UNIVERSIDAD
DE SANTIAGO
DE CHILE

unesp
UNIVERSIDADE ESTADUAL PAULISTA
"JÚLIO DE MESQUITA FILHO"
Câmpus de São José do Rio Preto

Laura Rezzieri Gambera

# Zero-one law for (a,k)-regularized resolvent families and the Blackstock-Crighton-Westervelt equation on Banach spaces

São José do Rio Preto
2020

Laura Rezzieri Gambera

# Zero-one law for (a,k)-regularized resolvent families and the Blackstock-Crighton-Westervelt equation on Banach spaces

Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Orientadora pela UNESP: Profa. Dra. Andréa Cristina Prokopczyk Arita
Orientador pela Usach: Prof. Dr. Carlos Enrique Lizama Yañez

Financiadora: CAPES

São José do Rio Preto
2020

Laura Rezzieri Gambera

# Zero-one law for (a,k)-regularized resolvent families and the Blackstock-Crighton-Westervelt equation on Banach spaces

Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

Financiadora: CAPES

Comissão Examinadora

Profa. Dra. Andréa Cristina Prokopczyk Arita
Orientadora - UNESP - Câmpus São José do Rio Preto

Prof. Dr. Carlos Enrique Lizama Yañez
Orientador - Universidad de Santiago de Chile

Prof. Dr. Waldemar Donizete Bastos
Departamento de Matemática - UNESP - Câmpus de São José do Rio Preto

Prof. Dr. Eduardo Alex Hernández Morales
Departamento de Computação e Matemática - USP - Câmpus de Ribeirão Preto

Prof. Dr. José Paulo Carvalho dos Santos
Departamento de Matemática - Universidade Federal de Alfenas

São José do Rio Preto
17 de Abril de 2020

*Dedico a minha família*

# ACKNOWLEDGMENT

To my advisors Prof. Dr. Andréa Cristina Prokopczyk Arita and Prof. Dr. Carlos Lizama, without whom this work would not have come to fruition.

To my parents Silma and Agostinho, my brothers Artur and José Leonardo, my sisters-in-law Mariela and Tatiana, and my niece Alice for all the love, support, good examples, encouragement and teachings in each stage of my life.

To Eric Busatto Santiago, for all the constancy, tolerance and patience with me. For the companionship, affection and support through all the days we shared.

To Master Raumsol for the teachings and to my dear fellow disciples for the numerous exchanges and shared experiences.

To every professor who was part of my academic life, from college to postgraduate years, for everything I have learned during all these years at Unesp in São José do Rio Preto. In particular, to Prof. Dr. Tatiana Miguel Rodrigues who, besides being a teacher, was a great friend during this period.

To professors Eduardo, Waldemar and José Paulo for accepting to compose the examining board.

To all my undergraduate and graduate friends, in particular, Marcelo Bongarti and Willian Nunes, for all the suggestions and assistance during the development of this work.

To all who were with me during my time living in Chile, especially Rocío, Facundo, Fabio, Natalia, Daniel, Sofía and Yanina. Without your affection I could not have won this stage.

This work was carried out with the support of the Coordination for the Improvement of Higher Education Personnel - Brazil (CAPES) - Financing Code 001, to which I am grateful.

*"Para triunfar es necesario vencer, para vencer es necesario luchar, para luchar es necesario estar preparado, para estar preparado es necesario proveerse de una gran entereza de ánimo y una paciencia a toda prueba."*

(Carlos Bernardo González Pecotche, 1957, p. 133)

## ***ABSTRACT***

*This work presents some results of the theory of the (a,k)-regularized resolvent families, that are the main tool used in this thesis. Related with this families, one result proved in this work is the zero-one law, providing new insights on the structural properties of the theory of (a,k)-regularized resolvent families including strongly continuous semigroups, strongly continuous cosine families, integrated semigroups, among others. Moreover, an abstract nonlinear degenerate hyperbolic equation is considered, that includes the Blackstock-Crighton-Westervelt equation. By proposing a new approach based on strongly continuous semigroups and resolvent families of operators, it is proved an explicit representation of the strong and mild solutions for the linearized model by means of a kind of variation of parameters formula. In addition, under nonlocal initial conditions, a mild solution of the nonlinear equation is established.*

*Keywords: (a,k)-regularized resolvent families, Zero-one law, Blackstock-Crighton-Westervelt equation, well-posedness.*

# RESUMO

Este trabalho apresenta alguns resultados da teoria de famílias resolvente (a,k)-regularizadas, que é a principal ferramenta utilizada nesta tese. Relacionado com estas famílias, um resultado provado neste trabalho é a lei zero-um, que fornece novas percepções de propriedades estruturais da teoria de famílias resolventes (a,k)-regularizadas, incluindo os semigrupos fortemente contínuos, as famílias cosseno fortemente contínuas, os semigrupos integrados, entre outras. Além disso, uma equação hiperbólica degenerada não-linear abstrata é considerada, a qual inclui a equação de Blackstock-Crighton-Westervelt. Propondo uma nova abordagem baseada em semigrupos fortemente contínuos e famílias resolvente, é demonstrada uma representação explícita das soluções forte e branda para a linearização do modelo por uma espécie de método de variação dos parâmetros. Por fim, sob condições iniciais não-locais, uma solução branda da equação não-linear é estabelecida.

Palavras-chave: famílias resolvente (a,k)-regularizadas, Lei zero-um, equação de Blackstock-Crighton-Westervelt, boa colocação.

# RESUMEN

Este trabajo presenta algunos resultados de la teoría de las familias resolvente (a,k)-regularizadas, que es la principal herramienta utilizada en esta tesis. Relacionado con estas familias, uno de los resultados demostrados en este trabajo es la ley cero-uno, proveyendo nuevas percepciones de propiedades estructurales de la teoría de las familias resolventes (a,k)-regularizdas, incluyendo los semigrupos fuertemente continuos, las familias coseno fuertemente continuas, los semigrupos integrados, entre otras. Además, una ecuación degenerada no lineal abstracta es considerada, la cual incluye la ecuación de Blackstock-Crighton-Westervelt. Proponiendo un nuevo enfoque basado en semigrupos fuertemente continuos y familias resolvente, es demostrada una representación explícita de las soluciones fuerte y débil de la linealización del modelo por una especie de método de variación de parámetros. Por fin, bajo condiciones iniciales no locales, una solucion débil de la ecuación no lineal es establecida.

Palabras clave: familias resolvente (a,k)-regularizadas, Ley cero-uno, equación de Blackstock-Crighton-Westervelt, buen planteamiento.

# Contents

# Introduction

Among the most used tools for the study of existence of solutions and qualitative properties of partial differential equations is the theory of operators. This aproach consists in transform a partial differential equation into an abstract problem by means of the transform theory (Laplace, Fourier), providing a more viable and transparent view of the original problem. The first and most famous tool that uses such idea is the theory of strongly continuous semigroup.

Later, the theory of strongly continuous cosine family appears. Concerning these two families of operators, there are uniqueness and qualitative properties of solutions for a large number of partial differential equations, but a lot of evolution equations are not well-posed. For this reason, many alternative tools have been in development, consisting of families of operators with some regularity.

The theory of $(\mathrm{a}, \mathrm{k})$-regularized resolvent families unify many of the theories of bounded linear operators, such as semigroups, cosine families, integrated semigroups, etc. One of its advantages is that it is an useful tool for many evolution equations, such as integral equations, differential equations and integrodifferential equations. This theory was introduced in the literature by Lizama in 2000, and it still is an ongoing research theme, which may provide many future works.

In this work the theory of $(\mathrm{a}, \mathrm{k})$-regularized resolvend families is used to study two problems that up to the present moment, by our knowledge, are not considered in the literature. The first one is related to a specific property of this family of bounded linear operators, and the other one consisting in apply this theory to study a particular differential equation in a Banach space.

This thesis is composed of four themed chapters. The first chapter presents some of the notation and results used throughout the work.

The second chapter provides an overview of the theory of the $(\mathrm{a}, \mathrm{k})$-regularized resolvent families, including definitions and classical theorems stated in the literature.

In the third chapter an intriguing structural property familiar of these families is studied. This property is known as zero-one law. The results of the first section of this chapter were published in [25].

The fourth chapter is concerned with the study of the Blackstock-Crighton-Westervelt equation in a generalized abstract form defined in a Banach space. This equation is an example of nonlinear acoustic models, which solutions describes the acoustic velocity potential in some fluid. Using the families presented on the second chapter it is possible to prove well-posedness and strong solution for this equation in Banach spaces, a local mild solutions to the semilinear problem, and a mild solution to this problem with nonlocal initial conditions. The results of this chapter have been submitted for publication.

# 1 Preliminaries

This chapter contains some of the notations used throughout this thesis and some results scattered in the literature which are important to the development of the theory in this work but is also of independent interest.

## 1.1 Notation

Most notation used throughout this thesis are fairly standard in the modern mathematical literature. For instance, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural, real and complex numbers respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\}$. If $(M, d)$ is a metric space and $N \subset M$, then $\overset{\circ}{N}$ and $\overline{N}$ designate the interior and the closure of $N$ respectively.

The capital letters $X$, $Y$ and $Z$ will denote Banach spaces endowed with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$, whith the subscripts being dropped when there is no possibility of confusion.

If $A$ is a linear operator defined on $X$, $D(A)$ denotes the domain of A, while $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of $A$ respectively. The spectrum can be decomposed into the point spectrum, the residual spectrum and the continuous spectrum of $A$, which are denoted by $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$, respectively.

Even when $A$ is a closed linear operator, its domain $D(A) \subset X$ is not necessarily closed when inherits the topology of $X$, but it is a Banach space when equipped with the graph norm of $A$, $\|x\|_A = \|x\| + \|Ax\|$, hencefoward denoted by $X_A$. Moreover, if A is, in addition, bounded, it admits a natural and norm preserving extension to the closure of $D(A)$, i.e, there exists $\widetilde{A} : \overline{D(A)} \to X$ such that $\|A\| = \|\widetilde{A}\|$. A proof of this result can be found in [12, Corollary 2.3, p. 33]

If a sequence $(x_n)_{n=0}^{\infty} \subset X$ converges to $x \in X$, the convergence is denoted by $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

The space of bounded linear operators from $X$ to $Y$ will be denoted by $\mathcal{L}(X, Y)$ ($\mathcal{L}(X) = \mathcal{L}(X, X)$ for short) and will be endowed with the uniform operator norm which makes it a Banach space. It follows from the definition of $\mathcal{L}(X, Y)$ and its norm that the uniform limit of a sequence of bounded operators is a bounded operator. However, the classic result mentioned below, a direct consequence of the Banach Steihauss Theorem, states that pointwise limit of a sequence of bounded operators also defines a bounded operator (see [12, Theorem 2.2, p. 33]).

If $(M, d)$ is a metric space, with $M$ compact and $X$ is a Banach space, then $C(M; X)$ denotes the space of all continuous functions $f : M \to X$. This space becomes a Banach space when endowed with the sup-norm

$$\|f\|_0 = \sup_{t \in M} \|f(t)\|.$$

The space of all functions $f : M \to X$ which are uniformly Lipschitz-continuous is denoted by $\mathrm{Lip}(M; X)$, and

$$\|f\|_{\mathrm{Lip}} = \sup_{t \neq s} \frac{\|f(t) - f(s)\|}{d(t,s)}.$$

If $(\Omega, \Sigma, \mu)$ is a measure space then $L^p(\Omega, \Sigma, \mu; X)$, $1 \leq p < \infty$, denotes the space of equivalence classes of all Bochner-measurable functions $f : \Omega \to X$ such that $\|f(\cdot)\|^p$ is integrable, where the equivalence relation is given by $f \sim g$, that is, $f$ equal $g$ almost everywhere. This space is also a well-known Banach space when equipped with the norm

$$\|f\|_p = \left( \int_\Omega \|f(t)\|^p d\mu(t) \right)^{\frac{1}{p}}.$$

Similarly, $L^\infty(\Omega, \Sigma, \mu; X)$ denotes the space of equivalence classes of Bochner-measurable essentially bounded functions $f : \Omega \to X$, and its norm is defined by

$$\|f\|_\infty = \operatorname{ess\,sup}_{t \in \Omega} \|f(t)\|.$$

In particular, for $\Omega \subset \mathbb{R}^n$ open, $\Sigma$ the Lebesgue $\sigma$-algebra and $\mu$ the Lebesgue measure, $L_p(\Omega; X)$ denotes the abbreviation of $L^p(\Omega, \Sigma, \mu; X)$.

Also, for $\Omega \subset \mathbb{R}^n$ open, $C^n(\overline{\Omega}; X)$ denotes the space of all functions $f : \overline{\Omega} \to X$ with continuous partial derivatives $\partial^m f$ in $\Omega$ that can be extended continuously to $\overline{\Omega}$, for each $|m| < n$, and $C^\infty(\overline{\Omega}; X)$ is the space $\bigcap_{n \geq 1} C^n(\overline{\Omega}; X)$.

Another space of interest is the space of all functions $f : [a, b] \to X$ of strongly bounded variation, i.e. the supremum

$$Var f|_{[a,b]} = \sup \left\{ \sum_{j=1}^{N} \|f(t_j) - f(t_{j-1})\|; t_0 < t_1 < \ldots < t_n,\ t_j \in [a, b] \right\}$$

is finite. This space is denoted by $BV([a, b]; X)$ and is of fundamental importance as it gives a sufficient condition to the almost everywhere differentiability and integrability of the respective derivative function. A proof of this fact can be found in [48, Corollary 6, p. 118].

The subscript loc sometimes assigned to any of the function spaces above stands for *locally* and it is used to drop the requirement that the membership property must be satisfied globally. Rigorously, if $W$ is any of the function spaces mentioned above, then $f : \Omega \to X$ belongs to $W_{loc}$ if and only if $f|_K \in W$ for each compact subsets $K$ of $\Omega$. It is clear that $W \subset W_{loc}$.

Moreover, if $X = \mathbb{K}$ is the underlying scalar field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, the second set in the function space notation introduced above will be dropped. For example $L^1_{loc}(\mathbb{R})$ denotes the space of all measurable scalar-valued functions which are integrable over each compact interval.

As usual, the star $*$ is employed for indicate the convolution of functions defined on the line and on the halfline, that is

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)ds, \ t \in \mathbb{R}, \tag{1.1}$$

e.g. for $f, g \in L^1(\mathbb{R})$, and

$$(f * g)(t) = \int_{0}^{t} f(t-s)g(s)ds, \ t \in \mathbb{R}_+, \tag{1.2}$$

e.g. for $f, g \in L^1(\mathbb{R}_+)$. Observe that (1.1) and (1.2) are equivalent for functions which vanish for $t < 0$.

If $f, g$ are integrable functions such that

$$\int_0^t f(t-s)g(s)ds = 0$$

almost everywhere in the interval $0 < s < \kappa$, then there exist $\lambda \geq 0$ and $\mu \geq 0$ satisfying $\lambda + \mu \geq \kappa$ such that $f(t) = 0$ almost everywhere in $0 < t < \lambda$ and $g(t) = 0$ almost everywhere in $0 < t < \mu$. This result is called the Titchmarsh convolution Theorem.

## 1.2 Laplace transform

A function $f \in L^1_{loc}(\mathbb{R}_+; X)$ is said to be of exponential growth or Laplace transformable if there exists $\omega \in \mathbb{R}_+$ such that $\int_0^{\infty} e^{-\omega t}|f(t)|dt < \infty$. The Laplace transform is then defined by

$$\widehat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t)dt, \ \operatorname{Re}\lambda \geq \omega.$$

**Theorem 1.1.** [35, Theorem 1.2] Let $M, \omega \geq 0$ and $f_n : [0, \infty) \to \mathcal{L}(X)$ be a sequence of functions such that $f_n(0) = 0$ and

$$\|f_n(t+h) - f_n(t)\| \leq Me^{\omega(t+h)},$$

for $t, h \geq 0$, $n \in \mathbb{N}_0$. Then the following assertions are equivalent:

(i) $\lim\limits_{n\to\infty} \int_0^{\infty} e^{-\lambda t} f_n(t)xdt = \int_0^{\infty} e^{-\lambda t} f_0(t)xdt$ for all $x \in X$ and $\lambda > \omega$;

(ii) $\lim\limits_{n\to\infty} f_n(t)x = f_0(t)x$ for all $x \in X$ and $t \geq 0$.

**Theorem 1.2.** [45, Theorem 0.2, p.6] Let $f : (0, \infty) \to X$. The following are equivalent:

(i) There exists $u \in \operatorname{Lip}(\mathbb{R}_+; X)$, $u(0) = 0$, such that $f(\lambda) = \widehat{u'}(\lambda)$ for all $\lambda > 0$;

(ii) $f \in C^{\infty}((0, \infty); X)$ and

$$\sup\left\{\frac{\lambda^{n+1}\|f^{(n)}(\lambda)\|}{n!}; \lambda > 0, n \in \mathbb{Z}_+\right\} =: m_{\infty}(f) < \infty.$$

In this case $\|u\|_{\operatorname{Lip}} = m_{\infty}(f)$.

## 1.3 Mittag-Leffler

The special functions

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\quad \alpha,\beta\in\mathbb{C},\quad \operatorname{Re}\alpha>0,\ \operatorname{Re}\beta>0,\ z\in\mathbb{C}, \tag{1.3}$$

are called Mittag-Leffler functions. They were introduced by Mittag-Leffler in connection with his method of summation of some divergent series. The main properties of these functions can be found in the book by Erdélyi [21, Section 18.1]. The Mittag-Leffler function arises naturally in the solution of fractional order integral equations or fractional order differential equations, and especially in the investigation of the fractional generalization of the kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems.

From the asymptotic expansion of the Mittag-Leffer function (see [26, eq. (6)] for details) one obtains that for $0<\alpha<2$ and arbitrary $\beta$:

$$E_{\alpha,\beta}(z)\sim\frac{1}{\alpha}z^{\frac{1-\beta}{\alpha}}e^{z^{\frac{1}{\alpha}}}\quad |z|\to\infty,\quad |\arg z|<\frac{\alpha\pi}{2}, \tag{1.4}$$

and this expression is used in the main result of Section 3.1 of Chapter 3.

Now, observe that

$$b_{\alpha,\beta}(\lambda,t):=\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})E_{\alpha,\beta}(\lambda\tau^{\alpha})d\tau=t^{\alpha+\beta-1}\frac{d}{dt}E_{\alpha,\beta}(\lambda t^{\alpha}), \tag{1.5}$$

which can be checked using the Laplace transform on both sides of the above equality and taking into account the following identity [44, p.21]:

$$\int_0^{\infty}e^{-\mu t}t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm\omega t^{\alpha})dt=\frac{k!\mu^{\alpha-\beta}}{(\mu^{\alpha}\mp\omega)^{k+1}},\quad Re(\mu)>|\omega|^{1/\alpha}. \tag{1.6}$$

Moreover, from [26, equations (38) and (43)] it follows that for $|z|>q$ ($q$ is a fixed number)

$$E'_{\alpha,\beta}(z)=\frac{1}{\alpha z}[E_{\alpha,\beta-1}(z)-(\beta-1)E_{\alpha,\beta}(z)]. \tag{1.7}$$

**Theorem 1.3.** [44, Theorem 1.6, p.35] If $\alpha<2$, $\beta$ is an arbitrary positive real number, $\mu$ is such that $\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$ and $C$ is a real constant, then

$$|E_{\alpha,\beta}(z)|\le\frac{C}{1+|z|},\quad \mu\le|\arg(z)|\le\pi,\quad |z|\ge 0. \tag{1.8}$$

## 1.4 Hausdorff measure of noncompactness

The next results and definitions are related to the Hausdorff measure of noncompactness, and are very much used in Section 4.4 of Chapter 4.

**Definition 1.4.** Let $Z$ be a bounded subset of a normed space $Y$. The Hausdorff measure of noncompactness of $Z$ is defined by

$$\eta(Z)=\inf\{\varepsilon>0: Z\text{ has a finite cover by balls of radius }\varepsilon\}.$$

This measure has some useful properties. For more general information of measure of noncompactness, the reader can consult [5, 6, 1].

**Lemma 1.5.** [6, Lemma 5.1, p.222] Let $X$ be a real Banach space and $B_1, B_2$ be bounded subsets of $X$. Then

(i) $\eta(B_1) = 0$ if and only if $B_1$ is totally bounded;

(ii) $\eta(B_1) \leq \eta(B_2)$ if $B_1 \subseteq B_2$;

(iii) $\eta(B_1) = \eta(\overline{B_1})\eta(\overline{co}(B_1))$, where $\overline{B_1}$ denotes the closure of $B_1$ and $\overline{co}(B_1)$ is the closed convex hull of $B_1$;

(iv) $\eta(B_1 \bigcup B_2) = \max\{\eta(B_1), \eta(B_2)\}$;

(v) $\eta(\lambda B_1) = |\lambda|\eta(B_1)$, with $\lambda \in \mathbb{R}$;

(vi) $\eta(B_1 + B_2) \leq \eta(B_1) + \eta(B_2)$, where $B_1 + B_2 = \{b_1 + b_2; b_1 \in B_1, b_2 \in B_2\}$.

In what follows, $\xi$ denotes the Hausdorff measure of noncompactness defined in $X$ and $\gamma$ denotes the Hausdorff measure of noncompactness on $C(I; X)$. Moreover, $\eta$ denotes the Hausdorff measure of noncompactness for general Banach spaces $Y$.

**Lemma 1.6.** [53, Property 1.1, p.10] Let $W \subseteq C(I; X)$ be a subset of continuous functions. If $W$ is bounded and equicontinuous, then the set $\overline{co}(W)$ is also bounded and equicontinuous.

Let $W$ be a set of functions from $I$ to $X$ and $t \in I$ fixed, and denote $W(t) = \{w(t) : w \in W\}$.

**Lemma 1.7.** [6, Lemma 5.3, p.224] Let $W \subseteq C(I; X)$ be a bounded set. Then $\xi(W(t)) \leq \gamma(W)$ for all $t \in I$. Furthermore, if $W$ is equicontinuous on $I$, then $\xi(W(t))$ is continuous on $I$, and

$$\gamma(W) = \sup\{\xi(W(t)) : t \in I\}.$$

A set of functions $W \subseteq L^1(I; X)$ is said to be uniformly integrable if there exists a positive funcion $\kappa \in L^1(I; \mathbb{R})$ such that $\|w(t)\| \leq \kappa(t)$ a.e. for all $w \in W$.

**Lemma 1.8.** [6, Lemma 5.4, p.224] If $\{u_n\}_{n=1}^{\infty} \subseteq L^1(I; X)$ is uniformly integrable, then for each $n \in \mathbb{N}$ the funcion $t \mapsto \xi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and

$$\xi\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \xi(\{u_n(s)\}_{n=1}^{\infty})ds.$$

The proof of the next result is in [11, Theorem 2].

**Lemma 1.9.** Let $Y$ be a Banach space. If $W \subseteq Y$ is a bounded subset, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq W$ such that

$$\eta(W) \leq 2\eta(\{u_n\}_{n=1}^{\infty}) + \varepsilon.$$

**Lemma 1.10.** [29, Lemma 2.4] Let $W$ be a closed and convex subset of a complex Banach space $Y$, and let $F : W \to W$ be a continuous operator such that $F(W)$ is a bounded set. Define

$$F^1(W) = F(W), \quad F^n(W) = F(\overline{co}(F^{n-1}(W))), \; n = 2, 3, \ldots .$$

If there exists a constant $0 \leq r < 1$ and $n_0 \in \mathbb{N}$ such that

$$\eta(F^{n_0}(W)) \leq r\eta(W), \tag{1.9}$$

then $F$ has a fixed point in W.

The following lemma is used in Section 4.4 of Chapter 4 and its proof can be found in [29, Theorem 3.1].

