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The Super-Diffusive Singular Perturbation Problem

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Abstract: In this paper we study a class of singularly perturbed defined abstract Cauchy problems. We investigate the singular perturbation problem $(\mathrm{P}_\epsilon)\epsilon^\alpha D_t^\alpha u_\epsilon(t) + u'_\epsilon(t) = Au_\epsilon(t), t \in [0,T]$, $1 < \alpha < 2, \epsilon > 0$, for the parabolic equation $(\mathrm{P})u'_0(t) = Au_0(t), t \in [0,T]$, in a Banach space, as the singular parameter goes to zero. Under the assumption that A is the generator of a bounded analytic semigroup and under some regularity conditions we show that problem (P_ϵ) has a unique solution $u_\epsilon(t)$ for each small $\epsilon > 0$. Moreover $u_\epsilon(t)$ converges to $u_0(t)$ as $\epsilon \to 0^+$, the unique solution of equation (P).

Keywords: singular perturbation; fractional partial differential equations; analytic semigroup; super-diffusive processes

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1. Introduction

In the last decade, fractional calculus has been recognized as one of the best tools to describe long memory processes. These models have not only interest for engineers and physicists, but also for pure mathematicians. The most important among these models are those described by partial differential equations containing fractional-order derivatives. Its evolution behaves in a much more complex way than in the classic case of the entire order and the study of the corresponding theory is an enormously demanding task. While some results of the qualitative analysis for partial differential equations can be similarly obtained, many classical methods are rarely directly applicable. Therefore, it is necessary to develop new theories and methods, which makes research on this topic more challenging.

From an abstract point of view, some works have been developed, including results on the existence and qualitative properties of abstract Cauchy problems [1,2]. However, to the best of our knowledge, until now singular perturbation problems have been unreported. One natural question in this regard is how perturbed super-diffusive processes are related with the parabolic limit when the perturbation becomes smaller. More precisely, our concern in this paper is to study under which conditions one can guarantee that the solutions of the super-diffusive initial value problem

$$\begin{cases} \epsilon^\alpha \partial_t^\alpha u_\epsilon(t,x) + \partial_t u_\epsilon(t,x) = \Delta u_\epsilon(t,x), \quad t \in [0,T]; \quad 1 < \alpha < 2, \quad x \in \Omega \subset \mathbb{R}^N; \\ u_\epsilon(0,x) = u_{0,\epsilon}(x); \\ u'_\epsilon(0,x) = u_{1,\epsilon}(x), \end{cases} \tag{1}$$

with small parameter $\epsilon > 0$, can be approximated by the solution of the diffusion equation

$$\begin{cases} \partial_t w(t,x) = \Delta w(t,x), \quad t \in [0,T]; \\ w(0,x) = w_0(x), \quad x \in \Omega \subset \mathbb{R}^N, \end{cases}$$

where ∂_t^α denotes the Caputo fractional partial derivative with respect to the variable t, of order $\alpha > 0$ fixed and Δ is the Laplacian operator.

When $\alpha = 2$, Equation (1) is named the Cattaneo equation (see Cattaneo [3]) which has been studied in many papers. We observe that for $1 < \alpha < 2$, this equation has been considered as one of the possible fractional generalizations of the Cattaneo equation, see for example the papers of Compte and Metzler [4] and Povstenko [5]. We notice that the time-fractional Cattaneo-type equations has been studied by many authors, as for instance Cascaval et al. [6], Gómez-Aguilar et al. [7], Ferreira et al. [8] and Eltayeb et al. [9]. However, to the best of our knowledge, none of them has treated the singular perturbation problem.

We note that the operator Δ is the generator of a bounded analytic semigroup as well as of other strongly continuous families of bounded and linear operators, e.g., cosine families [10]. Therefore, our study will be done in this general and abstract setting. Let $A : D(A) \subset X \to X$ be a closed and densely defined linear operator on a complex Banach space X. Assuming that A is the generator of a bounded analytic semigroup, we ask ourselves for the existence and uniqueness of solutions $u_\epsilon(t)$ of the abstract fractional singular perturbation problem

$$\begin{cases} \epsilon^\alpha D_t^\alpha u_\epsilon(t) + u_\epsilon'(t) = Au_\epsilon(t), \quad t \in [0,T], \quad 1 < \alpha < 2, \quad \epsilon > 0, \\ u_\epsilon(0) = u_{0,\epsilon}, \\ u_\epsilon'(0) = u_{1,\epsilon}; \end{cases} \tag{2}$$

and their convergence to a solution of the parabolic equation

$$\begin{cases} u_0'(t) = Au_0(t), \quad t \in [0,T], \\ u_0(0) = u_0. \end{cases} \tag{3}$$

