in [0, 2\pi] \\ u(0) &= u(2\pi) \end{aligned} \tag{2.9}$$

in terms of $R-$boundedness of the resolvent. Here A is not necessarily the generator of a C_0-semigroup. In order to study the periodic case, the authors establish a multiplier theorem, (see Theorem 2.24).
A strong L^p-solution of (2.9) is a function $u \in W^{1,p}_{2\pi}(\mathbb{R}, X) \cap L^p_{2\pi}(\mathbb{R}, X)$ such that (2.9) is satisfied a.e.
The main result in [9] is the following

Theorem 2.33 *Let A be a closed operator on a UMD space X and let $1 < p < \infty$. The following assertions are equivalent.*
(i) For all $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exists a unique strong L^p-solution of (2.9),
(ii) $i\mathbb{Z} \subset \rho(A)$ and the family $\{k(ik - A)^{-1} \ : \ k \in \mathbb{Z}\}$ is $R-$bounded.

Remark 2.34

1. Condition (ii), i.e. well-posedness of the periodic problem in the sense of strong L^p-solution is independent of p for $1 < p < \infty$.
2. Whereas no characterization of L^p-multipliers is available in general (if $1 < p < \infty$, $p \neq 2$), in the context of resolvents it is.
3. In [9], the authors also characterize maximal regularity of the second order Cauchy problem

$$u''(t) + Au(t) = f(t)$$

on a bounded interval with periodic, Dirichlet or Neumann boundary conditions.

2.6.2 Fourier Multipliers on periodic Besov spaces

The Marcinkiewicz type theorem stated in Theorem 2.22, enables one to study maximal regularity in vector-valued periodic Besov spaces for evolution equations with periodic boundary conditions as follows.
Let X be an arbitrary Banach space and A be a closed operator on X. Consider the periodic problem (2.9) with $f \in B^s_{p,q}(\mathbb{T}; X)$ for some $1 \leq p, q \leq \infty$ and $s > 0$. The problem (2.9) has $B^s_{p,q}-$maximal regularity if for each $f \in B^s_{p,q}(\mathbb{T}; X)$ there exists a unique $u \in B^{1+s}_{p,q}(\mathbb{T}; X)$ such that $u(t) \in D(A)$ and $u'(t) = Au(t) + f(t)$ for a.e. $t \in [0, 2\pi]$.

The authors prove in [10] the following result

Theorem 2.35 *Let A be a closed operator on X. The following assertions are equivalent*
(i) Problem (2.9) has $B^s_{p,q}-$maximal regularity for some (equivalently, for all) $s > 0$, $1 \leq p, q \leq \infty$.
(ii) $i\mathbb{Z} \subset \rho(A)$ and $\sup_{k \in \mathbb{Z}} ||k(ik - A)^{-1}|| < \infty$.

Theorem 2.22 may be also be applied to the second order problem with periodic boundary conditions giving necessary and sufficient conditions for such a problem to have $B^s_{p,q}-$maximal regularity as the following theorem shows, (see [10]).

Theorem 2.36 *Let A be a closed operator on X and let $1 \leq p, q \leq \infty$, $s > 0$. The following assertions are equivalent*
(i) For all $f \in B^s_{p,q}(\mathbb{T}; X)$ there exist a unique $u \in B^s_{p,q}(\mathbb{T}; [D(A)]) \cap B^{2+s}_{p,q}(\mathbb{T}; X)$ such that $u''(t) + Au(t) = f(t)$ for a.e.
(ii) $k^2 \in \rho(A)$ for all $k \in \mathbb{Z}$ and $\sup_{k \in \mathbb{Z}} ||k^2(k^2 - A)^{-1}|| < \infty$.

2.6.3 Integro-Differential Equations in Banach spaces

The operator-valued Fourier multiplier Theorems 2.22 and 2.24 have been used by Keyantuo and Lizama in [33] to establish maximal regularity results for an integro-differential equation with infinite delay in Banach spaces. The authors consider the following problem

$$\begin{aligned} u'(t) &= Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds \ + \ f(t) \qquad (2.10)\\ u(0) &= u(2\pi)\,, \end{aligned}$$

and examine this equation in various spaces of $2\pi-$periodic vector-valued functions: $L^p_{2\pi}(\mathbb{R};X)$, $C^\alpha(\mathbb{T};X)$, $B^s_{p,q}(\mathbb{T};X)$.
Suppose the kernel $a \in L^1_{loc}(\mathbb{R}_+)$ is such that $\tilde{a}(ik)$ exist for all $k \in \mathbb{Z}$, where $\tilde{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)\,dt$ denotes the Laplace transform of a.

Proposition 2.37 *[33, Proposition 2.8] Let A be a closed linear operator defined on the UMD space X. Let $\{d_k\}_{k\in\mathbb{Z}}$ be a 1-regular sequence such that $\{d_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent*
(i) $\{d_k\,(d_kI-A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier, $1<p<\infty$.
(ii) $\{d_k\,(d_kI-A)^{-1}\}_{k\in\mathbb{Z}}$ is $R-$bounded.

We adopt throughout the notations:

$$\begin{aligned} \tilde{c}_k &= \tilde{a}(ik) \qquad (2.11)\\ b_k &= \frac{ik}{1+\tilde{c}_k}\,, \quad \text{for all} \ \ k \in \mathbb{Z}\,, \qquad (2.12) \end{aligned}$$

and the following condition

$$(\mathbf{H1}) \quad \{\tilde{c}_k\}\,, \ \ \{k(\tilde{c}_{k+1}-\tilde{c}_k)\}\,, \quad \text{and} \quad \{1/(\tilde{c}_k+1)\} \ \text{are bounded sequences.} \qquad (2.13)$$

Denote by $H^{a,p}_{per}$ the space of all $u \in L^p_{2\pi}(\mathbb{R};X)$ for which there exists $v \in L^p_{2\pi}(\mathbb{R};X)$ such that $\hat{v}(k) = b_k\,\hat{u}(k)$ for all $k \in \mathbb{Z}$. A function $u \in H^{a,p}_{per}$ is called a strong L^p-solution of (2.10) if $u(t) \in D(A)$ and equation (2.10) holds for almost all $t \in [0,2\pi]$.
The following characterization of Keyantuo and Lizama establishes well-posedness for (2.10), extending the results of Arendt and Bu.

Theorem 2.38 *[33, Theorem 2.12] Let X be a UMD space and let $A : D(A) \subset X \to X$ be a closed linear operator. Assume that the sequence $\{\tilde{c}_k\}$ satisfies* **(H1)**. *Then the following assertions are equivalent for $1<p<\infty$.*

(*i*) *For all $f \in L^p_{2\pi}(\mathbb{R},X)$, there exists a unique strong L^p-solution of (2.10).*

(*ii*) $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$ *and* $\{b_k\,(b_k - A)^{-1}\}_{k\in\mathbb{Z}}$ *is an* L^p-*multiplier.*

(*iii*) $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$ *and* $\{b_k\,(b_k - A)^{-1}\}_{k\in\mathbb{Z}}$ *is* $R-$*bounded.*

For Besov spaces, they obtain the following fundamental result.

Proposition 2.39 *Let A be a closed linear operator defined on the Banach space X. Let $\{d_k\}_{k\in\mathbb{Z}}$ be a 2-regular sequence such that $\{d_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent*

(*i*) $\{d_k\,(d_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ *is a* $B^s_{p,q}-$*multiplier,* $1 \le p \le \infty$, $s \in \mathbb{R}$.

(*ii*) $\{d_k\,(d_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ *is bounded.*

Now, for $1 \le p \le \infty$ and $s > 0$, a function $u \in B^{1+s}_{p,q}(\mathbb{T};X)$ is called a strong $B^s_{p,q}-$solution of (2.10) if $u(t) \in D(A)$ and (2.10) holds for almost all $t \in [0,2\pi]$. The authors in [33], introduce the following condition

$$(\mathbf{H2}) \quad \{k\tilde{c}_k\},\ \{k^2(\tilde{c}_{k+1} - 2\tilde{c}_k + \tilde{c}_{k-1})\} \text{ are bounded sequences}. \tag{2.14}$$

Theorem 2.40 *Let $1 \le p \le \infty$ and $s > 0$. Let A be a closed linear operator on the Banach space X. Assume that $\{\tilde{c}_k\}$ satisfies* **(H2)** *and $\{b_k\}$ is 2-regular. Then the following assertions are equivalent*

(*i*) *For all $f \in B^s_{p,q}(\mathbb{T};X)$, there exists a unique strong $B^s_{p,q}-$solution of (2.10) such that u', Au and $a * Au \in B^s_{p,q}(\mathbb{T};X)$.*

(*ii*) $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} ||b_k\,(b_k - A)^{-1}|| < \infty$.

As a consequence the results in this thesis, we will see in Chapter 3, Section 3.4, that conditions **(H1)** and **(H2)** can be improved.

Chapter 3

Solutions of Second Order Integro-differential Equations on Periodic Besov Spaces.

3.1 Introduction

We consider the following integro-differential equation with infinite delay

$$\begin{cases} u''(t) + \alpha u'(t) = Au(t) + \displaystyle\int_{-\infty}^{t} c(t-s)Au(s)ds + f(t)\,, \quad 0 \le t \le 2\pi \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi), \end{cases} \tag{3.1}$$

where A is a closed linear operator defined on a Banach space X , $c \in L^1(\mathbb{R}_+)$ is a scalar-valued kernel, f is an $X-$valued function defined on $[0, 2\pi]$ and α is a real number.
We will study existence and uniqueness of solutions for (3.1) in the space of $2\pi-$periodic vector-valued functions $B^s_{pq}(\mathbb{T}; X)$.
We are able to obtain a very simple characterization of maximal regularity for (3.1) only in terms of the boundedness of $\{d_k(b_k - A)^{-1}\}_{k\in\mathbb{Z}}$ where $d_k = \frac{-k^2}{1+\tilde{c}(ik)}$, $b_k = \frac{\alpha ik-k^2}{1+\tilde{c}(ik)}$ and $\tilde{c}$ denotes the Laplace transform of c . We remark that the conditions that we impose on the kernel c are satisfied by a large class of functions appearing in the applications.
We also study a resonance case: we assume that there are $k_1, \dots, k_N \in \mathbb{Z}$ such that ik_j is a simple pole of $F(\lambda) = (\lambda^2 + \alpha\lambda - (1+\tilde{c}(\lambda))A)^{-1}$ for $j = 1, \dots, N$. In this case, we will show that equation (3.1) has a $B^s_{p,q}$-solution strong if and only if f satisfies suitable compatibility conditions (Theorem 3.23).
We remark that a similar case was studied in [33] for the first order integro differential equations for a general linear unbounded operator A. However, in [33] the resonance case was not considered. Our results extends those in [10, Theorem 5.3] where the case $\alpha = 0$ and $c \equiv 0$ was presented.

3.2 Maximal regularity on $B^s_{pq}(\mathbb{T};X)$

We denote by $\tilde{c}$ the Laplace transform of $c \in L^1(\mathbb{R}_+)$, $\tilde{c}(ik)$ exists for all k.

We adopt throughout the following notations

$$d_k = \frac{-k^2}{1+\tilde{c}_k}\,, \text{ for all } k \in \mathbb{Z} \tag{3.2}$$

$$b_k = \frac{\alpha i k - k^2}{1+\tilde{c}_k}\,, \text{ for all } k \in \mathbb{Z} \tag{3.3}$$

where $\tilde{c}_k := \tilde{c}(ik)$.

Remark 3.1

Note that by the Riemann Lebesgue lemma and the assumption that $\tilde{c}(ik) \neq -1$ exists for all $k \in \mathbb{Z}$ the sequences $\{\tilde{c}(ik)\}$ and $\{\frac{1}{1+\tilde{c}(ik)}\}$ are bounded.

Proposition 3.2 *If $\{\tilde{c}_k\}_{k\in\mathbb{Z}}$ satisfies a Marcinkiewicz estimate of order 2, then $\{\frac{1}{d_k}\}$ and $\{b_k\}$ defined by (3.2) and (3.3) verify the following:*

$$\{k\frac{1}{d_k}(b_{k+1}-b_k)\}_{k\in\mathbb{Z}\setminus\{0\}} \textit{ and } \{k^2\frac{1}{d_k}(b_{k+1}-2b_k+b_{k-1})\}_{k\in\mathbb{Z}\setminus\{0\}} \textit{ are bounded.}$$

Proof. We have the identities

$$k\,\frac{1}{d_k}\,(b_{k+1}-b_k) = k\,\frac{1+\tilde{c}_k}{-k^2}\left[\frac{-(k+1)^2+\alpha i(k+1)}{1+\tilde{c}_{k+1}} - \frac{-k^2+\alpha i k}{1+\tilde{c}_k}\right]$$

$$= \frac{-1}{1+\tilde{c}_{k+1}}k\,(\tilde{c}_{k+1}-\tilde{c}_k) - \frac{\alpha i}{1+\tilde{c}_{k+1}}(\tilde{c}_k-\tilde{c}_{k+1}) + \frac{-2k-1+\alpha i}{k}\,\frac{1+\tilde{c}_k}{1+\tilde{c}_{k+1}}.$$

By assumption and Remark 3.1 we obtain the first condition. In order to prove the second condition, we note the identities

$$k^2\,\frac{1}{d_k}\,(b_{k+1}-2b_k+b_{k-1})$$

$$= k^2\,\frac{1+\tilde{c}_k}{-k^2}\left[\frac{-(k+1)^2+\alpha i(k+1)}{1+\tilde{c}_{k+1}} - 2\frac{-k^2+\alpha ik}{1+\tilde{c}_k} + \frac{-(k-1)^2+\alpha i(k-1)}{1+\tilde{c}_{k-1}}\right]$$

$$= \frac{-1}{(1+\tilde{c}_{k+1})\,(1+\tilde{c}_{k-1})}\left[(1+\tilde{c}_{k+1})\,k^2\,(\tilde{c}_{k-1}-2\tilde{c}_k+\tilde{c}_{k+1})\right.$$

$$-\;k\,(\tilde{c}_{k+1}-\tilde{c}_{k-1})\;k\,(\tilde{c}_{k+1}-\tilde{c}_k) +\; 2(1+\tilde{c}_k)\,k\,(\tilde{c}_{k+1}-\tilde{c}_{k-1})$$

$$+\alpha i(1+\tilde{c}_{k-1})\,k\,(\tilde{c}_k-\tilde{c}_{k+1}) \;+\; \alpha i(1+\tilde{c}_{k+1})\,k\,(\tilde{c}_k-\tilde{c}_{k-1})$$

$$\left.+\;\;\alpha i(1+\tilde{c}_k)\,(\tilde{c}_{k-1}-\tilde{c}_{k+1}) \;-\; (1+\tilde{c}_{k-1})\,(1+\tilde{c}_k) - (1+\tilde{c}_{k+1})\,(1+\tilde{c}_k)\right].$$

Hence by assumption and Remark 3.1 we obtain the desired conclusion.

■

Proposition 3.3 *Let A be a closed linear operator defined on the Banach space X. Let $\{d_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ be defined by (3.2), (3.3) respectively. Assume that $\{\tilde{c}_k\}_{k\in\mathbb{Z}}$ satisfies a Marcinkiewicz estimate of order 2.*
If $b_k\in\rho(A)$ for all $k\in\mathbb{Z}$ and $\{d_k(b_k-A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded then $\{d_k(b_k-A)^{-1}\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}-$ multiplier, $1\le p\le\infty$.

Proof.
Denote by (M_k) the sequence $d_k\,(b_kI-A)^{-1}$. Note that M_0 is the null operator.

We will verify that the sequence $\{M_k\}$ satisfies a Marcinkiewicz estimate of order 2. Then, the result follows from Theorem 2.22. In fact, first we prove (2.6). We have the identity

$$k[M_{k+1}-M_k] \;=\; M_{k+1}\,k\,\frac{1}{d_{k+1}}\,[b_k-b_{k+1}]\,M_k \;+\; M_{k+1}\,k\,[1-\frac{d_k}{d_{k+1}}].$$

Note that $\dfrac{d_k}{d_{k+1}} = \dfrac{1+\tilde{c}_{k+1}}{1+\tilde{c}_k}\,(k/(k+1))^2$, hence for each $k\in\mathbb{Z}\setminus\{-1\}$ we have that

$k\left[1-\dfrac{d_k}{d_{k+1}}\right] = \left[\dfrac{2k^2+k}{(k+1)^2} + \dfrac{k^2}{(k+1)^2}\,\dfrac{1}{1+\tilde{c}_k}\,k\,(\tilde{c}_k-\tilde{c}_{k+1})\right]$ is bounded since $\{\tilde{c}_k\}$ verifies a Marcinkiewicz estimate of order 2.
Moreover, for all $k\in\mathbb{Z}\setminus\{-1\}$, by Proposition 3.2 we have that $\{k\,\dfrac{1}{d_{k+1}}\,(b_k-b_{k+1})\}$ is bounded. This, together with the boundedness of $\{M_k\}$ imply that

$$\sup_{k\in\mathbb{Z}}||k\,(M_{k+1}-M_k)||<\infty\;.$$

In order to verify the condition (2.7), with an analogous calculation as above we obtain

$$
\begin{aligned}
k^2\,(M_{k+1}-2M_k+M_{k-1}) \;&= k^2\,\frac{d_{k+1}-2d_k+d_{k-1}}{d_{k+1}}\;\; M_{k+1}\\
&-2\;\;k\left[1-\frac{d_{k-1}}{d_k}\right]\;k\,d_{k-1}\,(b_{k+1}-b_k)\;M_k\;M_{k-1}\\
&-\;\;k^2\frac{1}{d_k}(b_{k+1}-2b_k+b_{k-1})\;M_k\;M_{k-1}\\
&+2\;k\,\frac{1}{d_{k+1}}(b_{k+1}-b_k)\,k\,\frac{1}{d_{k-1}}(b_{k+1}-b_{k-1})\,M_{k+1}\,M_k\,M_{k-1}\\
&-k\,\frac{1}{d_k}(b_{k+1}-b_k)\;k\,\frac{1}{d_{k+1}}(b_{k+1}-b_{k-1})\;M_{k+1}\;M_k\;M_{k-1}\,,
\end{aligned}
$$

where, with a direct calculation, we have that

$$
\begin{aligned}
&k^2\,\frac{d_{k+1}-2d_k+d_{k-1}}{d_{k+1}} = \frac{k^2}{(k+1)^2(1+\tilde c_k)\,(1+\tilde c_{k-1})}\,[-(1+\tilde c_{k+1})\,k^2\,(\tilde c_{k+1}-2\tilde c_k+\tilde c_{k-1})\\
&+\,k\,(\tilde c_{k-1}-\tilde c_{k+1})\;k\,(\tilde c_k-\tilde c_{k+1})\;+\;2\,(1+\tilde c_k)\,k\,(\tilde c_{k-1}-\tilde c_{k+1})\\
&+\,(1+\tilde c_{k-1})\,(1+\tilde c_k)+(1+\tilde c_{k+1})\,(1+\tilde c_k)\,].
\end{aligned}
$$

Since $\{\tilde c_k\}$ verifies a Marcinkiewicz estimate of order 2, we conclude that the sequence $\{k^2\,\frac{d_{k+1}-2d_k+d_{k-1}}{d_{k+1}}\}$ is bounded for all $k\in\mathbb{Z}\setminus\{-1\}$. Hence, by Proposition 3.2 together with the boundedness of $\{M_k\}$, we obtain that $k^2\,(M_{k+1}-2M_k+M_{k-1})$ is bounded for all $k\in\mathbb{Z}\setminus\{-1,0,1\}$. Finally, since M_{-2}, M_2, M_{-1}, M_1 are well defined operators we obtain the claim. ∎

Lemma 3.4 *Let X be a Banach space, assume that the sequence $\{\tilde c_k\}_{k\in\mathbb{Z}}$ satisfies a Marcinkiewicz estimate of order 2. Then the sequences $\{(1+\tilde c_k)I\}_{k\in\mathbb{Z}}$ and $\{\frac{1}{1+\tilde c_k}I\}$ are $B^s_{p,q}-$multipliers.*

Proof. It is clear, directly from Marcinkiewicz estimates of order 1 and Theorem 2.22, that the sequence $\{(1+\tilde c_k)I\}$ is an $B^s_{p,q}-$multiplier.