**Lemma 1.11.** For all $0 \leq m \leq n$, denote $C_m^n = \binom{n}{m}$. If $0 < \epsilon < 1$, $h > 0$ and

$$S_n = \epsilon^n + C_1^n \epsilon^{n-1} h + C_2^n \epsilon^{n-2} \frac{h^2}{2!} + \ldots + C_{n-1}^n \epsilon^{n-(n-1)} \frac{h^{n-1}}{(n-1)!} + C_n^n \frac{h^n}{n!}, \; n \in \mathbb{N}, \tag{1.10}$$

then $\lim_{n \to \infty} S_n = 0$.

# 2 (a,k)-regularized resolvent families

Here is provided an overview of the theory of the $(\mathrm{a},\mathrm{k})$-regularized resolvent families, including definitions and classical theorems. This theory was introduced by Lizama in [36] and has been extensively studied in [38, 37]. Moreover, the $(\mathrm{a},\mathrm{k})$-regularized resolvent families include well-known families, such as $C_0$-semigroups, cosine and resolvent families of bounded linear operators.

This chapter is organized in the following way: the first section deals with the definitions and properties of these families as well as their relationship with the results of the other families present in the literature. Sections two, three, four and five contain classical results of $(\mathrm{a},\mathrm{k})$-regularized resolvent families and their comparison with similar results for other classes of operators.

## 2.1 Definitions and properties

This section presents the definitions and properties of the theory of the $(\mathrm{a},\mathrm{k})$-regularized resolvent families.

**Definition 2.1.** Let $\mathrm{k} \in C(\mathbb{R}_+)$, $\mathrm{k} \not\equiv 0$, $\mathrm{a} \in L^1_{loc}(\mathbb{R}_+)$, $\mathrm{a} \not\equiv 0$, and let $A : D(A) \subset X \to X$ be a closed and densely defined operator. A strongly continuous family $\{R_{\mathrm{a},\mathrm{k}}(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called an $(\mathrm{a},\mathrm{k})$-regularized resolvent family on $X$ having the operator $A$ as a generator if the following properties hold:

(i) $\displaystyle\lim_{t\to 0^+} \frac{R_{\mathrm{a},\mathrm{k}}(t)}{\mathrm{k}(t)}x = x$ for all $x \in X$;

(ii) $R_{\mathrm{a},\mathrm{k}}(t)x \in D(A)$ and $R_{\mathrm{a},\mathrm{k}}(t)Ax = AR_{\mathrm{a},\mathrm{k}}(t)x$ for all $x \in D(A)$ and $t \geq 0$;

(iii) $\displaystyle R_{\mathrm{a},\mathrm{k}}(t)x = \mathrm{k}(t)x + \int_0^t \mathrm{a}(t-s)AR_{\mathrm{a},\mathrm{k}}(s)xds$, $t \geq 0$, $x \in D(A)$.

**Example 2.2.** The choice of the pair $(\mathrm{a},\mathrm{k})$ classifies different families of bounded linear operators.

(1) If $\mathrm{k} \equiv 1$ and $\mathrm{a} \equiv 1$, $\{R_{\mathrm{a},\mathrm{k}}(t)\}_{t\geq 0}$ is a $C_0$-semigroup denoted as $\{T(t)\}_{t\geq 0}$ (see [4, Section 3.1]).

(2) If $\mathrm{k} \equiv 1$ and $\mathrm{a}(t) = t$, $\{R_{\mathrm{a},\mathrm{k}}(t)\}_{t\geq 0}$ is a strongly continuous cosine family, denoted by $\{C(t)\}_{t\geq 0}$ (see [4, Section 3.4]).

(3) If $\mathrm{k}(t) = t$ and $\mathrm{a}(t) = t$, $\{R_{\mathrm{a},\mathrm{k}}(t)\}_{t\geq 0}$ is a sine family.

(4) The case when $\mathrm{k} \equiv 1$ and $\mathrm{a}(t) \in L^1_{loc}(\mathbb{R}_+)$ was studied in [45] and $\{R_{\mathrm{a,k}}(t)\}_{t\geq 0}$ is called a resolvent family, denoted by $\{S(t)\}_{t\geq 0}$.

(5) Bazhlekova in [7] studied the case when $\mathrm{k} \equiv 1$ and $\mathrm{a}(t) = g_\alpha(t) := \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\alpha > 0$ and $\Gamma$ denotes the Gamma function. This family is called $\alpha$-resolvent family (also called fractional resolvent families [34] or solution families [33]) and it is denoted by $\{S_\alpha(t)\}_{t\geq 0}$.

(6) If $\mathrm{a}(t) = g_\alpha(t)$ and $\mathrm{k}(t) = g_\beta(t)$, $\alpha > 0$, $\beta > 0$, then $\{R_{\mathrm{a,k}}(t)\}_{t\geq 0}$ is an $(\alpha,\beta)$-resolvent of bounded linear operators, and it is denoted by $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$.

**Lemma 2.3.** Let $\{R_{\mathrm{a,k}}(t)\}_{t\geq 0}$ be an $(\mathrm{a},\mathrm{k})$-regularized resolvent family generated by $A$. Then, $(\mathrm{a} * R_{\mathrm{a,k}})(t)x \in D(A)$ for all $x \in X$ and $t \geq 0$, and

$$R_{\mathrm{a,k}}(t)x = \mathrm{k}(t)x + A\int_0^t \mathrm{a}(t-s)R_{\mathrm{a,k}}(s)x\,ds, \;\; x \in X, t \geq 0. \tag{2.1}$$

*Proof.* Let $x \in X$ and define $y = (\lambda - A)^{-1}x \in D(A)$, where $\lambda \in \rho(A)$ is fixed. Consider $z = (\mathrm{a} * R_{\mathrm{a,k}})(t)x$, $t \geq 0$. From *(ii)* and *(iii)* of Definition 2.1,

$$\begin{aligned} z &= (\lambda - A)(\mathrm{a} * R_{\mathrm{a,k}})(t)y = \lambda(\mathrm{a} * R_{\mathrm{a,k}})(t)y - (\mathrm{a} * AR_{\mathrm{a,k}})(t)y \\ &= \lambda(\mathrm{a} * R_{\mathrm{a,k}})(t)y - (R_{\mathrm{a,k}}(t)y - \mathrm{k}(t)y), \end{aligned}$$

that is, $z \in D(A)$ and

$$(\lambda - A)z = \lambda(\mathrm{a} * R_{\mathrm{a,k}})(t)x - (R_{\mathrm{a,k}}(t)x - \mathrm{k}(t)x) = \lambda z - (R_{\mathrm{a,k}}(t)x - \mathrm{k}(t)x),$$

which gives (2.1). □

**Corollary 2.4.** If $R_{\mathrm{a,k_1}}$ is an $(\mathrm{a},\mathrm{k}_1)$-regularized resolvent family and $R_{\mathrm{a,k_2}}$ is an $(\mathrm{a},\mathrm{k}_2)$-regularized resolvent family, then $(\mathrm{k}_1 * R_{\mathrm{a,k_2}})(t) = (\mathrm{k}_2 * R_{\mathrm{a,k_1}})(t)$ for all $t \geq 0$.

*Proof.* Let $x \in D(A)$. From (ii) and (iii) of Definition 2.1 it follows that

$$\begin{aligned} (\mathrm{k}_2 * R_{\mathrm{a,k_1}})(t)x &= \left((R_{\mathrm{a,k_2}} - (\mathrm{a} * AR_{\mathrm{a,k_2}})) * R_{\mathrm{a,k_1}}\right)(t)x \\ &= (R_{\mathrm{a,k_2}} * R_{\mathrm{a,k_1}})(t)x - (\mathrm{a} * R_{\mathrm{a,k_2}} * AR_{\mathrm{a,k_1}})(t)x \\ &= R_{\mathrm{a,k_2}} * (R_{\mathrm{a,k_1}} - (\mathrm{a} * AR_{\mathrm{a,k_1}}))(t)x \\ &= (R_{\mathrm{a,k_2}} * \mathrm{k}_1)(t)x = (\mathrm{k}_1 * R_{\mathrm{a,k_2}})(t)x. \end{aligned}$$

Now, let $\lambda \in \rho(A)$ and $y \in X$. Define $x = (\lambda - A)^{-1}y$. Since

$$(\lambda - A)(\mathrm{k}_2 * R_{\mathrm{a,k_1}})(t)x = (\lambda - A)(\mathrm{k}_1 * R_{\mathrm{a,k_2}})(t)x,$$

it follows that $(\mathrm{k}_2 * R_{\mathrm{a,k_1}})(t)y = (\mathrm{k}_1 * R_{\mathrm{a,k_2}})(t)y$ for each $y \in X$ and $t \geq 0$. □

**Remark 2.5.** (1) As consequence of the previous corollary and Titchmarsh convolution Theorem (see Section 1.1), the $(\mathrm{a},\mathrm{k})$-resolvent family is unique.

(2) If $\{R_{\mathrm{a,k_1}}(t)\}_{t\geq 0}$ is an $(\mathrm{a},\mathrm{k}_1)$-regularized resolvent family and also an $(\mathrm{a},\mathrm{k}_2)$-regularized resolvent family, then $\mathrm{k}_1 = \mathrm{k}_2$.

(3) Let $\{R_{\mathrm{a},\mathrm{k}_i}(t)\}_{t\geq 0}$ be an $(\mathrm{a},\mathrm{k}_i)$-regularized resolvent family, $i=1,2$.
Then $\{(R_{\mathrm{a},\mathrm{k}_1}+R_{\mathrm{a},\mathrm{k}_2})(t)\}_{t\geq 0}$ is an $(\mathrm{a},\mathrm{k}_1+\mathrm{k}_2)$-regularized resolvent family.

**Definition 2.6.** $\{R_{\mathrm{a},\mathrm{k}}(t)\}_{t\geq 0}$ is called **exponentially bounded** if there are constants $M\geq 1$ and $\omega\in\mathbb{R}$ such that

$$\|R_{\mathrm{a},\mathrm{k}}(t)\|\leq Me^{\omega t}, \text{ for all } t\geq 0; \tag{2.2}$$

or more precisely $(M,\omega)$ is called a **type** of $\{R_{\mathrm{a},\mathrm{k}}(t)\}_{t\geq 0}$.

**Remark 2.7.** The boundedness condition (2.2) can be proved for some families, see, for example [43, Theorem 2.2, p.4] in the case of $C_0$-semigroups and [24, Theorem 1.1, p.25] in the case of cosine and sine families.

The next result characterizes an $(\mathrm{a},\mathrm{k})$-regularized resolvent family.

**Proposition 2.8.** Let $\{R(t)\}_{t\geq 0}$ be an exponentially bounded and strongly continuous operator family in $\mathcal{L}(X)$ of type $(M,\omega)$ such that the Laplace transform $\widehat{R}(\lambda)$ exists for $\lambda>\omega$. Then, $\{R(t)\}_{t\geq 0}$ is an $(\mathrm{a},\mathrm{k})$-regularized resolvent family of type $(M,\omega)$ if and only if, for every $\lambda>\omega$, $(I-\widehat{\mathrm{a}}(\lambda)A)^{-1}$ exists in $\mathcal{L}(X)$ and

$$\widehat{\mathrm{k}}(\lambda)(I-\widehat{\mathrm{a}}(\lambda)A)^{-1}x=\int_0^\infty e^{-\lambda s}R(s)xds,\ \forall x\in X. \tag{2.3}$$

*Proof.* Suppose that $\{R(t)\}_{t\geq 0}$ is an $(\mathrm{a},\mathrm{k})$-regularized resolvent family. By assumption, the Laplace transform $H(\lambda)=\widehat{R}(\lambda)$ of the $(\mathrm{a},\mathrm{k})$-regularized resolvent family exists for $\lambda>\omega$. Then, $\frac{1}{\widehat{\mathrm{a}}}\in\rho(A)$ and $H(\lambda)=\widehat{\mathrm{k}}(\lambda)(I-\widehat{\mathrm{a}}(\lambda)A)^{-1}$.

In fact, from *(ii)* and *(iii)* of Definition 2.1 and Lemma 2.3 and the convolution Theorem, for $\operatorname{Re}\lambda>\omega$, it follows that

$$H(\lambda)x=\widehat{\mathrm{k}}(\lambda)x+\widehat{\mathrm{a}}(\lambda)H(\lambda)Ax, \tag{2.4}$$

for each $x\in D(A)$, and

$$H(\lambda)x=\widehat{\mathrm{k}}(\lambda)x+A\widehat{\mathrm{a}}(\lambda)H(\lambda)x, \tag{2.5}$$

for each $x\in X$. For each $x\in D(A)$,

$$H(\lambda)[I-\widehat{\mathrm{a}}(\lambda)A]x=\widehat{\mathrm{k}}(\lambda)x$$

can be obtained from (2.4), and for each $x\in X$

$$[I-\widehat{\mathrm{a}}(\lambda)A]H(\lambda)x=\widehat{\mathrm{k}}(\lambda)x$$

can be obtained from (2.5).

Thus the operators $I-\widehat{\mathrm{a}}(\lambda)A$ are invertible for all $\lambda\in\mathbb{C}$, with $\operatorname{Re}\lambda>\omega$ and

$$H(\lambda)=\widehat{\mathrm{k}}(\lambda)(I-\widehat{\mathrm{a}}(\lambda)A)^{-1}, \text{ for } \operatorname{Re}\lambda>\omega.$$

In particular, $\dfrac{1}{\widehat{\mathrm{a}}(\lambda)}\in\rho(A)$ for all such $\lambda$, provided $\widehat{\mathrm{a}}(\lambda)\neq 0$. To prove this fact assume that $\widehat{\mathrm{a}}(\lambda_0)=0$ for some $\lambda_0$ with $\operatorname{Re}\lambda_0>\omega$. Since $\widehat{\mathrm{a}}(\lambda)$ is holomorphic, $\lambda_0$ is an isolated zero of finite multiplicity, and $H(\lambda_0) = \widehat{\mathrm{k}}(\lambda_0)$. Choose a small circle $\Gamma$ around $\lambda_0$ which is entirely contained in the halfplane $\operatorname{Re}\lambda > \omega$ such that $\widehat{\mathrm{a}}(\lambda) \neq 0$ on $\Gamma$.

Notice that $\dfrac{H(\lambda)}{\widehat{\mathrm{k}}(\lambda)}$ is holomorphic on $\Gamma$, then

$$\frac{H(\lambda_0)}{\widehat{\mathrm{k}}(\lambda_0)} = \frac{1}{2\pi i}\int_\Gamma \frac{H(\lambda)}{(\lambda-\lambda_0)\widehat{\mathrm{k}}(\lambda)}d\lambda. \tag{2.6}$$

Now, since $A$ is closed, it follows that

$$A = A\frac{1}{\widehat{\mathrm{k}}(\lambda_0)}H(\lambda_0) = A\frac{1}{2\pi i}\int_\Gamma \frac{H(\lambda)}{(\lambda-\lambda_0)\widehat{\mathrm{k}}(\lambda)}d\lambda = \frac{1}{2\pi i}\int_\Gamma \frac{AH(\lambda)}{(\lambda-\lambda_0)\widehat{\mathrm{k}}(\lambda)}d\lambda.$$

Moreover,

$$AH(\lambda) = \frac{(H(\lambda)-\widehat{\mathrm{k}}(\lambda))}{\widehat{\mathrm{a}}(\lambda)}$$

is well-defined and holomorphic on $\Gamma$. Then

$$A = \frac{1}{2\pi i}\int_\Gamma \frac{H(\lambda)-\widehat{\mathrm{k}}(\lambda)}{(\lambda-\lambda_0)\widehat{\mathrm{a}}(\lambda)\widehat{\mathrm{k}}(\lambda)}d\lambda$$

is obtained by Cauchy Integral Formula, and so $A$ is bounded, which is a contradiction to the standing hypothesis. Thus $\widehat{\mathrm{a}}(\lambda) \neq 0$ for all $\operatorname{Re}\lambda > \omega$.

Conversely, let $\mu, \lambda > \omega$ and $x \in D(A)$. Then $x = (I - \widehat{\mathrm{a}}(\mu)A)^{-1}y$ for some $y \in X$. Since $(I - \widehat{\mathrm{a}}(\mu)A)^{-1}$ and $(I - \widehat{\mathrm{a}}(\lambda)A)^{-1}$ commute and $A$ is closed, it follows that

$$\begin{aligned}\int_0^\infty e^{-\lambda t}R(t)xdt &= \widehat{\mathrm{k}}(\lambda)(I-\widehat{\mathrm{a}}(\lambda)A)^{-1}(I-\widehat{\mathrm{a}}(\mu)A)^{-1}y\\ &= (I-\widehat{\mathrm{a}}(\mu)A)^{-1}\widehat{\mathrm{k}}(\lambda)(I-\widehat{\mathrm{a}}(\lambda)A)^{-1}y\\ &= (I-\widehat{\mathrm{a}}(\mu)A)^{-1}\widehat{R}(\lambda)y\\ &= \int_0^\infty e^{-\lambda t}(I-\widehat{\mathrm{a}}(\mu)A)^{-1}R(t)ydt.\end{aligned}$$

Hence, by uniqueness of the Laplace transform,

$$R(t)x = (I-\widehat{\mathrm{a}}(\mu)A)^{-1}R(t)(I-\widehat{\mathrm{a}}(\mu)A)x$$

for almost all $t \geq 0$, and then, $R(t)x \in D(A)$. Further, since $\widehat{\mathrm{a}}(\mu) \neq 0$ it follows from the above equality that $AR(t)x = R(t)Ax$ for every $t \geq 0$, $x \in D(A)$.

Now let $\lambda > \omega$ and $x \in D(A)$. From the convolution theorem,

$$\begin{aligned}\int_0^\infty e^{-\lambda t}k(t)xdt = \widehat{\mathrm{k}}(\lambda)x = \widehat{R}(\lambda)(I-\widehat{\mathrm{a}}(\lambda)A)x &= \widehat{R}(\lambda)x - \widehat{R}(\lambda)\widehat{\mathrm{a}}(\lambda)Ax\\ = \int_0^\infty e^{-\lambda t}\left[R(t)x - \int_0^t \mathrm{a}(t-s)R(s)Axds\right]dt.&\end{aligned}$$

The uniqueness of the Laplace transform and the strong continuity of $R(t)$ yield that

$$R(t)x = k(t)x + \int_0^t \mathrm{a}(t-s)R(s)Axds.$$

From the proof of Lemma 2.3, $R(0) = \mathrm{k}(0)I$ and the proof is complete. □

The next result presents a property of the generator $A$. For this, following assumption on $\mathrm{a} \in L^1_{loc}(\mathbb{R}_+)$ and $\mathrm{k} \in C(\mathbb{R}_+)$ is considered.

$(H_{\mathrm{a,k}})$ There exists $\epsilon_{\mathrm{a,k}} > 0$ and $t_{\mathrm{a,k}} > 0$ such that for all $0 < t \leq t_{\mathrm{a,k}}$

$$\left| \int_0^t \mathrm{a}(t-s)\mathrm{k}(s)ds \right| \geq \epsilon_{\mathrm{a,k}} \int_0^t |\mathrm{a}(t-s)\mathrm{k}(s)|ds.$$

**Theorem 2.9.** Suppose $A$ is the generator of an $(\mathrm{a},\mathrm{k})$-regularized resolvent family $\{R_{\mathrm{a,k}}(t)\}_{t\geq 0}$ such that $t \mapsto |\mathrm{k}(t)|$ is a nondecreasing function and satisfies

$$\limsup_{t\to 0^+} \frac{\|R_{\mathrm{a,k}}(t)\|}{|\mathrm{k}(t)|} < \infty. \tag{2.7}$$

Assume $(H_{\mathrm{a,k}})$. Then

(i) $D(A) = \left\{ x \in X; \lim\limits_{t\to 0^+} \dfrac{R_{\mathrm{a,k}}(t)x - k(t)x}{(\mathrm{k} * \mathrm{a})(t)} \text{ exists} \right\};$

(ii) $\lim\limits_{t\to 0^+} \dfrac{R_{\mathrm{a,k}}(t)x - \mathrm{k}(t)x}{(\mathrm{k} * \mathrm{a})(t)} = Ax$, for all $x \in D(A)$.

*Proof.* Let $z \in D(A)$. Then, items $(ii)$ and $(iii)$ of Definition 2.1, the strong continuity of $R_{\mathrm{a,k}}$ and the fact that $|\mathrm{k}(t)|$ is nondecreasing implies that

$$\begin{aligned} \left\| \frac{R_{\mathrm{a,k}}(t)z}{k(t)} - z \right\| &= \frac{1}{|\mathrm{k}(t)|} \left\| \int_0^t \mathrm{a}(t-s)AR_{\mathrm{a,k}}(s)zds \right\| \\ &\leq \left( \int_0^t |\mathrm{a}(t-s)| \frac{\|R_{\mathrm{a,k}}(s)\|}{|\mathrm{k}(s)|} ds \right) \|Az\|. \end{aligned}$$

Hence, for all $z \in D(A)$

$$\lim_{t\to 0^+} \left\| \frac{R_{\mathrm{a,k}}(t)z}{|\mathrm{k}(t)|} - z \right\| = 0.$$

The denseness of $D(A)$ and (2.7) imply that this actually holds for all $z \in X$. Thus, for every $z \in X$ and $\varepsilon > 0$, there exists $0 < t_{\varepsilon,z} < \min\{t_{\mathrm{a,k}}, 1\}$ such that

$$\left\| \frac{R_{\mathrm{a,k}}(t)z}{\mathrm{k}(t)} - z \right\| < \varepsilon, \quad \forall t \in (0, t_{\varepsilon,z}). \tag{2.8}$$

Now define the set

$$\tilde{D}(A) := \left\{ x \in X; \lim_{t\to 0^+} \frac{R_{\mathrm{a,k}}(t)x - \mathrm{k}(t)x}{(\mathrm{k} * \mathrm{a})(t)} \text{ exists} \right\}.$$

Let $x \in D(A)$ and define $z = Ax$. For all $t \in (0, t_{\varepsilon,Ax})$,

$$\left\| \frac{R_{\mathrm{a,k}}(t)Ax}{\mathrm{k}} - Ax \right\| < \varepsilon,$$

can be obtained from (2.8). Therefore, using (iii) of Definition 2.1 and $(H_{\mathrm{a,k}})$ for all $\tau \in (0, t_{\varepsilon,Ax})$, it follows that

$$\begin{aligned}\left\| \frac{R_{\mathrm{a,k}}(\tau)x - \mathrm{k}(\tau)x}{(\mathrm{k} * \mathrm{a})(\tau)} - Ax \right\| &= \frac{1}{|(\mathrm{k} * \mathrm{a})(\tau)|} \left\| \int_0^\tau \mathrm{a}(\tau - s)\mathrm{k}(s) \left[ \frac{R_{\mathrm{a,k}}(s)Ax}{\mathrm{k}(s)} - Ax \right] ds \right\| \\ &\leq \frac{1}{|(\mathrm{k} * \mathrm{a})(\tau)|} \int_0^\tau |\mathrm{a}(\tau - s)\mathrm{k}(s)| \varepsilon ds = \frac{\varepsilon}{\epsilon_{\mathrm{a,k}}}.\end{aligned}$$

Thus, $x \in \tilde{D}(A)$, that is, $D(A) \subset \tilde{D}(A)$ and condition (ii) holds.