In the bordeline case $\alpha = 2$ this question is referred as the hyperbolic singular perturbation problem, and has a large data. The abstract hyperbolic singular perturbation problem was first investigated by Kisynski in the reference [11]. In order to obtain his results, Kisynski imposed the following hypotheses on the operator A defined on a complex Hilbert space: positive definite and self-adjoint. After that, Sova, in 1970, investigated the same problem using the more general hypothesis that A generates a strongly continuous cosine function. However, the most relevant results were proved by Kisynski in [12] who showed explicit solutions using the approach of monotonic functions. Other references on the subject are [13–16]. The non-homogeneous case was investigated by Fattorini in ([17], Chapter VI). Lately, the singular perturbation for abstract non-densely defined Cauchy problems has been studied by Ducrot et al. [18]. An excellent monograph on the subject on singular perturbation is provided by Verhulst [19].

A remarkable and useful property that distinguishes the (Caputo) fractional singular perturbation problem with the integer case is provided by the fact that $D_t^\alpha u_\epsilon(0) = 0$ for every $1 < \alpha < 2$. This property is well known ([20], Theorem 3.1) but apparently has not been sufficiently exploited. A valuable consequence, for the fractional singular perturbation problem of System (2), is that the second initial condition $u_\epsilon'(0)$ is always completely determined by the first one. Namely,

$$u_\epsilon'(0) = Au_\epsilon(0).$$

Because of this fact, the abstract fractional singular perturbation problem (2) can be restated as follows:

$$\begin{cases} \epsilon^\alpha D_t^\alpha u_\epsilon(t) + u_\epsilon'(t) = Au_\epsilon(t), \quad t \in [0,T], \quad 1 < \alpha < 2, \quad \epsilon > 0, \\ u_\epsilon(0) = u_{0,\epsilon}, \\ u_\epsilon'(0) = Au_{0,\epsilon}. \end{cases} \tag{4}$$

This modeling of the abstract fractional singular perturbation problem, allows the first novelty of this work: in contrast with the approaches in the integer case $\alpha = 2$, we will use in this paper a completely different (and original) method. The classical approach to the abstract singular perturbation problem is to introduce a family of solution operators that is explicitly represented by means of Bessel functions [13,14,17], and whose specific properties are critical for the convergence. Instead, we show that A generates an abstract bounded resolvent family $\{S_{\alpha,\epsilon}(t)\}_{t\geq 0}$ associated with the problem (2) which is uniformly bounded with respect to $\epsilon > 0$ (see Theorem 2). After that, we show that the unique strong solution of (4) is given by

$$u_\epsilon(t) = S_{\alpha,\epsilon}(t)u_{0,\epsilon} + (E_\epsilon * S_{\alpha,\epsilon})(t)Au_{0,\epsilon},$$

provided that $u_{0,\epsilon} \in D(A^2)$. Here $E_\epsilon(t) := E_{\alpha-1,1}(-\epsilon^{-\alpha}t^{\alpha-1})$ denotes the Mittag–Leffler function evaluated at the point $-\epsilon^{-\alpha}t^{\alpha-1}$. This last fact, will play a central role in our findings.

As a consequence of the above result, we will derive our main theorem in this work that roughly speaking, asserts the convergence of the solution of (4) to the solution of (3) when A is the generator of a bounded analytic semigroup. The result, that corresponds to Theorem 4 in the text, reads as follows.

Theorem 4. *Let $1 < \alpha < 2$, $\epsilon > 0$ and assume that A generates a bounded analytic semigroup on a complex Banach space X. Suppose that $u_0(0), u_\epsilon(0) \in D(A^2)$. Then the solutions of Systems (3) and (4) exists and the following estimate holds:*

$$\begin{aligned}\|u_\epsilon(t) - u_0(t)\| \leq M^2C\frac{t^2\epsilon^\alpha}{\epsilon^\alpha + t^{\alpha-1}}\|A^2u_0(0)\| + MC\frac{t\epsilon^\alpha}{\epsilon^\alpha + t^{\alpha-1}}\|Au_0(0)\| + M\|u_\epsilon(0) - u_0(0)\| \\ + MC\frac{t\epsilon^\alpha}{\epsilon^\alpha + t^{\alpha-1}}\|Au_\epsilon(0)\|.\end{aligned}$$

where C and M are positive constants independent of $\epsilon > 0$. Moreover, if $u_\epsilon(0) \to u_0(0)$ in X and the set $\{Au_\epsilon(0)\}_{\epsilon>0}$ is bounded then the following convergence result hold true

$$\lim_{\epsilon\to 0^+} \sup_{t\in I} \|u_\epsilon(t) - u_0(t)\| = 0,$$

for each $I := [a, b] \subset [0, T]$.