Now, let $n_k := \frac{1}{1+\tilde c_k}$. The sequence $\{n_k\}$ is bounded and satisfies the identities

$$
k(n_{k+1}-n_k) = k[\tilde c_k-\tilde c_{k+1}]\;\frac{1}{1+\tilde c_k}\;\frac{1}{1+\tilde c_{k+1}}
$$

and

$$k^2(n_{k+1} - 2n_k + n_{k-1}) = \frac{-1}{(1+\tilde{c}_k)} \frac{1}{(1+\tilde{c}_{k-1})} k^2[\tilde{c}_{k+1} - 2\tilde{c}_k + \tilde{c}_{k-1}]$$
$$+ \frac{1}{1+\tilde{c}_{k+1}} \frac{1}{1+\tilde{c}_k} \frac{1}{1+\tilde{c}_{k-1}} k[\tilde{c}_{k+1} - \tilde{c}_{k-1}]\, k[\tilde{c}_{k+1} - \tilde{c}_k].$$

Hence the sequence verifies a Marcinkiewicz estimates of order 2, by Theorem 2.22 the lemma follows.

■

Definition 3.5 *Let* $1 \leq p, q \leq \infty$ *and* $s > 0$. *A function* $u \in B_{p,q}^{s+2}(\mathbb{T}; X)$, *is called a strong* $B_{p,q}^s$*−solution of (3.1) if* $u(t) \in D(A)$ *and (3.1) holds for almost every* $t \in [0, 2\pi]$.

We have the following result

Theorem 3.6 *Let* $1 \leq p, q \leq \infty$ *and* $s > 0$. *Let* A *be a closed linear operator defined on a Banach space* X. *If* $\{\tilde{c}_k\}_{k\in\mathbb{Z}}$ *satisfies a Marcinkiewicz estimate of order 2, then the following assertions are equivalent*

(i) $\{\frac{\alpha ik - k^2}{1+\tilde{c}_k}\}_{k\in\mathbb{Z}} \subset \rho(A)$ *and* $\sup_k \left\| \frac{-k^2}{1+\tilde{c}_k} \left(\frac{-k^2 + \alpha ik}{1+\tilde{c}_k} - A \right)^{-1} \right\| < \infty$.

(ii) For every $f \in B_{p,q}^s(\mathbb{T}; X)$, *there exist a unique strong* $B_{p,q}^s$*−solution of (3.1) such that* u'', u', $Au \in B_{p,q}^s(\mathbb{T}; X)$.

Proof. (ii) ⇒ (i) Let $x \in X$ be fixed. Define $f = e_k \otimes x$. Note that $f \in B_{p,q}^s(\mathbb{T}; X)$. Hence there exists $u \in B_{p,q}^{s+2}(\mathbb{T}; X)$ such that $u(t) \in D(A)$ and (3.1) holds for almost every $t \in [0, 2\pi]$.
Taking Fourier transforms on both sides we obtain that $\hat{u}(k) \in D(A)$ and

$$-k^2\hat{u}(k) + \alpha ik\hat{u}(k) = A\hat{u}(k) + \tilde{c}_k A\hat{u}(k) + \hat{f}(k).$$

Thus, $(-k^2 + \alpha ik - A - \tilde{c}_k A)\hat{u}(k) = \hat{f}(k) = x$ proving that $-k^2 + \alpha ik - A - \tilde{c}_k A$ is surjective.
Let $x \in D(A)$. If $(-k^2 + \alpha ik - A - \tilde{c}_k A)x = 0$, that is $Ax = \dfrac{-k^2 + \alpha ik}{1+\tilde{c}_k} Ix$, then $u(t) = e^{ikt}x$ define a periodic solution of

$$u''(t) + \alpha u'(t) = Au(t) + \int_{-\infty}^{t} c(t-s)Au(s)ds.$$

Hence $u = 0$ by the assumption of uniqueness and thus $x = 0$. Since A is closed, by Closed Graph Theorem we conclude that $\frac{\alpha ik - k^2}{1+\tilde{c}_k} \subset \rho(A)$, for all $k \in \mathbb{Z}$.

Next we claim that $\frac{-k^2}{1+\tilde{c}_k}\left(\frac{-k^2+\alpha ik}{1+\tilde{c}_k}-A\right)^{-1}$ is a $B^s_{p,q}-$ multiplier. Let $f\in B^s_{p,q}(\mathbb{T};X)$. By hypothesis, there exists a unique $u\in B^{s+2}_{p,q}(\mathbb{T};X)$ such that

$$u''(t)+\alpha u'(t)=Au(t)+\int_{-\infty}^{t}c(t-s)Au(s)ds+f(t).$$

Taking Fourier series on both sides we obtain that $\hat{u}(k)\in D(A)$ and

$$\hat{u}(k)=(-k^2+\alpha ik-(1+\tilde{c}_k)A)^{-1}\hat{f}(k)$$

or

$$-k^2\hat{u}(k)=-\frac{k^2}{1+\tilde{c}_k}\left(\frac{-k^2+\alpha ik}{1+\tilde{c}_k}-A\right)^{-1}\hat{f}(k).$$

By [10, Theorem 1.3], if $u\in B^{s+2}_{p,q}(\mathbb{T};X)$ then u' is differentiable almost everywhere and $u''\in B^s_{p,q}(\mathbb{T};X)$. Define $v=u''$. Then we obtain that

$$\hat{v}(k)=-\frac{k^2}{1+\tilde{c}_k}\left(\frac{-k^2+\alpha ik}{1+\tilde{c}_k}-A\right)^{-1}\hat{f}(k)\,,$$

proving the claim. It follows from the Closed Graph Theorem that there exist $C>0$ such that, for $f\in B^s_{p,q}(\mathbb{T};X)$, we have

$$\left\|\sum_{k\in\mathbb{Z}}e_kM_k\hat{f}(k)\right\|_{B^s_{p,q}}\leq C\,||f||_{B^s_{p,q}}.$$

Let $x\in X$ and defines $f(t)=e_n(t)x=e^{int}x$ for $n\in\mathbb{Z}$ fixed. Then the above inequality implies that $||e_n||_{B^s_{p,q}}||M_nx||=||e_nM_nx||\leq C||e_n||_{B^s_{p,q}}||x||$. Hence $||M_n||\leq C$.

(i) $\Rightarrow$ (ii) Let $M_k=-\frac{k^2}{1+\tilde{c}_k}\left(\frac{-k^2+\alpha ik}{1+\tilde{c}_k}-A\right)^{-1}$. By assumption we have that $\{M_k\}_{k\in\mathbb{Z}}$ is a bounded sequence. We define $N_k=\frac{1}{1+\tilde{c}_k}\left(\frac{-k^2+\alpha ik}{1+\tilde{c}_k}-A\right)^{-1}$.
First, we claim that the families $\{ik\,N_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are $B^s_{pq}-$multipliers. In order to see that, we will apply Theorem 2.22.

In fact, in order to verify condition (2.6), observe that $||ik\,N_k||\leq||k^2\,N_k||=||M_k||$ for all $k\in\mathbb{Z}$ and hence $\sup_{k\in\mathbb{Z}}||ik\,N_k||<\infty$.
Moreover we have the identity

$$k[\,(k+1)N_{k+1}-kN_k\,]\;=-M_{k+1}+M_k-(k+1)N_{k+1},$$

and hence condition (2.6) holds since $\{M_k\}$ is bounded.
To verify the condition (2.7), note that

$k^2[\,(k+1)N_{k+1} - 2kN_k + (k-1)N_{k-1}\,]$

$$= k[M_k - M_{k+1}] + k[M_k - M_{k-1}] - k[(k+1)N_{k+1} - kN_k] + k[(k-1)N_{k-1} - kN_k].$$

Since $\sup_{k\in\mathbb{Z}} ||M_k|| < \infty$, from the proof of Proposition 3.3, we see that the sequence $\{M_k\}$ satisfies the condition (2.6) of multiplier. Using this in the above identity, we conclude that the condition (2.7) hold for $\{ikN_k\}$. We claim is proved.
Secondly, we will prove that $\{N_k\}$ is a $B^s_{pq}-$multiplier. In fact, to verify condition (2.6) observe that $||N_k|| \leq ||k^2 N_k|| = ||M_k||$ for all $k \in \mathbb{Z}\setminus\{0\}$ hence $\sup_{k\in\mathbb{Z}} ||N_k|| < \infty$. Moreover we have

$$k[\,N_{k+1} - N_k\,] \;\; = (k+1)N_{k+1} + N_k - N_{k+1},$$

and since $\{kN_k\}$ and $\{N_k\}$ are bounded sequences we obtain condition (2.6).
In order to verify condition (2.7), note that

$k^2\,[\,N_{k+1} - 2N_k + N_{k-1}\,]$

$$= -M_{k+1} + 2M_k - M_{k-1} - (k+1)N_{k+1} + (k-1)N_{k-1} + N_{k-1} - N_{k+1},$$

and since $\{M_k\}$, $\{kN_k\}$ and $\{N_k\}$ are bounded sequences we obtain condition (2.7) and the claim follows.
Now, let $f \in B^s_{p,q}(\mathbb{T};X)$. Since $\{N_k\}$ is $B^s_{pq}-$multiplier, there exist $u \in B^s_{p,q}(\mathbb{T};X)$ such that

$$\hat{u}(k) = N_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}, \tag{3.4}$$

where we observe that $\hat{u}(k) \in D(A)$.
Since $\{ikN_k\}$ is a $B^s_{pq}-$multiplier there exists $v \in B^s_{p,q}(\mathbb{T};X)$ such that $\hat{v}(k) = ikN_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. From (3.4) we obtain that

$$ik\hat{u}(k) = \hat{v}(k). \tag{3.5}$$

By Lemma 2.1 of [9], u is differentiable a.e. with $u' = v$ and $u(0) = u(2\pi)$. By [10, Theorem 2.3] this implies that $u \in B^{s+1}_{p,q}(\mathbb{T};X)$.
By Proposition 3.3, we have that $\{M_k\}$ is a $B^s_{pq}-$multiplier, hence there exists $w \in B^s_{p,q}(\mathbb{T};X)$ such that $\hat{w}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. Using again equality (3.4) and equality (3.5) we have

$$-k^2\hat{u}(k) = ik\hat{v}(k) = \hat{w}(k).$$

By [9, Section 6] u' is differentiable a.e. with $w = u''$, $u'(0) = u'(2\pi)$ and $w = v' = u''$. By [10, Theorem 2.3] this implies that $u \in B^{s+2}_{p,q}(\mathbb{T};X)$.

We will show that $u(t) \in D(A)$. By (3.4), we have the identity

$$(-k^2 + \alpha ik - (1+\tilde{c}_k)A)\;\hat{u}(k) = \hat{f}(k) \tag{3.6}$$

for all $k \in \mathbb{Z}$, or equivalently

$$\begin{aligned} A\hat{u}(k) &= \frac{-k^2 + \alpha i k}{1+\tilde{c}_k}\,\hat{u}(k) - \frac{1}{1+\tilde{c}_k}\hat{f}(k) \\ &= \frac{1}{1+\tilde{c}_k}\,\hat{w}(k) + \frac{\alpha}{1+\tilde{c}_k}\hat{v}(k) - \frac{1}{1+\tilde{c}_k}\hat{f}(k)\,. \end{aligned} \tag{3.7}$$

Since $f, v, w \in B^s_{p,q}(\mathbb{T}; X)$ and by Lemma 3.4, the family $\{\frac{1}{1+\tilde{c}_k} I\}$ is a $B^s_{p,q}-$multiplier, there exists $g \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$A\hat{u}(k) = \hat{g}(k).$$

Then Lemma 3.1 of [9] implies that $u(t) \in D(A)$ and $Au(t) = g(t)$. Hence $Au \in B^s_{p,q}(\mathbb{T}; X)$.
Finally from (3.6), we have

$$(-k^2 + \alpha i k)\,\hat{u}(k) = A\hat{u}(k) + A\,\tilde{c}_k\hat{u}(k) + \hat{f}(k).$$

Define $h(t) = u''(t) + \alpha u'(t) - f(t)$. It is clear that $h \in B^s_{p,q}(\mathbb{T}; X)$. From the above equality, we obtain

$$\hat{h}(k) = A\hat{s}(k),$$

where $s(t) = u(t) + \int_{-\infty}^{t} c(t-s)u(s)ds$ and $s \in B^{s+2}_{p,q}(\mathbb{T}; X)$. From Lemma 3.1 of [9] we have $s(t) \in D(A)$, and then $\int_{-\infty}^{t} c(t-s)u(s)ds \in D(A)$. Since A is closed, we deduce that

$$u''(t) + \alpha u'(t) = Au(t) + \int_{-\infty}^{t} c(t-s)Au(s)ds + f(t)\,.$$

It remains to show uniqueness. Let $u \in B^s_{p,q}(\mathbb{T}; X)$ be such that

$$u''(t) + \alpha u'(t) - Au(t) - \int_{-\infty}^{t} c(t-s)Au(s)ds = 0.$$

Then $\hat{u}(k) \in D(A)$ and $[-k^2 + \alpha i k - (1+\tilde{c}_k)A]\hat{u}(k) = 0$. Since $\frac{-k^2+\alpha i k}{1+\tilde{c}_k} \in \rho(A)$, this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$.

∎

In the case where $p = q = \infty$ and $0 < s < 1$ we have that $B^s_{\infty,\infty}(\mathbb{T}; X)$ corresponds to the space $C^s(\mathbb{T}; X)$ of Hölder continuous functions. We state the corresponding result separately:

Corollary 3.7 *Let $0 < s < 1$. Let A be a closed linear operator defined on a Banach space X. Assume that $\{\tilde{c}_k\}_{k\in\mathbb{Z}}$ satisfies Marcinkiewicz estimates of order 2. The following assertions are equivalent*

(i) $\{\frac{\alpha ik - k^2}{1+\tilde{c}_k}\}_{k\in\mathbb{Z}} \subset \rho(A)$ *and* $\sup_k \left\| \frac{-k^2}{1+\tilde{c}_k}\left(\frac{\alpha ik - k^2}{1+\tilde{c}_k} - A\right)^{-1} \right\| < \infty$.

(ii) For every $f \in C^s(\mathbb{T}; X)$, there exist a unique strong C^s-solution of (3.1) such that u'', u', $Au \in C^s(\mathbb{T}; X)$.

Remark 3.8

Setting $\alpha = 0$ and $c = 0$ in the equation (3.1) we obtain the second order problem with periodic boundary conditions

$$\begin{cases} u''(t) = Au(t) + f(t) \quad 0 \le t \le 2\pi \\ u(0) = u(2\pi) \\ u'(0) = u'(2\pi) \end{cases} \tag{3.8}$$

and we may apply Theorem 3.6 to obtain a necessary and sufficient condition in order to such problem have maximal regularity in Besov spaces. In [9] Arendt and Bu studied the problem (3.8) for A a closed linear operator defined on UMD space X. They established conditions for maximal regularity in $L^p_{2\pi}(\mathbb{R}; X)$ in terms of R-boundedness of resolvents. In [10], the authors have obtained maximal regularity for (3.8) in periodic vector-valued Besov spaces as we have considered here.

3.3 The resonant case

We define

$$\rho_{d,e}(A) = \{\lambda \in \mathbb{C} \,:\, d(\lambda)I - e(\lambda)A \quad \text{is invertible and } (d(\lambda) - e(\lambda)A)^{-1} \in \mathcal{B}(X, [D(A)])\,\}$$

In what follows we will assume that $d(ik)$ and $e(ik)$ exist for all $k \in \mathbb{Z}$. We suppose that $\lambda \to d(\lambda)$ (resp. $e(\lambda)$) admits an analytical extension to a sector containing the imaginary axis, and still denote this extension by d (resp. e).
Denote by $\sigma_{d,e}(A)$ the set $\mathbb{C} \setminus \rho_{d,e}(A)$.
Now, we consider a resonant case: We assume that there are $k_1, \ldots, k_N \in \mathbb{Z}$ such that

$$\begin{cases} (i) & ik_j \in \sigma_{d,e}(A) & \text{for } j = 1, \ldots, N; \\ (ii) & ik \notin \sigma_{d,e}(A) & \text{for } k \in \mathbb{Z}, k \neq k_1, \ldots, k_N \\ (iii) & ik_j & \text{is a simple pole of } F(\cdot) \text{ for } j = 1, \ldots, N \end{cases} \tag{3.9}$$

where $F : \rho_{d,e}(A) \subset \mathbb{C} \to \mathcal{B}(X, [D(A)])$ is defined by $F(\lambda) = (d(\lambda)I - e(\lambda)A)^{-1}$.

We now give some preliminary results about the solvability of the equation

$$(d(\lambda_0)I - e(\lambda_0)A)x = y \tag{3.10}$$

where λ_0 is a simple pole of $F(\cdot)$.
We denote by Q the residue of $F(\cdot)$ at λ_0, that is,

$$Q = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)\, F(\lambda) \; = \; \frac{1}{2\pi i} \int_{B(\lambda_0, \varepsilon)} F(\lambda)\, d\lambda \tag{3.11}$$

where $\varepsilon > 0$ and $\Gamma(\lambda_0, \varepsilon) := \{\lambda \in \mathbb{C} \; : \; |\lambda - \lambda_0| < \varepsilon\}$.
We define

$$G(\lambda) = \begin{cases} (\lambda - \lambda_0)\, F(\lambda)\,, & 0 < |\lambda - \lambda_0| < \varepsilon \\ \\ Q\,, & \lambda = \lambda_0. \end{cases} \tag{3.12}$$

We note that $Q \in \mathcal{B}(X, [D(A)])$ is a non-zero operator which satisfies the following property.