On the other hand, let $x \in \tilde{D}(A)$. Then,

$$\lim_{t \to 0^+} \frac{R_{\mathrm{a,k}}(t)x - \mathrm{k}(t)x}{(\mathrm{k} * \mathrm{a})(t)} = y$$

for some $y \in X$. For given $\varepsilon > 0$ and all $t \in (0, t_{\varepsilon,x})$, by using (2.8) and $(H_{\mathrm{a,k}})$ it follows that

$$\begin{aligned}\left\| \frac{1}{(\mathrm{k} * \mathrm{a})(t)} \int_0^t \mathrm{a}(t - s)R_{\mathrm{a,k}}(s)x ds - x \right\| &= \frac{1}{|(\mathrm{k} * \mathrm{a})(t)|} \left\| \int_0^t \mathrm{a}(t - s)\mathrm{k}(s) \left[ \frac{R_{\mathrm{a,k}}(s)}{\mathrm{k}(s)} x - x \right] ds \right\| \\ &\leq \frac{\varepsilon}{|(\mathrm{k} * \mathrm{a})(t)|} \int_0^t |\mathrm{a}(t - s)\mathrm{k}(s)| ds \leq \frac{\varepsilon}{\epsilon_{\mathrm{a,k}}}.\end{aligned}$$

Then

$$\lim_{t \to 0^+} \frac{1}{(\mathrm{k} * \mathrm{a})(t)} \int_0^t \mathrm{a}(t - s)R_{\mathrm{a,k}}(s)x ds = x.$$

Now, from (iii) of Definition 2.1, observe that

$$\begin{aligned}\left\| A \left[ \frac{1}{(\mathrm{k} * \mathrm{a})(t)} \int_0^t \mathrm{a}(t - s)R_{\mathrm{a,k}}(s)x ds \right] - y \right\| &= \left\| \frac{1}{(\mathrm{k} * \mathrm{a})(t)} \int_0^t \mathrm{a}(t - s)AR_{\mathrm{a,k}}(s)x ds - y \right\| \\ &= \left\| \frac{R_{\mathrm{a,k}}(t)x - \mathrm{k}(t)x}{(\mathrm{k} * \mathrm{a})(t)} - y \right\|,\end{aligned}$$

where the right hand side goes to zero as $t \to 0^+$. Then, since $A$ is closed, $x \in D(A)$ and $Ax = y$, proving the theorem. □

**Example 2.10.** It is not difficult to find examples of functions a and k that satisfy assumption $(H_{\mathrm{a,k}})$.

(1) If a and k are positive functions, then

$$\left| \int_0^t \mathrm{a}(t - s)\mathrm{k}(s) ds \right| = \int_0^t |\mathrm{a}(t - s)\mathrm{k}(s)| ds.$$

That is, assumption $(H_{\mathrm{a,k}})$ is satisfied with $\epsilon_{\mathrm{a,k}} = 1$.

(2) If $\mathrm{a}(t) = -b - c^2 t$ and $\mathrm{k} \equiv 1$, then

$$\begin{aligned}|(\mathrm{a} * \mathrm{k})(t)| &= \left| \int_0^t -b - c^2(t - s) ds \right| = \left| \int_0^t -b - c^2 t + c^2 s ds \right| \\ &= \left| -bt - c^2 t^2 + c^2 \frac{t^2}{2} \right| = bt + c^2 \frac{t^2}{2} = \int_0^t | - b - c^2(t - s)| ds,\end{aligned}$$

that is, assumption $(H_{\mathrm{a,k}})$ is satisfied with $\epsilon_{\mathrm{a,k}} = 1$.

## 2.2 The Generation Theorem

This section presents a theorem known as the Generation Theorem.

Some general properties are well-known. For example, to relate different $\alpha$-resolvent families and its generators, Li, Chen and Li [34] shown that if the operator $-A$ generates a bounded $\alpha$-times resolvent family then, with some suitable $\beta$, $-A^\beta$ also generates an $\alpha$-times resolvent family. There also exists a principle of subordination (see [7, Chapter 3]). For instance, if an operator $A$ generates a cosine operator family then it also generates an $\alpha$-resolvent family for any $0 < \alpha < 2$, but the converse is not true. Moreover, considering for any $\alpha \in (0,2)$ and $\theta \in [0,\pi)$ the differential operator $B_\theta = e^{i\theta}\partial^2_{xx}$, with $D(B_\theta) = \{g \in W^{2,2}(0,1), g(0) = g(1) = 0\}$ on $X = L^2(0,1)$, then $B_\theta$ generates a bounded $\alpha$-resolvent family if and only if $|\theta| \le \left(1 - \frac{\alpha}{2}\right)\pi$. However, for $\frac{\pi}{2} < |\theta| \le \left(1 - \frac{\alpha}{2}\right)\pi$, the operator $B_\theta$ does not generates any $C_0$-semigroup [7, Section 2.2].

In the literature, this theorem was presented also in the particular cases of families of bounded linear operators:

(1) In [43, Theorem 3.1, p. 8] there is the Hille-Yosida theorem of generation of $C_0$-semigroups.

(2) For cosine families, [24, Theorem 2.1, p. 28]

(3) [45, Theorem 1.3, p. 43] for resolvent families.

(4) [7, Theorem 2.8, p. 23] for $\alpha$-resolvents.

**Theorem 2.11.** Let $A : D(A) \subset X \to X$ be a closed linear densely defined operator in a Banach space $X$. Then $A$ is the generator of a $(\mathrm{a},\mathrm{k})$-regularized resolvent family $\{R_{\mathrm{a,k}}(t)\}_{t\ge 0}$ of type $(M,\omega)$ if and only if the following conditions hold:

(i) $\widehat{\mathrm{a}}(\lambda) \neq 0$ and $\dfrac{1}{\widehat{\mathrm{a}}(\lambda)} \in \rho(A)$ for all $\lambda \in \mathbb{R}$, $\lambda > \omega$;

(ii) $H(\lambda) := \widehat{R}_{\mathrm{a,k}}(\lambda)$ satisfies $H(\lambda) = \widehat{\mathrm{k}}(\lambda)(I - \widehat{\mathrm{a}}(\lambda)A)^{-1}$ and

$$\|H^{(n)}(\lambda)\| \le \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad \lambda > \omega,\ n \in \mathbb{Z}_+. \tag{2.9}$$

*Proof.* If $\{R_{\mathrm{a,k}}(t)\}_{t\ge0}$ is an $(\mathrm{a},\mathrm{k})$-regularized resolvent family of type $(M,\omega)$, its Laplace transform

$$H(\lambda) = \widehat{R}_{\mathrm{a,k}}(\lambda) = \int_0^\infty e^{-\lambda s} R_{\mathrm{a,k}}(s)ds, \quad \operatorname{Re}\lambda > \omega,$$

is well-defined and holomorphic for $\operatorname{Re}\lambda > \omega$, and satisfies

$$\begin{aligned}\|H(\lambda)\| &\le \int_0^\infty \|e^{-\lambda s}R_{\mathrm{a,k}}(s)\|ds \le M\int_0^\infty \|e^{-\lambda s}\|e^{\omega s}ds \\ &\le M\int_0^\infty e^{-(\operatorname{Re}\lambda-\omega)s}ds = \frac{M}{(\operatorname{Re}\lambda-\omega)}.\end{aligned}$$

Also, note that for $\operatorname{Re}(\lambda) > \omega$,

$$\|H'(\lambda)\| = \left\|\int_0^\infty -se^{-\lambda s}R_{\mathrm{a,k}}(s)ds\right\| \le \int_0^\infty se^{-\operatorname{Re}\lambda s}Me^{\omega s}ds$$

$$\leq \int_0^\infty \frac{Me^{-(\operatorname{Re}\lambda-\omega)s}}{(\operatorname{Re}\lambda-\omega)}ds \leq \frac{M}{(\operatorname{Re}\lambda-\omega)^2}.$$

Now, suppose

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\operatorname{Re}\lambda-\omega)^{n+1}},\ \operatorname{Re}\lambda > \omega. \tag{2.10}$$

Then,

$$\begin{aligned}\|H^{(n+1)}(\lambda)\| &= \left\| \int_0^\infty (-s)^{n+1} e^{-\lambda s} R_{\mathrm{a,k}}(s)ds \right\| \\ &= \left\| \left[ \frac{(-s)^{n+1}}{(-\lambda)} e^{-\lambda s} R_{\mathrm{a,k}}(s) \right]_0^\infty + \frac{n+1}{\lambda} \int_0^\infty (-s)^n e^{-\lambda s} R_{\mathrm{a,k}}(s)ds \right\| \\ &\leq \frac{n+1}{|\lambda|} \|H^{(n)}(\lambda)\|.\end{aligned}$$

Using (2.10), it can be concluded the following inequality

$$\|H^{(n+1)}(\lambda)\| \leq M(n+1)!(\operatorname{Re}\lambda-\omega)^{-(n+2)}.$$

Then the following estimates holds

$$\|H^{(n)}(\lambda)\| \leq Mn!(\operatorname{Re}\lambda-\omega)^{-(n+1)},\quad \operatorname{Re}\lambda > \omega,\ n \in \mathbb{Z}_+. \tag{2.11}$$

Now, from (*ii*) and (*iii*) of Definition 2.1, Lemma 2.3 and the convolution theorem, for $\operatorname{Re}\lambda > \omega$, it follows that

$$H(\lambda)x = \widehat{\mathrm{k}}(\lambda)x + \widehat{\mathrm{a}}(\lambda)H(\lambda)Ax, \tag{2.12}$$

for each $x \in D(A)$, and

$$H(\lambda)x = \widehat{\mathrm{k}}(\lambda)x + A\widehat{\mathrm{a}}(\lambda)H(\lambda)x, \tag{2.13}$$

for each $x \in X$. Now, for each $x \in D(A)$

$$H(\lambda)[I - \widehat{\mathrm{a}}(\lambda)A]x = \widehat{\mathrm{k}}(\lambda)x$$

can be obtained from (2.12), and for each $x \in X$,

$$[I - \widehat{\mathrm{a}}(\lambda)A]H(\lambda)x = \widehat{\mathrm{k}}(\lambda)x$$

can be obtained from (2.13).

Thus the operators $I - \widehat{\mathrm{a}}(\lambda)A$ are invertible for all $\lambda \in \mathbb{C}$, with $\operatorname{Re}\lambda > \omega$, and

$$H(\lambda) = \widehat{\mathrm{k}}(\lambda)(I - \widehat{\mathrm{a}}(\lambda)A)^{-1},\ \text{for}\ \operatorname{Re}\lambda > \omega.$$

In particular, $\dfrac{1}{\widehat{\mathrm{a}}(\lambda)} \in \rho(A)$ for all such $\lambda$, provided $\widehat{\mathrm{a}}(\lambda) \neq 0$. In fact, assuming that $\widehat{\mathrm{a}}(\lambda_0) = 0$ for some $\lambda_0$ with $\operatorname{Re}\lambda_0 > \omega$.

Proceeding as in the proof of Proposition 2.8, it follows that $A$ is bounded, which is a contradiction to the standing hypothesis. Thus $\widehat{\mathrm{a}}(\lambda) \neq 0$ for all $\operatorname{Re}\lambda > \omega$.

On the other hand, assuming conditions $(i)$ and $(ii)$, by Theorem 1.2 there exists a Lipschitz family $\{U_\omega(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ with $U_\omega(0) = 0$ and $\widehat{U'}_\omega(\lambda) = H(\lambda+\omega)$, $\lambda > 0$. Then

$$\lambda\widehat{U}_\omega(\lambda) = \lambda\widehat{U}_\omega(\lambda) - U_\omega(0) = \widehat{U'}_\omega(\lambda) = H(\lambda+\omega),$$

that is,

$$\widehat{U}_\omega(\lambda) = \frac{H(\lambda+\omega)}{\lambda}.$$

Define $\{U(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ by

$$U(t) = e^{\omega t}U_\omega(t) - \omega\int_0^t e^{\omega s}U_\omega(s)ds, \quad t \geq 0. \tag{2.14}$$

Given $\gamma > 0$, let $s, t \in (0,\gamma)$ such that $s < t$.

Using the Mean Value Theorem for integrals, note that

$$\begin{aligned}\|U(\gamma) - U(0)\| &= \left\|e^{\omega\gamma}U_\omega(\gamma) - \omega\int_0^\gamma e^{\omega\tau}U_\omega(\tau)d\tau\right\| \\ &= \left\|\int_0^\gamma e^{\omega\tau}\frac{d}{d\tau}U_\omega(\tau)d\tau\right\| \\ &\leq \gamma e^{\omega\eta}\left\|\frac{d}{d\tau}U_\omega(\eta)\right\|,\end{aligned}$$

for some $\eta \in (0,\gamma)$.

Then, since $U_\omega(t)$ is Lipschitz and differentiable, its derivative is bounded, and using that $\eta < \gamma$, it follows that

$$\|U(\gamma) - U(0)\| \leq \gamma e^{\omega\gamma}M_\omega.$$

Then,

$$\begin{aligned}\|U(t) - U(s)\| &= \left\|\int_s^t e^{\omega\tau}\frac{d}{d\tau}U_\omega(\tau)d\tau\right\| \\ &\leq |t-s|e^{\omega\eta}\left\|\frac{d}{d\tau}U_\omega(\eta)\right\| \\ &\leq |t-s|e^{\omega\gamma}M_\omega = |t-s|M_\gamma,\end{aligned}$$

that is, $U(t)$ is locally Lipschitz.

Now, by First Shifting Property,

$$\begin{aligned}\widehat{U}(\lambda) &= \widehat{U}_\omega(\lambda-\omega) - \frac{\omega}{\lambda}\widehat{U}_\omega(\lambda-\omega) \\ &= \frac{H(\lambda)}{\lambda-\omega} - \frac{\omega H(\lambda)}{(\lambda-\omega)\lambda} \\ &= H(\lambda)\left(\frac{1}{\lambda-\omega} - \frac{\omega}{(\lambda-\omega)\lambda}\right) \\ &= \frac{H(\lambda)}{\lambda}.\end{aligned}$$

The definition of $H(\lambda)$ shows that $U(t)$ commutes with $A$ and yields the identity

$$\widehat{U}(\lambda) = \lambda^{-1}\widehat{\mathrm{k}}(\lambda) + \widehat{\mathrm{a}}(\lambda)\widehat{U}(\lambda)A,$$

i.e.

$$U(t)x = \int_0^t \mathrm{k}(s)xds + (\mathrm{a} * U)(t)Ax, \text{ for all } x \in D(A), t \geq 0.$$

For each $x \in D(A)$ let $g(t) = \displaystyle\int_0^t \mathrm{k}(s)ds$, then $g'(t) = k(t)x$. Given $[a,b] \subset \mathbb{R}_+$, it follows that

$$\|g(t)\| = \left\| \int_0^t k(s)xds \right\| \leq \int_0^t |k(s)| \|x\| ds \leq \infty,$$

that is, $g \in L^\infty_{loc}(\mathbb{R}_+; X)$. Similarly, $\|g'(t)\| = \|k(t)x\|$ and since $k \in C(\mathbb{R}_+)$, it follows that $g' \in L^\infty_{loc}(\mathbb{R}_+)$. Therefore, $g \in W^{1,\infty}_{loc}(\mathbb{R}_+; X)$.

Now, for each $x \in D(A)$, $f(t) = U(t)Ax$ is locally Lipschitz, i.e. $f \in BV_{loc}(\mathbb{R}_+; X)$. Then $f$ is continuous and, given $[a,b] \subset \mathbb{R}_+$, there exists $K = K_{a,b}$ such that $\|f(t)\|_X \leq K_{a,b}$, that is, $f \in L^\infty_{loc}(\mathbb{R}_+; X)$.

Note that $f$ is differentiable almost everywhere on $(a,b)$ for all $a, b \in \mathbb{R}_+$, $a < b$ because $f \in BV_{loc}(\mathbb{R}_+; X)$ (see Section 1.1 of Chapter 1).

So, since $f$ is locally Lipschitz, there exists $\tilde{K} = \tilde{K}_{a,b}$ such that $\|f'(t)\|_X \leq \tilde{K}_{a,b}$. Then, $f' \in L^\infty_{loc}(\mathbb{R}_+; X)$. Therefore, $f \in W^{1,\infty}_{loc}(\mathbb{R}_+; X)$.

Finally, let $t \in \mathbb{R}_+$ and $b > t$, then

$$\begin{aligned} \|(\mathrm{a} * f)(t)\|_X = \left\| \int_0^t \mathrm{a}(t-s)f(s)ds \right\|_X &\leq \int_0^b \|\mathrm{a}(t-s)f(s)\|_X ds \\ &\leq \|f\|_{L^\infty([a,b];X)} \int_0^b \|\mathrm{a}(t-s)x\| ds < \infty, \end{aligned}$$

and, similarly,

$$\|(\mathrm{a} * f')(t)\| \leq \|f'\|_{L^\infty([a,b];X)} \int_0^b \|\mathrm{a}(t-s)\| ds < \infty.$$

Therefore, $U(\cdot)x \in W^{1,\infty}_{loc}(\mathbb{R}_+; X)$ and

$$\frac{d}{dt}U(t)x = \mathrm{k}(t)x + \frac{d}{dt}(\mathrm{a} * f)(t) = \mathrm{k}(t)x + \mathrm{a} * \frac{d}{dt}f(t), \text{ for almost all } t \geq 0.$$

Notice that $\mathrm{a} * df(t) = \mathrm{a} * \dfrac{d}{dt}(U(t)Ax)$ and $U(t)Ax$ is only defined and consistent provided $x \in D(A^2)$ because

$$U(t)Ax = \int_0^t k(s)Axds + (\mathrm{a} * U)(t)A^2x.$$

This implies that $\left(\mathrm{a} * \dfrac{d}{dt}f\right)(t)$ is even continuous, from which in turn $U(t)x$ is continuously differentiable on $\mathbb{R}_+$ for each $x \in D(A^2)$.

Then, $\dfrac{d}{dt}U(t)$ is uniformly bounded for $t$ bounded, so

$$T_h := \frac{U(t+h) - U(t)}{h}$$

is uniformly bounded for $0 < h \leq 1$ and $t$ bounded. That is, $T_h \in \mathcal{L}(D(A^2); X)$ and $\sup\limits_{0<h\leq 1} \|T_h\|_{\mathcal{L}((D(A^2);X))} < \infty$.

Since $T_h x \to \dfrac{d}{dt}U(t)x$, by Proposition **??**, it follows that $\dfrac{d}{dt}U(t) \in \mathcal{L}(D(A^2);X)$ and

$$\left\| \frac{d}{dt}U(t) \right\|_{\mathcal{L}(D(A^2);X)} < \infty.$$

Now, once $D(A^2)$ is dense and $\dfrac{d}{dt}U(t)$ is bounded, it is possible to get an bounded extension of $\dfrac{d}{dt}U(t)$ in X, denoted as $\dfrac{d}{dt}U(t)$ (see Section 1.1 of Chapter 1). Then $\dfrac{d}{dt}U(t) : X \to X$ with

$$\left\| \frac{d}{dt}U(t) \right\|_{\mathcal{L}(X)} = \left\| \frac{d}{dt}U(t) \right\|_{\mathcal{L}(D(A^2);X)} < \infty.$$

Note that $\dfrac{d}{dt}U(t)x$ is continuous for $t$ bounded if $x \in D(A^2)$. From the fact that $D(A^2)$ is dense in $X$, for $y \in X$, $\xi$ close to $t$ and $x$ close to $y$ it is possible to obtain

$$\begin{aligned}
&\left\| \frac{d}{dt}U(t+\xi)y - \frac{d}{dt}U(t)y \right\| \\
&\leq \left\| \frac{d}{dt}U(t+\xi)(y-x) + \frac{d}{dt}U(t+\xi)x - \frac{d}{dt}U(t)x - \frac{d}{dt}U(t)(y-x) \right\| \\
&< \varepsilon.
\end{aligned}$$

So, $\dfrac{d}{dt}U(t)x$ exists for all $x \in X$ and $\dfrac{d}{dt}(U(t)x) \in C(\mathbb{R}_+;X)$. That is, $U(t)x$ is continuously differentiable on $\mathbb{R}_+$ for each $x \in X$.

Define $R_{\mathrm{a,k}}(t)x = \dfrac{d}{dt}U(t)x$, $t \geq 0$, $x \in X$. Then $R_{\mathrm{a,k}}(t)x$ is strongly continuous. Moreover, $R_{\mathrm{a,k}}(t)$ commutes with $A$ for all $x \in D(A)$ because

$$\frac{U(t+h)-U(t)}{h}Ax = A\left(\frac{U(t+h)-U(t)}{h}\right)x,$$

for $x \in D(A)$. Since $U(t)$ commutes with $A$ and using the fact that $A$ is closed, it follows that

$$\frac{d}{dt}U(t)Ax = \lim_{h\to 0}\left(\frac{U(t+h)-U(t)}{h}\right)Ax = A\left(\lim_{h\to 0}\frac{U(t+h)-U(t)}{h}\right)x = A\frac{d}{dt}U(t)x.$$

Finally, $\widehat{R}_{\mathrm{a,k}}(\lambda) = H(\lambda)$ for $\operatorname{Re}\lambda > \omega$. By assumption $(ii)$,

$$\widehat{R}_{\mathrm{a,k}}(\lambda) = \widehat{\mathrm{k}}(\lambda) + \widehat{\mathrm{a}}(\lambda)\widehat{R}_{\mathrm{a,k}}A,$$

that is,

$$R_{\mathrm{a,k}}(t)x = \mathrm{k}(t)x + \int_0^t \mathrm{a}(t-s)R_{\mathrm{a,k}}(s)Axds,$$

then the resolvent equation of Definition 2.1, item $(iii)$, is satisfied.

Moreover,

$$\|\widehat{R}_{\mathrm{a,k}}(\lambda)x\| = \|H(\lambda)x\| \leq \frac{M}{\lambda-\omega}\|x\|,$$

which implies that

$$\|R_{\mathrm{a,k}}(t)x\| \leq Me^{\omega t}\|x\|,$$

that is, $R_{\mathrm{a,k}}$ is of type $(M,\omega)$

Then the proof is complete. □

## 2.3 Approximation Theorem

This section presents a theorem of approximation for the $(\mathrm{a},\mathrm{k})$-regularized resolvent families. This result appeared in the literature for semigroups (see [43, Theorem 4.2, p. 85]), resolvent families (see [45, Theorem 6.3, p. 167]) and $\alpha$-resolvent families (see [7, Theorem 2.21, p. 30]).

**Theorem 2.12.** Let $(\mathrm{k}_n)_{n=0}^{\infty} \subset L^1_{loc}(\mathbb{R}_+)$ and $(\mathrm{a}_n)_{n=0}^{\infty} \subset AC_{loc}(\mathbb{R}_+)$ of type $(M,\omega)$, $\omega \geq 0$, such that $\widehat{\mathrm{a}}(\mu) \neq 0$ for $\mu > \omega$ and $\displaystyle\int_0^{\infty} e^{-\lambda s}\left|\frac{d}{ds}\mathrm{a}_n(s)\right| ds < \infty$. Let $(A_n)_{n=0}^{\infty}$ be closed and linear operators in $X$ such that $A_0$ is densely defined. Assume each $A_n$ generates an $(\mathrm{a}_n,\mathrm{k}_n)$-regularized resolvent family $\{R_{\mathrm{a}_n,\mathrm{k}_n}(t)\}_{t\geq 0}$ in $X$ for each $n \in \mathbb{N}$ and assume that

$$\sup_{n\in\mathbb{N}} \|R_{\mathrm{a}_n,\mathrm{k}_n}(t)\| \leq Me^{\omega t},\ t \in \mathbb{R}_+. \tag{2.15}$$

Suppose also $\mathrm{a}_n(t) \to \mathrm{a}_0(t)$ and $\mathrm{k}_n \to \mathrm{k}_0$ as $n \to \infty$. Then, the following statements are equivalent:

(i) $\lim_{n\to\infty} \widehat{\mathrm{k}}_n(\lambda)(I - \widehat{\mathrm{a}}_n(\lambda)A_n)^{-1}x = \widehat{\mathrm{k}}_0(\lambda)(I - \widehat{\mathrm{a}}_0(\lambda)A_0)^{-1}x$, for all $\lambda > \omega$, $x \in X$.

(ii) $\lim_{n\to\infty} R_{\mathrm{a}_n,\mathrm{k}_n}(t)x = R_{\mathrm{a}_0,\mathrm{k}_0}(t)x$, for all $x \in X$, $t \geq 0$. Moreover, the convergence is uniform in $t$ on every compact subset of $\mathbb{R}_+$.