The above theorem ensures the local uniform convergence of u_ϵ to u_0. Note that, in general, without specific assumption on the dynamical behaviour of the reduced problem (3), one cannot expect to get a more refined convergence property. However, we are able to show the convergence of integrals and derivatives, see Corollaries 1 and 2 below.

2. Preliminaries

Let $\alpha > 0$, $m = \lceil \alpha \rceil$ and $u : [0, \infty) \to X$, where X is a complex Banach space. We denote by $\mathbb{R}_+$ the closed interval $[0, \infty)$. The Caputo fractional derivative of u of order α is defined by

$$D_t^\alpha u(t) := \int_0^t g_{m-\alpha}(t-s)u^{(m)}(s)ds, \quad t > 0,$$

where

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \quad \beta > 0,$$

and in case $\beta = 0$ we set $g_0(t) := \delta_0$, the Dirac measure concentrated at the origin.

The Laplace transform of a function $f \in L^1(\mathbb{R}_+, X)$ is defined by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \lim_{T\to\infty} \int_0^T e^{-\lambda t} f(t)dt, \quad \mathrm{Re}(\lambda) > \omega,$$

when the limit exists.

In particular if f is such that $\int_0^t f(s)ds$ is is exponentially bounded, i.e., there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\| \int_0^t f(s)ds\| \leq Me^{\omega t}$ for all $t \geq 0$, then $\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt$ exists for $\mathrm{Re}(\lambda) > \omega$, and the integral is absolutely convergent. This remains true if we make the stronger assumption that f is exponentially bounded (see [10], Chapter I). We have

$$\widehat{D_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{m-1} f^{(k)}(0)\lambda^{\alpha-1-k}. \tag{5}$$

The power function λ^α is uniquely defined as $\lambda^\alpha = |\lambda|^\alpha e^{i\arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$. The Mittag–Leffler function (see, e.g., [21–23] is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_{H_a} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},$$

where H_a is a Hankel path. For a recent review, we refer to the references [21,24].

The next formula states a quite important property related with the Laplace transform of the Mittag–Leffler function (cf. [22], (A.27) p. 267):

$$\mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(-\rho^\alpha t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho^\alpha}, \quad \mathrm{Re}(\lambda) > |\rho|; \quad \alpha > 0,\ \beta > 0,\ \rho \in \mathbb{R}. \tag{6}$$

The following important lemma will be very useful for the proof of our results.

Lemma 1 ([25], Th. 1.6)**.** *If $0 < \alpha < 2$, $\beta \in \mathbb{R}$ and $\pi\frac{\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$, then*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad z \in \mathbb{C},\ \mu \leq |\arg(z)| \leq \pi \tag{7}$$

where the constant $C > 0$ depends on α, β and μ.

As consequence of the expansion series of $E_{\alpha,1}(-at^\alpha)$ and $tE_{\alpha,2}(-at^\alpha)$ we obtain the following result.

Lemma 2. *For $a > 0$, $\alpha > 0$ and $m \in \mathbb{Z}_+$, we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-at^\alpha) = -at^{\alpha-m} E_{\alpha,\alpha-m+1}(-at^\alpha), \tag{8}$$

and

$$\frac{d}{dt}[tE_{\alpha,2}(-at^\alpha)] = E_{\alpha,1}(-at^\alpha). \tag{9}$$

Proof. The power series defining $E_{\alpha,1}(-at^\alpha)$ for $t > 0$ admit the termwise differentiation any times, and the termwise differentiation and induction on m yields the conclusion for the first identity. The second identity is a direct consequence of ([25], Equation 1.83). □

Lemma 3 ([26])**.** *The Mittag–Leffler function $E_{\alpha,\beta}(-s)$ for $0 < \alpha \leq 1$, $\beta \geq \alpha$ and $s > 0$ is completely monotone, that is,*

$$(-1)^n \frac{d^n}{ds^n}\left[E_{\alpha,\beta}(-s)\right] \geq 0, \quad n = 0, 1, 2, \ldots.$$

Next, let us denote by $\mathcal{L}(X, Y)$ the space of bounded operators from X to Y, and by $\mathcal{L}(X)$ when $X = Y$.