Lemma 3.9 *With the notations as above, we have*

$$Q \; = \; Q\,[d'(\lambda_0)I - e'(\lambda_0)A]\,Q.$$

Proof.
For λ, μ belonging to $\Gamma(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ with $|\lambda - \lambda_0| > |\mu - \lambda_0|$ we have

$$\begin{aligned} F(\lambda) - F(\mu) \; &= F(\lambda)\,[\, d(\mu)I - e(\mu)A - d(\lambda)I + e(\lambda)A\,]\, F(\mu) \\ &= F(\lambda)\; [\, (\, d(\mu) - d(\lambda)\,)I + (\, e(\lambda) - e(\mu)\,)A\,]\; F(\mu). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{F(\lambda) - F(\mu)}{\lambda - \mu}(\lambda - \lambda_0)(\mu - \lambda_0) \\ &= (\lambda - \lambda_0)F(\lambda)\; \left[\frac{d(\mu) - d(\lambda)}{\lambda - \mu} I + \frac{e(\lambda) - e(\mu)}{\lambda - \mu} A \right]\; (\mu - \lambda_0)\, F(\mu) \end{aligned}$$

and using (3.12) we have

$$G(\lambda)\frac{\mu - \lambda_0}{\lambda - \mu} - G(\mu)\frac{\lambda - \lambda_0}{\lambda - \mu} \; = G(\lambda) \left[\frac{d(\mu) - d(\lambda)}{\lambda - \mu} I + \frac{e(\lambda) - e(\mu)}{\lambda - \mu} A \right]\; G(\mu).$$

Since $A \in \mathcal{B}([D(A)], X)$, letting $\mu \to \lambda_0$ we obtain

$$-Q \quad = G(\lambda)\left[\frac{d(\lambda_0)-d(\lambda)}{\lambda-\lambda_0}I+\frac{e(\lambda)-e(\lambda_0)}{\lambda-\lambda_0}A\right]Q.$$

Letting $\lambda \to \lambda_0$ we get

$$Q \quad = Q\,[\,d'(\lambda_0)I - e'(\lambda_0)A]\,Q.$$

This proves the Lemma.

■

The following result is the key for results on existence of solutions in the resonance case.

Proposition 3.10 *Let* λ_0 *be a simple pole of* $F(\cdot)$ *and let* $Q \in \mathcal{B}(X, [D(A)])$ *be defined by* (3.11) *. Then*

$$Ker(d(\lambda_0)I - e(\lambda_0)A) = Q(X). \tag{3.13}$$

Moreover, for any $y \in X$ *such that* $Qy = 0$ *, all solutions of* (3.10) *are given by*

$$x = G'(\lambda_0)y - Q\,A(e'G)'(\lambda_0)y + Q\,(d'G)'(\lambda_0)y. \tag{3.14}$$

Proof.
First we show (3.13). For any sufficiently small $\varepsilon > 0$ and $0 < |\lambda - \lambda_0| < \varepsilon$ we have

$$\begin{aligned}(d(\lambda_0)I - e(\lambda_0)A)G(\lambda) = \quad & (\lambda-\lambda_0) - (d(\lambda)I - e(\lambda)A)G(\lambda) + (d(\lambda_0)I - e(\lambda_0)A)G(\lambda)\\ = \quad & (\lambda-\lambda_0) + (d(\lambda_0) - d(\lambda))G(\lambda) + (e(\lambda) - e(\lambda_0))AG(\lambda).\end{aligned}$$

Since $A \in \mathcal{B}([D(A)], X)$, letting $\lambda \to \lambda_0$ we obtain $(\,d(\lambda_0)I - e(\lambda_0)A\,)\;Q \;=\; 0$, so that $Q(X)$ is contained in $Ker(\,d(\lambda_0)I - e(\lambda_0)A\,)$. Let now $x \in D(A)$ be such that $(\,d(\lambda_0)I - e(\lambda_0)A\,)\;x \;=\; 0$, then for $0 < |\lambda - \lambda_0| < \varepsilon$ with ε small, we have

$$F(\lambda)\;(\,d(\lambda_0)I - e(\lambda_0)A\,)\;x \;=\; 0. \tag{3.15}$$

For each $x \in X$, we have the identity $x - F(\lambda)\,(\,d(\lambda)I - e(\lambda)A\,)\;x \;=\; 0$, or equivalently

$$x + F(\lambda)\,[d(\lambda_0) - d(\lambda)]\,x + F(\lambda)[e(\lambda) - e(\lambda_0)]\,Ax - F(\lambda)\,[d(\lambda_0)I - e(\lambda_0)A]\,x \;=\; 0\,.$$

It follows from this and (3.15) that

$$x - (\lambda-\lambda_0)F(\lambda)\,\frac{d(\lambda)-d(\lambda_0)}{\lambda-\lambda_0}\,x + (\lambda-\lambda_0)F(\lambda)\frac{e(\lambda)-e(\lambda_0)}{\lambda-\lambda_0}\,Ax \;=\; 0\,,$$

that is, using (3.12)

$$x - G(\lambda)\,\frac{d(\lambda)-d(\lambda_0)}{\lambda-\lambda_0}\,x + G(\lambda)\frac{e(\lambda)-e(\lambda_0)}{\lambda-\lambda_0}\,Ax \;=\; 0.$$

Letting $\lambda \to \lambda_0$ we get

$$x - Q\,d'(\lambda_0)\,x + Q\,e'(\lambda_0)\,Ax \;=\; 0\,,$$

so that x belongs to $Q(X)$ proving (3.13).
Let us now show (3.14). First we claim that

$$\lim_{\lambda\to\lambda_0} F(\lambda)\,[I+(\,e'(\lambda_0)A-d'(\lambda_0)I\,)\,Q\,] \;=\; G'(\lambda_0)-Q\,A(e'G)'(\lambda_0)+Q\,(d'G)'(\lambda_0). \quad (3.16)$$

In fact, proceeding as in the proof of Lemma 3.9, we have

$$\begin{aligned}
G'(\lambda) \quad &= F(\lambda) - (\lambda-\lambda_0)\,F(\lambda)\,[\,d'(\lambda)I - e'(\lambda)A\,]\,F(\lambda)\\
&= F(\lambda) - (\lambda-\lambda_0)\,F(\lambda)\,d'(\lambda)F(\lambda) + (\lambda-\lambda_0)\,F(\lambda)\,e'(\lambda)A\,F(\lambda)\\
&= F(\lambda)\,[I+(\,e'(\lambda_0)A-d'(\lambda_0)I\,)Q\,] - F(\lambda)\,e'(\lambda_0)\,A\,Q + F(\lambda)\,d'(\lambda_0)\,Q\\
&- F(\lambda)\,d'(\lambda)(\lambda-\lambda_0)\,F(\lambda) + F(\lambda)\,e'(\lambda)A\,(\lambda-\lambda_0)\,F(\lambda)\\
&= F(\lambda)\;[\;I+(\,e'(\lambda_0)A-d'(\lambda_0)I\,)Q\;] \;+\; F(\lambda)\,A\,[\;e'(\lambda)\,G(\lambda)-e'(\lambda_0)\,Q\;]\\
&- F(\lambda)\,[\;d'(\lambda)\,G(\lambda)-d'(\lambda_0)Q\;]\\
&= F(\lambda)\;[\;I+(\,e'(\lambda_0)A-d'(\lambda_0)I\,)Q\;] \;+\; (\lambda-\lambda_0)\,F(\lambda)\,A\,[\frac{e'(\lambda)\,G(\lambda)-e'(\lambda_0)\,Q}{\lambda-\lambda_0}]\\
&- (\lambda-\lambda_0)\,F(\lambda)\,[\frac{d'(\lambda)\,G(\lambda)-d'(\lambda_0)Q}{\lambda-\lambda_0}].
\end{aligned}$$

Since $A \in \mathcal{B}([D(A)], X)$, letting $\lambda \to \lambda_0$ in the above identity, we obtain the claim.
On the other hand, using Lemma 3.9 we obtain

$$\begin{aligned}
&\lim_{\lambda\to\lambda_0}[d(\lambda_0)I - e(\lambda_0)A\,]\,F(\lambda)\,[\,I+(\,e'(\lambda_0)A-d'(\lambda_0)I\,)\,Q\,]\\
&= \lim_{\lambda\to\lambda_0}[d(\lambda)I - e(\lambda)A + e(\lambda)A - d(\lambda)I + d(\lambda_0)I - e(\lambda_0)A\,]\\
&F(\lambda)[\,I+(\,e'(\lambda_0)A-d'(\lambda_0)I\,)\,Q\,]
\end{aligned}$$

$$= \lim_{\lambda \to \lambda_0} [\, I + \{ \frac{e(\lambda) - e(\lambda_0)}{\lambda - \lambda_0} A - \frac{d(\lambda) - d(\lambda_0)}{\lambda - \lambda_0} I \,\}(\lambda - \lambda_0) F(\lambda) \,]$$

$$[I + (\, e'(\lambda_0)A - d'(\lambda_0)I \,)\, Q \,]$$

$$= [I + (e'(\lambda_0)A - d'(\lambda_0)I)Q] \; [I + (e'(\lambda_0)A - d'(\lambda_0)I)Q]$$

$$= I + 2(e'(\lambda_0)A - d'(\lambda_0)I)Q + (e'(\lambda_0)A - d'(\lambda_0)I)Q(e'(\lambda_0)A - d'(\lambda_0)I)Q$$

$$= I + 2(e'(\lambda_0)A - d'(\lambda_0)I)Q - (e'(\lambda_0)A - d'(\lambda_0)I)Q$$

$$= I + (e'(\lambda_0)A - d'(\lambda_0)I)Q.$$

Due to (3.16) and the fact that A belongs to $\mathcal{B}([D(A)], X)$ we have

$$[d(\lambda_0) - e(\lambda_0)A] \; [\, G'(\lambda_0) - Q\, A(e'G)'(\lambda_0) + Q\,(d'G)'(\lambda_0) \,] = I + (e'(\lambda_0)A - d'(\lambda_0)I)Q. \tag{3.17}$$

Therefore, if $y \in X$ is such that $Qy = 0$, equation (3.10) is solvable, and the solution is given by

$$w = \; G'(\lambda_0)y - Q\, A(e'G)'(\lambda_0)y + Q\,(d'G)'(\lambda_0)y.$$

■

Now, arguing as in the proof of Theorem 3.6, we find that, if $f \in B^s_{p,q}(\mathbb{T}; X)$, and $u \in B^{s+2}_{p,q}(\mathbb{T}; X)$ is a strong $B^s_{p,q}$–solution of (3.1), then

$$(-k^2 + \alpha i k - (1 + \tilde{c}_k)A) \; \hat{u}(k) = \hat{f}(k) \,, \quad k \in \mathbb{Z} \,. \tag{3.18}$$

We suppose that $\lambda \to \tilde{c}(\lambda)$ admits an analytical extension to a sector containing the imaginary axis, and still denote this extension by $\tilde{c}$.
Substituting $d(\lambda) := \lambda^2 + \alpha\lambda$ and $e(\lambda) := 1 + \tilde{c}(\lambda)$, we have that

$$F(\lambda) = (\lambda^2 + \alpha\lambda - (1 + \tilde{c}(\lambda))A)^{-1} \;\text{ for all } \lambda \in \rho_{d,e}(A).$$

Now, we assume that there are $k_1, \ldots, k_N \in \mathbb{Z}$ such that 3.9 hold.
For each $k \neq k_n$, $n = 1, \ldots, N$, equation (3.18) can be uniquely solved, with

$$\hat{u}(k) = (-k^2 + \alpha i k - (1 + \tilde{c}_k)A)^{-1} \hat{f}(k).$$

For k_n, $n = 1, \ldots, N$, by Proposition 3.10 equation (3.18) is solvable if and only if

$$Q_n \, \hat{f}(k_n) = 0, \tag{3.19}$$

where Q_n is the residue of $F(\cdot)$ at $\lambda = i k_n$. If (3.19) holds, then by (3.14), the Fourier coefficients of the solution to (3.18) in k_n, $n = 1, \ldots, N$ are given by

$$\hat{u}(k_n) = [G_n'(ik_n) \, - Q_n \, A(\tilde{c}' G_n)'(ik_n) + Q_n \, (d' G_n)'(ik_n)] \, \hat{f}(k_n), \tag{3.20}$$

where $G_n : B(ik_n, \varepsilon) \to \mathcal{B}(X, [D(A)])$ is the analytic function defined by

$$G_n(\lambda) = \begin{cases} (\lambda - ik_n) \, F(\lambda) \, , & 0 < |\lambda - ik_n| < \varepsilon \\ \\ Q_n \, , & \lambda = ik_n \end{cases} \tag{3.21}$$

for any $\varepsilon > 0$ sufficiently small.

Now, define the family of operators $\{N_k\}$ by

$$N_k = \begin{cases} (-k^2 + \alpha ik - (1 + \tilde{c}_k)A)^{-1} & k \in \mathbb{Z} \setminus \{ \, k_1, \ldots, k_N \, \} \\ \\ G_j'(ik_j) \, - Q_j \, A(\tilde{c}' G_j)'(ik_j) + Q_j \, (d' G_j)'(ik_j) & j = 1, \ldots, N. \end{cases} \tag{3.22}$$

where $ik \in \rho_{d,e}(A)$ for all $k \in \mathbb{Z} \setminus \{k_1, \ldots, k_N\}$. Note that $\{N_k\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X)$.
The following main theorem give compatibility conditions on f which are necessary and sufficient for the existence of a strong $B^s_{p,q}$- solution of (3.1).

Theorem 3.11 *Let $1 \leq p, q \leq \infty$ and $s > 0$. Let A be a closed linear operator defined on a Banach space X. Suppose that* (3.9) *holds and that $\{\tilde{c}_k\}_{k \in \mathbb{Z}}$ satisfies Marcinkiewicz estimates of order 2. If $\sup_{k \in \mathbb{Z}} ||k^2 N_k|| < \infty$, then for every $f \in B^s_{p,q}(\mathbb{T}; X)$ the equation (3.1) has a strong $B^s_{p,q}-$solution if and only if $Q_n \hat{f}(k_n) = 0$, for every $n = 1, \ldots, N$.*
In this case, all the strong solutions of (3.1) *are given by*

$$\begin{aligned} u(t) = \lim_{n \to \infty} & \sum_{\substack{k = -n \\ k \neq k_1, \ldots, k_N}}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt} (-k^2 + \alpha ik - (1 + \tilde{c}_k)A)^{-1} \hat{f}(k) \\ & + \sum_{j=1}^{N} e^{ik_j t} \, [G_j'(ik_j) \, - Q_j \, A(\tilde{c}' G_j)'(ik_j) + Q_j \, (d' G_j)'(ik_j)] \, \hat{f}(k_j). \end{aligned} \tag{3.23}$$

Proof.
First we assume that for every $f \in B^s_{p,q}(\mathbb{T}; X)$ there exists $v \in B^{s+2}_{p,q}(\mathbb{T}; X) \cap B^s_{p,q}(\mathbb{T}; [D(A)])$ which is a strong $B^s_{p,q}-$solution of the equation (3.1). Taking Fourier series on both sides in (3.1) we obtain that $\hat{v}(k) \in D(A)$ and

$$(-k^2 + \alpha ik - (1 + \tilde{c}_k)A)\hat{v}(k) = \hat{f}(k) \, , \text{ for all } k \in \mathbb{Z}.$$

For $\lambda \in \rho_{d,e}(A)$ and $k_1, k_2, ...k_N$ we have that

$$(\lambda - ik_j)F(\lambda) \, [\lambda^2 + \alpha\lambda - (1 + \tilde{c}(\lambda))A] \, \hat{v}(k_j) = (\lambda - ik_j) \, \hat{v}(k_j).$$

Letting $\lambda \to ik_j$ it follows that

$$\lim_{\lambda \to ik_j} (\lambda - ik_j)\; F(\lambda)\; [\lambda^2 + \alpha\lambda - (1+\tilde{c}(\lambda))A]\; \hat{v}(k_j) = 0.$$

Since both limits $\lim_{\lambda \to ik_j} (\lambda - ik_j)F(\lambda)$ and $\lim_{\lambda \to ik_j} [\lambda^2 + \alpha\lambda - (1+\tilde{c}(\lambda))A]\; \hat{v}(k_j)$ exist, we obtain that

$$Q_j\,(-k_j^2 + \alpha i k_j - (1+\tilde{c}(ik_j))A)\hat{v}(k_j) = 0,$$

or equivalently, $Q_j \hat{f}(k_j) = 0$, for all k_j, $j = 1, \dots, N$. Hence by Proposition 3.10 equation (3.18) is solvable and

$$\hat{v}(k) = \begin{cases} (-k^2 + \alpha i k - (1+\tilde{c}_k)A)^{-1}\hat{f}(k) & k \in \mathbb{Z} \setminus \{\, k_1, \dots, k_N \,\} \\ \\ \{G_j'(ik_j) - Q_j\, A(\tilde{c}' G_j)'(ik_j) + Q_j\,(d' G_j)'(ik_j)\}\, \hat{f}(k) & j = 1, \dots, N. \end{cases} \tag{3.24}$$

from which (3.23) follows.

Conversely, assume that $f \in B^s_{p,q}(\mathbb{T}; X)$ and $Q_n \hat{f}(k_n) = 0$ for $n = 1, \dots, N$. We define $u(t)$ by (3.23). Then

$$\hat{u}(k) = N_k\, \hat{f}(k) \tag{3.25}$$

for all $k \in \mathbb{Z}$, where N_k is defined by (3.22). Note that $\hat{u}(k) \in D(A)$ for all $k \in \mathbb{Z}$.

For each $k \in \mathbb{Z}$, we define $M_k := -k^2 N_k$. By hypothesis $\{M_k\}_{k\in\mathbb{Z}}$ is bounded. We observe that $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ and $\{k^2(M_{k+1} - 2M_k + M_{k-1})\}_{k\in\mathbb{Z}}$ are bounded which can be proved following the same lines as the proof of Proposition 3.3. Then by Theorem 2.22 we have that $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{pq}-$multiplier.

In a similar way as in the proof of Theorem 3.6 it follows that the family $\{ik\, N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{pq}-$multiplier. Hence, there exist $v, w \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$-k^2 \hat{u}(k) = ik\hat{v}(k) = \hat{w}(k)\,.$$

By [9, Lemma 2.1] and [9, Section 6] u, u' are differentiable a.e. with $u' = v$, $w = v' = u''$ and $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$. By [10, Theorem 1.3], this implies that $u \in B^{s+2}_{p,q}(\mathbb{T}; X)$.