*Proof.* Notice that

$$\widehat{\mathrm{k}}_n(\lambda)(I - \widehat{\mathrm{a}}_n(\lambda)A_n)^{-1}x = \int_0^{\infty} e^{-\lambda s}R_{\mathrm{a}_n,\mathrm{k}_n}(s)x\,ds.$$

Then, assuming $(ii)$, by Lebesgue's Dominated Convergente Theorem, $(i)$ holds.
Conversely, define $K_n(t) := (\mathrm{a}_n * R_{\mathrm{a}_n,\mathrm{k}_n})(t)$. From the hypothesis, it follows that

$$\begin{aligned}
\|K_n(t+h) - K_n(t)\| &= \left\|\int_t^{t+h} \frac{d}{ds}K_n(s)ds\right\| \\
&\leq \int_t^{t+h} \left\|\mathrm{a}_n(0)R_{\mathrm{a}_n,\mathrm{k}_n}(s) + \left(\left(\frac{d}{ds}\mathrm{a}_n\right) * R_{\mathrm{a}_n,\mathrm{k}_n}\right)(s)\right\| ds \\
&\leq |\mathrm{a}_n(0)| \int_t^{t+h} \|R_{\mathrm{a}_n,\mathrm{k}_n}(s)\|ds + \int_t^{t+h} \left\|\left(\left(\frac{d}{ds}\mathrm{a}_n\right) * R_{\mathrm{a}_n,\mathrm{k}_n}\right)(s)\right\| ds \\
&\leq Che^{w(t+h)},\ \forall t,h \geq 0.
\end{aligned}$$

Denote $H_n(\lambda) := \widehat{\mathrm{k}}_n(\lambda)(I - \widehat{\mathrm{a}}(\lambda)A)^{-1}$. By Proposition 2.8 and assumption $(i)$ it follows that

$$\begin{aligned}
\lim_{n\to\infty} \widehat{K}_n(\lambda)x &= \lim_{n\to\infty} \widehat{\mathrm{a}}_n(\lambda)\widehat{R}_{\mathrm{a}_n,\mathrm{k}_n}(\lambda)x = \lim_{n\to\infty} \widehat{\mathrm{a}}H_n(\lambda)x \\
&= \widehat{\mathrm{a}}_0(\lambda)H_0(\lambda)x = \widehat{\mathrm{a}}_0(\lambda)\widehat{R}_{\mathrm{a}_0,\mathrm{k}_0}(\lambda)x = \widehat{K}_0(\lambda)x,
\end{aligned}$$

for all $\lambda > \omega$ and $x \in X$.

Therefore, by Theorem 1.1

$$\lim_{n\to\infty} K_n(t)x = k_0(t)x, \quad \forall x \in X,$$

where the convergence is uniform in $t$ on compact subsets of $\mathbb{R}_+$, for a fixed $x \in X$.

Now let $y \in D(A_0)$ be fixed and let $x = [H_0(\lambda)]^{-1}y$, where $\lambda > \omega$. Then,

$$\begin{aligned}\|R_{\mathrm{a}_n,\mathrm{k}_n}(t)y - R_{\mathrm{a}_0,\mathrm{k}_0}(t)y\| \le \|R_{\mathrm{a}_n,\mathrm{k}_n}(t)(H_0(\lambda)x - H_n(\lambda)x)\| \\ + \|R_{\mathrm{a}_n,\mathrm{k}_n}(t)H_n(\lambda)x - R_{\mathrm{a}_0,\mathrm{k}_0}(t)H_0(\lambda)x\| \end{aligned} \tag{2.16}$$

and, since (2.15) and $(i)$ hold,

$$\lim_{n\to\infty} \|R_{\mathrm{a}_n,\mathrm{k}_n}(t)(H_0(\lambda)x - H_n(\lambda)x)\| = 0.$$

By the fact that $H_n(\lambda) - \widehat{\mathrm{k}}_n(\lambda) = \widehat{\mathrm{a}}_n(\lambda)A_nH_n(\lambda)$, from $(iii)$ of Definition 2.1 it follows that

$$\begin{aligned} R_{\mathrm{a}_n,\mathrm{k}_n}(t)H_n(\lambda)x &= \mathrm{k}_n(t)H_n(\lambda)x + \int_0^t \mathrm{a}_n(t-s)R_{\mathrm{a}_n\mathrm{k}_n}(s)H_n(\lambda)xds \\ &= \mathrm{k}_n(t)H_n(\lambda)x + K_n(t)A_nH_n(\lambda)x \\ &= \mathrm{k}_n(t)H_n(\lambda)x + \frac{1}{\widehat{\mathrm{a}}_n(\lambda)}K_n(t)[H_n(\lambda) - \widehat{\mathrm{k}}_n(\lambda)] \\ &= \mathrm{k}_n(t)H_n(\lambda)x + \frac{1}{\widehat{\mathrm{a}}_n(\lambda)}K_n(t)H_n(\lambda)x - \frac{\widehat{\mathrm{k}}_n(\lambda)}{\widehat{\mathrm{a}}_n(\lambda)}K_n(t)x, \end{aligned}$$

for $x \in X$, $\lambda > \omega$, $t \ge 0$ and $n \in \mathbb{N}$.

Thus,

$$\lim_{n\to\infty} \|R_{\mathrm{a}_n,\mathrm{k}_n}(t)H_n(\lambda)x - R_{\mathrm{a}_0,\mathrm{k}_0}(t)H_0(\lambda)x\| = 0.$$

Once $D(A_0)$ is dense, the assertion follows. □

## 2.4 Perturbation Theorem

Let $B : D(A) \to X$ be a linear operator. The objective of this section is to study conditions in order to guarantee the existence of an $(\mathrm{a},\mathrm{k})$-regularized resolvent family generated by $A + B$. This result is called the Perturbation Theorem.

This theorem appeared in the literature for $C_0$-semigroups (see [46], [43, Theorem 1.1, p. 76]), cosine families (see [41, Theorem 1]), resolvent and sine families (see [47]) and $\alpha$-resolvent families [7, Theorem 2.25, p.35].

**Theorem 2.13.** Let $A : D(A) \subset X \to X$ be the generator of an $(\mathrm{a},\mathrm{k})$-regularized resolvent family $\{R_{\mathrm{a},\mathrm{k}}(t)\}_{t\ge0}$ of type $(M,\omega)$. Suppose that

(i) there exists $\mathrm{b} \in L^1_{loc}(\mathbb{R}_+)$ such that $(\mathrm{b} * \mathrm{k})(t) = \mathrm{a}(t)$, for all $t \ge 0$;

(ii) there exists constants $\mu > \omega$ and $\gamma \in [0,1)$ such that

$$\int_0^\infty e^{-\mu\tau} \left\| B\int_0^\tau \mathrm{b}(\tau - s)R_{\mathrm{a},\mathrm{k}}(s)xds \right\| d\tau \le \gamma\|x\|, \ \forall x \in D(A).$$

Then, $A + B$ generates an $(\mathrm{a},\mathrm{k})$-regularized resolvent family $\{S_{\mathrm{ak}}(t)\}_{t\ge0}$ on $X$ such that

$$\|S_{\mathrm{a},\mathrm{k}}(t)\| \le \frac{M}{1-\gamma}e^{\mu t}.$$

In addition,

$$S_{\mathrm{a,k}}(t)x = R_{\mathrm{a,k}}(t)x + \int_0^t S_{\mathrm{a,k}}(t-\tau)B\int_0^\tau \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xdsd\tau,\ x\in D(A).$$

*Proof.* Let $T_0(t) := R_{\mathrm{a,k}}(t)$, then $T_0(t)\in\mathcal{L}(X)$, $t\mapsto T_0(t)$ is strongly continuous and

$$\|T_0(t)\| \le Me^{\omega t} \le \gamma^0 Me^{\mu t},\quad t\ge 0.$$

Suppose that there exists operators $T_j(t)\in\mathcal{L}(X)$, $j=0,1,\ldots,n$, $n\in\mathbb{N}$, with the following properties:

(a) $t\mapsto T_j(t)$ is strongly continuous;

(b) $\|T_j(t)\|\le \gamma^j Me^{\mu t}$, $t\ge 0$.

For $x\in D(A)$ define

$$T_{n+1}(t)x := \int_0^t T_n(t-\tau)B\int_0^\tau \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xdsd\tau.$$

Then, $t\mapsto T_{n+1}(t)x$ is continuous and, since $T_n(t)$ satisfies $(b)$, it follows that

$$\begin{aligned}\|T_{n+1}(t)x\| &= \left\|\int_0^t T_n(t-\tau)B\int_0^\tau \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xdsd\tau\right\|\\ &\le \int_0^t \|T_n(t-\tau)\|\left\|B\int_0^\tau \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xds\right\|d\tau\\ &\le \int_0^t \gamma^n Me^{\mu(t-\tau)}\left\|B\int_0^\tau \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xds\right\|d\tau\\ &\le \gamma^n Me^{\mu t}\int_0^t e^{-\mu\tau}\left\|B\int_0^\tau \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xds\right\|d\tau,\end{aligned}$$

and, by hypothesis $(ii)$,

$$\|T_{n+1}(t)x\| \le \gamma^n Me^{\mu t}\gamma\|x\| = \gamma^{n+1}Me^{\mu t}\|x\|.$$

Since $D(A)\subset X$ is dense, $T_{n+1}(t)$ can be uniquely extended to an operator $\widetilde{T}_{n+1}(t)$, also denoted as $T_{n+1}(t)$, and which satisfies $(a)$ and $(b)$.

Then, there exists operators $T_n(t)\in\mathcal{L}(X)$, $n=0,1,2,\ldots$, $t\ge 0$, with the properties $(a)$ and $(b)$.

Let $S_{\mathrm{a,k}}(t) := \sum_{n=0}^{\infty} T_n(t)$. Note that $S_{\mathrm{a,k}}$ is well defined because

$$\sum_{n=0}^{\infty}\|T_n(t)\| \le \sum_{n=0}^{\infty}\gamma^n Me^{\mu t} = \frac{M}{1-\gamma}e^{\mu t}.$$

Moreover,

$$\|S_{\mathrm{a,k}}\| \le \frac{M}{1-\gamma}e^{\mu t}.$$

For each $x \in D(A)$, using $(a)$ and $(b)$, it follows that the map $t \mapsto S_{\mathrm{a,k}}(t)x$ is continuous and

$$\begin{aligned} S_{\mathrm{a,k}}(t)x &= \sum_{n=0}^{\infty} T_n(t)x = T_0(x) + \sum_{n=1}^{\infty} T_n(t)x = R_{\mathrm{a,k}}(t)x + \sum_{n=0}^{\infty} T_{n+1}(t)x \\ &= R_{\mathrm{a,k}}(t)x + \sum_{n=0}^{\infty}\left(\int_0^t T_n(t-\tau)B\int_0^{\tau} \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xdsd\tau\right) \\ &= R_{\mathrm{a,k}}(t)x + \int_0^t \sum_{n=0}^{\infty} T_n(t-\tau)B\int_0^{\tau} \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xdsd\tau \\ &= R_{\mathrm{a,k}}(t)x + \int_0^t S_{\mathrm{a,k}}(t-\tau)B\int_0^{\tau} \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xdsd\tau. \end{aligned}$$

In particular, $S_{\mathrm{a,k}}(0)x = R_{\mathrm{a,k}}(0)x = k(0)x$ for all $x \in D(A)$, and as $D(A)$ is dense, $S_{\mathrm{a,k}}(0) = \mathrm{k}(0)I$.

Now, let $x \in X$ and define

$$H(\lambda)x = \int_0^{\infty} e^{-\lambda t} S_{\mathrm{a,k}}(t)xdt =: \widehat{S}_{\mathrm{a,k}}(\lambda)x$$

and

$$H(\lambda; A)x = \int_0^{\infty} e^{-\lambda t} R_{\mathrm{a,k}}(t)xdt =: \widehat{R}_{\mathrm{a,k}}(\lambda)x = \widehat{\mathrm{k}}(\lambda)(I - \widehat{\mathrm{a}}(\lambda)A)^{-1}x.$$

Then, consider $H_{\mathrm{k}}(\lambda)x = \dfrac{1}{\lambda\widehat{\mathrm{k}}(\lambda)}H(\lambda)x$ and $H_{\mathrm{k}}(\lambda; A)x = \dfrac{1}{\lambda\widehat{\mathrm{k}}(\lambda)}H(\lambda; A)x$.

Note that $H_k(\lambda)$ is a bounded operator and

$$\begin{aligned} \|H_{\mathrm{k}}(\lambda)\| &= \frac{1}{\lambda|\widehat{\mathrm{k}}(\lambda)|}\left\|\int_0^{\infty} e^{-\lambda t} S_{\mathrm{a,k}}(t)dt\right\| \le \frac{1}{\lambda|\widehat{\mathrm{k}}(\lambda)|}\int_0^{\infty} e^{-\lambda t}\|S_{\mathrm{a,k}}(t)\|dt \\ &\le \frac{1}{\lambda|\widehat{\mathrm{k}}(\lambda)|}\frac{1}{1-\gamma}\int_0^{\infty} e^{-(\lambda-\mu)t}dt = \frac{M}{(1-\gamma)(\lambda-\mu)\lambda|\widehat{\mathrm{k}}(\lambda)|}. \end{aligned}$$

Now observe that, for $x \in D(A)$,

$$H_{\mathrm{k}}(\lambda)x - H_{\mathrm{k}}(\lambda; A)x = \frac{1}{\lambda\widehat{\mathrm{k}}(\lambda)}(H(\lambda)x - H(\lambda; A)x)$$

and

$$\begin{aligned} H(\lambda)x - H(\lambda; A)x &= \widehat{S}_{\mathrm{a,k}}(\lambda)x - \widehat{R}_{\mathrm{a,k}}(\lambda)x \\ &= \widehat{R}_{\mathrm{a,k}}(\lambda)x + \widehat{S}_{\mathrm{a,k}}(\lambda)B\widehat{\mathrm{b}}(\lambda)\widehat{R}_{\mathrm{a,k}}(\lambda)x - \widehat{R}_{\mathrm{a,k}}(\lambda)x \\ &= \widehat{\mathrm{b}}(\lambda)H(\lambda)BH(\lambda; A)x. \end{aligned}$$

Then,

$$\begin{aligned} H_{\mathrm{k}}(\lambda)x - H_{\mathrm{k}}(\lambda; A)x &= \frac{\widehat{\mathrm{b}}(\lambda)}{\lambda\widehat{\mathrm{k}}(\lambda)}H(\lambda)BH(\lambda; A)x = \frac{H(\lambda)}{\lambda\widehat{\mathrm{k}}(\lambda)}\widehat{\mathrm{b}}(\lambda)BH(\lambda; A)x \\ &= H_{\mathrm{k}}(\lambda)\widehat{\mathrm{b}}(\lambda)BH(\lambda; A)x. \end{aligned}$$

So, since $D(A)$ is dense on $X$, it follows that

$$H_{\mathrm{k}}(\lambda) - H_{\mathrm{k}}(\lambda; A) = H_{\mathrm{k}}(\lambda)\widehat{\mathrm{b}}(\lambda)BH(\lambda; A).$$

But $H(\lambda; A) = \widehat{R}_{\mathrm{a,k}}(\lambda)$, then, for $x \in D(A)$,

$$\begin{aligned}\|\widehat{\mathrm{b}}(\lambda)BH(\lambda;A)x\| = \|B\widehat{R}_{\mathrm{a,k}}(\lambda)\widehat{\mathrm{b}}(\lambda)x\| &= \left\|B\int_0^\infty e^{-\lambda\tau}\int_0^\tau \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xdsd\tau\right\| \\ &\le \int_0^\infty e^{-\mu\tau}\left\|B\int_0^\tau \mathrm{b}(\tau-s)R_{\mathrm{a,k}}(s)xds\right\|d\tau \\ &\le \gamma\|x\|, \ \gamma < 1.\end{aligned}$$

Thus $(I - \widehat{\mathrm{b}}(\lambda)BH(\lambda;A))^{-1}$ exists and is bounded.
So, $H_{\mathrm{k}}(\lambda) = H_{\mathrm{k}}(\lambda;A)(I - \widehat{\mathrm{b}}(\lambda)BH(\lambda;A))^{-1}$ implies that

$$\begin{aligned}&[\lambda - \lambda\widehat{\mathrm{a}}(\lambda)(A+B)]H_{\mathrm{k}}(\lambda) \\ &= [\lambda - \lambda\widehat{\mathrm{a}}(\lambda)(A+B)]H_{\mathrm{k}}(\lambda)(I - \widehat{\mathrm{b}}(\lambda)BH(\lambda;A))^{-1} \\ &= [\lambda(I - \widehat{\mathrm{a}}(\lambda))AH_{\mathrm{k}}(\lambda;A) - \lambda\widehat{\mathrm{a}}(\lambda)BH_{\mathrm{k}}(\lambda;A)](I - \widehat{\mathrm{b}}(\lambda)BH(\lambda;A))^{-1} \\ &= \left[\frac{(I - \widehat{\mathrm{a}}(\lambda)A)}{\widehat{\mathrm{k}}(\lambda)}H(\lambda;A) - \lambda\widehat{\mathrm{a}}(\lambda)BH_{\mathrm{k}}(\lambda;A)\right](I - \widehat{\mathrm{b}}(\lambda)BH(\lambda;A))^{-1} \\ &= \left[\frac{(I - \widehat{\mathrm{a}}(\lambda)A)}{\widehat{\mathrm{k}}(\lambda)}\widehat{R}_{\mathrm{a,k}}(\lambda) - \lambda\widehat{\mathrm{a}}(\lambda)BH_{\mathrm{k}}(\lambda;A)\right]\left(I - \frac{\widehat{\mathrm{a}}(\lambda)}{\widehat{\mathrm{k}}(\lambda)}BH(\lambda;A)\right)^{-1} \\ &= [I - \lambda\widehat{\mathrm{a}}(\lambda)BH_{\mathrm{k}}(\lambda;A)](I - \lambda\widehat{\mathrm{a}}(\lambda)BH_{\mathrm{k}}(\lambda;A))^{-1} \\ &= I.\end{aligned}$$

This proves that $(\lambda - \lambda\widehat{\mathrm{a}}(\lambda)(A+B))$ is invertible and satisfies

$$(I - \widehat{\mathrm{a}}(\lambda)(A+B))^{-1}x = \frac{1}{\widehat{\mathrm{k}}(\lambda)}\int_0^\infty e^{-\lambda t}S_{\mathrm{a,k}}(t)xdt, \ x \in X.$$

Using Proposition 2.8 the proof is complete. □

## 2.5 Spectral properties

This section presents some spectral properties of the generator of an $(\mathrm{a},\mathrm{k})$-regularized resolvent family. This result for $C_0$-semigroups can be found in [43, Theorem 2.3, p. 45], and for cosine families can be found in [41].

For more results of spectral properties of $(\mathrm{a},\mathrm{k})$-regularized resolvent families the reader can see [38].

For each $\lambda \in \mathbb{C}$, $\mathrm{s}_\lambda(t)$ denotes the unique solution of the scalar valued convolution equation

$$\mathrm{s}_\lambda(t) = \mathrm{a}(t) + \lambda\int_0^t \mathrm{a}(t-\tau)\mathrm{s}_\lambda(\tau)d\tau, t \ge 0. \tag{2.17}$$

It is also defined

$$\mathrm{r}_\lambda(t) := \mathrm{k}(t) + \lambda\int_0^t \mathrm{s}_\lambda(t-\tau)\mathrm{k}(\tau)d\tau. \tag{2.18}$$

**Theorem 2.14.** Let $\{R_{\mathrm{a,k}}(t)\}_{t\geq 0}$ be an $(\mathrm{a},\mathrm{k})$-regularized resolvent family with generator $A$. Then,

$$\sigma(R_{\mathrm{a,k}}(t)) \supset \mathrm{r}_{\sigma(A)}(t),\ t \geq 0.$$

*Proof.* Let $x \in D(A)$. Then *(iii)* of Definition 2.1 and Lemma 2.3 show that

$$\begin{aligned}(\mathrm{s}_\lambda * (\lambda - A)R_{\mathrm{a,k}})(t)x &= \lambda(\mathrm{s}_\lambda * R_{\mathrm{a,k}})(t)x - (\mathrm{s}_\lambda * AR_{\mathrm{a,k}})(t)x \\ &= \lambda(\mathrm{s}_\lambda * R_{\mathrm{a,k}})(t)x - ([\mathrm{a} + \lambda(\mathrm{a} * \mathrm{s}_\lambda)] * AR_{\mathrm{a,k}})(t)x \\ &= \lambda(\mathrm{s}_\lambda * R_{\mathrm{a,k}})(t)x - (\mathrm{a} * AR_{\mathrm{a,k}})(t)x - \lambda(\mathrm{a} * \mathrm{s}_\lambda * AR_{\mathrm{a,k}})(t)x \\ &= \lambda(\mathrm{s}_\lambda * R_{\mathrm{a,k}})(t)x - [R_{\mathrm{a,k}} - \mathrm{k}](t)x - \lambda(\mathrm{s}_\lambda * [R_{\mathrm{a,k}} - \mathrm{k}])(t)x \\ &= \mathrm{k}(t)x + \lambda(\mathrm{s}_\lambda * \mathrm{k})(t)x - R_{\mathrm{a,k}}(t)x \\ &= \mathrm{r}_\lambda(t)x - R_{\mathrm{a,k}}(t)x,\end{aligned}$$

for all $\lambda \in \mathbb{C}$ and $t \geq 0$.

From the closedness of $A$, it follows that

$$(\lambda - A)(\mathrm{s}_\lambda * R_{\mathrm{a,k}})(t)x = \mathrm{r}_\lambda(t)x - R_{\mathrm{a,k}}(t)x,$$

for all $x \in X$, $\lambda \in \mathbb{C}$ and $t \geq 0$.

Suppose $\mathrm{r}_\lambda \in \rho(R_{\mathrm{a,k}}(t))$ for some $\lambda \in \mathbb{C}$ and $t \geq 0$, and denote $L_{\lambda,t} := (\mathrm{r}_\lambda - R_{\mathrm{a,k}}(t))^{-1}$.

Since $L_{\lambda,t}$ commutes with $R_{\mathrm{a,k}}(t)$, and hence also with $A$, it follows that

$$(\lambda - A)\int_0^t \mathrm{s}_\lambda(t-\tau)R_{\mathrm{a,k}}(\tau)L_{\lambda,t}d\tau = x,$$

for all $x \in X$, and

$$\int_0^t \mathrm{s}_\lambda(t-\tau)R_{\mathrm{a,k}}(\tau)L_{\lambda,t}(\lambda - A)xd\tau = x,$$

for all $x \in D(A)$.

Define $B_\lambda x := \displaystyle\int_0^t \mathrm{s}_\lambda(t-\tau)R_{\mathrm{a,k}}(\tau)L_{\lambda,t}xd\tau$. Then, $B_\lambda$ is a bounded operator and is a two-sided inverse of $(\lambda - A)$. Thus, $\lambda \in \rho(A)$.

Observe that if $\theta \in \sigma(A)$, then $\theta \notin \rho(A)$ and from the previous assertions, it follows that $\mathrm{r}_\theta(t) \notin \rho(R_{\mathrm{a,k}}(t))$, that is, $r_\theta \in \sigma(R_{\mathrm{a,k}}(t))$.

Further, $\mathrm{r}_{\sigma(A)} \subset \sigma(R_{\mathrm{a,k}}(t))$. □

The next results present spectral inclusions for the point and residual spectrum.

**Theorem 2.15.** Let $\{R_{\mathrm{a,k}}(t)\}_{t\geq 0}$ be an $(\mathrm{a},\mathrm{k})$-regularized resolvent family with generator $A$. Then,

$$\sigma_p(R_{\mathrm{a,k}}(t)) \supset \mathrm{r}_{\sigma_p(A)}(t),\ t \geq 0.$$

*Proof.* Let $\lambda \in \sigma_p(A)$ and $x \in D(A)$ be an eigenvector corresponding to $\lambda$, so

$$\int_0^t \mathrm{s}_\lambda(t-\tau)R_{\mathrm{a,k}}(\tau)(\lambda - A)xd\tau = \mathrm{r}_\lambda(t)x - R_{\mathrm{a,k}}(t)x,\ \forall x \in D(A), \tag{2.19}$$

shows that $R_{\mathrm{a,k}}(t)x = \mathrm{r}_\lambda(t)x$. Then $\mathrm{r}_\lambda(t)$ is an eigenvalue of $R_{\mathrm{a,k}}(t)$ with eigenvector $x \in D(A)$.