Definition 1. *Let $f : \mathbb{R}_+ \to X$ be an integrable function (as a Bochner integral) and let $T : \mathbb{R}_+ \to \mathcal{L}(X, Y)$ be strongly continuous. Then the convolution of T and f is defined by*

$$(T * f)(t) := \int_0^t T(t-s)f(s)ds, \quad t \in \mathbb{R}_+.$$

It is well-known that $T * f : \mathbb{R}_+ \to Y$ exists (as a Bochner integral) and defines a continuous function (see [10], Prop. 1.3.4). Analogously, we define $(a * f)(t)$ when a is a real or complex-valued function defined on $\mathbb{R}_+$.

We recall some useful properties of convolutions that will be frequently used throughout the paper. For every $f \in C(\mathbb{R}_+; X)$, $k \in \mathbb{N}$, $\alpha > 0$ we have that for any $t \geq 0$,

$$\frac{d^k}{dt^k}[(g_{k+\alpha} * f)(t)] = (g_\alpha * f)(t).$$

Let $u \in C(\mathbb{R}_+; X)$ and $v \in C^1(\mathbb{R}_+; X)$. Then for every $t \geq 0$,

$$\frac{d}{dt}[(u * v)(t)] = u(t)v(0) + (u * v')(t). \tag{10}$$

Let us define

$$E_\epsilon(t) := E_{\alpha-1,1}(-\epsilon^{-\alpha}t^{\alpha-1})$$

for $1 < \alpha \leq 2,\ t > 0$ and $\epsilon > 0$ given. The next lemma shows some interesting properties of $E_\epsilon(t)$ that we will use in what follows.

Proposition 1. *Let $1 < \alpha \leq 2$, $t > 0$ and $\epsilon > 0$ be given. Then*

1. $E'_\epsilon(t) = -\epsilon^{-\alpha}t^{\alpha-2}E_{\alpha-1,\alpha-1}(-\epsilon^{-\alpha}t^{\alpha-1})$ *and* $E'_\epsilon(t) \leq 0$ *for all* $t > 0$.
2. $0 \leq E_\epsilon(t) \leq 1$.
3. $$\widehat{E}_\epsilon(\lambda) = \frac{\epsilon^\alpha\lambda^{\alpha-2}}{\epsilon^\alpha\lambda^{\alpha-1}+1} = \frac{\epsilon^\alpha\lambda^{\alpha-1}}{\epsilon^\alpha\lambda^\alpha+\lambda}, \quad \lambda > 0.$$
4. $\epsilon^\alpha(g_{2-\alpha} * E'_\epsilon)(t) = -E_\epsilon(t)$.
5. *Let*
$$m_\epsilon(t) := \frac{\epsilon^\alpha}{\epsilon^\alpha + t^{\alpha-1}}, \quad t > 0.$$
Then
$$E_\epsilon(t) \leq C \cdot m_\epsilon(t),$$
where $C > 0$ is given in Lemma 1 and does not depend on $\epsilon > 0$. In particular, $E_\epsilon(t) \to 0$ as $\epsilon \to 0$.
6. $$\int_0^t E_\epsilon(s)ds \leq C \cdot t m_\epsilon(t).$$

Proof. The proof of each item is sketched as follows.

1. Follows from Lemmas 2 and 3.
2. Follows from Lemma 3, the identity $(1 * E'_\epsilon)(t) = E_\epsilon(t) - 1$ and (a).
3. Follows from (6).
4. Using (c), we observe that the Laplace transform of the left hand side of (d) gives
$$\frac{\epsilon^\alpha}{\lambda^{2-\alpha}}[\lambda\hat{E}_\epsilon(\lambda) - 1] = \frac{\epsilon^\alpha}{\lambda^{2-\alpha}}\Big[\frac{\epsilon^\alpha\lambda^\alpha}{\epsilon^\alpha\lambda^\alpha+\lambda} - 1\Big] = -\frac{\epsilon^\alpha\lambda^{\alpha-1}}{\epsilon^\alpha\lambda^\alpha+\lambda}$$
which is precisely the Laplace transform of the right hand side of (d).
5. Follows from Lemma 1, taking into account that $\arg(-\epsilon^{-\alpha}t^{\alpha-1}) = \pi$.

6. Note that by (9) we have

$$\int_0^t E_\epsilon(s)ds = \int_0^t \frac{d}{ds}[sE_{\alpha-1,2}(-\epsilon^{-\alpha}s^{\alpha-1})]ds = tE_{\alpha-1,2}(-\epsilon^{-\alpha}t^{\alpha-1})$$
$$\leq t \cdot \frac{C}{1+\epsilon^{-\alpha}t^{\alpha-1}} = C \cdot tm_\epsilon(t),$$

where in the last inequality we used Lemma 1. This finish the proof.