Now, we will show that $u(t) \in D(A)$. Since $Q_n \hat{f}(k_n) = 0$ for all $n = 1, \dots, N$, by Proposition 3.10 we have that

$$(-k^2 + \alpha i k - (1+\tilde{c}_k)A)\; N_k\, \hat{f}(k) = \hat{f}(k) \tag{3.26}$$

for all $k \in \mathbb{Z}$, or equivalently

$$\begin{aligned} AN_k \hat{f}(k) &= \frac{-k^2 + \alpha i k}{1+\tilde{c}_k}\, N_k \hat{f}(k) \;-\; \frac{1}{1+\tilde{c}_k}\hat{f}(k) \\ &= \frac{1}{1+\tilde{c}_k}\,(-k^2 N_k)\hat{f}(k) \;+\; \frac{\alpha}{1+\tilde{c}_k} ik N_k \hat{f}(k) - \frac{1}{1+\tilde{c}_k}\hat{f}(k) \\ &= \frac{1}{1+\tilde{c}_k}\,\hat{w}(k) \;+\; \frac{\alpha}{1+\tilde{c}_k}\hat{v}(k) - \frac{1}{1+\tilde{c}_k}\hat{f}(k)\,. \end{aligned} \tag{3.27}$$

Since $f, v, w \in B^s_{p,q}(\mathbb{T}; X)$ and by Corollary 3.4 the family $\{\frac{1}{1+\tilde{c}_k} I\}$ is a $B^s_{p,q}-$multiplier, there exists $g \in B^s_{p,q}(\mathbb{T}; X)$ such that

$$AN_k \hat{f}(k) = \hat{g}(k).$$

From (3.25) we obtain $A\hat{u}(k) = \hat{g}(k)$, and Lemma 3.1 of [9] implies that $u(t) \in D(A)$. By (3.26) we have

$$\begin{aligned}
\hat{w}(k) = -\,k^2\,\hat{u}(k) = \;& -\alpha\hat{v}(k) + [1+\tilde{c}_k]\,AN_k\hat{f}(k) + \hat{f}(k) \\
& = -\alpha ik\hat{u}(k) + [1+\tilde{c}_k]\,A\hat{u}(k) + \hat{f}(k) \\
& = -\alpha ik\hat{u}(k) + A\hat{u}(k) + \tilde{c}_k\,A\hat{u}(k) + \hat{f}(k)\,.
\end{aligned} \tag{3.28}$$

It follows from the uniqueness theorem of Fourier coefficients that $u(t)$ defined by (3.23) satisfies (3.1) for almost all $t \in [0, 2\pi]$. ■

3.4 Notes and comments

In [33] Keyantuo and Lizama establish maximal regularity results in Besov spaces for an integro-differential equation with infinite delay, see Chapter 2 Section 2.6. For this, they introduce the conditions (2.13) and (2.14). Now, using the previous results, one can improve Theorems 2.38 and 2.40, corresponding to [33, Theorem 2.12 and Theorem 3.9] replacing these conditions in terms of Marcinkiewicz estimates. For the L^p case, we only need that $\{\tilde{c}_k\}$ satisfy a Marcinkiewicz estimate of order 1 and the Besov case we only need that $\{\tilde{c}_k\}$ satisfy a Marcinkiewicz estimate of order 2. We reformulate Theorem 2.40 as follows.

Theorem 3.12 *Let $1 \le p \le \infty$ and $s > 0$. Let A be a closed linear operator on the Banach space X. Assume that $\{\tilde{c}_k\}$ satisfies a Marcinkiewicz estimate of order 2. Then the following assertions are equivalent*

(*i*) *For all $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique strong $B^s_{p,q}-$solution of (2.10) such that u', Au and $a * Au \in B^s_{p,q}(\mathbb{T}; X)$.*

(*ii*) $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} ||b_k\,(b_k - A)^{-1}|| < \infty$.

Proof. Since the sequence $\{1+\tilde{c}_k\}$ satisfies a Marcinkiewicz estimate of order 2 and $\{1/(1+\tilde{c}_k)\}$ is bounded by Theorem 2.31 we obtain that $\{1+\tilde{c}_k\}$ is a 2-regular sequence. It is clear that $\{ik\}$ is 2-regular, hence by Lemma 2.27 it follows that $\{b_k\}$ is 2-regular sequence.
$(ii) \Rightarrow (i)$. Let $M_k = \frac{ik}{1+\tilde{c}_k}(b_k - A)^{-1}$. Define $N_k = \frac{i}{1+\tilde{c}_k}(b_k - A)^{-1}$. We claim that the family $\{N_k\}$ is a $B^s_{p,q}-$multiplier.

In fact, to verify that $\{N_k\}$ satisfies Marcinkiewicz estimate of order 2, note that $||N_k|| \leq ||kN_k|| = ||M_k||$ for all $k \in \mathbb{Z} \setminus \{0\}$ and hence $sup_{k\in\mathbb{Z}}||N_k|| < \infty$. Moreover, we have

$$k[N_{k+1} - N_k] = (k+1)N_{k+1} - kN_k - N_{k+1}\,,$$

and

$$k^2[N_{k+1} - 2N_k + N_{k-1}] = k(M_{k+1} - M_k) + k(M_{k-1} - M_k) + M_{k-1} - M_{k+1} + N_{k-1} + N_{k+1}\,,$$

since $\{M_k\}$ satisfies Marcinkiewicz estimate of order 2 and $\{N_k\}$ is a bounded sequence by Theorem 2.24, we obtain the claim.
Now, let $f \in B^s_{p,q}(\mathbb{T}, X)$. Since $\{N_k\}$ is a $B^s_{p,q}-$multiplier, there exists $u \in B^s_{p,q}(\mathbb{T}, X)$ such that

$$\hat{u}(k) = N_k \hat{f}(k)\,, \quad \text{for all} \quad k \in \mathbb{Z}\,, \tag{3.29}$$

we observe that $\hat{u}(k) \in D(A)$.
Since $\{M_k\}$ is a $B^s_{p,q}-$multiplier, there exists $v \in B^s_{p,q}(\mathbb{T}, X)$ such that $\hat{v}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. From (3.29), we obtain that

$$ik\hat{u}(k) = \hat{v}(k)\,, \quad \text{for all} \quad k \in \mathbb{Z}\,, \tag{3.30}$$

By Lemma 2.1 of [9], u is differentiable a.e. with $u' = v$ and $u(0) = u(2\pi)$. By [10, Theorem 2.3], this implies that $u \in B^{s+1}_{p,q}(\mathbb{T}; X)$.
Apply again the fact that $\{1/(1+\tilde{c}_k)\}$ is a $B^s_{p,q}-$multipliers, there exists $w_1 \in B^s_{p,q}(\mathbb{T}, X)$ such that $\hat{w_1}(k) = \dfrac{1}{1+\tilde{c}_k}\hat{f}(k)$ for all $k \in \mathbb{Z}$. From (3.29) have the identity

$$A\hat{u}(k) = \frac{ik}{1+\tilde{c}_k}\hat{u}(k) - \frac{1}{1+\tilde{c}_k}\hat{f}(k) = \hat{w}(k) - \hat{w_1}(k) \tag{3.31}$$

from [9, Lemma 3.1] this implies that $u(t) \in D(A)$ and $Au(t) = w(t) - w_1(t)$. Hence $Au \in B^s_{p,q}(\mathbb{T}, X)$.
Since A is closed, from (3.31), we deduce that (2.10) holds. We have proved that u is a strong $B^s_{p,q}-$solution of (2.10). It remains to establish uniqueness.
Let $u \in B^s_{p,q}(\mathbb{T}, [D(A)])$ be such that

$$u'(t) - Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)\,ds = 0\,,$$

then $\hat{u}(k) \in D(A)$ and $(ikI - (1+\tilde{c}_k)A)\hat{u}(k) = 0$. Since $b_k \in \rho(A)$ this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$.
$(i) \Rightarrow (ii)$. Is the proof of Keyantuo and Lizama in [33].

∎

Chapter 4

Additive Perturbation for Integro-differential Equations and Maximal Regularity

4.1 Introduction

In this chapter we study existence and uniqueness of periodic solutions for the following integral equation with infinite delay

$$u(t) = \int_{-\infty}^{t} a(t-s)Au(s)ds + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \tag{4.1}$$

where $a(\cdot)$, $b(\cdot) \in L^1(\mathbb{R}_+)$ are scalar-valued kernels, A and B are closed linear operators defined on a UMD space, such that $D(A) \subset D(B)$. In contrast with many papers on the subject of integrodifferential equations, in this work we will study directly the full problem (4.1) by mean of Theorem 2.24.
If we assume that B is relatively bounded with respect to the unperturbed operator A, then we are able to obtain sufficient conditions for maximal regularity in terms of R-boundedness of

$$\{(I - \tilde{b}(ik)B - \tilde{a}(ik)A)^{-1}\}_{k\in\mathbb{Z}}. \tag{4.2}$$

We remark that the $R-$boundedness assumption is satisfied by a large number of examples. We refer to the recent monographs by Denk, Hieber and Prüss [25] and Kunstmann and Weiss [36] for the corresponding developments.
We observe that the present results on perturbation of R-boundedness (see [25, Proposition 4.3]) are not sufficient to directly handle the case studied here.
In contrast with to all the above papers dealing with this subject, we obtain very simple conditions for well-posedness. Among the conditions that we impose on a and b is one of $k-$regularity. Furthermore, we do not make any parabolicity assumption on the operator, not even that A generates a semigroup. In fact, we give examples showing that the condition that A be the generator of a semigroup is not necessary.

4.2 Additive Perturbation and R-boundedness

In this section we consider the behaviour of R-boundedness with respect to perturbations. For this purpose, we recall the following definition (see [26, Definition 2.1]).

Definition 4.1 *Let* $A : D(A) \subset X \longrightarrow X$ *be a linear operator on a Banach space* X *. An operator* $B : D(B) \subset X \longrightarrow X$ *is called* $A-$*bounded if* $D(A) \subset D(B)$ *and if there exist constants* $c \geq 0$, $d \geq 0$ *such that*

$$||Bx|| \leq c||Ax|| + d||x|| \tag{4.3}$$

for all $x \in D(A)$ *. The* $A-$*bound of* B *is*

$$c_0(A) := inf\{\, c \geq 0 \;\; : \;\; \textit{there exists } d > 0 \textit{ such that } (4.3) \textit{ holds}\}.$$

Lemma 4.2 *Let* $\{\alpha_k\}_{k\in I\subseteq \mathbb{Z}} \in \mathbb{C}$ *be a bounded sequence. Let* A *and* B *be closed linear operators defined on* X *. Assume that* B *is* $A-$*bounded. Then* $\{\alpha_k B\}_{k\in I\subseteq \mathbb{Z}} \subset \mathcal{B}([D(A)], X)$ *is* $R-$*bounded and*

$$R_p[\{\alpha_k B\}_{k\in I\subseteq \mathbb{Z}}] \leq 2(c+d) \sup_{k\in I} |\alpha_k|. \tag{4.4}$$

Proof.
This is a direct consequence of the Kahane contraction principle and $R-$boundedness of products. However we will give a direct proof. Denote $B_k := \alpha_k B$, $k \in I \subseteq \mathbb{Z}$. Using Definition 2.1 and the inequality (4.3), we have

$$\begin{aligned} ||\sum_{j=1}^{m} r_j\, B_{k_j} x_j||^p_{L^p(0,1;X)} &= \int_0^1 ||\sum_{j=1}^{m} r_j(t)\alpha_{k_j} B x_j||^p_X \, dt \\ &\leq (c+d)^p \int_0^1 ||\sum_{j=1}^{m} r_j(t)\alpha_{k_j} x_j||^p_{[D(A)]} \, dt \\ &= (c+d)^p ||\sum_{j=1}^{m} r_j\, \alpha_{k_j} x_j||^p_{L^p(0,1;[D(A)])}, \end{aligned}$$

for all $k_1, ..., k_m \in I \subseteq \mathbb{Z}$, $x_1, ..., x_m \in [D(A)]$ and $m \in \mathbb{N}$, where $1 \leq p < \infty$. By Kahane's contraction principle one has

$$||\sum_{j=1}^{m} r_j\, \alpha_{k_j} x_j||_{L^p(0,1;[D(A)])} \leq 2 \max_{j=1,\ldots,m} |\alpha_{k_j}|\, ||\sum_{j=1}^{m} r_j\, x_j||_{L^p(0,1;[D(A)])} \,,$$

for all $\alpha_{k_j} \in \mathbb{C}$ and $x_j \in D(A)$, $j = 1, \ldots, m$. Since $\{\alpha_k\}_{k\in I\subseteq \mathbb{Z}}$ is bounded, we have that

$$||\sum_{j=1}^{m} r_j\, B_{k_j} x_j||_{L^p(0,1;X)} \leq 2(c+d) \sup_{k\in I} |\alpha_k|\, ||\sum_{j=1}^{m} r_j\, x_j||_{L^p(0,1;[D(A)])}.$$

■

The following is the main result of this section.

Theorem 4.3 *Let A and B be closed linear operators defined on a UMD space X. Assume that B is $A-$bounded. Suppose that the sequence $\{b_k\}_{k\in\mathbb{Z}}$ satisfies a Marcinkiewicz estimate of order 1 and $\{a_k\}_{k\in\mathbb{Z}}$ is a 1-regular.*
If $1\in\rho(a_kA+b_kB)$ for all $k\in\mathbb{Z}$, then the following assertions are equivalent

(i) $\{(I-a_kA-b_kB)^{-1}\}_{k\in\mathbb{Z}}$ *is an $L^p_{X,[D(A)]}-$ multiplier,* $1<p<\infty$.

(ii) $\{(I-a_kA-b_kB)^{-1}\}_{k\in\mathbb{Z}}\subseteq\mathcal{B}(X,[D(A)])$ *and is R-bounded.*

Proof.
$(ii)\Rightarrow(i)$ Define $B_k:=\frac{1}{a_k}(I-b_kB)$. We first claim that the family

$$\{k\,a_{k+1}\,(B_k-B_{k+1})\}_{k\in\mathbb{Z}}$$

is $R-$bounded. In fact, we have that

$$ka_{k+1}[B_k-B_{k+1}]\quad=k[\frac{a_{k+1}-a_k}{a_k}]I+k[b_{k+1}-b_k]\frac{a_{k+1}}{a_k}B-k[\frac{a_{k+1}-a_k}{a_k}]b_{k+1}B$$

Since $\{a_k\}$ is a 1-regular sequence, is follows that $|\dfrac{a_{k+1}}{a_k}-1|<\dfrac{1}{|k|}\to 0$ when $|k|\to\infty$, hence $\{\frac{a_{k+1}}{a_k}\}$ is bounded.
Setting $\alpha_k=k[\dfrac{a_{k+1}-a_k}{a_k}]$, $\beta_k=k[b_{k+1}-b_k]\dfrac{a_{k+1}}{a_k}$ and $\gamma_k=k[\dfrac{a_{k+1}-a_k}{a_k}]b_{k+1}$, the sequences $\{\alpha_k\}_{k\in\mathbb{Z}}$, $\{\beta_k\}_{k\in\mathbb{Z}}$ and $\{\gamma_k\}_{k\in\mathbb{Z}}$ are bounded by hypothesis. The claim follows from Lemma 4.2 and Proposition 2.5.
Let $N_k=(I-a_kA-b_kB)^{-1}=\dfrac{1}{a_k}\,(B_kI-A)^{-1}$. In order to prove (i) it is sufficient to show, by Theorem 2.24, that the set $\{k(N_{k+1}-N_k)\}_{k\in\mathbb{Z}}$ is R-bounded. In fact,

$$\begin{aligned}k[N_{k+1}-N_k]\quad&=\quad\frac{1}{a_{k+1}}\,(B_{k+1}-A)^{-1}\,[ka_{k+1}(B_k-B_{k+1})]\,\frac{1}{a_k}(B_k-A)^{-1}\\&\quad-\frac{1}{a_{k+1}}\,(B_{k+1}-A)^{-1}\,\frac{k(a_{k+1}-a_k)}{a_k}\end{aligned}$$

and the result follows from Proposition 2.5.
$(i)\Rightarrow(ii)$ Since $\rho(a_kA+b_kB)\neq\emptyset$ for each $k\in\mathbb{Z}$, the operators a_kA+b_kB are closed. The result follows from the Closed Graph Theorem and Remark 2.18.

∎

The next corollary extends Proposition 2.37.

Corollary 4.4 *Let A be a closed linear operator on a UMD space X. Let $\{a_k\}_{k\in\mathbb{Z}}$, $\{m_k\}_{k\in\mathbb{Z}}$ be 1-regular sequences such that*

$$\{a_km_k\}_{k\in\mathbb{Z}}\ \textit{is bounded.}$$

Suppose $\{m_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. *Then the following assertions are equivalent*

(i) $\{\frac{1}{a_k}(m_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ *is an* $L^p_{X,[D(A)]}$ *multiplier,* $1 < p < \infty$.

(ii) $\{\frac{1}{a_k}(m_kI - A)^{-1}\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X, [D(A)])$ *and is R-bounded.*

Proof.
Apply the above theorem with $B = I$ and $b_k = 1 - m_k a_k$, $k \in \mathbb{Z}$. Moreover

$$k(b_{k+1} - b_k) = m_k a_k \frac{k(a_k - a_{k+1})}{a_k} + m_{k+1}a_{k+1}\frac{k(m_k - m_{k+1})}{m_{k+1}},$$

and

$$I - a_kA - b_kB = a_k(m_k - A),$$

from which the assertions follows.

■

Proposition 4.5 *Let* $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}\setminus\{0\}$ *and* $\{b_k\}_{k\in\mathbb{Z}}$ *be sequences with* $\lim_{k\to\infty} b_k = 0$. *Let* A *be a closed linear operator defined on a Banach space* X *such that* $\{\frac{1}{a_k}\} \subset \rho(A)$ *and*

$$R_p[\{(I - a_kA)^{-1}\}_{k\in\mathbb{Z}}] =: M < \infty. \tag{4.5}$$

Assume that B *is* $A-$*bounded. Then there exists* $N \in \mathbb{N}\cup\{0\}$ *such that* $1 \in \rho(a_kA + b_kB)$ *for all* $|k| \geq N$ *and*

$$R_p[\{(I - a_kA - b_kB)^{-1}\}_{|k|\geq N}] < \infty. \tag{4.6}$$

Proof. Since B is A-bounded, there exists constants $c, d \geq 0$ such that (4.3) holds. By hypothesis, there exists $N \in \mathbb{N}\cup\{0\}$ such that

$$|b_k| \leq \frac{1}{4M(c+d)} \text{ for all } |k| \geq N.$$

By Lemma 4.2, the family of operators $\{b_kB\}_{|k|\geq N} \subset \mathcal{B}([D(A)], X)$ is R-bounded and

$$R_p[\{b_kB\}_{|k|\geq N}] \leq 2(c+d)\sup_{|k|\geq N}|b_k| \leq \frac{1}{2M}. \tag{4.7}$$

Since the family $\{(I-a_kA)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded, we have by properties of R-boundedness (see chapter 2) that the family $\{b_kB(I - a_kA)^{-1}\}_{|k|\geq N}$ is R-bounded with

$$R_p[\{b_kB(I - a_kA)^{-1}\}_{|k|\geq N}] \leq R_p[\{b_kB\}_{|k|\geq N}]R_p[\{(I - a_kA)^{-1}\}_{|k|\geq N}] \leq \frac{1}{2}. \tag{4.8}$$

In particular, the family $\{b_kB(I - a_kA)^{-1}\}_{|k|\geq N}$ is uniformly bounded, that is

$$||b_kB(I - a_kA)^{-1}|| \leq 1/2 \text{ for all } |k| \geq N. \tag{4.9}$$

We decompose $I - a_kA - b_kB$ as the product

$$I - a_kA - b_kB = [I - b_kB(I - a_kA)^{-1}]\,[I - a_kA]$$

and observe that $I - a_kA$ is a bijection from $D(A)$ onto X, while $B(I - a_kA)^{-1}$ is bounded on X since B is $A-$bounded. By (4.9) we obtain that the operator $I - b_kB(I - a_kA)^{-1}$ is invertible for each $|k| \geq N$ fixed, with inverse

$$(I - a_kA - b_kB)^{-1} = (I - a_kA)^{-1} \sum_{n=0}^{\infty} (b_kB(I - a_kA)^{-1})^n. \tag{4.10}$$

Using induction over n, we have by properties of R-bounded families and (4.8),

$$\begin{aligned} R_p[\{(I - a_kA)^{-1}\}\{(b_kB(I - a_kA)^{-1})\}^n] &\leq R_p[\{(I - a_kA)^{-1}\}]R_p[\{b_kB(I - a_kA)^{-1}\}]^n \\ &\leq M(\frac{1}{2})^n. \end{aligned}$$

Finally, taking into account that R-boundedness is preserved by convergence in the strong operator topology, one has

$$R_p(\{(I - a_kA - b_kB)^{-1}\}_{|k| \geq N}) \leq 2M. \tag{4.11}$$

This proves that $\{(I - a_kA - b_kB)^{-1}\}_{|k| \geq N}$ is R-bounded.