Thus, $\mathrm{r}_{\sigma_p(A)}(t) \subset \sigma_p(R_{\mathrm{a,k}}(t))$, $t \geq 0$. □

**Theorem 2.16.** [38, Theorem 5.6] Let $\{R_{\mathrm{a,k}}(t)\}_{t\geq 0}$ be an $(\mathrm{a},\mathrm{k})$-regularized resolvent family with generator $A$. Then,

$$\sigma_r(R_{\mathrm{a,k}}(t)) \supset \mathrm{r}_{\sigma_r(A)}(t),\ t \geq 0.$$

# 3 Zero-one law

Let $A$ be a closed linear operator with domain $D(A)$ defined on a complex Banach space $X$. This chapter studies an intriguing structural property of the class of strongly continuous family of bounded linear operators $\{R_{\mathrm{a,k}}\}$ and in particular $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$, with $\alpha > 0$, $\beta > 0$.

This family contains several important classes of well-known subfamilies, as seen in Example 2.2. Although several qualitative properties are well known for the class of semigroups and cosine families, much less has been reported in the setting of integrated semigroups and, specially, $\alpha$-times resolvent families and $(\mathrm{a}, \mathrm{k})$-regularized resolvent families.

Concerning the differences of structure among the various subfamilies $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$, it has been recently proven that, if the set of all bounded strongly continuous cosine families is treated as a metric space under the metric of the uniform convergence associated with the operator norm on the space $\mathcal{L}(X)$ of all bounded linear operators on $X$, then the isolated points of this set are precisely the scalar cosine families [9]. By definition, a scalar cosine family is a family whose members are all scalar multiples of the identity operator. Remarkably, this picture changes dramatically considering to semigroups of operators. In such case, the isolated points constitute only a small fraction of the set of all scalar semigroups. The proof of this and related properties relies on the fact that if the distance between cosine families (resp. semigroups) and their scalar counterparts is less than a certain bound, say $\gamma$, then the cosine family (resp. semigroup) must be scalar. They are called $0-\gamma$ Laws. Only recently the problem to determine the optimal bound for cosine families was solved, obtaining $\gamma = \frac{8}{3\sqrt{3}}$ [10, 19, 18, 23, 22, 51]. It is surprising that a corresponding result for integrated semigroups and sine families ($\alpha = \beta = 2$) has just been discovered in 2017 [8]. Motivated by the above earlier works it is natural to ask the following:

(Q) It is posible to find $0-\gamma$ Laws for the class of strongly continuous families of bounded operators?

This question is answered in the first section of this chapter at least for the case $\gamma = 1$. As a second contribution, an example is given to show that the optimal bound, say $\theta$, in case of $\beta \geq \alpha$ and $0 < \alpha \leq 1$ is strictly less than 1, which interpretes the fact that, roughly speaking, $\beta$-times integrated $\alpha$-resolvent families have more regularity at $t = 0$ than $(\alpha, \beta)$-resolvent families, when they are over the diagonal $\beta = \alpha$, and in this way widely improving a recent result for the very special case $\alpha = 1, \beta = 2$ (see [8]). The results of the first section were published in [25].

The second section presents the zero-one law for $(\mathrm{a}, \mathrm{k})$-regularized resolvent families.

## 3.1 Zero-one law for $(\alpha,\beta)$-resolvent families

The concept of $\alpha$-times resolvent families, or solution operators, plays an important role in the theory of fractional abstract Cauchy problems, that models several physical phenomena. One example is the fractional diffusion-wave equation

$$D_t^\alpha u(x,t) = k^2 u_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0, \quad 0 < \alpha < 2, \quad k \in \mathbb{R}, \qquad (3.1)$$

with initial conditions $u(x,0) = f(x)$, $u_t(x,0) = 0$ (the last one only when $1 < \alpha < 2$). As seen in [7, Example 3.6], the explicit form of the $\alpha$-times resolvent family is

$$R_{\alpha,1}(t)f(x) = \frac{1}{2|k|t^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} \phi_{\frac{\alpha}{2}}\left(\frac{|s|}{|k|t^{\frac{\alpha}{2}}}\right) f(x-s)ds,$$

where $\phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}$ is a function of Wright type.

The fractional diffusion equation (3.1) ($\alpha \in (0,1)$) has been introduced by Nigmatullin in [42] to describe diffusion in special types of porous media. Mainardi [39] has shown that the fractional wave equation (3.1) ($\alpha \in (1,2)$) governs the propagation of mechanical diffusive waves in viscoelastic media.

First a purely algebraic notion of the theory of $(\alpha,\beta)$-resolvents of bounded linear operators is presented and more details can be found on [37].

The $(\alpha,1)$-resolvent families are called *$\alpha$-resolvent families,* or solution operator, or fractional resolvent family/operator in the current literature. For $0 < \alpha = \beta < 1$ the above definition was studied by Li and Peng [32]. This concept was introduced earlier [2], but without reference to the condition near to zero given in $(a)$.

The linear operator $A$ defined by

$$D(A) := \{x \in X \ : \ \lim_{t\to 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ exists } \}$$

and

$$Ax := \lim_{t\to 0^+} \frac{R_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)} \text{ for } x \in D(A)$$

is called the *generator* of the $(\alpha,\beta)$-resolvent family $\{R_{\alpha,\beta}(t)\}_{t\geq 0}$.

For example, if $A$ is a bounded operator, then

$$R_{\alpha,\beta}(t) := \sum_{n=0}^{\infty} g_{\alpha n+\beta}(t)A^n = t^{\beta-1}\sum_{n=0}^{\infty} \frac{A^n t^{\alpha n}}{\Gamma(\alpha n+\beta)} = t^{\beta-1}E_{\alpha,\beta}(At^\alpha), \quad t > 0,$$

defines a uniformly continuous $(\alpha,\beta)$-resolvent family. Given $\beta > 1$, observe that the family $\{R_{\alpha,\beta}(t)\}_{t>0}$ is $(\beta-1)$-times integrated with respect to $\{R_{\alpha,1}(t)\}_{t\geq 0}$ as the identity

$$R_{\alpha,\beta}(t) = (g_{\beta-1} * R_{\alpha,1})(t) = J_t^{\beta-1}R_{\alpha,1}(t), \quad t > 0, \qquad (3.2)$$

holds. The following characterization is often used as definition.

**Theorem 3.1.** [37, Theorem 3.1 and Theorem 4.3] Let $\alpha > 0$ and $\beta > 0$ be given. A strongly continuous family $\{R_{\alpha,\beta}(t)\}_{t>0} \subset \mathcal{L}(X)$ of bounded linear operators in $X$ is an $(\alpha,\beta)$-resolvent family generated by $A$ if and only if the following conditions are satisfied

(i) $\lim_{t\to 0} t^{1-\beta}R_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)}I$ if $0<\beta<1$, $R_{\alpha,1}(0)=I$ and $R_{\alpha,\beta}(0)=0$ if $\beta>1$.

(ii) $R_{\alpha,\beta}(t)x \in D(A)$ and $R_{\alpha,\beta}(t)Ax = AR_{\alpha,\beta}(t)x$ for all $x\in D(A)$ and $t\geq 0$;

(iii) $R_{\alpha,\beta}(t)x = g_\beta(t)x + \int_0^t g_\alpha(t-s)AR_{\alpha,\beta}(s)xds, \quad t\geq 0, \quad x\in D(A).$

If $\beta>1$, $A$ is closed, but not necessarily densely defined [17, Proposition 3.10]. In the diagonal case $\alpha=\beta$ this notion appears by the first time in [2, Definition 2.3]. If $0<\alpha=\beta<1$ then $A$ must be densely defined [32, Theorem 3.1].

Roughly speaking, the notion of $(\alpha,1)$-resolvent families is associated with the Caputo fractional derivative, whereas the notion of $(\alpha,\alpha)$-resolvent family is linked with the Riemann-Liouville fractional derivative. Other relevant cases are $(\alpha,\gamma+(1-\gamma)\alpha)$-resolvent families with $0<\alpha<1,\ 0\leq\gamma\leq 1$, see [27], and $(\alpha,\alpha+\gamma(2-\alpha))$-resolvent families with $1<\alpha<2,\ 0\leq\gamma\leq 1$, see [40], because they are related with the notion of Hilfer fractional derivative that interpolates between the Caputo and Riemann-Liouville fractional derivative (for $0<\alpha<1$ take $\gamma=1$ and $\gamma=0$, respectively).

Assuming that $A$ is the generator of an $\gamma$-times integrated semigroup ($\gamma\geq 0$), i.e. an $(1,\gamma+1)$-resolvent family, then $(\alpha,\alpha\gamma+1)$-resolvent families and $(\alpha,\alpha(\gamma+1))$-resolvent families are important for $0<\alpha<1$ because these are the key for the treatment of existence, regularity and representation of fractional diffusion equations (see [30]). The same happens with $(\alpha,\frac{\alpha\gamma}{2}+1)$-resolvent families and $(\alpha,\alpha(\frac{\gamma}{2}+1))$-resolvent families for $1\leq\alpha\leq 2$ because these are present in the theoretical analysis of fractional wave equations (see [31]).

The following theorem is the main result of this section.

**Theorem 3.2.** Let $0<\alpha\leq 2$ and $\beta>0$ be given and let $(R_{\alpha,\beta}(t))_{t\geq 0}$ be an $(\alpha,\beta)$-resolvent family generated by $A$. If

$$\sup_{t>0}\left\|\frac{1}{t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)}R_{\alpha,\beta}(t)-I\right\| =: \theta < 1, \tag{3.3}$$

then $R_{\alpha,\beta}(t)=t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)I$ for all $t>0$ and $\lambda\geq 0$.

*Proof.* For all $t\geq 0$, $x\in X$ and $\lambda\geq 0$ as given in the hypothesis, define

$$B(t)x := \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)R_{\alpha,\beta}(\tau)xd\tau.$$

From Theorem 2.14, it follows that $B(t)x\in D(A)$ and

$$(\lambda-A)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)R_{\alpha,\beta}(\tau)xd\tau = t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)x - R_{\alpha,\beta}(t)x. \tag{3.4}$$

Denote

$$b_{\alpha,\beta}(\lambda,t) := \int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)E_{\alpha,\beta}(\lambda\tau^\alpha)d\tau.$$

Since $\lambda\geq 0$, $b_{\alpha,\beta}(\lambda,t)>0$ for $t>0$ can be obtaind from (1.3) and it follows the estimate

$$\left\|x-\frac{1}{b_{\alpha,\beta}(\lambda,t)}B(t)x\right\|$$

$$= \left\| \frac{1}{b_{\alpha,\beta}(\lambda,t)} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)(\tau^{\beta-1}E_{\alpha,\beta}(\lambda\tau^\alpha)x - R_{\alpha,\beta}(\tau)x)d\tau \right\|$$
$$= \left\| \frac{1}{b_{\alpha,\beta}(\lambda,t)} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha)\tau^{\beta-1}E_{\alpha,\beta}(\lambda\tau^\alpha) \left[ x - \frac{R_{\alpha,\beta}(\tau)x}{\tau^{\beta-1}E_{\alpha,\beta}(\lambda\tau^\alpha)} \right] d\tau \right\|$$
$$\leq \frac{1}{b_{\alpha,\beta}(\lambda,t)} \sup_{\tau\geq 0} \left\| I - \frac{R_{\alpha,\beta}(\tau)}{\tau^{\beta-1}E_{\alpha,\beta}(\lambda\tau^\alpha)} \right\| \|x\| \int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1} E_{\alpha,\alpha}(\lambda(t-\tau)^\alpha) E_{\alpha,\beta}(\lambda\tau^\alpha)d\tau$$
$$\leq \theta\|x\|.$$

Since $\theta < 1$, the operator $\frac{1}{b_{\alpha,\beta}(\lambda,t)}B(t)$ is boundedly invertible for all $t > 0$ and

$$\left\|b_{\alpha,\beta}(\lambda,t)(B(t))^{-1}\right\| \leq \sum_{k=0}^{\infty} \left\| I - \frac{1}{b_{\alpha,\beta}(\lambda,t)}B(t) \right\|^k \leq \sum_{k=0}^{\infty} \theta^k = \frac{1}{1-\theta},$$

which implies that

$$\|(B(t))^{-1}\| \leq \frac{1}{(1-\theta)b_{\alpha,\beta}(\lambda,t)}, \tag{3.5}$$

in the norm of $\mathcal{L}(X)$. From (3.3) it follows that

$$\begin{aligned} \|(\lambda - A)B(t)x\| &= \|R_{\alpha,\beta}(t)x - t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)x\| \\ &= \left\| t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha) \left[ \frac{R_{\alpha,\beta}(t)}{t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)} - I \right] x \right\| \\ &\leq \theta t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\|x\|, \end{aligned}$$

for all $t > 0$ and $x \in X$. Hence, for each $x \in D(A)$

$$\|(\lambda - A)x\| = \|(\lambda - A)B(t)(B(t)^{-1}x)\| \leq \frac{\theta\, t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)}{(1-\theta)b_{\alpha,\beta}(\lambda,t)}\|x\|, \ \forall t > 0 \tag{3.6}$$

can be obtained from (3.5).

Inserting (1.5) in (3.6) implies that

$$\|(\lambda - A)x\| \leq \frac{\theta\, E_{\alpha,\beta}(\lambda t^\alpha)}{(1-\theta)t^\alpha E'_{\alpha,\beta}(\lambda t^\alpha)}\|x\|, \quad \forall t > 0. \tag{3.7}$$

From (1.4) and (1.7) it follows that $E'_{\alpha,\beta}(z) \sim \frac{1}{\alpha^2} z^{\frac{1-\beta-\alpha}{\alpha}} e^{z^{\frac{1}{\alpha}}}[(1-\beta) + z^{\frac{1}{\alpha}}]$. In particular,

$$\frac{E_{\alpha,\beta}(z)}{E'_{\alpha,\beta}(z)} \sim \frac{\alpha z}{(1-\beta) + z^{\frac{1}{\alpha}}} \quad |z| \to \infty, \quad |\arg z| < \frac{\alpha\pi}{2}.$$

Consequently,

$$\frac{E_{\alpha,\beta}(\lambda t^\alpha)}{t^\alpha E'_{\alpha,\beta}(\lambda t^\alpha)} \sim \frac{\alpha\lambda}{(1-\beta) + \lambda^{\frac{1}{\alpha}} t}, \quad t \to \infty.$$

Taking $t \to \infty$ in (3.7), it follows that $Ax = \lambda x$ for all $x \in D(A)$. Since $B(t)x \in D(A)$ for each $x \in X$ and $t > 0$, then (3.4) implies that $R_{\alpha,\beta}(t)x = t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)x$, $x \in X, t \geq 0$. □

Considering the special case $\lambda = 0$, it follows this important consequence, stated as a Theorem.

**Theorem 3.3.** Let $0 < \alpha \leq 2$ and $\beta > 0$ be given and let $(R_{\alpha,\beta}(t))_{t\geq 0}$ be an $(\alpha,\beta)$-resolvent family generated by $A$. If

$$\sup_{t>0} \| \frac{1}{g_\beta(t)} R_{\alpha,\beta}(t) - I \| < 1,$$

then $R_{\alpha,\beta}(t) = g_\beta(t)I$ for all $t \geq 0$.

Observe that the result is dependent of $\beta > 0$, which interpretes the fact that the property takes into account the regularizing effect of the parameter $\beta$, and hence of the family of operators under consideration, near to zero.

the zero-one law for $C_0$-semigroups is a corollary of Theorem 3.13 (with $\alpha = \beta = 1$) and can be found for example in [51, Theorem 3.2] (see also [52, Remark 3.1.4]).

**Corollary 3.4.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup generated by $A$. Suppose that

$$\sup_{t>0} \|T(t) - I\| < 1.$$

Then $T(t) = I$ for all $t \geq 0$.

From the literature, a zero-two law [51] for cosine families, and also a $0 - \frac{3}{2}$ law [3, Theorem 1.1] for cosine families on general Banach spaces was proved without considering strong continuity (see also the reference [18]) In [19], Chojnacki gives an extension of the results from [51] in the case of cosine families, not necessarily continuous, in a normed algebra.

Observe that a zero-one law for strongly continuous cosine families was proved in [50, Theorem 1.1]. Such zero-one law for cosine families is a consequence of Theorem 3.13 (with $\alpha = 2, \beta = 1$) as follows.

**Corollary 3.5.** Let $(C(t))_{t\geq 0}$ be a strongly continuous cosine family generated by $A$. Suppose that

$$\sup_{t>0} \|C(t) - I\| < 1.$$

Then $C(t) = I$ for all $t \geq 0$.

**Remark 3.6.** The case $\alpha = 1$ and $\beta = 2$ in Theorem 3.13 is of particular interest. In such case the family $(R_{1,2}(t))_{t\geq 0}$ corresponds to an integrated semigroup and hence Theorem 3.13 coincides with recent results of Bobrowski [8, Theorem 2.3]. The findings of Bobrowski are stated in an arbitrary unital Banach Algebra.

It is interesting to observe that for the range $\beta \geq \alpha$ and $0 < \alpha \leq 1$ in Theorem 3.13, the bound $\theta = 1$ is optimal. The following example inspired in [8, Example 2.4] shows this fact:

**Example 3.7.** For $\beta \geq 1$, $0 < \alpha \leq 1$ and $\lambda > 0$ consider the scalar $(\alpha,\beta)$-resolvent family defined by

$$R_{\alpha,\beta}(t) := \int_0^t g_{\beta-\alpha}(t-s) E_{\alpha,1}(-\lambda s^\alpha) ds,$$

and using (1.6) it is possible to verify the following identity

$$R_{\alpha,\beta}(t) = t^{\beta-\alpha} E_{\alpha,\beta-\alpha+1}(-\lambda t^\alpha).$$

In other words, $R_{\alpha,\beta}(t)$ is an $(\alpha, \beta-\alpha+1)$-resolvent family. For all $t>0$ it follows that

$$\left|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1\right| = \left|\frac{1}{g_\beta(t)}\int_0^t g_{\beta-\alpha}(t-s)E_{\alpha,1}(-\lambda s^\alpha)ds-1\right|$$
$$= \left|\frac{1}{g_\beta(t)}t^{\beta-\alpha}E_{\alpha,\beta-\alpha+1}(-\lambda t^\alpha)-1\right|.$$

From [49], the function $x \to E_{\alpha,\beta}(-x)$ is completely monotone if and only if $0<\alpha\le 1$ and $\beta\ge\alpha$. Therefore $E_{\alpha,\beta-\alpha+1}(-\lambda t^\alpha)\ge 0$ for all $t\ge 0$. In particular, $R_{\alpha,\beta}(t)\ge 0$.

The identity

$$R_{\alpha,\beta}(t)=g_\beta(t)-\lambda\int_0^t g_\alpha(t-s)R_{\alpha,\beta}(s)ds,$$

implies that

$$\frac{1}{g_\beta(t)}t^{\beta-\alpha}E_{\alpha,\beta-\alpha+1}(-\lambda t^\alpha)-1=-\frac{\lambda}{g_\beta(t)}\int_0^t g_\alpha(t-s)R_{\alpha,\beta}(s)ds\le 0.$$

Consequently

$$\left|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1\right| = 1-\frac{1}{g_\beta(t)}t^{\beta-\alpha}E_{\alpha,\beta-\alpha+1}(-\lambda t^\alpha)\le 1.$$

Then, $\sup_{t>0}\left|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1\right|\le 1.$

Conversely, from Theorem 1.3 for any $0<\alpha<2$ and $\beta>0$ there exists a constant $C>0$ such that, for each $\frac{\alpha\pi}{2}<\mu<\min\{\pi,\alpha\pi\}$ the estimate

$$|E_{\alpha,\beta-\alpha+1}(z)|\le\frac{C}{1+|z|}\qquad \mu<|\arg(z)|\le\pi,$$

holds. This shows that $\lim_{t\to\infty}E_{\alpha,\beta-\alpha+1}(-\lambda t^\alpha)=0$. Hence

$$\sup_{t>0}\left|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1\right| = \sup_{t>0}|\Gamma(\beta)t^{1-\alpha}E_{\alpha,\beta-\alpha+1}(-\lambda t^\alpha)-1|$$
$$\ge \lim_{t\to\infty}|\Gamma(\beta)t^{1-\alpha}E_{\alpha,\beta-\alpha+1}(-\lambda t^\alpha)-1|=1.$$

Therefore,

$$\sup_{t>0}\left|\frac{1}{g_\beta(t)}R_{\alpha,\beta}(t)-1\right|=1,$$

while $R_{\alpha,\beta}(t)\ne g_\beta(t)$, proving that $\theta<1$ is optimal.

Now consider $\alpha$-resolvent families $S_\alpha(t)$ (*i.e.* $0<\alpha\le 2, \beta=1$). The following zero-one law is a simple corollary of Theorem 3.13 and constitutes a completely new result.

**Corollary 3.8.** Let $0<\alpha\le 2$ and $(S_\alpha(t))_{t\ge 0}$ be an $\alpha$-resolvent family generated by $A$. Suppose that

$$\sup_{t>0}\|S_\alpha(t)-I\|<1.$$

Then $S_\alpha(t)=I$ for all $t\ge 0$.

For the next results $\lambda \neq 0$. They are some of the consequences of Theorem 3.2. Beginning with the following extension of Corollary 3.4, which seems to be new in the present form although is also an easy consequence of Wallen's formula, see [9, Lemma 10].

**Corollary 3.9.** Let $(T(t))_{t\geq 0}$ be a $C_0$-semigroup generated by $A$. Let $\lambda \geq 0$ be given and suppose that

$$\sup_{t>0} \|e^{-\lambda t}T(t) - I\| < 1.$$

Then $T(t) = e^{\lambda t}I$ for all $t \geq 0$.

The following corollary is a new result about the zero-one law for cosine families (see [50]).

**Corollary 3.10.** Let $(C(t))_{t\geq 0}$ be a strongly continuous cosine family generated by $A$, and $\lambda > 0$ be given. Suppose that

$$\sup_{t\geq 0} \left\| \frac{1}{\cosh(\sqrt{\lambda}t)} C(t) - I \right\| < 1.$$

Then $C(t) = \cosh(\sqrt{\lambda}t)I$ for all $t \geq 0$.

The following is an extension of the zero-one law to $\alpha$-resolvent families. This result is also new.

**Corollary 3.11.** Let $0 < \alpha \leq 2$ and $(S_\alpha(t))_{t\geq 0}$ be a $\alpha$-resolvent family generated by $A$. Given $\lambda > 0$ suppose that

$$\sup_{t\geq 0} \left\| \frac{1}{E_{\alpha,1}(\lambda t^\alpha)} S_\alpha(t) - I \right\| < 1.$$

Then $S_\alpha(t) = E_{\alpha,1}(\lambda t^\alpha)I$ for all $t \geq 0$. Moreover, $\theta < 1$ is optimal in the range $0 < \alpha \leq 1$.

## 3.2 Zero-one law for (a,k)-regularized resolvent families

This sections presents a general version of the zero-one law for $(\mathrm{a},\mathrm{k})$-regularized resolvent families.

**Theorem 3.12.** Let $(R_{\mathrm{a},\mathrm{k}}(t))_{t\geq 0}$ be an $(\mathrm{a},\mathrm{k})$-regularized resolvent generated by $A$. Let $\lambda \in \mathbb{C}$ be such that $r_\lambda \neq 0$ and

$$\lim_{t\to\infty} \frac{\mathrm{r}_\lambda(t)}{(\mathrm{s}_\lambda * \mathrm{r}_\lambda)(t)} = 0 \tag{3.8}$$

Finally, because for each $x \in X$ and $t > 0$ and consider

$$\epsilon_{\mathrm{r},\mathrm{s}} = \sup_{t>0} \frac{\displaystyle\int_0^t |\mathrm{s}_\lambda(t-\tau)\mathrm{r}_\lambda(\tau)|d\tau}{\left|\displaystyle\int_0^t \mathrm{s}_{\lambda(t-\tau)}\mathrm{r}_\lambda(\tau)d\tau\right|}.$$

If

$$\sup_{t>0}\left\|\frac{1}{\mathrm{r}_\lambda(t)}R_{\mathrm{a,k}}(t)-I\right\| =: \theta, \tag{3.9}$$

with $\theta\epsilon_{\mathrm{r,s}}<1$, then $R_{\mathrm{a,k}}(t)=\mathrm{r}_\lambda(t)I$ for all $t\geq 0$.