□

Remark 1. *Some properties of the function $m_\epsilon(t)$ are the following:*

1. $0 < m_\epsilon(t) < 1, \quad t > 0.$
2. $\lim_{\epsilon \to 0} t \cdot m_\epsilon(t) = 0$ *for t in compact subsets of $\mathbb{R}$.*
3. $\int_0^t s^k m_\epsilon(s)ds \leq \epsilon^\alpha t^{k+1-\alpha}, \quad t > 0, k = 1, 2 \ldots.$

We recall the following definition.

Definition 2 ([27])**.** *A strongly measurable family of operators $\{R(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is said to be uniformly integrable if $\int_0^\infty ||R(t)||dt < \infty$.*

From now on, we will denote the norm of any uniformly integrable family of operators by $\|R\|$, i.e., $\|R\| := \int_0^\infty ||R(t)||dt$.

The following definition will help us to give an operator theoretical approach to the problem (2).

Definition 3 ([27])**.** *Let A be a closed and densely defined linear operator with domain $D(A)$ on a complex Banach space X and let $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable. We say that A is the generator of a resolvent family if there exist $\omega \in \mathbb{R}$ and a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{L}(X)$ such that $\{\frac{1}{\hat{a}(\lambda)} : Re(\lambda) > \omega\} \subset \rho(A)$ and*

$$H(\lambda)x := \frac{1}{\lambda\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1}x = \int_0^\infty e^{-\lambda t}S(t)xdt, \quad Re(\lambda) > \omega, \quad x \in X.$$

In such case we say that $\{S(t)\}_{t\geq 0}$ is the resolvent family generated by A.

As a particular case, we propose the following definition.

Definition 4. *Let $\epsilon > 0$ and $1 \leq \alpha \leq 2$ be given. Let A be a closed and densely defined linear operator on a complex Banach space X. We say that A the generator of an (α, ϵ)-resolvent family if A is the generator of a resolvent family for $a(t) := 1 - E_\epsilon(t)$. In such case, we denote by $\{S_{\alpha,\epsilon}(t)\}_{t\geq 0}$ the (α, ϵ)-resolvent family generated by A. In the limit case $\epsilon = 0$ and $\alpha = 1$, we consider $a(t) \equiv 1$ and we denote by $S_{1,0}(t)$ the C_0-semigroup generated by A.*

Remark 2. *As a consequence of Proposition 1, part (b), we have $a(t) \geq 0$. Moreover, by definition, $a(0) = 0$.*

From ([28], Proposition 3.1 and Lemma 2.2) we obtain directly the following properties.

Proposition 2. *Let $1 < \alpha \leq 2$ and $\epsilon > 0$ be given. Suppose that A is the generator of an (α, ϵ)-resolvent family $\{S_{\alpha,\epsilon}(t)\}_{t\geq 0}$ on X. Then the following assertions hold true:*

1. *$S_{\alpha,\epsilon}(t)$ is strongly continuous and $S_{\alpha,\epsilon}(0) = I$.*
2. *For all $x \in D(A)$ and $t \geq 0$ we have $S_{\alpha,\epsilon}(t)x \in D(A)$ and $AS_{\alpha,\epsilon}(t)x = S_{\alpha,\epsilon}(t)Ax$.*

3. *Let $x \in X$ and $t \geq 0$. Then $\int_0^t (1 - E_\epsilon)(t-s) S_{\alpha,\epsilon}(s) x ds \in D(A)$ and*

$$S_{\alpha,\epsilon}(t)x = x + A\int_0^t S_{\alpha,\epsilon}(s)xds - A\int_0^t E_\epsilon(t-s)S_{\alpha,\epsilon}(s)xds. \tag{11}$$

4. *For all $x \in D(A^2)$ we have $S_{\alpha,\epsilon}(\cdot)x \in C^2(\mathbb{R}_+; X)$. Moreover,*

$$S'_{\alpha,\epsilon}(t)x = -\int_0^t E'_\epsilon(t-s)S_{\alpha,\epsilon}(s)Axds$$

for all $x \in D(A)$, $t \geq 0$ and

$$S''_{\alpha,\epsilon}(t)x = -E'_\epsilon(t)Ax - \int_0^t S'_{\alpha,\epsilon}(t-s)E'_\epsilon(s)Axds$$

for all $x \in D(A^2)$ and $t > 0$.