∎

4.3 An Integral Equation of Hyperbolic type

Consider the following integral equation with infinite delay

$$\begin{cases} u(t) &= \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)ds + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t) \\ u(0) &= u(2\pi) \end{cases} \tag{4.12}$$

where a, $b \in L^1(\mathbb{R}_+)$ are scalar kernels, and A, B are closed linear operators defined on a UMD space X, such that $D(A) \subset D(B)$.
In this section, we give sufficient conditions for the maximal regularity for periodic solutions for the equation (4.12) in the vector valued Lebesgue spaces.
We define

$$\rho(A,B) = \{\lambda \in \mathbb{C} : I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B \text{ is invertible and } (I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B)^{-1} \in \mathcal{B}(X, [D(A)])\}$$

where $\tilde{a}(\lambda)$, $\tilde{b}(\lambda)$ are the Laplace transforms of a and b respectively.
We suppose that $\lambda \to \tilde{a}(\lambda)$ (resp. $\tilde{b}(\lambda)$) admits an analytical extension to a sector containing the imaginary axis, and still denote this extension by $\tilde{a}$ (resp. $\tilde{b}$). In what follows we will assume that $\tilde{a}(ik)$, $\tilde{b}(ik)$ exist for all $k \in \mathbb{Z}$ and use the notation $\tilde{a}_k = \tilde{a}(ik)$ and $\tilde{b}_k = \tilde{b}(ik)$.
Denote by $\sigma(A, B)$ the set $\mathbb{C} \setminus \rho(A, B)$.

Definition 4.6 *Let* $1 < p < \infty$. *A function* u *is called a strong* L^p-*solution of (4.12) if* $u \in L^p_{2\pi}(\mathbb{R};[D(A)])$ *and equation (4.12) holds for almost all* $t \in [0, 2\pi]$.

The following is the main result of this section.

Theorem 4.7 *Let* $a, b \in L^1(\mathbb{R}_+)$ *be functions such that the sequence* $\{\tilde{b}_k\}$ *satisfies a Marcinkiewicz estimate of order 1 and* $\{\tilde{a}_k\}$ *is 1-regular. Let* A *and* B *be closed linear operators defined on an* UMD *space* X *and assume that* B *is* $A-$*bounded. If* $\{ik\}_{k\in\mathbb{Z}} \in \rho(A, B)$ *and* $\{(I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X, [D(A)])$ *is R-bounded, then for every* $f \in L^p_{2\pi}(\mathbb{R}, X)$ *there exists a unique strong* L^p-*solution of* (4.12).

Proof.
Let $f \in L^p_{2\pi}(\mathbb{R}, X)$. By Theorem 4.3, we have that there is $u \in L^p_{2\pi}(\mathbb{R};[D(A)])$ such that

$$\hat{u}(k) = (I - \tilde{b}_k B - \tilde{a}_k A)^{-1}\hat{f}(k), \quad \text{for all} \quad k \in \mathbb{Z}.$$

We conclude that $\hat{u}(k) \in D(A) \subset D(B)$ and

$$(I - \tilde{b}_k B - \tilde{a}_k A)\hat{u}(k) = \hat{f}(k). \tag{4.13}$$

On the other hand, since $\{\tilde{b}_k\}$ satisfies a Marcinkiewicz estimate of order 1, by Lemma 4.2 we have that $\{\tilde{b}_k B\}_{k\in\mathbb{Z}}$ and $\{\, k\,(\,\tilde{b}_{k+1} - \tilde{b}_k)B\,\}_{k\in\mathbb{Z}}$ are $R-$bounded. By Theorem 2.24 it follows that $\{\tilde{b}_k B\}_{k\in\mathbb{Z}}$ is an $L^p_{[D(A)],X}-$ multiplier. Hence for each $g \in L^p_{2\pi}(\mathbb{R}, [D(A)])$ there exists $h \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{h}(k) = \tilde{b}_k B\hat{g}(k)$, for all $k \in \mathbb{Z}$. In particular, for $g := u \in L^p_{2\pi}(\mathbb{R}, [D(A)])$ we obtain $\hat{h}(k) = B\tilde{b}_k\hat{u}(k)$. Since B is closed, from Lemma 3.1 in [9] we conclude that $(b*u)(t) = \displaystyle\int_{-\infty}^{t} b(t-s)u(s)ds \in D(B)$ and $B(b*u)(t) = h(t)$.
By (4.13) we have $\tilde{a}_k A\hat{u}(k) = \hat{u}(k) - \tilde{b}_k B\hat{u}(k) - \hat{f}(k)$ and then $A\tilde{a}_k\hat{u}(k) = \hat{u}(k) - \hat{h}(k) - \hat{f}(k)$. Hence from Lemma 3.1 in [9] it follows that $(a*u)(t) = \displaystyle\int_{-\infty}^{t} a(t-s)u(s)ds \in D(A)$ and

$$A(a*u)(t) = u(t) - h(t) - f(t) = u(t) - B(b*u)(t) - f(t). \tag{4.14}$$

(cf. [33, equation (2.1)]). It follows from the closedness of A and B, and from the uniqueness theorem of Fourier coefficients that (4.12) holds for almost all $t \in [0, 2\pi]$. We have proved that u is a strong L^p-solution of (4.12). It remains to show uniqueness.
Let $u \in L^p_{2\pi}(\mathbb{R};[D(A)])$ such that $u(t) - \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)ds - \int_{-\infty}^{t} b(t-s)Bu(s)ds = 0$, then $\hat{u}(k) \in D(A)$ and $(I - (\tilde{a}_k A + \tilde{b}_k B))\hat{u}(k) = 0$. Since $\{ik\}_{k\in\mathbb{Z}} \in \rho(A, B)$ this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$.

∎

Remark 4.8

In the context of Theorem 4.7 we have $Au, a*Au, b*Bu \in L^p_{2\pi}(\mathbb{R}, X)$. Moreover, by the Closed Graph Theorem there exists a constant $C > 0$, independent of $f \in L^p_{2\pi}(\mathbb{R}, X)$ such that

$$||u||_{L^p_{2\pi}(\mathbb{R},X)} + ||a * Au||_{L^p_{2\pi}(\mathbb{R},X)} + ||b * Bu||_{L^p_{2\pi}(\mathbb{R},X)} \leq C||f||_{L^p_{2\pi}(\mathbb{R},X)}.$$

Example 4.9

Let $X = l^2(\mathbb{Z})$ and $0 < \beta < 1$. Consider the system

$$u_n = (n - i\beta)a * u_n + f_n, \quad n \in \mathbb{Z}, \tag{4.15}$$

the Fourier series version of the boundary value problem

$$\begin{cases} u(t,x) = -\displaystyle\int_{-\infty}^{t} a(t-s)(iu_x(s,x) + i\beta u(s,x))ds + f(t,x), \quad x \in [0, 2\pi], \quad t \geq 0 \\ \\ u(t,0) = u(t,2\pi), \quad t \geq 0. \end{cases}$$

This problem is of the form (4.12) with

$$(Au)_n = (n - i\beta)u_n, \qquad D(A) = \{(u_n) \in l^2(\mathbb{Z}) : (n \cdot u_n) \in l^2(\mathbb{Z})\},$$

and $b(t) = 0$ for all $t \in \mathbb{R}_+$. Note that A does not generate a C_0-semigroup since $\sigma(A) = \{n - i\beta : n \in \mathbb{Z}\}$ is not contained in any left halfplane. Define $a(t) = e^{-\alpha t}$, $\alpha > 0$. Clearly the sequence $\tilde{a}_k = \frac{1}{ik+\alpha}$ is 1-regular and $\{ik+\alpha\}_{k\in\mathbb{Z}} \subset \rho(A)$. Moreover, for each $x = (x_n) \in l^2(\mathbb{Z})$ we have

$$\begin{aligned} ||(I - \tilde{a}_k A)^{-1}x|| &= ||(ik+\alpha)(ik+\alpha-A)^{-1}x|| \\ &= \sum_{n\in\mathbb{Z}} \left| \frac{ik+\alpha}{ik+\alpha-n+i\beta} x_n \right|^2 \\ &\leq \sum_{n\in\mathbb{Z}} \frac{k^2+\alpha^2}{(\alpha-n)^2+(\beta+k)^2} |x_n|^2 \\ &\leq \sum_{n\in\mathbb{Z}} \frac{k^2+\alpha^2}{(k+\beta)^2} |x_n|^2. \end{aligned}$$

Since $0 < \beta < 1$, we obtain for all $k \in \mathbb{Z}$

$$||(I - \tilde{a}_k A)^{-1}x|| \leq \max\{\frac{\alpha^2}{\beta^2}, \frac{\alpha^2+1}{(\beta-1)^2}\} \sum_{n\in\mathbb{Z}} |x_n|^2 =: M||x||,$$

where, as indicated, the constant M depends only on α and β. Then, the hypotheses of Theorem 4.7 are satisfied and we conclude that for every $f \in L^p_{2\pi}(\mathbb{R}, l^2(\mathbb{Z}))$ there exists a unique strong L^p-solution of the boundary value problem.

4.4 The resonant case

In the Section 4.3 we considered the nonresonance case: $ik \notin \sigma(A,B)$ for all $k \in \mathbb{Z}$, and we proved that, for every $f \in L^p_{2\pi}(\mathbb{R}, X)$ there exists a unique strong L^p−solution of (4.12).

Now, we consider a resonant case: We assume that there are $k_1, \dots, k_N \in \mathbb{Z}$ such that

$$\begin{cases} (i)\ \ ik_j \in \sigma(A,B) & \text{for } j = 1, \dots, N; \\ (ii)\ \ ik \notin \sigma(A,B) & \text{for } k \in \mathbb{Z}, k \neq k_1, \dots, k_N \\ (iii)\ \ ik_j & \text{is a simple pole of } F(\cdot) \text{ for } j = 1, \dots, N \end{cases} \tag{4.16}$$

where $F : \rho(A,B) \subset \mathbb{C} \to \mathcal{B}(X, [D(A)])$ is defined by $F(\lambda) = (I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B)^{-1}$.

We now give some preliminary results about the solvability of the equation

$$(I - \tilde{a}(\lambda_0)A - \tilde{b}(\lambda_0)B)x = y \tag{4.17}$$

where λ_0 is a simple pole of $F(\cdot)$.
From Section 3.3 we recall that $Q \in \mathcal{B}(X, [D(A)])$ is the residue of $F(\cdot)$ at λ_0 and $G(\lambda)$ is defined by (3.12).

Lemma 4.10 *Suppose that B is A-bounded. With the notations as above, we have*

$$Q \;=\; Q\,[-\tilde{a}'(\lambda_0)A - \tilde{b}'(\lambda_0)B]\,Q$$

Proof. We proceed analogously as in the proof of Lemma 3.9. For each λ, μ belonging to $B(\lambda_0, \varepsilon) \setminus \{\lambda_0\}$ with $|\lambda - \lambda_0| > |\mu - \lambda_0|$ we have

$$F(\lambda) - F(\mu) \;= F(\lambda)\;[\,(\,\tilde{a}(\lambda) - \tilde{a}(\mu)\,)A + (\,\tilde{b}(\lambda) - \tilde{b}(\mu)\,)B\,]\;F(\mu).$$

Hence

$$\begin{aligned} &\frac{F(\lambda) - F(\mu)}{\lambda - \mu}(\lambda - \lambda_0)(\mu - \lambda_0) \\ &= (\lambda - \lambda_0)F(\lambda)\,\left[\frac{\tilde{a}(\lambda) - \tilde{a}(\mu)}{\lambda - \mu}A + \frac{\tilde{b}(\lambda) - \tilde{b}(\mu)}{\lambda - \mu}B\right]\,(\mu - \lambda_0)\,F(\mu) \end{aligned}$$

and using (3.12) we have

$$G(\lambda)\frac{\mu - \lambda_0}{\lambda - \mu} - G(\mu)\frac{\lambda - \lambda_0}{\lambda - \mu} \;= G(\lambda)\,\left[\frac{\tilde{a}(\lambda) - \tilde{a}(\mu)}{\lambda - \mu}A + \frac{\tilde{b}(\lambda) - \tilde{b}(\mu)}{\lambda - \mu}B\right]\,G(\mu).$$

Since B is A-bounded, we have

$$-Q \quad = G(\lambda)\left[\frac{\tilde{a}(\lambda)-\tilde{a}(\lambda_0)}{\lambda-\lambda_0}A+\frac{\tilde{b}(\lambda)-\tilde{b}(\lambda_0)}{\lambda-\lambda_0}B\right]Q$$

as $\mu \to \lambda_0$. Letting $\lambda \to \lambda_0$ we get

$$Q \quad = Q\left[-\tilde{a}'(\lambda_0)A-\tilde{b}'(\lambda_0)B\right]Q.$$

This proves the Lemma.

■

The following result is analogous to Proposition 3.10.

Proposition 4.11 *Let λ_0 be a simple pole of $F(\cdot)$ and let Q be defined by* (3.11). *Suppose that B is A-bounded. Then*

$$Ker(I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B)=Q(X). \tag{4.18}$$

Moreover, for any $y \in X$ such that $Qy=0$, all solutions of (4.17) *are given by*

$$x=G'(\lambda_0)y-Q\,A(\tilde{a}'G)'(\lambda_0)y-Q\,B(\tilde{b}'G)'(\lambda_0)y\,. \tag{4.19}$$

Proof. First we show (4.18). For any sufficiently small $\varepsilon>0$ and $0<|\lambda-\lambda_0|<\varepsilon$ we have

$$(I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B\,)\,G(\lambda)=(\lambda-\lambda_0)+(\tilde{a}(\lambda)-\tilde{a}(\lambda_0))AG(\lambda)+(\tilde{b}(\lambda)-\tilde{b}(\lambda_0))BG(\lambda)$$

Since B is A-bounded and $A \in \mathcal{B}([D(A)],X)$, letting $\lambda \to \lambda_0$ we obtain $(\,I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B\,)\;Q\;=\;0$, so that $Q(X)$ is contained in $Ker(\,I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B\,)$. Let now $x \in D(A)$ be such that $(\,I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B\,)\;x\;=\;0$, then for $0<|\lambda-\lambda_0|<\varepsilon$ with ε small, we have

$$F(\lambda)\;(\,I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B\,)\;x\;=\;0. \tag{4.20}$$

Since $x-F(\lambda)\,(\,I-\tilde{a}(\lambda)A-\tilde{b}(\lambda)B\,)\;x\;=\;0$, that is,

$$x-F(\lambda)\,x+F(\lambda)\,\tilde{a}(\lambda)Ax+F(\lambda)\,\tilde{b}(\lambda)B\,x\;=\;0$$

and then

$$x+F(\lambda)\,[\tilde{a}(\lambda)-\tilde{a}(\lambda_0)]\,Ax+F(\lambda)[\tilde{b}(\lambda)-\tilde{b}(\lambda_0)]\,Bx-F(\lambda)\,[I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B]\,x\;=\;0.$$

It follows from (4.20) that

$$x+(\lambda-\lambda_0)F(\lambda)\,\frac{\tilde{a}(\lambda)-\tilde{a}(\lambda_0)}{\lambda-\lambda_0}\,Ax+(\lambda-\lambda_0)F(\lambda)\frac{\tilde{b}(\lambda)-\tilde{b}(\lambda_0)}{\lambda-\lambda_0}\,Bx\;=\;0\,,$$

that is,

$$x + G(\lambda)\,\frac{\tilde a(\lambda) - \tilde a(\lambda_0)}{\lambda - \lambda_0}\,Ax + G(\lambda)\frac{\tilde b(\lambda) - \tilde b(\lambda_0)}{\lambda - \lambda_0}\,Bx \;=\; 0.$$

Letting $\lambda \to \lambda_0$ we get

$$x + Q\,\tilde a'(\lambda_0)\,Ax + Q\,\tilde b'(\lambda_0)\,Bx \;=\; 0\,,$$

so that x belongs to $Q(X)$.

Let us show now (4.19). First we claim that

$$\lim_{\lambda\to\lambda_0} F(\lambda)\,[1 + (\,\tilde a'(\lambda_0)A + \tilde b'(\lambda_0)B\,)\,Q\,] \;=\; G'(\lambda_0) - Q\,A\,(\tilde a' G)'(\lambda_0) - Q\,B\,(\tilde b' G)'(\lambda_0) \tag{4.21}$$

In fact, with direct computations, we obtain

$$\begin{aligned}
G'(\lambda) \;&= F(\lambda) - (\lambda-\lambda_0)\,F(\lambda)\,[-\tilde a'(\lambda)A - \tilde b'(\lambda)B\,]\,F(\lambda)\\
&= F(\lambda)\,[1 + (\,\tilde a'(\lambda_0)\,A + \tilde b'(\lambda_0)\,B\,)Q\,] - F(\lambda)\,\tilde a'(\lambda_0)\,A\,Q - F(\lambda)\,\tilde b'(\lambda_0)\,B\,Q\\
&\quad + F(\lambda)\,\tilde a'(\lambda)A(\lambda-\lambda_0)\,F(\lambda) + F(\lambda)\,\tilde b'(\lambda)B\,(\lambda-\lambda_0)\,F(\lambda)\\
&= F(\lambda)\;[\;1 + (\,\tilde a'(\lambda_0)\,A \;+\; \tilde b'(\lambda_0)\,B\,)Q\;] \;+\; F(\lambda)\,A\,[\;\tilde a'(\lambda)\,G(\lambda) - \tilde a'(\lambda_0)\,Q\;]\\
&\quad + F(\lambda)\,B\,[\;\tilde b'(\lambda)\,G(\lambda) - \tilde b'(\lambda_0)Q\;]\\
&= F(\lambda)\;[\;1 + (\,\tilde a'(\lambda_0)\,A \;+\; \tilde b'(\lambda_0)\,B\,)Q\;] \;+\; (\lambda-\lambda_0)\,F(\lambda)\,A\,\frac{\tilde a'(\lambda)\,G(\lambda) - \tilde a'(\lambda_0)\,G(\lambda_0)}{\lambda-\lambda_0}\\
&\quad + (\lambda-\lambda_0)\,F(\lambda)\,B\,\frac{\tilde b'(\lambda)\,G(\lambda) - \tilde b'(\lambda_0)G(\lambda_0)}{\lambda-\lambda_0}.
\end{aligned}$$

Therefore

$$\begin{aligned}
G'(\lambda) \;&= F(\lambda)\;[\;1 + (\,\tilde a'(\lambda_0)\,A \;+\; \tilde b'(\lambda_0)\,B\,)Q\;] \;+\; G(\lambda)\,A\,\frac{\tilde a'(\lambda)\,G(\lambda) - \tilde a'(\lambda_0)\,G(\lambda_0)}{\lambda-\lambda_0}\\
&\quad + G(\lambda)\,B\,\frac{\tilde b'(\lambda)\,G(\lambda) - \tilde b'(\lambda_0)G(\lambda_0)}{\lambda-\lambda_0}.
\end{aligned}$$

Using the fact that B is A-bounded and $A \in \mathcal{B}([D(A)], X)$ we let $\lambda \to \lambda_0$ and obtain (4.21).