*Proof.* For all $t\geq 0$, $x\in X$ and $\lambda\in\mathbb{C}$ given as in the hypothesis, define

$$B(t)x := \int_0^t s_\lambda(t-\tau)R(\tau)x d\tau.$$

From Theorem 2.14, it follows that $B(t)x\in D(A)$ and

$$(\lambda-A)\int_0^t \mathrm{s}_\lambda(t-\tau)R_{\mathrm{a,k}}(\tau)xd\tau = \mathrm{r}_\lambda(t)x - R_{\mathrm{a,k}}(t)x. \tag{3.10}$$

Therefore

$$\begin{aligned}
\left\|x-\frac{1}{(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)}B(t)x\right\| &= \left\|\frac{1}{(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)}\int_0^t \mathrm{s}_\lambda(t-\tau)(\mathrm{r}_\lambda(\tau)x-R_{\mathrm{a,k}}(\tau)x)d\tau\right\|\\
&= \left\|\frac{1}{(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)}\int_0^t \mathrm{s}_\lambda(t-\tau)\mathrm{r}_\lambda(\tau)\left[x-\frac{R_{\mathrm{a,k}}(\tau)}{\mathrm{r}_\lambda(\tau)}x\right]d\tau\right\|\\
&\leq \frac{1}{|(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)|}\sup_{\tau\geq 0}\left\|I-\frac{R_{\mathrm{a,k}}(\tau)}{\mathrm{r}_\lambda(\tau)}\right\|\|x\|\int_0^t|\mathrm{s}_\lambda(t-\tau)\mathrm{r}_\lambda(\tau)|d\tau\\
&\leq \frac{\theta(|\mathrm{s}_\lambda*\mathrm{r}_\lambda|)(t)}{|(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)|}\|x\|\\
&\leq \theta\epsilon_{\mathrm{r,s}}\|x\|.
\end{aligned}$$

Thus, since $\theta\epsilon_{\mathrm{r,s}}<1$, it follows that the operator $\frac{1}{(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)}B(t)$ is boundedly invertible for all $t>0$ and

$$\left\|(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)(B(t))^{-1}\right\| \leq \sum_{k=0}^\infty\left\|I-\frac{1}{(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)}B(t)\right\|^k \leq \sum_{k=0}^\infty(\epsilon_{\mathrm{r,s}}\theta)^k = \frac{1}{1-\epsilon_{\mathrm{r,s}}\theta},$$

which implies that

$$\|(B(t))^{-1}\| \leq \frac{1}{(1-\epsilon_{\mathrm{r,s}}\theta)(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)}, \tag{3.11}$$

in the norm of $\mathcal{L}(X)$. From (3.9) it follows that

$$\|(\lambda-A)B(t)x\| = \|R_{\mathrm{a,k}}(t)x-\mathrm{r}_\lambda(t)x\| = \left\|\mathrm{r}_\lambda(t)\left[\frac{R_{\mathrm{a,k}}(t)}{\mathrm{r}_\lambda(t)}-I\right]x\right\| \leq \epsilon_{\mathrm{r,s}}\theta|\mathrm{r}_\lambda(t)|\|x\|,$$

for all $x\in X$. Hence, for any $x\in D(A)$

$$\|(\lambda-A)x\| = \|(\lambda-A)B(t)(B(t)^{-1}x)\| \leq \frac{\epsilon_{\mathrm{r,s}}\theta\mathrm{r}_\lambda(t)}{(1-\epsilon_{\mathrm{r,s}}\theta)(\mathrm{s}_\lambda*\mathrm{r}_\lambda)(t)}\|x\|,\ \forall t>0 \tag{3.12}$$

can be obtained by (3.11).

In consequence, by hypothesis (3.8), taking $t\to\infty$ in (3.12) it follows that $Ax=\lambda x$ for all $x\in D(A)$. Since $B(t)x\in D(A)$ for each $x\in X$ and $t\geq 0$, (3.10) implies that $R_{\mathrm{a,k}}(t)x=\mathrm{r}_\lambda(t)x$, $x\in X, t\geq 0$. □

The following Theorem is obtained considering $\lambda = 0$.

**Theorem 3.13.** Let $(R_{\mathrm{a,k}}(t))_{t\geq 0}$ be an $(\mathrm{a},\mathrm{k})$-regularized resolvent generated by $A$, with $\mathrm{k}(t) \neq 0$ satisfying

$$\lim_{t\to\infty} \frac{\mathrm{k}(t)}{(\mathrm{k} * \mathrm{a})(t)} = 0 \tag{3.13}$$

and consider

$$\epsilon_{\mathrm{a,k}} = \sup_{t>0} \frac{\displaystyle\int_0^t |\mathrm{a}(t-\tau)\mathrm{k}(\tau)|d\tau}{\left|\displaystyle\int_0^t \mathrm{a}(t-\tau)\mathrm{k}(\tau)d\tau\right|}.$$

If

$$\sup_{t\geq 0} \left\| \frac{1}{\mathrm{k}(t)} R_{\mathrm{a,k}}(t) - I \right\| = \theta, \tag{3.14}$$

with $\epsilon_{\mathrm{a,k}}\theta < 1$ then $R_{\mathrm{a,k}}(t) = k(t)I$ for all $t \geq 0$.

The following example shows a family that satisfies condition (3.13) and such that $\epsilon_{\mathrm{a,k}} = 1$.

**Example 3.14.** Given $b, c > 0$, consider the $(\mathrm{a},\mathrm{k})$-regularized resolvent family generated by $-A$ and with $\mathrm{a}(t) = b + c^2 t$ and $\mathrm{k} \equiv 1$.

Note that

$$\lim_{t\to\infty} \frac{1}{\int_0^t b + c^2 s ds} = \lim_{t\to\infty} \frac{2}{2bt + c^2 t} = 0$$

and, since $\mathrm{a}(t) \geq 0$, $\mathrm{k}(t) > 0$ for $t \geq 0$, it follows that

$$\int_0^t |\mathrm{a}(t-\tau)\mathrm{k}(\tau)|d\tau = \left|\int_0^t \mathrm{a}(t-\tau)\mathrm{k}(\tau)d\tau\right|,$$

that is, $\epsilon_{\mathrm{a,k}} = 1$.

# 4 Blackstock-Crighton-Westervelt equation

The classical models in nonlinear acoustics are partial differential equations of second order in time and characterized by the presence of a viscoelastic damping. The most general of these popular models is Kuznetsov's equation

$$u_{tt} - c^2\Delta u - b\Delta u_t = \left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_t, \tag{4.1}$$

where $u$ denotes the acoustic velocity potential, $c > 0$ is the speed of sound, $b \geq 0$ is the diffusivity of sound and $B/A$ is the parameter of nonlinearity. Neglecting local nonlinear effects one arrives at the Westervelt equation

$$u_{tt} - b\Delta u_t - c^2\Delta u = \left(\frac{1}{c^2}\left(1 + \frac{B}{2A}\right)(u_t)^2\right)_t. \tag{4.2}$$

The Kuznetsov equation can be regarded in some sense as a simplification of the following higher order model

$$(a\Delta - \partial_t)(u_{tt} - c^2\Delta u - b\Delta u_t) = \left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_{tt}, \quad a > 0, \tag{4.3}$$

which is called the Blackstock-Crighton-Kuznetsov equation. The constant $a$ is the heat conductivity of the fluid. Neglecting local nonlinear effects as its done when reducing the Kuznetsov to the Westervet equation, one arrives at the Blackstock-Crighton-Westervelt equation

$$(a\Delta - \partial_t)(u_{tt} - c^2\Delta u - b\Delta u_t) = \left(\frac{1}{c^2}\left(1 + \frac{B}{2A}\right)(u_t)^2\right)_{tt}. \tag{4.4}$$

The seminal mathematical study of this equation was initiated in 2014 by Brunnhuber and Kaltenbacher [13]. These authors used the theory of $C_0$-semigroups in order to investigate the linearization of the model, proving that the underlying semigroup is analytic. That leads to exponential decay results for the linear homogeneous equation. Moreover, it was proved local in time well-posedness of the model under the assumption that initial data are sufficiently small and a fixed point argument. Global in time well-posedness was also obtained, by performing energy estimates and using the classical barrier method, again for sufficiently small initial data. Additionally, Brunnhuber and Kaltenbacher provided results concerning exponential decay of solutions of the nonlinear equation.

Later, equation (4.3) was studied by Brunnhuber and Meyer in the reference [14] to show optimal regularity and exponential stability in $L_p$-spaces with Dirichlet and Newmann boundary conditions. In such reference, it was also proved long-time well-posedness and exponential stability for sufficiently small data.

More recently, in the reference [16], Celik and Kyed considered the Blackstock-Crighton- Westervelt equation in a threedimensional bounded domain with both non-homogeneous Dirichlet and Neumann boundary values. Existence of a solution was obtained via a fixed-point argument based on appropriate a priori estimates for the linearized equations.

However, up to date, there is no research on the abstract modeling of equation (4.3), i.e. replacing the Laplace operator $-\Delta$ by a general closed linear operator $A$ defined on a Banach space.

This chapter is concerned with the study of the Blackstock-Crighton-Westervelt equation in a generalized abstract form

$$(-aA - D_t)(u''(t) + c^2Au(t) + bAu'(t)) = f(t, u, u_t), \; t \geq 0, \tag{4.5}$$

defined in a Banach space $X$ with initial conditions $x = u(0)$, $y = u'(0)$, $z = u''(0)$, and $A : D(A) \subset X \to X$ a closed linear densely defined operator that satisfy appropriate conditions described later. Moreover, in (4.5) $D_t$ denotes the differentiation operator of order 1 with respecto to the temporal variable $t$.

As intimate before, this approach in the setting of Banach spaces is completely new and has been not studied until now. The main advantage is that the abstract model can serve as prototype for other common operators $A$, like e.g. the fractional Laplacian, among others.

One of our main and surprising results obtained using this abstract approach, use the theory of $C_0$-semigroups of operators $\{T(t)\}_{t\geq 0}$, combined with the theory of resolvent families $\{S(t)\}_{t\geq 0}$, see [45], to solve *explicitly* the linearized equation (4.5) which provides new insights even in case that $A = -\Delta$, the negative Laplacian. Namely, it is proved that the solution of the linearized equation can be represented as:

$$\begin{aligned} u(t) \; = \; & e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T(s)xds + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)xds \\ & + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)xds - \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)xds \\ & + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)xd\tau ds + R(t)y - \frac{a+b}{a-b}\int_0^t T(s)yds \\ & + \frac{a+b}{a-b}\int_0^t S(s)yds - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)yds + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)yd\tau ds \\ & + \int_0^t R(s)zds + \int_0^t\int_0^s R(s-\tau)f(\tau)d\tau ds, \end{aligned} \tag{4.6}$$

where $R(t) = (S * T)(t)$ is the finite convolution of a resolvent family $\{S(t)\}_{t\geq 0}$ generated by $-A$ with kernel $\mathrm{a}(t) = b + c^2t$, and $\{T(t)\}_{t\geq 0}$ a $C_0$-semigroup generated by $-aA$.

Our second contribution in this chapter is that, assuming the existence of the above representation, and certain hypothesis on the nonlinearity $f$, the existence of at least one mild solution of the nonlinear model (4.5) can be guaranteed.

This chapter is organized as follows: the first section is concerned with the preliminary results of the theory os analytic semigroups and resolvent families. In the second section it is shown an explicit representation of the solution presenting conditions for a mild solution to be strong. A local mild solution for the Blackstock-Crighton-Westevelt equation is proved in section 3. And section 4 is concerned with the mild solution of the equation (4.5) with nonlocal inicial conditions.

## 4.1 Some results of analytic semigroups and resolvent families

This section presents some definitions and results of the theory of analytic semigroups and resolvent families. These results are essential to the work developed in the other sections. Starting with the following definition of resolvent family due to Pruss [45].

**Definition 4.1.** [45, Definition 1.3 p. 32] A family $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ of bounded linear operators in $X$ is called a resolvent family if the following conditions are satisfied:

(S1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;

(S2) $S(t)$ commutes with $A$, which means that $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(S3) the resolvent equation holds

$$S(t)x = x + \int_0^t \mathrm{a}(t-s)AS(s)xds, \text{ for all } x \in D(A),\ t \geq 0,$$

where $\mathrm{a} \in L^1_{loc}(\mathbb{R}_+)$ is a scalar kernel $\not\equiv 0$. In this case $A$ is called the generator of the resolvent family $\{S(t)\}_{t\geq 0}$.

The following definition was introduced by Pruss [45] and have ultimate importance in the development of our main results.

**Definition 4.2.** [45, Definition 3.2 p. 68] Let $\mathrm{a} \in L^1_{loc}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. The kernel $\mathrm{a}(t)$ is called $k$-regular if there is a constant $C > 0$ such that

$$|\lambda^n \hat{\mathrm{a}}^{(n)}(\lambda)| \leq C|\hat{\mathrm{a}}(\lambda)|, \quad \text{for all} \quad Re(\lambda) > 0,\ 0 \leq n \leq k. \tag{4.7}$$

The following examples are given to illustrate the above definition and they are very useful later.

**Example 4.3.** Define $\mathrm{a}(t) = b + c^2 t$. Then, $\mathrm{a}(t)$ is $k$-regular for all $k \in \mathbb{N}$. Indeed, given $k \in \mathbb{N}$ notice that for each $n \in \{0, 1, ..., k\}$ it follows that

$$\lambda^n \hat{\mathrm{a}}^{(n)}(\lambda) = (-1)^n n! b\lambda^{-1} + (-1)^n (n+1)! c^2 \lambda^{-2},$$

then,

$$\frac{\lambda^n \hat{\mathrm{a}}^{(n)}(\lambda)}{\hat{\mathrm{a}}(\lambda)} = \frac{(-1)^n n! [b\lambda^{-1} + c^2\lambda^{-1}] + (-1)^n n(n!) c^2 \lambda^{-2}}{-b\lambda^{-1} - c^2\lambda^{-2}}$$

$$= (-1)^{n+1}n! + (-1)^{n+1}n(n!)\frac{c^2}{b\lambda + c^2},$$

which is bounded, for all $\Re(\lambda) > 0$ and $0 \le n \le k$. Therefore $\mathrm{a}(t)$ is $k$-regular, for all $k \in \mathbb{N}$.

The following result will be very useful in this paper. And its proof is very similar to the proof of [45, Theorem 3.1, p. 73]

**Theorem 4.4.** Let $\mathrm{a} \in L^1_{loc}(\mathbb{R}_+)$ be $k$-regular and suppose that $A$ is closed and densely defined operator on a Banach space $X$ such that $\hat{\mathrm{a}}(\lambda) \neq 0$ and $\frac{1}{\hat{\mathrm{a}}(\lambda)} \in \rho(A)$ for all $Re(\lambda) > \omega$. Suppose that

$$\|\lambda^{-1}(I - \hat{\mathrm{a}}(\lambda)A)^{-1}\| \le \frac{M}{|\lambda - \omega|}, \quad \text{for all } \; Re(\lambda) > \omega. \tag{4.8}$$

Then there exists a resolvent family $\{S(t)\}_{t\ge 0}$ such that $S \in C^{k-1}((0,\infty), \mathcal{L}(X))$ and $\|S(t)\| \le Me^{wt}$.

**Remark 4.5.** It is posible to obtain spatial regularity of the resolvent $\{S(t)\}_{t\ge0}$, provided $\mathrm{a}(t)$ is $k$-regular for some large enough $k$. It follows the estimate

$$\|t^k AS(t)\| \le Me^{wt}, \quad t \ge 0, \tag{4.9}$$

i.e. $S(t)X \subset D(A)$ for all $t > 0$ (see [45, Comment (f) p. 82]).

**Example 4.6.** The following is an example of a kernel $\mathrm{a} \in L^1_{loc}(\mathbb{R}_+)$ which is $k$-regular and such that $|\arg(\hat{\mathrm{a}}(\lambda))| \le \theta_0$, for all $Re(\lambda) > \omega := \frac{c^2}{b} \in \mathbb{R}$ for some $\theta_0 < \pi$.

Consider $\mathrm{a}(t) = -b - c^2 t$. Then there exists $\theta_0 < \pi$ such that $|\arg(\hat{\mathrm{a}}(\lambda))| < \theta_0$ for all $Re(\lambda) > \frac{c^2}{b}$. Indeed, for $\lambda = re^{i\theta}$ with $Re(\lambda) > \frac{c^2}{b}$ and $\theta < \pi$ it follows that

$$\begin{aligned}
\arg(\hat{\mathrm{a}}(re^{i\theta})) &= \operatorname{Im}\log(\hat{\mathrm{a}}(re^{i\theta})) = \operatorname{Im}\int_0^\theta \frac{d}{dt}\log(\hat{\mathrm{a}}(re^{it}))dt \\
&= \operatorname{Im}\int_0^\theta \frac{\hat{\mathrm{a}}'(re^{it})ire^{it}}{\hat{\mathrm{a}}(re^{it})}dt \\
&= \operatorname{Im}\int_0^\theta 1i + \frac{c^2}{br(\cos t + i\sin t) + c^2}dt \\
&= \operatorname{Im}\theta i + \operatorname{Im}\int_0^\theta \frac{c^2 i(c^2 + br\cos t - ibr\sin t)}{(c^2 + br\cos t)^2 + (br\sin t)^2}dt \\
&= \theta + c^2\int_0^\theta \frac{c^2 + br\cos t}{2brc^2\cos t + c^4 + b^2r^2}dt.
\end{aligned}$$

Using [28, 2554 2, p. 173], then

$$\begin{aligned}
\arg(\hat{\mathrm{a}}(re^{i\theta})) &= \theta + c^2\left[\frac{brt}{2brc^2}\right]_0^\theta + c^2\frac{2brc^4 - (b^3r^3 + brc^4)}{2brc^2}\int_0^\theta \frac{dt}{b^2r^2 + c^4 + 2brc^2\cos t} \\
&= \theta + \frac{\theta}{2} - \frac{b^2r^2 - c^4}{2}\int_0^\theta \frac{dt}{b^2r^2 + c^4 + 2brc^2\cos t}
\end{aligned}$$

From [28, 2553 3, p. 172] it is possible to conclude that

$$\begin{aligned}\arg(\hat{a}(re^{i\theta})) &= \frac{\theta}{2} - \frac{b^2r^2-c^4}{2}\left[\frac{2}{\sqrt{(b^2r^2-c^4)^2}}\arctan\left(\frac{(br-c^2)^2}{\sqrt{(b^2r^2-c^4)^2}}\tan\left(\frac{\theta}{2}\right)\right)\right] \\ &= \frac{3\theta}{2} - \arctan\left(\frac{br-c^2}{br+c^2}\tan\left(\frac{\theta}{2}\right)\right).\end{aligned}$$

Since $Re(\lambda) > \frac{c^2}{b}$, then $|\theta| < \frac{\pi}{2}$. Let $\theta_1 > 0$ be such that $|\theta| < \frac{\theta_1}{2} < \frac{\pi}{2}$. Then $|\tan\left(\frac{\theta}{2}\right)| < 1$. Notice that

$$\left|\frac{br-c^2}{br+c^2}\right| \leq 1$$

then

$$\tan\left(-\frac{\pi}{4}\right) = -1 < \frac{br-c^2}{br+c^2}\tan\left(\frac{\theta}{2}\right) < 1 = \tan\left(\frac{\pi}{4}\right),$$

that is

$$-\frac{\pi}{4} < \arctan\left(\frac{br-c^2}{br+c^2}\tan\left(\frac{\theta}{2}\right)\right) < \frac{\pi}{4},$$

therefore

$$-\frac{3\theta_1}{4} - \frac{\pi}{4} < \frac{3\theta}{2} - \arctan\left(\frac{br-c^2}{br+c^2}\tan\left(\frac{\theta}{2}\right)\right) < \frac{3\theta_1}{4} + \frac{\pi}{4} =: \theta_0 < \pi.$$

Then, $|\arg\hat{a}(\lambda)| < \pi$ for all $Re(\lambda) > \frac{c^2}{b}$.

**Example 4.7.** From Example 4.6, note that $\frac{1}{\hat{a}(\lambda)} \in \Sigma(0,\pi)$ for all $Re(\lambda) > \frac{c^2}{b}$.

Let $-A$ be a closed and densely defined operator on $X$. If $\rho(-A) \supset \Sigma(0,\pi)$ for $Re(\lambda) > \frac{c^2}{b}$ and

$$\|\lambda(\lambda^2 I + (b\lambda + c^2)A)^{-1}\| \leq \frac{M}{|\lambda - \frac{c^2}{b}|}, \quad \text{for all} \quad Re(\lambda) > \frac{c^2}{b}, \tag{4.10}$$

then, $-A$ satisfies the hypotheses of Theorem 4.4 and there exists a resolvent family $\{S(t)\}_{t\geq 0}$ such that $S \in C^\infty((0,\infty),\mathcal{L}(X))$ and $\|S(t)\| \leq Me^{\frac{c^2}{b}t}$. Moreover, from Remark 4.5, $S(t)X \subset D(-A)$ for all $t > 0$.

## 4.2 Well-posedness and strong solutions

Let $X$ be a Banach space. This section is concerned with the study the well posedness for the abstract equation (4.5), that is rewritten including their initial conditions as follows

$$\begin{cases} u'''(t) + (a+b)Au''(t) + (abA + c^2)Au'(t) + ac^2A^2u(t) = f(t), \ t \geq 0 \\ u(0) = x, \quad u'(0) = y \quad u''(0) = z, \end{cases} \tag{4.11}$$

where $x, y, z \in X$ and $A : D(A) \subset X \to X$ is a operator that satisfy appropriate conditions which described later. First, the notion of solution used in this chapter is introduced.

**Definition 4.8.** A function $u : \mathbb{R}_+ \to X$ is called a strong solution of (4.11) if satisfies

(i) $u \in C(\mathbb{R}_+; D(A^2)) \cap C^3(\mathbb{R}_+; X)$;

(ii) $u' \in C(\mathbb{R}_+; D(A^2))$;

(iii) $u'' \in C(\mathbb{R}_+; D(A))$;

(iv) (4.11) holds on $\mathbb{R}_+$.

A closed linear densely defined operator $A$ satisfies **hypothesis** $(H)$ if

(i) $-A$ is the generator of a analytic semigroup $\{T(t)_{t\geq 0}\}$ uniformly bounded, that is $\|T(t)\| \leq M$.

(ii) $-A$ generates a resolvent family $\{S(t)\}_{t\geq 0}$ with kernel $\mathrm{a}(t) = b + c^2 t$ and satisfying $\|S(t)\| \leq M e^{\omega t}$ and $S(t)X \subset D(A)$.

In such case, $T_a(t)$ denotes the semigroup generated by $-aA$, and

$$R(t) := (S * T_a)(t) := \int_0^t S(t-s) T_a(s) ds, \quad t \geq 0, \tag{4.12}$$

the finite convolution. Here the integral is understood in the Bochner sense. Note that $\|R(t)\| \leq K e^{\omega t}$ for some $K > 0$ and $\omega \in \mathbb{R}$.