The next result is the corresponding Hille–Yosida type Theorem for (α, ϵ)-resolvent families. The proof is a particular case of earlier results that can be found in [27] or ([28], Theorem 3.4).

Theorem 1. *Let A be a closed linear densely defined operator in a complex Banach space X. Let $1 < \alpha \leq 2$ and $\epsilon > 0$. Then the following assertions are equivalent.*

(i) *The operator A is the generator of an (α, ϵ)-resolvent family $(S_{\alpha,\epsilon}(t))_{t\geq 0}$ satisfying $||S_{\alpha,\epsilon}(t)|| \leq M_{\alpha,\epsilon}e^{\omega_{\alpha,\epsilon}t}$ for all $t \geq 0$ and for some constants $M_{\alpha,\epsilon} > 0$ and $\omega_{\alpha,\epsilon} \in \mathbb{R}$.*
(ii) *There exist constants $\omega_{\alpha,\epsilon} \in \mathbb{R}$ and $M_{\alpha,\epsilon} > 0$ such that*

(P1) *$\epsilon^\alpha \lambda^\alpha + \lambda \in \rho(A)$ for all λ with $Re(\lambda) > \omega_{\alpha,\epsilon}$ and*
(P2) *$H_{\alpha,\epsilon}(\lambda) := (\epsilon^\alpha \lambda^{\alpha-1} + 1)(\epsilon^\alpha \lambda^\alpha + \lambda - A)^{-1}$ satisfies the estimates*

$$||H^{(n)}_{\alpha,\epsilon}(\lambda)|| \leq \frac{M_{\alpha,\epsilon} n!}{(\lambda - \omega_{\alpha,\epsilon})^{n+1}}, \quad \lambda > \omega_{\alpha,\epsilon}, \quad n = 0, 1, 2.... $$

The next proposition is taken from ([27], Proposition 0.1). It will be helpful in order to obtain our main result.

Proposition 3. *Let Y be a complex Banach space. Suppose $h : \mathbb{C}_+ \to Y$ is analytic and verifies*

$$||\lambda h(\lambda)|| + ||\lambda^2 h'(\lambda)|| \leq M$$

for all $Re(\lambda) > 0$ and some $M > 0$. Then

$$||h^{(n)}(\lambda)|| \leq \frac{Mn!}{\lambda^{n+1}}$$

for all $\lambda > 0$ and $n = 0, 1, 2....$

Finally, let us remember that a closed and densely defined operator A is called ω-sectorial of angle θ if there exist $\theta \in [0, \pi/2)$, $M_0 > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(\lambda)| < \frac{\pi}{2} + \theta\} \setminus \{\omega\}, \tag{12}$$

and

$$||(\mu - A)^{-1}|| \leq \frac{M_0}{|\mu - \omega|}, \quad \mu \in \omega + S_\theta. \tag{13}$$

These are generators of analytic semigroups. In case $\omega = 0$ we simply say that A is sectorial of angle $\theta + \pi/2$. We should point out that in the general theory of sectorial operators, it is not essential that (13) holds in a sector of angle $\pi/2$.

Sufficient conditions to obtain generators of an (α, ϵ)-resolvent family are given in the following general result. The proof is similar to ([29], Theorem 3.2).

Theorem 2. *Let $1 < \alpha < 2$, $\epsilon > 0$ and A be a sectorial operator of angle $\pi/2$. Then A generates an (α, ϵ)-resolvent family $\{S_{\alpha,\epsilon}(t)\}_{t\geq 0}$ and there exists $\tilde{M} > 0$ (which does not depends of ϵ) such that $\|S_{\alpha,\epsilon}(t)\| \leq \tilde{M}$ for all $t \geq 0$ and for all $\epsilon > 0$.*

Proof. Let us define $g_\epsilon(\lambda) := \epsilon^\alpha \lambda^\alpha + \lambda$ where $\lambda = re^{i\theta}$ with $|\theta| < \pi/2$ and $r > 0$. We observe that

$$\arg(g_\epsilon(re^{i\theta})) = \operatorname{Im} \log(g_\epsilon(re^{i\theta})) = \operatorname{Im} \int_0^\theta \frac{d}{dt} \log(g_\epsilon(re^{it}))dt = \operatorname{Im} \int_0^\theta \frac{g'_\epsilon(re^{it}) ire^{it}}{g_\epsilon(re^{it})} dt,$$

where a direct computation gives

$$\lambda \frac{g'_\epsilon(\lambda)}{g_\epsilon(\lambda)} = \alpha - 1 + \frac{(2-\alpha)\epsilon^{-\alpha}}{\lambda^{\alpha-1} + \epsilon^{-\alpha}} + \frac{\lambda^{\alpha-1}}{\lambda^{\alpha-1} + \epsilon^{-\alpha}}.$$