On the other hand, using (4.21) we obtain

$$\lim_{\lambda\to\lambda_0}[I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B\,]\,F(\lambda)\,[\,1+(\tilde{a}'(\lambda_0)A+\tilde{b}'(\lambda_0)B)Q\,]$$

$$=\lim_{\lambda\to\lambda_0}[I-\tilde{a}(\lambda)A-\tilde{b}(\lambda)B+\tilde{a}(\lambda)A+\tilde{b}(\lambda)B-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B\,]$$

$$F(\lambda)[\,1+(\tilde{a}'(\lambda_0)A+\tilde{b}'(\lambda_0)B)Q\,]$$

$$=\lim_{\lambda\to\lambda_0}[\,I+\{\,(\tilde{a}(\lambda)-\tilde{a}(\lambda_0))\,A+(\,\tilde{b}(\lambda)-\tilde{b}(\lambda_0))\,B\,\}F(\lambda)\,]\;\;[\,I+(\tilde{a}'(\lambda_0)A+\tilde{b}'(\lambda_0)B)Q\,]$$

$$=I+(\tilde{a}'(\lambda_0)A+\tilde{b}'(\lambda_0)B)Q$$

Due to (4.21) and the fact that $I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B$ belongs to $\mathcal{B}([D(A)],X)$ we have

$$[I-\tilde{a}(\lambda_0)A-\tilde{b}(\lambda_0)B]\,[\,G'(\lambda_0)-Q\,A\,(\tilde{a}'G)'(\lambda_0)-Q\,B\,(\tilde{b}'G)'(\lambda_0)\,]=I+(\tilde{a}'(\lambda_0)A+\tilde{b}'(\lambda_0)B)Q \tag{4.22}$$

Therefore, if $y\in X$ is such that $Qy=0$, the equation (4.17) is solvable, and the solution is given by

$$w=\;G'(\lambda_0)y-Q\,A\,(\tilde{a}'G)'(\lambda_0)y-Q\,B\,(\tilde{b}'G)'(\lambda_0)y$$

The proof is complete. ■

If $f\in L^p_{2\pi}(\mathbb{R},X)$ and $u\in L^p_{2\pi}(\mathbb{R},[D(A)])$ is a strong L^p-solution of (4.12), taking Fourier series on both sides of (4.12) we obtain

$$(I-\tilde{a}_kA-\tilde{b}_kB)\;\hat{u}(k)=\hat{f}(k)\,,\quad k\in\mathbb{Z}\,. \tag{4.23}$$

The equation (4.23) can be uniquely solved for each $k\neq k_n$, $n=1,\ldots,N$, with

$$\hat{u}(k)=(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}\hat{f}(k)$$

For k_n, $n=1,\ldots,N$, by Proposition 4.11 the equation (4.23) is solvable if and only if

$$Q_n\,\hat{f}(k_n)=0 \tag{4.24}$$

where Q_n is the residue of $F(\cdot)$ at $\lambda=ik_n$. If (4.24) holds, then by (4.19), the Fourier coefficients of the solution to (4.23) in k_n, $n=1,\ldots,N$ are given by

$$\hat{u}(k_n)=[G_n'(ik_n)\;-Q_n\,A(\tilde{a}'G_n)'(ik_n)-Q_n\,B(\tilde{b}'G_n)'(ik_n)]\,\hat{f}(k_n) \tag{4.25}$$

where G_n is the analytic function defined by (3.20).
We define the family of operators

$$M_k = \begin{cases} (I - \tilde{a}_k A - \tilde{b}_k B)^{-1} & k \in \mathbb{Z} \setminus \{ k_1, \dots, k_N \} \\ \\ G_j'(ik_j) - Q_j\, A(\tilde{a}' G_j)'(ik_j) - Q_j\, B(\tilde{b}' G_j)'(ik_j) & j = 1, \dots, N. \end{cases} \tag{4.26}$$

Since $ik \in \rho(A,B)$ for all $k \in \mathbb{Z} \setminus \{k_1, \dots, k_N\}$, $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, [D(A)])$.
The following main theorem give compatibility conditions on f which are necessary and sufficient for the existence of a strong L^p- solution of (4.12).

Theorem 4.12 *Let $a, b \in L^1(\mathbb{R}_+)$ be functions such that $\{\tilde{b}_k\}$ satisfies a Marcinkiewicz estimate of order 1 and $\{\tilde{a}_k\}$ is 1-regular. Suppose that* (4.16) *holds. Let A and B be closed linear operators defined on a UMD space X such that B is A-bounded. If $\{M_k\}_{k\in\mathbb{Z}}$, defined by* (4.26)*, is $R-$bounded then for every $f \in L^p_{2\pi}(\mathbb{R}, X)$ equation* (4.12) *has a strong L^p-solution if and only if $Q_n \hat{f}(k_n) = 0$, for every $n = 1, \dots, N$. In this case, all the strong solutions of* (4.12) *are given by*

$$u(t) = \lim_{n\to\infty} \sum_{\substack{k\,=\,-n \\ k\,\neq\, k_1,\dots,k_N}}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt}(I - \tilde{a}_k A - \tilde{b}_k B)^{-1} \hat{f}(k)$$

$$+ \sum_{j=1}^{N} e^{ik_j t}\, [G_j'(ik_j) - Q_j\, A(\tilde{a}' G_j)'(ik_j) - Q_j\, B(\tilde{b}' G_j)'(ik_j)]\, \hat{f}(k_j). \tag{4.27}$$

Proof. First we assume that for every $f \in L^p_{2\pi}(\mathbb{R}, X)$ there exists a function $v \in L^p_{2\pi}(\mathbb{R}, [D(A)])$ which is a strong L^p-solution equation (4.12). Taking Fourier series on both sides in (4.12) we obtain that $\hat{v}(k) \in D(A)$ and that

$$(I - \tilde{a}_k A - \tilde{b}_k B)\hat{v}(k) = \hat{f}(k), \text{ for all } k \in \mathbb{Z}.$$

For $\lambda \in \rho(A, B)$, and $k_1, k_2, ...k_N$ we have that

$$(\lambda - ik_n)F(\lambda)(I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B)\hat{v}(k_n) = (\lambda - ik_n)\,\hat{v}(k_n).$$

Letting, $\lambda \to ik_n$ it follows that

$$\lim_{\lambda\to ik_n} (\lambda - ik_n)F(\lambda)(I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B)\hat{v}(k_n) = 0$$

Since both limits $\lim\limits_{\lambda\to ik_n} (\lambda - ik_n)F(\lambda)$ and $\lim\limits_{\lambda\to ik_n} (I - \tilde{a}(\lambda)A - \tilde{b}(\lambda)B)\hat{v}(k_n)$ exist, we obtain that

$$Q_n\,(I - \tilde{a}(ik_n)A - \tilde{b}(ik_n)B)\hat{v}(k_n) = 0,$$

or, equivalently, $Q_n\hat{f}(k_n) = 0$, for all k_j, $j = 1, \ldots, N$. Hence by Proposition 4.11 equation (4.23) is solvable and

$$\hat{v}(k) = \begin{cases} (I - \tilde{a}_k A - \tilde{b}_k B)^{-1}\hat{f}(k) & k \in \mathbb{Z} \setminus \{ k_1, \ldots, k_N \} \\ \\ [G_j'(ik_j) - Q_j\, A(\tilde{a}'G_j)'(ik_j) - Q_j\, B(\tilde{b}'G_j)'(ik_j)]\hat{f}(k_j) & j = 1, \ldots, N \end{cases}$$

from which (4.27) follows.

Conversely, assume that $f \in L^p_{2\pi}(\mathbb{R}, X)$ and $Q_n\hat{f}(k_n) = 0$. We define $u(t)$ by (4.27). Then

$$\hat{u}(k) = M_k\, \hat{f}(k) \tag{4.28}$$

for all $k \in \mathbb{Z}$, where M_k is defined by (4.26). Note that $\hat{u}(k) \in D(A)$ for all $k \in \mathbb{Z}$. Since $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded, we claim that $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is also R-bounded. In fact, note that any finite family of operators is R-bounded, and for all $|k| > k_N$ we have

$$k\,(M_{k+1} - M_k) = k\,[(I - \tilde{a}_{k+1}A - \tilde{b}_{k+1}B)^{-1} - (I - \tilde{a}_k A - \tilde{b}_k B)^{-1}]$$

$$= F((k+1)i)\; k\,(\tilde{b}_{k+1} - \tilde{b}_k)\; B\; F(ki) + F((k+1)i)\; k\,\frac{\tilde{a}_{k+1} - \tilde{a}_k}{\tilde{a}_k}\; [(I - \tilde{b}_k B)F(ki) - I].$$

Since B is $A-$bounded, $\{\tilde{b}_k\}$ satisfies a Marcinkiewicz estimate of order 1 and $\{\tilde{a}_k\}$ is $1-$regular, the claim follows by Lemma 4.2 and properties of R-boundedness. By Theorem 2.24 we conclude that $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier and then there exists $v \in L^p_{2\pi}(\mathbb{R}; [D(A)])$ such that $\hat{v}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. Then the uniqueness theorem shows that $u = v$ for $t - a.e.$ It follows that $u \in L^p_{2\pi}(\mathbb{R}; [D(A)])$.
It remains to show that u satisfies equation (4.12). In order to simplify the notation we write

$$S_n[h(k)] := \sum_{\substack{k\,=\,-n \\ k\,\neq\,k_1,\ldots,k_N}}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt}h(k)$$

and

$$S_{N,n}[h(k_j)] := \sum_{j=1}^{N} \left(1 - \frac{|k_j|}{n+1}\right) e^{ik_jt}h(k_j)$$

and note that $\lim\limits_{n\to\infty} S_{N,n}[h(k_j)] = \sum\limits_{j=1}^{N} e^{ik_jt}h(k_j)$ and

$$S_n[h(k)] + S_{N,n}[h(k)] = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt}h(k) =: \sigma_n[h(k)]$$

Using the identity

$$(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}=I+\tilde{a}_kA(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}+\tilde{b}_kB(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}$$

valid for all $k\in\mathbb{Z}\setminus\{\,k_1,\dots,k_N\,\}$ we obtain

$$\begin{aligned}
u(t) = & \lim_{n\to\infty} S_n[(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}\,\hat{f}(k)] \\
& + \sum_{j=1}^{N} e^{ik_jt}\,[G_j'(ik_j)\,-Q_j\,A(\tilde{a}'G_j)'(ik_j)-Q_j\,B(\tilde{b}'G_j)'(ik_j)]\,\hat{f}(k_j) \\
= & \lim_{n\to\infty} S_n[\hat{f}(k)]+\lim_{n\to\infty} S_n[\tilde{a}_kA(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}\hat{f}(k)] \\
& + \lim_{n\to\infty} S_n[\tilde{b}_kB(I-\tilde{a}_kA-\tilde{b}_kB)^{-1}\,\hat{f}(k)] \\
& + \sum_{j=1}^{N} e^{ik_jt}\,[G_j'(ik_j)\,-Q_j\,A(\tilde{a}'G_j)'(ik_j)-Q_j\,B(\tilde{b}'G_j)'(ik_j)]\,\hat{f}(k_j)
\end{aligned}$$

Using (4.28) we have

$$\begin{aligned}
u(t) = & \lim_{n\to\infty}\{S_n[\hat{f}(k)] \;+\; S_{N,n}[\hat{f}(k_j)]\} \;-\; \lim_{n\to\infty} S_{N,n}[\hat{f}(k_j)] \\
& + \lim_{n\to\infty}\;\{S_n[\tilde{a}_k\,A\,\hat{u}(k)]\;+\; S_{N,n}[\tilde{a}(ik_j)\,A\,\hat{u}(k_j)]\} \\
& - \lim_{n\to\infty}\; S_{N,n}[\tilde{a}(ik_j)\,A\,\hat{u}(k_j)] \\
& + \lim_{n\to\infty}\;\{S_n[\tilde{b}_k\,B\,\hat{u}(k)]\;+\; S_{N,n}[\tilde{b}(ik_j)\,B\,\hat{u}(k_j)]\} \\
& - \lim_{n\to\infty}\; S_{N,n}[\tilde{b}(ik_j)\,B\,\hat{u}(k_j)]+\;\sum_{j=1}^{N} e^{ik_jt}\,\hat{u}(k_j) \\
= & \lim_{n\to\infty}\sigma_n[\hat{f}(k)]+\lim_{n\to\infty}\sigma_n[\tilde{a}_k\,A\,\hat{u}(k)]+\lim_{n\to\infty}\sigma_n[\tilde{b}_k\,B\,\hat{u}(k)] \\
& -\sum_{j=1}^{N} e^{ik_jt}\hat{f}(k_j)\;-\;\sum_{j=1}^{N} e^{ik_jt}[\tilde{a}(ik_j)\,A+\tilde{b}(ik_j)\,B]\,\hat{u}(k_j)\;+\;\sum_{j=1}^{N} e^{ik_jt}\,\hat{u}(k_j) \\
= & \; f(t)\;+\;(a*Au)(t)\;+\;(b*Bu)(t)-\sum_{j=1}^{N} e^{ik_jt}\hat{f}(k_j) \\
& + \sum_{j=1}^{N} e^{ik_jt}[I-\tilde{a}(ik_j)\,A-\tilde{b}(ik_j)\,B]\,\hat{u}(k_j)
\end{aligned}$$

Since $Q_k\hat{f}(k_j) = 0$, $j = 1, \ldots, N$, it follows from equality (4.28) and Proposition 4.11 (see also (4.22)) that

$$[I - \tilde{a}(ik_j)\, A - \tilde{b}(ik_j)\, B]\; [G_j'(ik_j) \; - Q_j\, A(\tilde{a}' G_j)'(ik_j) - Q_j\, B(\tilde{b}' G_j)'(ik_j)]\, \hat{f}(k_j) = \hat{f}(k_j)\,.$$

Hence

$$u(t) = f(t) \; + \; (a * Au)(t) \; + \; (b * Bu)(t),$$

proving the claim and the theorem.

■

Example 4.13

Let $X = l^2(\mathbb{Z})$ and define $Ax_n = (n + in)x_n$ with maximal domain. Clearly A does not generate a C_0-semigroup. We take $b(t) \equiv 0$ and $a(t) = e^{-t}$ in equation (4.12). Clearly $\tilde{a}_k = \frac{1}{ik+1}$ is 1-regular and $\frac{1}{\tilde{a}_k} = ik + 1 \in \rho(A)$ for all $k \in \mathbb{Z} \setminus \{1\}$. Moreover $\lambda_0 = i$ is a simple pole of $F(\lambda) = (I - \tilde{a}(\lambda)A)^{-1}$. It remains to show that the set $\{I - \tilde{a}_k A)^{-1}\}_{k \in \mathbb{Z}\setminus\{1\}}$ is bounded. In fact, for each $x = (x_n) \in l^2(\mathbb{Z})$ and $k \in \mathbb{Z} \setminus \{1\}$ we have

$$\begin{aligned} ||(I - \tilde{a}_k A)^{-1} x||^2 &= ||(ik+1)(ik+1-A)^{-1}x||^2 \\ &= \sum_{n \in \mathbb{Z}} \left| \frac{ik+1}{ik+1-n-in} x_n \right|^2 \\ &\le \sum_{n \in \mathbb{Z}} \frac{k^2+1}{(1-n)^2 + (k-n)^2} |x_n|^2 \\ &\le \sum_{n \in \mathbb{Z}} 2\frac{k^2+1}{(k-1)^2} |x_n|^2, \end{aligned}$$

then we obtain

$$\sup_{k \in \mathbb{Z}\setminus\{1\}} ||(I - \tilde{a}_k A)^{-1}|| \le 10.$$

We conclude by Theorem 4.12 that for every $f \in L^p_{2\pi}(\mathbb{R}, l^2(\mathbb{Z}))$ the equation

$$u(t,x) = \int_{-\infty}^{t} e^{-(t-s)}(u_x(s,x) - iu_x(s,x))ds + f(t,x) \quad x \in [0, 2\pi], \quad t \ge 0$$

with boundary values $u(t,0) = u(t,2\pi)$, has a strong L^p-solution if and only if $Q_1\,\hat{f}(1) = 0$.
To calculate Q_1 we note that $F(\lambda)x_n = \frac{\lambda+1}{\lambda+1-n-in} x_n$ and hence

$$(\lambda - i)F(\lambda)x_n = \frac{(\lambda-i)(\lambda+1)}{(\lambda-in)+(1-n)} x_n = \begin{cases} (\lambda+1)x_1 & n = 1 \\ \frac{(\lambda-i)(\lambda+1)}{(\lambda-in)+(1-n)} x_n & n \ne 1. \end{cases}$$

Then

$$Q_1 x_n := \lim_{\lambda \to i} (\lambda - i) F(\lambda) x_n = \begin{cases} (i+1)x_1 & n = 1 \\ 0 & n \neq 1. \end{cases}$$

Therefore if $f(t) = (f_n(t))$, then $Q_1 \hat{f}(1) = 0$ if and only if

$$(i+1) \int_0^{2\pi} e^{-it} f_1(t) dt = 0.$$

Chapter 5

Maximal Regularity of Delay Equations on the Real Line

5.1 Introduction

Partial differential equations with delay are a subject which has been extensively studied and there is an enormous literature on the subject. In an abstract way they can be written as

$$u'(t) = Au(t) + Fu_t + f(t), \qquad t \in \mathbb{R}, \tag{5.1}$$

where $(A, D(A))$ is a (unbounded) linear operator on a Banach space X, $u_t(\cdot) = u(t+\cdot)$ on $[-r, 0]$, $r > 0$, and the delay operator F is supposed to belong to $\mathcal{B}(C([-r, 0], X), X)$. In this chapter we are able to obtain necessary and sufficient conditions in order to guarantee well-posedness of the delay equation (5.1) in the Hölder spaces $C^\alpha(\mathbb{R}, X)$ $(0 < \alpha < 1)$, and under the condition that X is a B-convex space. However we stress that here A is not necessarily the generator of a C_0-semigroup.