**Remark 4.9.** If $-A$ generates an analytic semigroup uniformly bounded such that $\Sigma(0, \pi) \subset \rho(A)$ for $Re(\lambda) > 0$, then $A$ satisfies hypothesis $(H)$. Indeed, item $(i)$ is clear and it follows that

$$\|\lambda(\lambda I + A)^{-1}\| \leq \frac{M}{|\lambda|}, \quad \forall\, Re(\lambda) > 0$$

and then, since $Re(\lambda^2(b\lambda + c^2)) > 0$,

$$\begin{aligned} \|\lambda(\lambda^2 I + (b\lambda + c^2)A)^{-1}\| &= \|\lambda(b\lambda + c^2)(\lambda^2(b\lambda + c^2)I + A)^{-1}\| \\ &\leq \frac{|\lambda(b\lambda + c^2)|M}{|\lambda^2(b\lambda + c^2)|} = \frac{M}{|\lambda|} \\ &\leq \frac{M}{\left|\lambda - \frac{c^2}{b}\right|}. \end{aligned}$$

From Example 4.7 it is possible to conclude that $-A$ generates a resolvent family $\{S(t)\}_{t\geq 0}$ of type $\left(M, \frac{c^2}{b}\right)$ with kernel $\mathrm{a}(t) = b^2 + c^2 t$ and such that $S(t)X \subset D(-A)$.

**Example 4.10.** If $1 < p < \infty$, the Laplacian operator $\Delta$ in $L^p(\mathbb{R}^n)$, i.e. the operator $\Delta_p$ with domain $D(\Delta_p) = \{f \in L^p(\mathbb{R}^n);\ \Delta f \in L^p(\mathbb{R}^n)\}$ satisfies hypothesis $(H)$. Indeed, $-\Delta_p$ generates an analytic semigroup uniformly bounded and $\Sigma(0, \pi) \in \rho(-\Delta_p)$. (see [15, Theorem 2.3.3 p. 40 and A.7.6 p. 329])

The next Lemma concerns the definition of $R(t)$ given in (4.12).

**Lemma 4.11.** Let $A$ be a closed linear operator satisfying hypothesis $(H)$. Suppose $a \neq b$, $a > 0$, then $R(t)x \in D(A)$, for all $x \in X$ and satisfies

$$AR(t)x = \frac{1}{a-b}\left[S(t)x - T_a(t)x + \frac{c^2}{a}\int_0^t S(s)xds - \frac{c^2}{a}R(t)x\right], \quad \forall x \in X. \tag{4.13}$$

Moreover, $AR(t)x \in D(A)$ for all $x \in X$.

*Proof.* Notice that by hypothesis, $S(t)X \subset D(A)$, and $T_a(t)X \subset D(A)$ where $\{T_a(t)\}_{t\geq 0}$ is an analytic semigroup generated by $-aA$ (see [20, Theorem 4.6 (c) p. 101]). Since $-A$ is closed, for all $x \in X$ it follows that

$$R(t)x = (T_a * S)(t)x = \int_0^t T_a(t-s)S(s)xds \in D(A).$$

Since $T_a(t)$ is uniformly bounded and $S(t)$ and $R(t)$ are exponentially bounded the Laplace transform can be applied, and then, for all $Re(\lambda) > \omega$ and $x \in X$

$$\begin{aligned} A\hat{R}(\lambda)x &= A\hat{S}(\lambda)\hat{T}_a(\lambda)x = \hat{S}(\lambda)\left[-\frac{\lambda}{a}\hat{T}_a(\lambda) + \frac{1}{a}I\right]x \\ &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)\lambda\hat{S}(\lambda)x = \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)[\hat{S'}(\lambda) + I]x. \end{aligned}$$

By $(S3)$ of Definition 4.1 with a$(t) = b + c^2t$, it follows that

$$S'(t)x = -bAS(t)x - \int_0^t c^2AS(s)xds,$$

and therefore

$$\begin{aligned} A\hat{R}(\lambda)x &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)\left[-bA\hat{S}(\lambda) - \frac{c^2}{\lambda}A\hat{S}(\lambda)\right]x \\ &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}A\hat{R}(\lambda)x + \frac{c^2}{a\lambda}A\hat{T}_a(\lambda)\hat{S}(\lambda)x \\ &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}A\hat{R}(\lambda)x + \frac{c^2}{a\lambda}\left[-\frac{\lambda}{a}\hat{T}_a(\lambda) + \frac{1}{a}I\right]\hat{S}(\lambda)x \\ &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}A\hat{R}(\lambda)x - \frac{c^2}{a^2}\hat{R}(\lambda)x + \frac{c^2}{a^2\lambda}\hat{S}(\lambda)x. \end{aligned}$$

Then

$$\frac{a-b}{a}A\hat{R}(\lambda)x = \frac{1}{a}\left[\hat{S}(\lambda) - \hat{T}_a(\lambda) - \frac{c^2}{a}\hat{R}(\lambda) + \frac{c^2}{a\lambda}\hat{S}(\lambda)\right].$$

So, applying the inversion of the Laplace transform, and by the uniqueness theorem, it follows that

$$AR(t)x = \frac{1}{a-b}\left[S(t)x - T_a(t)x + \frac{c^2}{a}\int_0^t S(s)xds - \frac{c^2}{a}R(t)x\right].$$

Since $S(t)X \subset D(A)$, $T_a(t)X \subset D(A)$ and $A$ is closed, the last assertion of the lemma can be concluded. □

The following lemma gives an additional property.

**Lemma 4.12.** Let $A$ be a closed linear operator satifying hypothesis $(H)$. Suppose $a > 0$, $a \neq b$. Then,

$$\begin{aligned} A^2 \int_0^t R(s)xds &= \frac{1}{a(a-b)}T_a(t)x - \frac{1}{b(a-b)}S(t)x - \frac{c^2}{a(a-b)^2}\int_0^t S(s)xds \\ &+\frac{c^2}{a(a-b)^2}\int_0^t T_a(s)xds + \frac{c^4}{a^2(a-b)^2}\int_0^t R(s)xds \\ &-\frac{c^4}{a^2(a-b)^2}\int_0^t\int_0^s S(\tau)xd\tau ds \\ &+\frac{1}{ab}e^{-\frac{c^2}{b}t}x + \frac{c^2}{ab^2}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)xds \end{aligned} \tag{4.14}$$

for all $x \in X$.

*Proof.* Given $x \in X$, define by $G(t)x$ the right hand side of (4.14). Notice that $G(t)x$ is well defined for all $t \geq 0$. Moreover, it follows the identity

$$-\frac{1}{b(a-b)}S(t)x = -\left[\frac{1}{a(a-b)} + \frac{1}{ab}\right]S(t)x. \tag{4.15}$$

Then, replacing (4.15) in $G(t)x$ and applying the Laplace transform, which is possible since $T_a(t)$ is uniformly bounded, and $S(t)$ and $R(t)$ are exponentially bounded,it follows that

$$\begin{aligned} \hat{G}(\lambda)x &= \frac{1}{a(a-b)}\hat{T}_a(\lambda)x - \frac{1}{a(a-b)}\hat{S}(\lambda)x + \frac{c^2}{a(a-b)^2\lambda}\hat{T}_a(\lambda)x - \frac{c^2}{a(a-b)^2\lambda}\hat{S}(\lambda)x \\ &+\frac{c^4}{a^2(a-b)^2\lambda}\hat{R}(\lambda)x - \frac{c^4}{a^2(a-b)^2\lambda^2}\hat{S}(\lambda)x + \frac{1}{ab}\frac{1}{\lambda+\frac{c^2}{b}}x \\ &+\frac{c^2}{ab^2}\frac{1}{\lambda+\frac{c^2}{b}}\hat{S}(\lambda)x - \frac{1}{ab}\hat{S}(\lambda)x \\ &= \frac{1}{a(a-b)}[\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{c^2}{a(a-b)^2\lambda}\hat{T}_a(\lambda)x - \frac{c^2}{a(a-b)^2\lambda}\hat{S}(\lambda)x \\ &+\frac{c^4}{a^2(a-b)^2\lambda}\hat{R}(\lambda)x - \frac{c^4}{a^2(a-b)^2\lambda^2}\hat{S}(\lambda)x + \frac{1}{a(b\lambda+c^2)}[I - \lambda\hat{S}(\lambda)]x \\ &= \frac{1}{a(a-b)}[\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{1}{a(b\lambda+c^2)}[I - \lambda\hat{S}(\lambda)]x \\ &+\frac{c^2}{a(a-b)^2\lambda}\left[\hat{T}_a(\lambda) - \hat{S}(\lambda) + \frac{c^2}{a}\hat{R}(\lambda) - \frac{c^2}{a\lambda}\hat{S}(\lambda)\right]x. \end{aligned}$$

By Lemma 4.11 and since $\hat{a}(\lambda) = \frac{b}{\lambda} + \frac{c^2}{\lambda^2}$ it follows that

$$\hat{G}(\lambda)x = \frac{1}{a(a-b)}[\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{1}{a\hat{a}(\lambda)}\left[\frac{1}{\lambda^2}I - \frac{1}{\lambda}\hat{S}(\lambda)\right]x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x.$$

Moreover, by definition of $S(t)$,

$$A\hat{S}(\lambda)x = \frac{1}{\lambda\hat{a}(\lambda)}x - \frac{\hat{S}(\lambda)}{\hat{a}(\lambda)}x.$$

Then,

$$\begin{aligned}
\hat{G}(\lambda)x &= \frac{1}{a(a-b)\lambda}[\lambda\hat{T}_a(\lambda) - \lambda\hat{S}(\lambda)]x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x \\
&= \frac{1}{a(a-b)\lambda}[\hat{T}_a'(\lambda) - \hat{S}'(\lambda)]x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x \\
&= \frac{1}{a(a-b)\lambda}\left[\hat{T}_a'(\lambda) + \frac{c^2}{\lambda}A\hat{S}(\lambda) + bA\hat{S}(\lambda)\right]x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x \\
&= \frac{1}{(a-b)\lambda}\left[\frac{1}{a}\hat{T}_a'(\lambda) + \frac{c^2}{a\lambda}A\hat{S}(\lambda) + A\hat{S}(\lambda) - \frac{c^2}{a}A\hat{R}(\lambda)\right]x
\end{aligned}$$

Using Lemma 4.11 and the fact that $-aA$ is the generator of $T_a(t)$, it is concluded that

$$\begin{aligned}
\hat{G}(\lambda)x &= \frac{1}{(a-b)\lambda}\left[A\hat{S}(\lambda) - A\hat{T}_a(\lambda) + \frac{c^2}{a\lambda}A\hat{S}(\lambda) - \frac{c^2}{a}A\hat{R}(\lambda)\right]x \\
&= A^2\frac{1}{\lambda}\hat{R}(\lambda)x.
\end{aligned}$$

Notice that from Lemma 4.11, $AR(t)x \in D(A)$ for $x \in X$, and then $A^2R(t)x$ is well defined for $x \in X$. So, applying the inversion of the Laplace transform and by the uniqueness theorem,

$$G(t)x = A^2\int_0^t R(s)xds,$$

for all $x \in X$. This finishes the proof. □

**Remark 4.13.** Considering

$$R(t)x = (T_a * S)(t)x = \int_0^t T_a(t-s)S(s)xds, \quad x \in X,$$

then

$$R'(t)x = S(t)x - aAR(t)x,$$

for all $x \in X$;

$$\begin{aligned}
R''(t)x &= S'(t)x - aAR'(t)x = S'(t)x - aAS(t)x + a^2A^2R(t)x \\
&= -bAS(t)x - c^2A\int_0^t S(s)xds - aAS(t)x + a^2A^2R(t)x \\
&= -(a+b)AS(t)x - c^2A\int_0^t S(s)xds + a^2A^2R(t)x
\end{aligned}$$

for all $x \in X$; and

$$
\begin{aligned}
R'''(t)x =& -(a+b)AS'(t)x - c^2AS(t)x + a^2A^2R'(t)x\\
=& -(a+b)AS'(t)x - c^2AS(t)x + a^2A^2S(t)x - a^3A^3R(t)x\\
=& -(a+b)AS'(t)x + (a^2-c^2)AS(t)x - a^3A^3R(t)x,
\end{aligned}
$$

for all $x \in D(A)$. Moreover

$$
\begin{aligned}
\|R(t)\| &\leq \int_0^t \|T_a(t-s)\|\|S(s)\|ds \leq M^2\int_0^t e^{\omega s}ds\\
&= \frac{M^2}{\omega}[e^{\omega t}-1] \leq \frac{M^2}{\omega}e^{\omega t}
\end{aligned}
$$

Considering $K = \dfrac{M^2}{\omega}$, then

$$
\|R(t)\| \leq Ke^{\omega t}.
$$

The next theorem is the main result of this section.

**Theorem 4.14.** Let $A$ be a closed linear operator satisfying hypothesis $(H)$, $a > 0$, $a \neq b$. If $f \in L^1_{loc}(\mathbb{R}_+, D(A))$, $x \in D(A^2)$, $y \in D(A^2)$ and $z \in D(A)$, then $u(t)$ given by

$$
\begin{aligned}
u(t) &= \left[S(t) + bAR(t) + (abA + c^2)A\int_0^t R(s)ds\right]x + \left[R(t) + (a+b)A\int_0^t R(s)ds\right]y\\
&+ \int_0^t R(s)zds + \int_0^t (R*f)(s)ds. \qquad (4.16)
\end{aligned}
$$

is a strong solution of (4.11).

*Proof.* For $x, y, z \in X$ and $f \in L^1_{loc}(\mathbb{R}_+; X)$ consider

$$
\begin{aligned}
u(t) &= \left[S(t) + bAR(t) + (abA + c^2)A\int_0^t R(s)ds\right]x + \left[R(t) + (a+b)A\int_0^t R(s)ds\right]y\\
&+ \int_0^t R(s)zds + \int_0^t (R*f)(s)ds.
\end{aligned}
$$

Then $u(t)$ is a strong solution of (4.11). Indeed, since $f \in L^1_{loc}(\mathbb{R}_+, D(A))$, $x \in D(A^2)$, $y \in D(A^2)$ and $z \in D(A)$ then $u \in C(\mathbb{R}_+; D(A^2))$ and is differentiable.

Moreover

$$
u(0) = [S(0) + bAR(0)]x + R(0)y + (R*f)(0) = x.
$$

Now, for $x, y, z \in X$,

$$
\begin{aligned}
u'(t) =& [S'(t) + bAR'(t) + (abA + c^2)AR(t)]x + [R'(t) + (a+b)AR(t)]y\\
&+ R(t)z + (R*f)(t)\\
=& [S'(t) + bA[S(t) - aAR(t)] + (abA + c^2)AR(t)]x\\
&+ [S(t) - aAR(t) + (a+b)AR(t)]y + R(t)z + (R*f)(t)\\
=& [S'(t) + bAS(t) + c^2AR(t)]x\\
&+ [S(t) + bAR(t)]y + R(t)z + (R*f)(t)
\end{aligned}
$$

and by hypothesis it follows that $u' \in C(\mathbb{R}_+; D(A^2))$ and is differentiable. In particular, it implies that,

$$u'(0) = [S'(0) + bAS(0) + c^2AR(0)]x + [S(0) + bAR(0)]y + R(0)z + (R * f)(0) = y.$$

Further, for $x, y, z \in X$,

$$\begin{aligned} u''(t) =& [S''(t) + bAS'(t) + c^2AR'(t)]x + [S'(t) + bAR'(t)]y + R'(t)z + (R' * f)(t) \\ =& [-c^2AS(t) + c^2AR'(t)]x + [S'(t) + bAR'(t)]y + R'(t)z + (R' * f)(t) \end{aligned}$$

and since $f \in L^1_{loc}(\mathbb{R}_+, D(A))$, $x \in D(A^2)$, $y \in D(A^2)$, $z \in D(A)$, then $u'' \in C(\mathbb{R}_+; D(A))$ and is differentiable. In addition, for $x, y \in D(A)$, $z \in X$, it follows that

$$u''(0) = [-c^2AS(0) + c^2AR'(0)]x + [S'(0) + bAR'(0)]y + R'(0)z + (R' * f)(0) = z.$$

At last,

$$\begin{aligned} u'''(t) =& [-c^2AS'(t) + c^2AR''(t)]x + [S''(t) + bAR''(t)]y + R''(t)z + (R'' * f)(t) + f(t) \\ =& [-c^2aA^2R'(t)]x + [-c^2AS(t) - abA^2R'(t)]y + R''(t)z + (R'' * f)(t) + f(t). \end{aligned}$$

Then, it is possible to conclude that the conditions (i), (ii) and (iii) of Definition 4.8 are satisfied.

To verify that (4.11) holds first notice that

$$\begin{aligned} S'''(t)x &+ (a+b)AS''(t)x + (abA + c^2)AS'(t)x + ac^2A^2S(t)x \\ &= \{(b^2A - c^2)AS'(t) + (a+b)A[-bAS'(t) - c^2AS(t)] + ac^2A^2S(t) \\ &\quad +(abA + c^2)AS'(t) + c^2bA^2S(t)\}x \\ &= 0, \end{aligned}$$

for all $x \in D(A^2)$. Moreover, it follows that

$$\begin{aligned} R'''(t)x &+ (a+b)AR''(t)x + (abA + c^2)AR'(t)x + ac^2A^2R(t)x \\ &= -(a+b)AS'(t)x + (a^2A - c^2)AS(t)x + (a+b)A[S'(t) - aAS(t) \\ &+ a^2A^2R(t)]x - a^3A^3R(t)x + (abA + c^2)A[S(t) - aAR(t)]x + ac^2A^2R(t)x \\ &= 0, \end{aligned}$$

for all $x \in D(A^2)$. Now, taking

$$h(t)x = R''(t)x + (a+b)AR'(t)x + (abA + c^2)AR(t)x + ac^2A^2\int_0^t R(s)xds, \quad x \in X,$$

then $h'(t)x = 0$ as seen above. But, $h(0)x = 0$, then $h \equiv 0$. Then $u(t)$ given by (4.16) satisfies (4.11) and is a strong solution. This concludes the proof. □

**Remark 4.15.** By Lemmas 4.11 and 4.12, an equivalent representation of (4.16) is

$$\begin{aligned} u(t) &= e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)xds + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)xds \\ &\quad + \frac{c^2(a^2 - ab - b^2)}{a(a-b)^2}\int_0^t S(s)xds - \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)xds \end{aligned}$$

$$+\frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)xd\tau ds + R(t)y - \frac{a+b}{a-b}\int_0^t T_a(s)yds \tag{4.17}$$
$$+\frac{a+b}{a-b}\int_0^t S(s)yds - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)yds + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)yd\tau ds$$
$$+\int_0^t R(s)zds + \int_0^t\int_0^s R(s-\tau)f(\tau)d\tau ds$$

## 4.3 The semilinear problem

Let $A$ be a closed linear operator satifying hypothesis $(H)$. For $(x,y,z)\in X\times X\times X$ and $a\neq b$, $a>0$, consider the semilinear problem

$$\begin{cases} u'''(t)+(a+b)Au''(t)+(abA+c^2)Au'(t)+ac^2A^2u(t)=f(t,u(t)),\ t\geq 0 \\ u(0)=x,\quad u'(0)=y,\quad u''(0)=z. \end{cases} \tag{4.18}$$

The purpose of this section is to prove that under certain conditions the semilinear problem (4.18) has a mild solution.

First introducing the following definition.

**Definition 4.16.** Given $(x,y,z)\in X\times X\times X$, a continuous function $u(t,x,y,z)$ that satisfies

$$\begin{aligned} u(t) \quad = \quad & e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)xds + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)xds \\ & +\frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)xds - \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)xds \\ & +\frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)xd\tau ds + R(t)y - \frac{a+b}{a-b}\int_0^t T_a(s)yds \\ & +\frac{a+b}{a-b}\int_0^t S(s)yds - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)yds + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)yd\tau ds \\ & +\int_0^t R(s)zds + \int_0^t\int_0^s R(s-\tau)f(\tau,u(\tau))d\tau ds \end{aligned} \tag{4.19}$$

is called mild solution of the problem (4.18).

The next main result shows that there exists a local mild solution of the problem (4.18).

**Theorem 4.17.** Let $A$ be a closed linear operator satifying hypothesis $(H)$. If $f:[0,+\infty)\times X\to X$ satisfies a Lipschitz condition in $x$ uniformly in $t\in\mathbb{R}$, with Lipschitz constant $L>0$, then there is a unique mild solution of (4.18) in $[0,T]$.

*Proof.* Consider $R(t)=(S*T_a)(t)$. There exists constants $K>0$ and $\omega>0$ such that

$$\|R(t)\|\leq Ke^{\omega t},\ \forall t\in[0,+\infty).$$

Let $T>0$ be given and consider the space $C([0,T],X)$ of the continuous functions from $[0,T]$ to $X$, with the norm

$$\|u\|_{\tilde{L}} = \max_{t\in[0,T]}\{e^{-(\tilde{L}+\omega)t}\|u(t)\|\},$$

where $u \in C([0,T],X)$ and $\tilde{L} > \dfrac{-\omega + \sqrt{\omega^2 + 4KL}}{2} > 0$ is arbitrary but fixed.

First note that that the norm $\|\cdot\|_{\tilde{L}}$ is equivalent with the standard norm $\|\cdot\|_\infty$, which is defined by

$$\|u\|_\infty = \max_{t\in[0,T]} \|u(t)\|.$$

Indeed, since the function $e^{-\left(\tilde{L}+\omega\right)t}$ is decreasing in $[0,T]$, for all $t \in [0,T]$ it follows that

$$e^{-\left(\tilde{L}+\omega\right)T}\|u(t)\| \le e^{-(\tilde{L}+\omega)t}\|u(t)\| \le \|u(t)\|.$$

Then,

$$e^{-\left(\tilde{L}+\omega\right)T}\|u\|_\infty \le \|u\|_{\tilde{L}} \le \|u\|_\infty.$$

Therefore the norms are equivalent and $C([0,T],X)$ is a Banach space with the norm $\|\cdot\|_{\tilde{L}}$. Let $x,y,z \in X$ be fixed and define the operator $\Gamma : C([0,T],X) \to C([0,T],X)$ by

$$\begin{aligned}
\Gamma u(t) \;=\; & e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)xds + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)xds \\
& + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)xds - \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)xds \\
& + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)xd\tau ds + R(t)y - \frac{a+b}{a-b}\int_0^t T_a(s)yds \qquad (4.20) \\
& + \frac{a+b}{a-b}\int_0^t S(s)yds - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)yds + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)yd\tau ds \\
& + \int_0^t R(s)zds + \int_0^t\int_0^s R(s-\tau)f(\tau,u(\tau))d\tau ds
\end{aligned}$$

Notice that given $u \in C([0,T],X)$, the function $s \mapsto f(s,u(s))$ is continuous in $[0,T]$ and therefore integrable in $[0,t]$ for all $t \in [0,T]$. Then $\Gamma u$ is a continuous function from $[0,T]$ to $X$, implying that $\Gamma$ is well defined.