Next, observe that for $\epsilon^{-\alpha} > 0$ and all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$ we have $\frac{\epsilon^{-\alpha}}{|z+\epsilon^{-\alpha}|} \leq 1$ and $\frac{|z|}{|z+\epsilon^{-\alpha}|} \leq 1$. Moreover, since $1 < \alpha \leq 2$ we have $\mathrm{Re}\,(\lambda^{\alpha-1}) \geq 0$. It follows that

$$\left| \operatorname{Im} \int_0^\theta \frac{\epsilon^{-\alpha}}{r^{\alpha-1} e^{i(\alpha-1)t} + \epsilon^{-\alpha}} dt \right| \leq \int_0^\theta \left| \frac{\epsilon^{-\alpha}}{r^{\alpha-1} e^{i(\alpha-1)t} + \epsilon^{-\alpha}} \right| dt \leq \theta,$$

and

$$\left| \operatorname{Im} \int_0^\theta \frac{r^{\alpha-1} e^{i(\alpha-1)t}}{r^{\alpha-1} e^{i(\alpha-1)t} + \epsilon^{-\alpha}} dt \right| \leq \theta.$$

Therefore

$$|\arg(g_\epsilon(re^{i\theta}))| \leq (\alpha-1)|\theta| + (2-\alpha)|\theta| + |\theta| < \frac{\pi}{2} + \frac{\pi}{2}. \tag{14}$$

We conclude that $g_\epsilon(\lambda) \in S_{\frac{\pi}{2}}$ for all $\mathrm{Re}(\lambda) > 0$. The previous discussion implies that

$$H_\epsilon(\lambda) = \frac{1}{\lambda} g_\epsilon(\lambda)(g_\epsilon(\lambda) - A)^{-1},$$

is well defined and, by (13) with $\omega = 0$ and $\theta = \pi/2$, satisfies the estimate

$$||\lambda H_\epsilon(\lambda)|| \leq M_0 \quad \text{for all} \quad \mathrm{Re}(\lambda) > 0,$$

where M_0 does not depend on $\epsilon > 0$. For $H'_\epsilon(\lambda)$ one obtains that

$$\begin{aligned} \lambda^2 H'_\epsilon(\lambda) = & \frac{(\alpha-1)\epsilon^\alpha \lambda^{\alpha-1}}{\epsilon^\alpha \lambda^{\alpha-1} + 1} \lambda H_\epsilon(\lambda) - \frac{\alpha \epsilon^\alpha \lambda^{\alpha-1}}{\epsilon^\alpha \lambda^{\alpha-1} + 1} \lambda^2 H(\lambda)^2 \\ & - \frac{1}{\epsilon^\alpha \lambda^{\alpha-1} + 1} (\lambda H_\epsilon(\lambda))(\lambda H_\epsilon(\lambda)), \end{aligned}$$

and hence we conclude that for all $\mathrm{Re}(\lambda) > 0$

$$\begin{aligned} ||\lambda^2 H'_\epsilon(\lambda)|| &\leq (\alpha-1)||\lambda H_\epsilon(\lambda)|| + \alpha||\lambda^2 H_\epsilon(\lambda)^2|| + ||\lambda H_\epsilon(\lambda)|| ||\lambda H_\epsilon(\lambda)|| \\ &\leq (\alpha-1)M_0 + \alpha M_0^2 + M_0^2 =: M_1. \end{aligned}$$

Let $\tilde{M} := M_0 + M_1$. Then

$$||\lambda H_\epsilon(\lambda)|| + ||\lambda^2 H'_\epsilon(\lambda)|| \leq \tilde{M}.$$

Proposition 3 now gives (P2) of Theorem 1 with $\omega = 0$. So we arrive at the conclusion. □

Remark 3. *Since sectorial operators generate bounded analytic semigroups, we have that the analytic semigroup associated to A, denoted by $\{S_{1,0}(t)\}$ is also bounded, i.e., satisfies the following: there exists $K_0 > 0$ such that $\|S_{1,0}(t)\| \leq K_0$ for all $t \geq 0$ and the unique strong solution of* (3) *is given by*

$$u_0(t) = S_{1,0}(t)u_0,$$

whenever $u_0 \in D(A)$. In order to simplify computations, we set $M := \max\{\tilde{M}, K_0\}$. Hence $\|S_{1,0}(t)\| \leq M$ and $\|S_{\alpha,\epsilon}(t)\| \leq M$ for all $t > 0$, where M does not depend on $\epsilon > 0$.