We remark that the Fourier multiplier approach used allows us to give a direct treatment of the equation, in contrast with the approach using the correspondence between (5.1) and the solutions of the abstract Cauchy problem

$$\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t) \qquad t \geq 0,$$

where $\mathcal{A} = \begin{pmatrix} A & F \\ 0 & d/d\sigma \end{pmatrix}$. In this case the question of well-posedness of the delay equation reduces to the question whether or not the operator $(\mathcal{A}, D(\mathcal{A}))$ generates a C_0-semigroup; see [13, 14, 48] and the references therein.

5.2 Periodic case

In [41] Lizama, characterized existence and uniqueness of periodic solutions of delay equations (5.1), here the operator F is assumed to belong to $\mathcal{B}(L^p([-2\pi N, 0], X), X)$

for $1 \leq p < \infty$, $N \in \mathbb{N}$, and u_t is an element of $L^p([-2\pi N, 0], X)$ which is defined by $u_t(\theta) = u(t+\theta)$.
Denote by $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{B_\lambda\}_{\lambda\in\mathbb{R}} \subset \mathcal{B}(X)$ by $B_\lambda x = F(e_\lambda x)$. Defining the real spectrum of (5.1) by

$$\sigma(\triangle) = \{s \in \mathbb{R} \ : \ isI - A - B_s \in \mathcal{B}([D(A)], X) \ \text{ is not invertible } \}.$$

The author proved in [41] the following result.

Proposition 5.1 *Let A be a closed linear operator defined on a UMD space X. Suppose that $\mathbb{Z} \cap \sigma(\triangle) = \emptyset$. Then the following assertions are equivalent.*
(i) $\{ik(ikI - A - B_k)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.
(ii) $\{ik(ikI - A - B_k)^{-1}\}_{k\in\mathbb{Z}}$ is $R-$bounded.

We denote

$$H^{1,p}(\mathbb{T}; X) = \{u \in L^p(\mathbb{T}, X) \ : \ \exists\, v \in L^p(\mathbb{T}, X)\,,\ \hat{v}(k) = ik\hat{u}(k) \ \text{ for all } \ k \in \mathbb{Z}\}.$$

A function $u \in H^{1,p}(\mathbb{T}; X)$ is called a strong L^p-solution of (5.1) if $u(t) \in D(A)$ and equation (5.1) holds for almost all $t \in [0, 2\pi)$.
The main result in [41], says the following

Theorem 5.2 *Let X be a UMD space and let $A : D(A) \subset X \to X$ be a closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.*
(i) For every $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p-solution of (5.1).
(ii) $\mathbb{Z} \cap \sigma(\triangle) = \emptyset$ and $\{ik(ikI - A - B_k)^{-1}\}_{k\in\mathbb{Z}}$ is $R-$bounded.

5.3 Multipliers on the Real Line

Let X, Y be Banach spaces and let $0 < \alpha < 1$. We denote by $\dot{C}^\alpha(\mathbb{R}, X)$ the spaces

$$\dot{C}^\alpha(\mathbb{R}, X) = \{f : \mathbb{R} \to X : f(0) = 0, ||f||_\alpha < \infty\}$$

normed by

$$||f||_\alpha = \sup_{t\neq s} \frac{||f(t) - f(s)||}{|t-s|^\alpha}.$$

Let $\Omega \subset \mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all C^∞-functions in $\Omega \subseteq \mathbb{R}$ having compact support in Ω.
We denote by $\mathcal{F}f$ or $\hat{f}$ the Fourier transform, i.e.

$$(\mathcal{F}f)(s) := \int_{\mathbb{R}} e^{-ist} f(t) dt$$

$(s \in \mathbb{R}, f \in L^1(\mathbb{R}; X))$.
Following [8], we define $C^\alpha-$multipliers.

Definition 5.3 *Let $M : \mathbb{R}\backslash\{0\} \rightarrow \mathcal{B}(X,Y)$ be continuous. We say that M is a $\dot{C}^\alpha-$multiplier if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R},X) \rightarrow \dot{C}^\alpha(\mathbb{R},Y)$ such that*

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot M))(s)f(s)ds \tag{5.2}$$

for all $f \in C^\alpha(\mathbb{R},X)$ and all $\phi \in C_c^\infty(\mathbb{R}\backslash\{0\})$.

Here $(\mathcal{F}(\phi \cdot M))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)M(t)dt \in \mathcal{B}(X,Y)$. Note that L is well defined, linear and continuous (cf. [8, Definition 5.2]).
Define the space $C^\alpha(\mathbb{R},X)$ as the set

$$C^\alpha(\mathbb{R},X) = \{f : \mathbb{R} \rightarrow X : ||f||_{C^\alpha} < \infty\}$$

with the norm

$$||f||_{C^\alpha} = ||f||_\alpha + ||f(0)||.$$

Let $C^{\alpha+1}(\mathbb{R},X)$ be the Banach space of all $u \in C^1(\mathbb{R},X)$ such that $u' \in C^\alpha(\mathbb{R},X)$, equipped with the norm

$$||u||_{C^{\alpha+1}} = ||u'||_{C^\alpha} + ||u(0)||.$$

Observe from Definition 5.3 and

$$\int_{\mathbb{R}} (\mathcal{F}(\phi M))(s)ds = 2\pi(\phi M)(0) = 0,$$

that for $f \in C^\alpha(\mathbb{R},X)$ we have $Lf \in C^\alpha(\mathbb{R},X)$. Moreover, if $f \in C^\alpha(\mathbb{R},X)$ is bounded then Lf is bounded as well (see [8, Remark 6.3]).
The following multiplier theorem is due to Arendt-Batty and Bu [8, Theorem 5.3].

Theorem 5.4 *Let $M \in C^2(\mathbb{R}\backslash\{0\},\mathcal{B}(X,Y))$ be such that*

$$\sup_{t\neq 0} ||M(t)|| + \sup_{t\neq 0} ||tM'(t)|| + \sup_{t\neq 0} ||t^2M''(t)|| < \infty. \tag{5.3}$$

Then M is a $\dot{C}^\alpha$-multiplier.

Remark 5.5

If X is B-convex, in particular if X is a UMD space, Theorem 5.4 remains valid if condition 5.3 is replaced by the following weaker condition

$$\sup_{t\neq 0} ||M(t)|| + \sup_{t\neq 0} ||tM'(t)|| < \infty, \tag{5.4}$$

where $M \in C^1(\mathbb{R}\backslash\{0\},\mathcal{B}(X,Y))$ (cf. [8, Remark 5.5]).

We use the symbol $\mathcal{C}f$ for the Carleman transform:

$$(\mathcal{C}f)(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} f(t)dt & Re\lambda > 0 \\ \displaystyle -\int_{-\infty}^0 e^{-\lambda t} f(t)dt & Re\lambda < 0, \end{cases}$$

where $f \in L^1_{loc}(\mathbb{R}, X)$ is of subexponential growth; by this we mean

$$\int_{-\infty}^{\infty} e^{-\epsilon|t|}\|f(t)\|dt < \infty, \qquad \text{for each } \epsilon > 0.$$

For details of Carleman transform and examples see [7] and [44]. We remark that if $u' \in L^1_{loc}(\mathbb{R}, X)$ is of subexponential growth, then

$$(\mathcal{C}u')(\lambda) = \lambda(\mathcal{C}u)(\lambda) - u(0), \qquad Re\lambda \neq 0.$$

5.4 A Characterization

We consider in this section the equation

$$u'(t) = Au(t) + Fu_t + f(t), \qquad t \in \mathbb{R}, \tag{5.5}$$

where $A : D(A) \subseteq X \to X$ is a linear, closed operator; $f \in C^\alpha(\mathbb{R}, X)$ and, for $r > 0$, $F : C([-r, 0], X) \to X$ is a linear, bounded operator. Moreover u_t is an element of $C([-r, 0], X)$ which is defined as $u_t(\theta) = u(t + \theta)$ for $-r \leq \theta \leq 0$.

Example 5.6 Let $\mu : [-r, 0] \to \mathcal{B}(X)$ be of bounded variation. Let $F : C([-r, 0], X) \to X$ be the bounded operator given by the Riemann-Stieltjes integral

$$F(\phi) = \int_{-r}^0 \phi d\mu \text{ for all } \phi \in C([-r, 0], X).$$

An important special case consists of operators F defined by

$$F(\phi) = \sum_{k=0}^n C_k \phi(\tau_k), \qquad \phi \in C([-r, 0], X),$$

where $C_k \in \mathcal{B}(X)$ and $\tau_k \in [-r, 0]$ for $k = 0, 1, ..., n$. For concrete equations dealing with the above classes of delays operators see the monograph of Bátkai and Piazzera [13, Chapter 3].

Definition 5.7 *We say that (5.1) is C^α-well posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there is a unique function $u \in C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ such that (5.1) is satisfied.*

Denote by $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{F_\lambda\}_{\lambda\in\mathbb{R}} \subseteq \mathcal{B}(X)$ by

$$F_\lambda x = F(e_\lambda x), \text{ for all } \lambda \in \mathbb{R} \text{ and } x \in X. \tag{5.6}$$

We define the *real spectrum* of (5.5) by

$$\sigma(\Delta) = \mathbb{C} \setminus \{s \in \mathbb{R} : isI - F_s - A \text{ is invertible }\}.$$

Proposition 5.8 *Let X be a Banach space and let $A : D(A) \subset X \to X$ be a closed linear operator. Suppose that (5.5) is C^α-well posed. Then*

(i) $\mathbb{R} \cap \sigma(\Delta) = \emptyset$,

(ii) $\{i\eta(i\eta I - A - F_\eta)^{-1}\}_{\eta\in\mathbb{R}}$ *is bounded.*

Proof. Let $x \in D(A)$ and let $u(t) = e^{i\eta t}x$ for $\eta \in \mathbb{R}$. Then $u_t(s) = e^{it\eta}e^{is\eta}x$. Thus

$$F(u_t) = e^{it\eta}F(e_\eta x) = e^{it\eta}F_\eta x. \tag{5.7}$$

Now if $(i\eta - A - F_\eta)x = 0$, then $u(t)$ is a solution of equation (5.1) when $f \equiv 0$. Hence by uniqueness it follows that $x = 0$.
Now let $L : C^\alpha(\mathbb{R}, X) \to C^{\alpha+1}(\mathbb{R}, X)$ be the bounded operator which associate to each $f \in C^\alpha(\mathbb{R}, X)$ the unique solution $u \in C^{\alpha+1}(\mathbb{R}, X)$ of (5.1). In order to show surjectivity, let $y \in X$ and $f(t) = e^{it\eta}y$, $t \in \mathbb{R}$. Let $u(t)$ be the unique solution of (5.1) such that $L(f) = u$ and $s_0 \in \mathbb{R}$ be fixed. Next, we claim that $v(t) := u(t + s_0)$ and $w(t) := e^{i\eta s_0}u(t)$ both satisfy equation

$$\tau'(t) = A\tau(t) + F\tau_t + e^{is_o\eta}f(t), \qquad t \in \mathbb{R}, \tag{5.8}$$

First we notice that

$$v_t(s) = u(t + s_0 + s) = u_{t+s_0}(s)\,, \quad \text{with} \quad s \in [-r, 0]\,.$$

Hence $Fv_t = Fu_{t+s_0}$. Then an easy computation shows that $v(t)$ satisfies equation (5.8). On the other hand,

$$w_t(s) = w(t + s) = e^{i\eta s_0}u(t + s) = e^{i\eta s_0}u_t(s)\,, \quad \text{with} \quad s \in [-r, 0]\,.$$

Hence $Fw_t = e^{is_0\eta}Fu_t$. Thus

$$e^{i\eta s_0}u'(t) = e^{i\eta s_0}(Au(t) + F(u_t) + f(t)) = Aw(t) + Fw_t + e^{is_0\eta}f(t).$$

Thus $w(t)$ satisfies equation (5.8). By uniqueness again, we have that

$$u(t + s_0) = e^{i\eta s_0}u(t)$$

for all $t, s_0 \in \mathbb{R}$. In particular when $t = 0$ we obtain that

$$u(s_0) = e^{i\eta s_0}u(0),\ s_0 \in \mathbb{R}.$$

Now let $x = u(0) \in D(A)$. Since $u(t) = e^{i\eta t}x$ satisfy (5.1), by (5.7), we have

$$i\eta u(t) = Au(t) + Fu_t + e^{i\eta t}y = Au(t) + e^{i\eta t}F_\eta x + e^{i\eta t}y.$$

In particular if $t = 0$ we obtain that

$$i\eta x = Ax + F_\eta x + y,$$

since $x = u(0)$. Thus

$$(i\eta I - A - F_\eta)x = y \tag{5.9}$$

and hence $i\eta I - A - F_\eta$ is bijective. This shows assertion (i) of the proposition.
Next we notice that $u(t) = e^{i\eta t}(i\eta - A - F_\eta)^{-1}y$ by (5.9). Since $||e_\eta\, x||_\alpha = \gamma_\alpha|\eta|^\alpha||x||$, where $\gamma_\alpha = ||e_1||_\alpha = 2\sup_{t>0} t^{-\alpha}sin(t/2)$, thus

$$\begin{aligned}
\gamma_\alpha|\eta|^\alpha||i\eta(i\eta - A - F_\eta)^{-1}y|| &= ||e_\eta\; i\eta(i\eta - A - F_\eta)^{-1}y||_\alpha = ||u'||_\alpha \\
&\leq ||u||_{1+\alpha} = ||Lf||_{1+\alpha} \leq ||L||||f||_\alpha \\
&\leq ||L||(||f||_\alpha + ||f(0)||) \\
&= ||L||(||e_\eta\; y||_\alpha + ||y|| \\
&\leq ||L||(\gamma_\alpha|\eta|^\alpha + 1)||y||.
\end{aligned}$$

Hence it follows that

$$\sup_{|\eta|>1} ||i\eta(i\eta - A - F_\eta)^{-1}|| \leq ||L|| \sup_{|\eta|>1}(1 + \frac{1}{\gamma_\alpha|\eta|^\alpha}) < \infty$$

and since $\sup_{|\eta|\leq 1} ||i\eta(i\eta - A - F_\eta)^{-1}|| < \infty$ by continuity, it follows that (ii) holds.

■

Recall that a Banach space X has *Fourier type* p, where $1 \leq p \leq 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; X)$, where q is the conjugate index of p. For example, the space $L^p(\Omega)$, where $1 \leq p \leq 2$ has Fourier type p; X has Fourier type 2 if and only if X is isomorphic to a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1; X is B-convex if it has Fourier type p for some $p > 1$. Every uniformly convex space is B-convex.
Our main result in this Section, establishes that the converse of Proposition 5.8 is true.

Theorem 5.9 *Let A be a closed linear operator defined on a B-convex space X. Then the following assertions are equivalent*

(i) Equation (5.1) is C^α-well posed.

(ii) $\mathbb{R} \cap \sigma(\Delta) = \emptyset$ and $\sup_{\eta\in\mathbb{R}} ||i\eta(i\eta I - A - F_\eta)^{-1}|| < \infty$.

Proof.
$(ii) \Rightarrow (i)$. Define the operator $M(t) = (B_t - A)^{-1}$, with $B_t = itI - F_t$. Note that by hypothesis (ii) $M \in C^1(\mathbb{R}, \mathcal{B}(X, [D(A)]))$.
We claim that M is a $C^\alpha-$multiplier. In fact, by hypothesis it is clear that $\sup_{t\in\mathbb{R}} ||M(t)||_X < \infty$. On the other hand, we have

$$M'(t) = -M(t)\, B'_t\, M(t)$$

with $B'_t = iI - F'_t$ and $F'_t(x) = F(e'_t x)$ where $e'_t(s) = ise^{ist}$. Note that for each $x \in X$

$$||F_t x||_X \leq ||F(e_t x)||_X \leq ||F||\, ||e_t x||_\infty \leq \; ||F||\, ||x||_X\,, \tag{5.10}$$

and

$$||F'_t x||_X \leq ||F(e'_t x)||_X \leq ||F||\, ||e'_t x||_\infty \leq r||F|| \;\; ||x||_X. \tag{5.11}$$

Hence B'_t is uniformly bounded with respect to $t \in \mathbb{R}$ and we conclude from the hypothesis that

$$\sup_{t\in\mathbb{R}} ||tM'(t)||_X = \sup_{t\in\mathbb{R}} ||\, [tM(t)]\, B'_t\, M(t)||_X < \infty\,, \tag{5.12}$$

Note that $||\, [tM(t)]\, B'_t\, M(t)||_{[D(A)]} = ||\, M(t)\, B'_t\, tM(t)||_X + ||\, AM(t)\, B'_t\, tM(t)||_X$ but

$$||AM(t)B'_t tM(t)||_X = ||(B_t M(t) - I_X)B'_t tM(t)||_X \leq ||B_t M(t)B'_t tM(t)||_X + ||B'_t\, tM(t)||_X$$

from hypothesis (ii) we obtain that $\sup_{t\in\mathbb{R}} ||tM'(t)||_{[D(A)]} < \infty$.
Analogous $||M(t)||_{[D(A)]} < \infty$ and hence the claim follows from Theorem 5.4 and Remark 5.5.
Now, define $N \in C^1(\mathbb{R}, \mathcal{B}(X))$ by $N(t) = (id \cdot M)(t)$, where $id(t) := it$ for all $t \in \mathbb{R}$. We will prove that N is a $C^\alpha-$multiplier. In fact, with a direct calculation, we have

$$\begin{aligned} t\, N'(t) &= itM(t) + it^2 M'(t) \; = itM(t) + i[itM(t)]\, B'_t\, [itM(t)] \\ &= N(t) + iN(t)\, B'_t\, N(t). \end{aligned}$$

By hypothesis (ii) and (5.11) it follows that

$$\sup_{t\in\mathbb{R}} ||tN'(t)||_X \; \leq \; \sup_{t\in\mathbb{R}} ||N(t)||_X + \sup_{t\in\mathbb{R}} ||N(t)\, B'_t\, N(t)||_X < \; \infty\,,$$

hence from Theorem 5.4 and Remark 5.5 the claim is proved.
A similar calculation prove that $P \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{B}(X))$ defined by $P(t) = F_t\, M(t)$ is a $C^\alpha-$multiplier.
In fact, we have $t\, P'(t) = F'_t N(t) + F_t\, t\, M'(t)$, and hence from (5.10), (5.11) and (5.12) we obtain that $\sup_{t\in\mathbb{R}} ||P(t)||_X + \sup_{t\in\mathbb{R}} ||tP'(t)||_X < \infty$.