Consider $u,v \in C([0,T],X)$ and $t \in [0,T]$. Then

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\| &= \left\|\int_0^t\int_0^s R(s-\tau)[f(\tau,u(\tau)) - f(\tau,v(\tau))]d\tau ds\right\| \\
&\le \int_0^t\int_0^s \|R(s-\tau)\|\|f(\tau,u(\tau)) - f(\tau,v(\tau))\|d\tau ds,
\end{aligned}$$

then

$$\begin{aligned}
e^{-(\tilde{L}+\omega)t}\|\Gamma u(t) - \Gamma v(t)\| &\le e^{-(\tilde{L}+\omega)t}\int_0^t\int_0^s \|R(s-\tau)\|\|f(\tau,u(\tau)) - f(\tau,v(\tau))\|d\tau ds \\
&\le e^{-(\tilde{L}+\omega)t}\int_0^t\int_0^s Ke^{\omega(s-\tau)}L\|u(\tau)-v(\tau)\|d\tau ds \\
&= e^{-(\tilde{L}+\omega)t}KL\int_0^t e^{\omega s}\int_0^s e^{\tilde{L}\tau}e^{-\omega\tau}e^{-\tilde{L}\tau}\|u(\tau)-v(\tau)\|d\tau ds \\
&\le KLe^{-(\tilde{L}+\omega)t}\int_0^t e^{\omega s}\int_0^s e^{\tilde{L}\tau}\|u-v\|_{\tilde{L}}d\tau ds
\end{aligned}$$

$$
\begin{aligned}
&\leq KLe^{-(\tilde{L}+\omega)t}\|u-v\|_{\tilde{L}}\int_0^t e^{\omega s}\int_0^s e^{\tilde{L}\tau}d\tau ds\\
&= \frac{KL}{\tilde{L}}\|u-v\|_{\tilde{L}}e^{-(\tilde{L}+\omega)t}\int_0^t e^{\omega s}[e^{\tilde{L}s}-1]ds\\
&\leq \frac{KL}{\tilde{L}}\|u-v\|_{\tilde{L}}e^{-(\tilde{L}+\omega)t}\int_0^t e^{\left(\omega+\tilde{L}\right)s}ds\\
&= \frac{KL}{\tilde{L}(\omega+\tilde{L})}\|u-v\|_{\tilde{L}}e^{-(\tilde{L}+\omega)t}[e^{(\omega+\tilde{L})t}-1]\\
&\leq \frac{KL}{\tilde{L}(\omega+\tilde{L})}\|u-v\|_{\tilde{L}}.
\end{aligned}
$$

Since $\tilde{L} > \dfrac{-\omega+\sqrt{\omega^2+4KL}}{2} > 0$, then $\dfrac{KL}{\tilde{L}(\omega+\tilde{L})} < 1$. So $\Gamma$ is a contraction. By the Banach Fixed Point Theorem, $\Gamma$ has a unique fixed point, that is, there exists $u \in C([0,T],X)$ such that $u(t) = \Gamma u(t)$, so

$$
\begin{aligned}
u(t) \;=\; & e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)xds + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)xds\\
&+\frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)xds - \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)xds\\
&+\frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)xd\tau ds + R(t)y - \frac{a+b}{a-b}\int_0^t T_a(s)yds\\
&+\frac{a+b}{a-b}\int_0^t S(s)yds - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)yds + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)yd\tau ds\\
&+\int_0^t R(s)zds + \int_0^t\int_0^s R(s-\tau)f(\tau,u(\tau))d\tau ds,
\end{aligned}
$$

for all $t \in [0,T]$. By the uniqueness of the fixed point, the problem (4.18) has a unique solution in $[0,T]$. □

## 4.4 Mild solutions with nonlocal initial conditions

In this section, it is considered the Blackstock-Crighton-Westervelt with nonlocal initial conditions:

$$
\begin{cases}
u'''(t) + (a+b)Au''(t) + (abA+c^2)Au'(t) + ac^2A^2u(t) = f(t,u(t)), \;\; t \in I\\
u(0) = g_1(u), \quad u'(0) = g_2(u), \quad u''(0) = g_3(u).
\end{cases}
\tag{4.21}
$$

where $A$ is a closed linear operator satifying hypothesis $(H)$ and considering $I = [0,1]$. The functions $f : I \times X \to X$ and $g_1, g_2, g_3 : C(I;X) \to X$ are $X$-valued functions that satisfy appropriate conditions described later.

Consider the function $U : I \to \mathcal{L}(X)$ given by $U(t) = (1*R)(t) = \int_0^t R(s)ds$. Notice that $U$ is uniformly continuous in $\mathcal{L}(X)$.

Considering $I = [0,1]$, let $A$ be a closed linear operator satifying the hypothesis $(H)$ and the problem (4.21). For the following result the following assertions will be assumed.

(H1) The functions $g_1, g_2, g_3 : C(I; X) \to X$ are compact maps.

(H2) The function $f : I \times X \to X$ satisfies the Carathéodory type conditions; that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in I$.

(H3) There exist a function $m \in L^1(I; \mathbb{R}^+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leq m(t)\Phi(\|x\|)$$

for all $x \in X$ and almost all $t \in I$.

(H4) There exists a function $G \in L^1(I; \mathbb{R}^+)$ such that for any bounded $S \subseteq X$

$$\xi(f(t,S)) \leq G(t)\xi(S)$$

for almost all $t \in I$.

**Remark 4.18.** Assuming that a function $g$ satisfies hypothesis $(H1)$ it is clear that $g$ takes bounded sets into bounded sets. For this reason, denote $g_J = \sup\{\|g(u)\| : \|u\|_\infty \leq J\}$ for each $J \geq 0$.

For the following result consider

$$L = \max\left\{1, \sup_{t\in I}\{\|S(t)\|\}, \sup_{t\in I}\{\|R(t)\|\}, \sup_{t\in I}\{\|T_a(t)\|\}, \sup_{t\in I}\|U(t)\|\right\}. \tag{4.22}$$

**Theorem 4.19.** Suppose $0 < a < b$. If the hypotheses (H1)-(H4) are satisfied and there exists a constant $J \geq 0$ such that

$$\begin{aligned} J \ \geq\ & \left(1 + \frac{c^2(2b^2 - a^2)}{ab(a-b)^2} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) Lg_{1J} + \left(1 + \frac{4b(a+c^2)}{a(a-b)}\right) Lg_{2J} \\ & + Lg_{3J} + L\Phi(J)\int_0^1 m(s)ds, \end{aligned}$$

where L is given by (4.22), then the problem (4.21) has at least one mild solution.

*Proof.* Given $x, y, z \in X$, define $F : C(I;X) \to C(I;X)$ by

$$\begin{aligned} (Fu)(t) \ =\ & e^{-\frac{c^2}{b}t}g_1(u) - \frac{bc^2}{a(a-b)}R(t)g_1(u) + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)g_1(u)ds \\ & + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)g_1(u)ds + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)g_1(u)ds \\ & - \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)g_1(u)ds + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)g_1(u)d\tau ds \\ & + R(t)g_2(u) - \frac{a+b}{a-b}\int_0^t T_a(s)g_2(u)ds + \frac{a+b}{a-b}\int_0^t S(s)g_2(u)ds \\ & - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)g_2(u)ds + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)g_2(u)d\tau ds \\ & + \int_0^t R(s)g_3(u)ds + \int_0^t U(t-s)f(s,u(s))ds, \end{aligned}$$

for all $u \in C(I;X)$.

Note that $F$ is a continuous map. Indeed, let $\{u_n\}_{n=1}^{\infty} \subseteq C(I;X)$ such that $u_n \to u$ as $n \to \infty$ (in the norm of $C(I;X)$). Since $0 < a < b$, it follows that $a^2 - ab - b^2 < 0$, and then

$$\begin{aligned}
\|F(u_n) - F(u)\| \;\le\; & \left(1 + \frac{2c^2(-a^2+ab+b^2)}{ab(a-b)^2} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) L\|g_1(u_n) - g_1(u)\| \\
& + \left(1 + \frac{2(a+b)(a+c^2)}{a(b-a)}\right) L\|g_2(u_n) - g_2(u)\| + L\|g_3(u_n) - g_3(u)\| \\
& + L\int_0^1 \|f(s,u_n(s)) - f(s,u(s))\|ds \\
\le\; & \left(1 + \frac{c^2(-a^2+2b^2)}{ab(a-b)^2} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) L\|g_1(u_n) - g_1(u)\| \\
& + \left(1 + \frac{4b(a+c^2)}{a(a-b)}\right) L\|g_2(u_n) - g_2(u)\| + L\|g_3(u_n) - g_3(u)\| \\
& + L\int_0^1 \|f(s,u_n(s)) - f(s,u(s))\|ds.
\end{aligned}$$

By hyphoteses (H1) and (H2) and by the dominated convergence theorem it follows that $\|F(u_n) - F(u)\| \to 0$ when $n \to \infty$.

Denote $B_J = \{u \in C(I;X) : \|u(t)\| \le J, \forall t \in I\}$ and note that for any $u \in B_J$

$$\begin{aligned}
\|(Fu)(t)\| \;\le\; & \|e^{-\frac{c^2}{b}t} g_1(u)\| + \frac{bc^2}{a(b-a)}\|R(t)g_1(u)\| + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t \|T_a(s)g_1(u)\|ds \\
& + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t \|R(s)g_1(u)\|ds + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t \|S(s)g_1(u)\|ds \\
& + \frac{c^2}{b}\int_0^t |e^{-\frac{c^2}{b}(t-s)}|\|S(s)g_1(u)\|ds + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s \|S(\tau)g_1(u)\|d\tau ds \\
& + \|R(t)g_2(u)\| + \frac{a+b}{b-a}\int_0^t \|T_a(s)g_2(u)\|d + \frac{a+b}{b-a}\int_0^t \|S(s)g_2(u)\|ds \\
& + \frac{c^2(a+b)}{a(b-a)}\int_0^t \|R(s)g_2(u)\|ds + \frac{c^2(a+b)}{a(b-a)}\int_0^t\int_0^s \|S(\tau)g_2(u)\|d\tau ds \\
& + \int_0^t \|R(s)g_3(u)\|ds + \int_0^t\int_0^s \|R(s-\tau)f(\tau,u(\tau))\|d\tau ds \\
\le\; & \left(1 + \frac{c^2(2b^2-a^2)}{ab(a-b)^2} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) Lg_{1J} + \left(1 + \frac{4b(a+c^2)}{a(a-b)}\right) Lg_{2J} \\
& + Lg_{3J} + L\Phi(J)\int_0^1 m(s)ds \le J.
\end{aligned}$$

Therefore $F$ maps $B_J$ into itself and $F(B_J)$ is a bounded set. Moreover, by continuity of the functions $t \to R(t)$, $t \to T_a(t)$, $t \to S(t)$ and $t \to U(t)$ on $[0,1]$, it follows that the set $F(B_J)$ is an equicontinuous set of functions.

Define $\mathcal{B} = \overline{co}(F(B_J))$ the closed convex hull of the set $F(B_J)$. It follows from Lemma 1.6 that the set $\mathcal{B}$ is equicontinuous. In addition, the operator $F : \mathcal{B} \to \mathcal{B}$ is continuous and $F(\mathcal{B})$ is a bounded set of functions.

Let $\varepsilon > 0$ be given. For $t \in I$, recall the notation $F(\mathcal{B})(t) = \{v(t); v \in F(\mathcal{B})\}$. By Lemma 1.9 there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset F(\mathcal{B})$ such that

$$\xi(F(\mathcal{B})(t)) \leq 2\xi(\{v_n(t)\}_{n=1}^{\infty}) + \varepsilon$$

and since the functions $g_1, g_2, g_3$ are compacts maps, it follows that

$$2\xi(\{v_n(t)\}_{n=1}^{\infty}) + \varepsilon \leq 2\xi\left(\int_0^t \{U(t-s)f(s,u_n(s))\}_{n=1}^{\infty} ds\right) + \varepsilon$$

By hypothesis (H3), $\|U(t-s)f(s,u_n(s))\| \leq L\Phi(J)m(s)$ for each $t \in I$. And then, from Lemma 1.8

$$2\xi\left(\int_0^t \{U(t-s)f(s,u_n(s))\}_{n=1}^{\infty} ds\right) \leq 4L\int_0^t \xi(\{f(s,u_n(s))\}_{n=1}^{\infty}).$$

Therefore, by condition (H4)

$$\begin{aligned}
\xi(F(\mathcal{B}(t))) &\leq 4L\int_0^t \xi(\{f(s,u_n(s))\}_{n=1}^{\infty})ds + \varepsilon \leq 4L\int_0^t G(s)\xi(\{u_n(s)\}_{n=1}^{\infty})ds + \varepsilon \\
&\leq 4L\gamma(\mathcal{B})\int_0^t G(s)ds + \varepsilon.
\end{aligned}$$

By hypothesis (H4), $G \in L^1(I;\mathbb{R}^+)$. Then for $\kappa < \frac{1}{4L}$ there exists $\varphi \in C(I;\mathbb{R}^+)$ satisfying $\int_0^1 |G(s) - \varphi(s)|ds < \alpha$, where $\alpha = 4\kappa L$. Hence,

$$\begin{aligned}
\xi(F(\mathcal{B})(t)) &\leq 4L\gamma(\mathcal{B})\left[\int_0^t |G(s)-\varphi(s)|ds + \int_0^t \varphi(s)ds\right] + \varepsilon \\
&\leq 4L\gamma(\mathcal{B})[\kappa + Nt] + \varepsilon,
\end{aligned}$$

where $N = \|\varphi\|_\infty$. Since $\varepsilon > 0$ is arbitrary,

$$\xi(F(\mathcal{B})(t)) \leq (\alpha + \beta t)\gamma(\mathcal{B}), \quad \text{where} \quad \beta = 4LN. \tag{4.23}$$

Let $\varepsilon > 0$ be given. Since the functions $g_1, g_2, g_3$ are compact maps and applying Lemma 1.9 there exists a sequence $\{w_n\}_{n=1}^{\infty} \subseteq \overline{co}(F(\mathcal{B}))$ such that

$$\begin{aligned}
\xi(F^2(\mathcal{B})(t)) &\leq 2\xi\left(\int_0^t \{U(t-s)f(s,w_n(s))\}_{n=1}^{\infty} ds\right) + \varepsilon \\
&\leq 4L\int_0^t \xi(\{f(s,w_n(s))\}_{n=1}^{\infty})ds + \varepsilon \\
&\leq 4L\int_0^t G(s)\xi(\overline{co}(F(\mathcal{B})(s))) + \varepsilon
\end{aligned}$$

and by item (iii) of Lemma 1.5,

$$\xi(F^2(\mathcal{B})(t)) \leq 4L\int_0^t G(s)\xi(F(\mathcal{B})(s))ds + \varepsilon.$$

Using the inequality (4.23) it follows that

$$\begin{aligned}
\xi(F^2(\mathcal{B})(t)) &\leq 4L\int_0^t G(s)(\alpha+\beta s)\gamma(\mathcal{B})ds+\varepsilon \\
&\leq 4L\int_0^t [|G(s)-\varphi(s)|+|\varphi(s)|](\alpha+\beta s)\gamma(\mathcal{B})ds+\varepsilon \\
&\leq 4L(\alpha+\beta t)\gamma(\mathcal{B})\int_0^t |G(s)-\varphi(s)|ds+4LN\gamma(\mathcal{B})\left(\alpha t+\frac{\beta t^2}{2}\right)+\varepsilon \\
&\leq \left[\alpha(\alpha+\beta t)+\beta\left(\alpha t+\frac{bt^2}{2}\right)\right]\gamma(\mathcal{B})+\varepsilon \\
&\leq \left(\alpha^2+2\beta t+\frac{(\beta t)^2}{2}\right)\gamma(\mathcal{B})+\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\xi(F^2(\mathcal{B})(t)) \leq \left(\alpha^2+2\beta t+\frac{(\beta t)^2}{2}\right)\gamma(\mathcal{B}).$$

By an inductive process, for all $n \in \mathbb{N}$, it holds

$$\xi(F^n(\mathcal{B})(t)) \leq \left(\alpha^n + C_1^n\alpha^{n-1}\beta t + C_2^n\alpha^{n-2}\frac{(\beta t)^2}{2!}+\cdots+\frac{(\beta t)^n}{n!}\right)\gamma(\mathcal{B}),$$

where, for $0 \leq m \leq n$, the symbol $C_m^n$ denotes the binomial coefficient $\binom{n}{m}$.

In addition, for all $n \in \mathbb{N}$ the set $F^n(\mathcal{B})$ is an equicontinuous set of functions. Therefore, using the Lemma 1.7 it is possible to conclude that

$$\gamma(F^n(\mathcal{B})) \leq \left(\alpha^n + C_1^n\alpha^{n-1}\beta + C_2^n\alpha^{n-2}\frac{\beta^2}{2!}+\cdots+\frac{\beta^n}{n!}\right)\gamma(\mathcal{B}).$$

Since $0 \leq \alpha < 1$ and $\beta > 0$, it follows from Lemma 1.11 that there exists $n_0 \in \mathbb{N}$ such that

$$\left(\alpha^{n_0} + C_1^{n_0}\alpha^{n_0-1}\beta + C_2^{n_0}\alpha^{n_0-2}\frac{\beta^2}{2!}+\cdots+\frac{\beta^n}{n_0!}\right) = r < 1.$$

Consequently, $\gamma(F^{n_0}(\mathcal{B})) \leq r\gamma(\mathcal{B})$. It follows from Lemma 1.10 that $F$ has a fixed point in $\mathcal{B}$, and this fixed point is a mild solution of (4.21).

□

# Bibliography


[1] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskii. *Measures of noncompactness and Condensing Operators.* Springer, Basel, 1992.

[2] D. Araya and C. Lizama. Almost automorphic mild solutions to fractional differential equations. *Nonlinear Anal.*, 69 (11):3692 – 3705, 2008.

[3] W. Arendt. A $0 - 3/2$ - Law for Cosine Functions. *Ulmer Seminare, Funktionalanalysis und Evolutionsgleichungen*, 17:349 – 350, 2012.

[4] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems.* Birkhäuser/Springer Basel AG, 2011.

[5] J. Banaś and K. Goebel. *Measures of noncompactness in Banach spaces (Lecture Notes in Pure and Applied Mathematics 60).* Marcel Dekker, New York, 1980.

[6] J. Banaś, M. Jleli, M. Mursaleen, B. Samet, and C. Vetro. *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness.* Springer, Singapore, 2017.

[7] E. G. Bazhlekova. *Fractional Evolution Equations in Banach Spaces.* PhD thesis, Eindhoven University of Technology, 2001.

[8] A. Bobrowski. On Wallen-type formulae for integrated semigroups and sine functions. *Arch. Math.*, 108 (4):395 – 404, 2017.

[9] A. Bobrowski and W. Chojnacki. Isolated points of some sets of bounded cosine families, bounded semigroups, and bounded groups on a Banach space. *Studia Math.*, 217 (3):219 – 241, 2013.

[10] A. Bobrowski, W. Chojnacki, and A. Gregosiewicz. On close-to-scalar one-parameter cosine families. *J. Math. Anal. Appl.*, 429 (1):383 – 394, 2015.

[11] D. Bothe. Multivalued perturbation of $m$-accretive differential inclusions. *Israel J. Math.*, 108:109 – 138, 1998.

[12] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, 2010.

[13] R. Brunnhuber and B. Kaltenbacher. Well-posedness and asymptotic behaviour of solutions for the Blackstock-Crighton-Westervelt equation. *Discrete Contin. Dyn. Syst.*, 34 (11):4515 – 4535, 2014.

[14] R. Brunnhuber and S. Meyer. Optimal regularity and exponential stability for the Blackstock-Crighton equations in $L_p$-spaces with Dirichlet and Newmann boundary conditions. *J. Evol. Equ.*, 16:945 – 981, 2016.

[15] C. Martínez Carracedo and M. Sanz Alix. *The Theory of Fractional Powers of Operators.* North Holland, 2001.

[16] A. Celik and M. Kyed. Nonlinear acoustics: Blackstock-Crighton equations with a periodic forcing term. *J. Math. Fluid Mech.*, 21 (3):21 – 45, 2019.

[17] C. Chen and M. Li. On fractional resolvent operator functions. *Semigroup Forum*, 80:121 – 142, 2010.

[18] W. Chojnacki. On cosine families close to scalar cosine families. *J. Aust. Math. Soc.*, 99 (2):166 – 174, 2015.

[19] W. Chojnacki. Around Schwenninger and Zwart's zero-two law for cosine families. *Arch. Math.*, 106 (6):561 – 571, 2016.

[20] K-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Springer, 1995.

[21] A. Erdelyi. *Higher Transcendental Functions Vol. III.* McGray-Hill, New York, 1953.

[22] J. Esterle. Bounded cosine functions close to continuous scalar bounded cosine functions. *Arch. Math.*, 105 (4):381 – 387, 2015.

[23] J. Esterle. Bounded cosine functions close to continuous scalar bounded cosine functions. *Integral Equations Operator Theory*, 85 (3):347 – 357, 2016.

[24] H. O. Fattorini. *Second order linear differential equations in Banach spaces.* North-Holland, Amsterdam, Netherlands, 1985.

[25] L. R. Gambera, C. Lizama, and A. Prokopczyk. On close to scalar families for fractional evolution equations: zero-one law. *Semigroup Forum*, 99 (1):140 – 152, 2019.

[26] R. Gorenflo, J. Loutchko, and Y. Luchko. Computation of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its derivatives. *Fract. Calc. Appl. Anal.*, 5 (4):491– 518, 2002.

[27] H. Gou and Li. Existence of mild solutions for fractional nonautonomous evolution equations of Sobolev type with delay. *J. Inequal. Appl.*, 252:1– 20, 2017.

[28] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products.* Elsevier, 7th edition, 2007.

[29] F. Guo, S. Liu, C. Wu, and Y. Wu. Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. *J. Math. Anal. Appl.*, 309 (2):638 – 649, 2005.

[30] V. Keyantuo, C. Lizama, and M. Warma. Existence, regularity and representation of solutions of time fractional diffusion equations. *Adv. Differential Equations*, 21 (9):837 – 886, 2016.

[31] V. Keyantuo, C. Lizama, and M. Warma. Existence, regularity and representation of solutions of time fractional wave equations. *Electron. J. Differential Equations*, 222:1 – 42, 2017.

[32] K. Li and J. Peng. Fractional resolvents and fractional evolution equations. *Appl. Math. Lett.*, 25 (5):808 – 812, 2012.

[33] K. Li, J. Peng, and J. Jia. Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives. *J. Funct. Anal.*, 263:476 – 510, 2012.

[34] M. Li, C. Chen, and F.B. Li. On fractional powers of generators of fractional resolvent families. *J. Funct. Anal.*, 259 (10):2702 – 2726, 2010.

[35] C. Lizama. On the convergence and approximation of integrated semigroups. *J. Math. Anal. Appl.*, 181:89 – 103, 1994.

[36] C. Lizama. Regularized solutions for abstract Volterra equations. *J. Math. Anal. Appl.*, 243 (2):278 – 292, 2000.

[37] C. Lizama and F. Poblete. On a functional equation associated with $(a, k)$-regularized resolvent families. *Abstr. Appl. Anal.*, 2012 (2), 2012.

[38] C. Lizama and H. Prado. On duality and spetral properties of (a,k)-regularized resolvents. *Proc. Roy. Soc. Edinburgh Sect. A*, 139:507 – 517, 2009.

[39] F. Mainardi and P. Paradisi. Fractional diffusive waves. *J. Comput. Acoust.*, 9 (4):1417 – 1436, 2001.

[40] Z. D. Mei, J. G. Peng, and J. H. Gao. General fractional differential equations of order $\alpha \in (1, 2)$ and type $\beta \in [0, 1]$ in Banach spaces. *Semigroup Forum*, 94 (3):712– 737, 2017.

[41] B. Nagy. On cosine operator functions in Banach spaces. *Acta Sci. Math.*, 36 (3):281 – 290, 1973.

[42] R. R. Nigmatullin. The realization of the generalized transfer equation in a medium with fractal geometry. *Physica Status Solidi*, (b) 133 (1):425 – 430, 1986.

[43] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer, New York, 1983.

[44] I. Podlubny. *Fractional Differential Equations.* Academic Press, San Diego, 1999.

[45] J. Prüss. *Evolutionary Integral Equations ans Applications.* Birkhäuser Verlag, Switzerland, 1993.

[46] A. Rhandi. Dyson-Phillips expansion and unbounded perturbations of linear $C_0$-semigroups. *J. Comput. Appl. Math.*, 44:339 – 349, 1992.

[47] A. Rhandi. Positive perturbations of linear Volterra equations and since functions of operators. *J. Integral Equations Appl.*, 4:409 – 420, 1992.

[48] H. L. Royden and P. M. Fitzpatrick. *Real Analysis.* Prentice Hall, 4th edition, 2011.

[49] W. R. Schneider. Completely monotone generalized Mittag-Leffler functions. *Expo. Math.*, 14 (1):3 – 16, 1996.

[50] F. L. Schwenninger and H. Zwart. Less than one implies zero. *Studia Math.*, 229 (2):181 – 188, 2015.

[51] F. L. Schwenninger and H. Zwart. Zero-two law for cosine families. *J. Evol. Equ.*, 15:559 – 569, 2015.

[52] O. Staffans. *Well-Posed Linear Systems (Encyclopedia of Mathematics and its Applications 103).* Cambridge University Press, Cambridge, 2005.

[53] Y. Zhou, R-N Wang, and L. Peng. *Topological Structure of the Solution Set for Evolution Inclusions.* Springer, Singapore, 2017.