Let $f \in C^\alpha(\mathbb{R}, X)$. Since M, N and P are $C^\alpha-$multipliers, there exist $\bar{u} \in C^\alpha(\mathbb{R}, [D(A)])$, $v \in C^\alpha(\mathbb{R}, X)$ and $w \in C^\alpha(\mathbb{R}, X)$, respectively, such that

$$\int_{\mathbb{R}} \bar{u}(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi \cdot M)(s)f(s)\, ds\,, \tag{5.13}$$

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\psi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\psi \cdot id \cdot M)(s)f(s)\, ds\,, \tag{5.14}$$

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot F_{.} M)(s)f(s)\, ds\,, \tag{5.15}$$

for all ϕ, ψ, $\varphi \in C_c^\infty(\mathbb{R})$.
Note that for $x \in X$ and $\phi \in C_c^\infty(\mathbb{R})$ we have

$$\mathcal{F}(\phi\, F_{.}\, M)(s)\, x = \int_{\mathbb{R}} e^{-ist}\phi(t)\, F_t\, M(t)\, x\, dt = \int_{\mathbb{R}} e^{-ist}\phi(t)\, F(e_t\, M(t)\, x)\, dt\,. \tag{5.16}$$

where $\int_{\mathbb{R}} e^{-ist}\phi(t)\, e_t\, M(t)\, x\, dt \;\in C([-r, 0], X)$. Now, for all $\theta \in [-r, 0]$ we have

$$\left\| \int_{\mathbb{R}} e^{-ist}\phi(t)\, e_t(\theta)\, M(t)\, x\, dt \right\|_X \leq \int_{\mathbb{R}} |\, \phi(t)\, |\; ||M(t)\, x\, ||_X dt.$$

Since F is bounded, we deduce that

$$\mathcal{F}(\phi \cdot F_{.}\, M)(s)x = F(\mathcal{F}(\phi \cdot e_{.}\, M)(s)\, x)\,. \tag{5.17}$$

Furthermore, observe that for $\theta \in [-r, 0]$ fixed we have that $e_{.}(\theta)\phi \in C_c^\infty(\mathbb{R})$. Using (5.13) we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \bar{u}(s+\theta)(\mathcal{F}\phi)(s)ds &= \int_{\mathbb{R}} \bar{u}(s+\theta) \int_{\mathbb{R}} e^{-ist}\phi(t)\, dt\;\, ds \\
&= \int_{\mathbb{R}} \bar{u}(s+\theta) \int_{\mathbb{R}} e^{-i(s+\theta)t} e_t(\theta)\phi(t)\, dt\;\, ds \\
&= \int_{\mathbb{R}} \bar{u}(s+\theta)(\mathcal{F}e_{.}(\theta)\phi)(s+\theta)\;\, ds \\
&= \int_{\mathbb{R}} \bar{u}(s)(\mathcal{F}e_{.}(\theta)\phi)(s)\;\, ds \\
&= \int_{\mathbb{R}} \mathcal{F}(e_{.}(\theta)\phi \cdot M)(s)f(s)\;\, ds,
\end{aligned}$$

hence $\int_{\mathbb{R}} \bar{u}_s(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(e_{.}\phi \cdot M)(s)f(s)\;\, ds\,.$
Since the function $\theta \to \int_{\mathbb{R}} \bar{u}_s(\theta)(\mathcal{F}\phi)(s)ds \;\in C([-r, 0], X)$ (see [8, p.3]), due to the boundedness of F and (5.17) it follows that

$$\int_{\mathbb{R}} \mathcal{F}(\phi \cdot\ F_{\cdot} M)(s) f(s) ds = \int_{\mathbb{R}} F \mathcal{F}(\phi \cdot\ e_{\cdot} M)(s) f(s) ds = \int_{\mathbb{R}} F \bar{u}_s \, (\mathcal{F}\phi)(s) ds, \qquad (5.18)$$

for all $\phi \in C_c^\infty(\mathbb{R})$. Since $F_{\cdot} M$ is $C^\alpha-$multiplier, we obtain from (5.15)

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\phi)(s) ds = \int_{\mathbb{R}} F \bar{u}_s \, (\mathcal{F}\phi)(s) ds \, .$$

for all $\phi \in C_c^\infty(\mathbb{R})$. We conclude that there exists $y_1 \in X$ satisfying $w(t) = F\bar{u}_t + y_1$, proving that $F\bar{u}_{\cdot} \in C^\alpha(\mathbb{R}, X)$.
Choosing $\phi = id \cdot \psi$ in (5.13) we obtain from (5.14) that

$$\int_{\mathbb{R}} \bar{u}(s) \mathcal{F}(id \cdot \psi)(s) \, ds = \int_{\mathbb{R}} v(s) \, (\mathcal{F}\psi)(s) \, ds \, , \qquad (5.19)$$

and it follows from Lemma 6.2 in [8] that $\bar{u} \in C^{\alpha+1}(\mathbb{R}, X)$ and $\bar{u}' = v + y_2$ for some $y_2 \in X$.
Since $(id \, I - F_{\cdot} - A) \, M = I$ we have $id \cdot M = I + F_{\cdot} M + AM$ and replacing in (5.14) gives

$$\begin{aligned} \int_{\mathbb{R}} v(s) \, (\mathcal{F}\phi)(s) \, ds \;\; &= \int_{\mathbb{R}} \mathcal{F}(\phi \cdot (I + F_{\cdot} M + AM))(s) \, f(s) \, ds \\ &= \int_{\mathbb{R}} (\mathcal{F}\phi)(s) \, f(s) \, ds + \int_{\mathbb{R}} \mathcal{F}(\phi \cdot F_{\cdot} \, M)(s) \, f(s) \, ds \qquad (5.20) \\ &+ \int_{\mathbb{R}} \mathcal{F}(\phi \cdot A \, M)(s) \, f(s) \, ds \, , \end{aligned}$$

for all $\phi \in C_c^\infty(\mathbb{R})$.
Since $\bar{u}(t) \in D(A)$ and $\mathcal{F}(\phi \cdot M)(s)x \in D(A)$ for all $x \in X$, using the fact that A is closed and setting (5.13) and (5.18) in (5.20) we obtain that

$$\begin{aligned} \int_{\mathbb{R}} v(s) \, (\mathcal{F}\phi)(s) \, ds \;\; &= \int_{\mathbb{R}} F\bar{u}_s \, (\mathcal{F}\phi)(s) ds + \int_{\mathbb{R}} A\bar{u}(s) (\mathcal{F}\phi)(s) \, f(s) \, ds \\ &+ \int_{\mathbb{R}} f(s) \, (\mathcal{F}\phi)(s) \, ds \, , \end{aligned} \qquad (5.21)$$

for all $\phi \in C_c^\infty(\mathbb{R})$.
By Lemma 5.1 in [8] this implies that for some $y_3 \in X$ one has

$$v(t) = F\bar{u}_t + A\bar{u}(t) + f(t) + y_3, \quad t \in \mathbb{R} \, .$$

Consequently, $\bar{u}'(t) = v(t) + y_2 = F\bar{u}_t + A\bar{u}(t) + f(t) + y$ where $y = y_2 + y_3$. In particular $A\bar{u} \in C^\alpha(\mathbb{R}, X)$. Now, by hypothesis we can define $x = (A + F)^{-1} y \in D(A)$, and then is clear that $u(t) := \bar{u}(t) + x$ is in $C^{\alpha+1}(\mathbb{R}, X) \cap C^{\alpha}(\mathbb{R}, [D(A)])$ and satisfies (5.1). We have shown that a solution of (5.1) exists.

In order to shown uniqueness, suppose that

$$u'(t) = Au(t) + Fu_t\,, \qquad t \in \mathbb{R}, \tag{5.22}$$

where $u \in C^{\alpha+1}(\mathbb{R}, X) \cap C^{\alpha}(\mathbb{R}, [D(A)])$ and, as showed, $Au, Fu. \in C^{\alpha}(\mathbb{R}, X)$.

We claim that $(\mathcal{C}u.)(\lambda) \in C([-r,0], X)$ for $Re\lambda \neq 0$. In fact, let $Re\lambda > 0$. Then

$$\begin{aligned} ||e^{-\lambda t}u_t||_\infty &= \sup_{\theta\in[-r,0]} ||e^{-\lambda t}u(t+\theta)||_X \leq \sup_{\theta\in[-r,0]} e^{-Re\lambda t}(1+|t+\theta|^\alpha) \\ &\leq e^{-Re\lambda t}(1+(|t|+r)^\alpha). \end{aligned}$$

Since $e^{-Re\lambda t}(1+(|t|+r)^\alpha) \in L^1(\mathbb{R}_+)$ applying the dominated convergence theorem, we obtain the claim. Analogously we obtain the claim for $Re\lambda < 0$.
Now, note that for $Re\lambda > 0$ and $\theta \in [-r, 0]$

$$\begin{aligned} \int_0^\infty e^{-\lambda t}u_t(\theta)dt &= \int_0^\infty e^{-\lambda t}u(t+\theta)dt \\ &= \int_\theta^\infty e^{-\lambda(t-\theta)}u(t)dt \\ &= e^{\lambda\theta}\int_\theta^\infty e^{-\lambda t}u(t)dt \\ &= e^{\lambda\theta}\left(\int_0^\infty e^{-\lambda t}u(t)dt + \int_\theta^0 e^{-\lambda t}u(t)dt\right) \\ &= e^{\lambda\theta}(\mathcal{C}u)(\lambda) + e^{\lambda\theta}\int_\theta^0 e^{-\lambda t}u(t)dt. \end{aligned}$$

Analogously if $Re\lambda < 0$ and $\theta \in [-r, 0]$, then

$$\begin{aligned} -\int_{-\infty}^0 e^{-\lambda t}u_t(\theta)dt &= -\int_{-\infty}^0 e^{-\lambda t}u(t+\theta)dt \\ &= -\int_{-\infty}^\theta e^{-\lambda(t-\theta)}u(t)dt \\ &= -e^{\lambda\theta}\left(\int_{-\infty}^0 e^{-\lambda t}u(t)dt - \int_\theta^0 e^{-\lambda t}u(t)dt\right) \\ &= e^{\lambda\theta}(\mathcal{C}u)(\lambda) + e^{\lambda\theta}\int_\theta^0 e^{-\lambda t}u(t)dt. \end{aligned}$$

Since F is bounded, we obtain that

$$(\mathcal{C}Fu.)(\lambda) = F(\mathcal{C}u.)(\lambda) = Fg(\mathcal{C}u)(\lambda) + Fgh, \quad \text{for } Re(\lambda) \neq 0 \tag{5.23}$$

where $g(\theta) = e^{\lambda\theta}$ and $h(\theta) = \displaystyle\int_\theta^0 e^{-\lambda t}u(t)dt$. Note that $gh \in C([-r,0], X)$.

Since $(\mathcal{C}u')(\lambda) = \lambda(\mathcal{C}u)(\lambda) - u(0)$ for $Re(\lambda) \neq 0$, one has $(\mathcal{C}u)(\lambda) \in D(A)$ and

$$(\mathcal{C}u')(\lambda) = (\mathcal{C}Au)(\lambda) + (\mathcal{C}Fu.)(\lambda)\,, \quad \text{for} \quad Re(\lambda) \neq 0\,. \tag{5.24}$$

Using the fact that A is closed, from (5.23) and (5.24) we get

$$(\lambda I - Fg - A)\,(\mathcal{C}u)(\lambda) = u(0) + Fgh \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus i\mathbb{R}.$$

Since $i\mathbb{R} \subset \rho(A)$, it follows that the Carleman spectrum $sp_{\mathcal{C}}(u)$ of u is empty. Hence $u \equiv 0$ by [7, Theorem 4.8.2]. ∎

We denote by $\mathcal{K}_F(X)$ the class of operators in X satisfying (ii) in the above theorem. If $A \in \mathcal{K}_F(X)$ we have $u', Au, Fu. \in C^\alpha(\mathbb{R}, X)$, and hence we deduce the following result.

Corollary 5.10 *Let X be B-convex and $A \in \mathcal{K}_F(X)$. Then*

(i) (5.1) has a unique solution in $Z := C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ if and only if $f \in C^\alpha(\mathbb{R}, X)$.

(ii) There exists a constant $M > 0$ independent of $f \in C^\alpha(\mathbb{R}, X)$ such that

$$\|u'\|_{C^\alpha(\mathbb{R},X)} + \|Au\|_{C^\alpha(\mathbb{R},X)} + \|Fu.\|_{C^\alpha(\mathbb{R},X)} \leq M\|f\|_{C^\alpha(\mathbb{R},X)}. \tag{5.25}$$

Remark 5.11 The inequality (5.25) is a consequence of the closed graph theorem and known as the *maximal regularity property* for equation (5.1). From it we deduce that the operator L defined by

$$D(L) = Z$$
$$(Lu)(t) = u'(t) - Au(t) - Fu_t$$

is an isomorphism onto. In fact, since A is closed, the space Z becomes a Banach space under the norm

$$||\ u\ ||_Z \ := \|u\|_{C^\alpha(\mathbb{R},X)} + \|u'\|_{C^\alpha(\mathbb{R},X)} + \|Au\|_{C^\alpha(\mathbb{R},X)}.$$

Such isomorphisms are crucial for the treatment of nonlinear versions of (5.1) by means of an argument using the implicit function theorem (see [5]).

A second way to study semilinear problems is the following. Assume X be B-convex and $A \in \mathcal{K}_F(X)$ and consider the semilinear problem

$$u'(t) = Au(t) + Fu_t + f(t, u(t)), \quad t \geq 0. \tag{5.26}$$

Define the Nemytskii's superposition operator $N : Z \to C^\alpha(\mathbb{R}, X)$ given by $N(v)(t) = f(t, v(t))$ and the linear operator

$$S : C^\alpha(\mathbb{R}, X) \to Z$$

by $S(g) = u$ where u is the unique solution of the linear problem

$$u'(t) = Au(t) + Fu_t + g(t).$$

Then we have to show that the operator $H : Z \to Z$ defined by $H = SN$ has a fixed point. Note that Z is defined as a subspace of $C^\alpha(\mathbb{R}, X)$.
For example, if we assume that S is a compact operator, and we suppose that for some $M > 0$,

$$\sup_{\|u\|_{C^\alpha} \leq M} \|f(\cdot, u(\cdot))\|_{C^\alpha(\mathbb{R},X)} \leq M/\|S\|,$$

then one may apply Schauder's fixed point theorem to H in the ball $\{u \in Z : \|u\|_{C^\alpha} \leq M\}$ to get existence of a strong solution, i.e. $u \in Z$ such that (5.26) is satisfied. This way one obtain the existence of global solutions.

A third way is to show that H is a strict contraction on an interval $(0, \tau)$ if $\tau > 0$ is small enough and f satisfies some condition of Lipschitz type. Thus the Banach fixed point theorem shows that H has a fixed point which is a strong (local) solution of (5.26). For related information on this subject we refer to Amann [4] where results in quasilinear delay equations involving the method of maximal regularity are presented.

We finish this chapter with the following result which give us a useful criterion to verify condition (ii) in the above theorem.

Theorem 5.12 *Let X be a B-convex space and let $A : D(A) \subset X \to X$ be a closed linear operator such that $i\mathbb{R} \subset \rho(A)$ and $\sup_{s\in\mathbb{R}} \|A(isI - A)^{-1}\| =: M < \infty$. Suppose that*

$$||F|| < \frac{1}{||A^{-1}||M}. \tag{5.27}$$

Then for each $f \in C^\alpha(\mathbb{R}, X)$ there is a unique function $u \in C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ such that (5.1) is satisfied.

Proof. From the identity

$$isI - A - F_s = (I - F_s(isI - A)^{-1})(isI - A) \quad s \in \mathbb{R},$$

it follows that $isI - A - F_s$ is invertible whenever $||F_s(isI - A)^{-1}|| < 1$. Next observe that

$$||F_s|| \leq ||F||, \tag{5.28}$$

and hence

$$||F_s(isI - A)^{-1}|| = ||F_s A^{-1} A(isI - A)^{-1}|| \leq ||F|| ||A^{-1}|| M =: \alpha.$$

Therefore, under the condition (5.27) we obtain that $\mathbb{R} \cap \sigma(\Delta) = \emptyset$, and the identity

$$(isI - A - F_s)^{-1} = (isI - A)^{-1}(I - F_s(isI - A)^{-1})^{-1} = (isI - A)^{-1}\sum_{n=0}^{\infty}[F_s(isI - A)^{-1}]^n. \quad (5.29)$$

For all $n \in \mathbb{N}$ we have
$\|is(isI - A)^{-1}[F_s(isI - A)^{-1}]^n\|$

$$\begin{aligned} &\leq \|is(isI - A)^{-1}\|[\|F_sA^{-1}A(isI - A)^{-1}\|]^n \\ &\leq \|is(isI - A)^{-1})[\|F_sA^{-1}\|]^n[\|(A(is - A)^{-1}\|]^n \\ &\leq \|is(isI - A)^{-1}\|\|A^{-1}\|^n[\|F_s\|]^n[\|A(isI - A)^{-1}\|]^n. \end{aligned}$$

By (5.28) we obtain

$$\begin{aligned} \|is(isI - A)^{-1}[F_s(isI - A)^{-1}]^n\| &\leq \|is(isI - A)^{-1}\|\|A^{-1}\|^n\|F\|^n M^n \\ &= \|is(isI - A)^{-1}\|\alpha^n. \end{aligned}$$

Finally by (5.29), one has

$$\|is(isI - A - F_s)^{-1}\| \leq \|is(isI - A)^{-1}\|\frac{1}{1-\alpha} \leq \frac{M+1}{1-\alpha}.$$

This proves that $\{is(isI - A - F_s)^{-1}\}$ is bounded and the conclusion follows from Theorem 5.9.

■

Chapter 6

Summary

The modern extension of the classical theory of Fourier multiplier to operator-valued multipliers give us tools for study a variety of integro-differential equations. In order to efficiently apply the abstract machinery to obtain Fourier multipliers theorems for vector-valued functions, some conditions on the geometry of the Banach spaces are required. We studied UMD spaces and give some theorems related to them. We also review the notion of $R-$boundedness of operators families. This concept is very important to characterize operator-valued multipliers. Furthermore, we have given examples of the modern multiplier results, which combine the UMD theory and the notion of $R-$boundedness.

We establish and prove new properties of $k-$regular sequences $(k = 1, 2, 3)$. For the scalar case, we prove certain equivalences with sequences that satisfies Marcinkiewicz estimates. This characterization plays a fundamental role in the proof of our main theorems.

We obtain solution with maximal regularity of (1.1) on periodic vector-valued Besov spaces. For this, we use the Fourier multipliers technique to characterize periodic solutions solely in terms of spectral properties of the data. Note that in this case, only conditions of boundedness over the resolvent are required. In comparison with [33], our assumptions on the kernel are weaker. We also reformulate and give a new proof for the existence of solutions. (See Theorem 2.12 in [33]) .

Additionally, we characterize existence and uniqueness of periodic solutions for the linear perturbed Volterra equation (1.2), in vector-valued Lebesgue spaces. One difference with problem (1.1), is that here the result involves $UMD-$spaces and $R-$boundedness. These assumptions are fundamental for the resolvent operator to be a multiplier.

For equations (1.1) and (1.2) we obtain a new formula for the solution if a finite amount of ik, $k \in \mathbb{Z}$ does not belong to the resolvent set of A.

Finally, we characterize existence and uniqueness of solutions, but now on the line, for the inhomogeneous abstract delay equation (1.3) in Hölder spaces and under the condition that X is a B-convex space. The main tool used here is the theory of operator valued Fourier multipliers on the line. We finish the work with the study of a semilinear case associated to equation (1.3).

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