3. Main Results

First of all, we investigate the existence and uniqueness of solutions for the linear fractional evolution equation

$$\begin{cases} \epsilon^\alpha D_t^\alpha u_\epsilon(t) + u'_\epsilon(t) = Au_\epsilon(t), \quad t \in [0,T], \quad 1 < \alpha < 2, \quad \epsilon > 0, \\ u_\epsilon(0) = u_{0,\epsilon}, \\ u'_\epsilon(0) = Au_{0,\epsilon}. \end{cases} \tag{15}$$

Here we will assume that A is a closed and densely linear operator which is a generator of an exponentially bounded (α,ϵ)-resolvent family.

As natural, and recalling that D_t^α denotes the Caputo fractional derivative of order $\alpha \in (1,2)$, we will say that $u_\epsilon \in C^2([0,T];X)$ is a strong solution of Equation (15) if $u_\epsilon(t) \in D(A)$ for all $t \in [0,T]$ and verifies (15).

Observe that the above definition at $t = 0$ of strong solution forces $u_\epsilon(0) \in D(A)$ and the second initial condition $u'_\epsilon(0) = Au_\epsilon(0)$, because $D_t^\alpha u_\epsilon(0) = 0$ when $\alpha \in (1,2)$, see, e.g., ([20], Theorem 3.1). This is the reason why we consider only one unknown initial value, namely $u_{0,\epsilon}$, in the problem (15).

Our first result, gives a representation of the unique solution of the initial value problem (15) in terms of the (α,ϵ)-resolvent family generated by A and the function E_ϵ analyzed in Proposition 1.

Theorem 3. *Let $1 < \alpha < 2$, $\epsilon > 0$ and A be the generator of an (α,ϵ)-resolvent family $S_{\alpha,\epsilon}(t)$. Then there exists a unique strong solution of Equation* (15) *which can be represented by*

$$u_\epsilon(t) = S_{\alpha,\epsilon}(t)u_\epsilon(0) + (E_\epsilon * S_{\alpha,\epsilon})(t)Au_\epsilon(0), \quad t \in [0,T], \tag{16}$$

provided $u_\epsilon(0) \in D(A^2)$.

Proof. It is enough to show that $u_\epsilon(t)$ defined as before, verifies (15). In fact, by Proposition 2 part (d) and (10) we have for $u_\epsilon(0) \in D(A)$ and all $t \in [0,T]$ the identity

$$u'_\epsilon(t) = -(E'_\epsilon * S_{\alpha,\epsilon})(t)Au_\epsilon(0) + S_{\alpha,\epsilon}(t)Au_\epsilon(0) + (E'_\epsilon * S_{\alpha,\epsilon})(t)Au_\epsilon(0) = S_{\alpha,\epsilon}(t)Au_\epsilon(0),$$

holds. Using that $u_\epsilon(0) \in D(A^2)$ we obtain again by part (d) of Proposition 2, and (10), the identity

$$u''_\epsilon(t) = -(E'_\epsilon * S_{\alpha,\epsilon})(t)A^2u_\epsilon(0), \tag{17}$$

valid for all $t \in (0, T]$. Then, convoluting with $g_{2-\alpha}$ and using the identity in Proposition 1 part (d) we obtain for all $t \in [0, T]$:

$$\epsilon^{\alpha}(g_{2-\alpha} * u_{\epsilon}'')(t) = -\epsilon^{\alpha}(g_{2-\alpha} * E_{\epsilon}' * S_{\alpha,\epsilon})(t)A^2u_{\epsilon}(0) = (E_{\epsilon} * S_{\alpha,\epsilon})(t)A^2u_{\epsilon}(0), \tag{18}$$

where in the last identity we applied (10). On the other hand, we have

$$\begin{aligned} Au_{\epsilon}(t) - u_{\epsilon}'(t) &= S_{\alpha,\epsilon}(t)Au_{\epsilon}(0) + (E_{\epsilon} * S_{\alpha,\epsilon})(t)A^2u_{\epsilon}(0) - S_{\alpha,\epsilon}(t)Au_{\epsilon}(0) \\ &= (E_{\epsilon} * S_{\alpha,\epsilon})(t)A^2u_{\epsilon}(0), \ t \in [0, T]. \end{aligned} \tag{19}$$

Since $\epsilon^{\alpha}D_t^{\alpha}u_{\epsilon}(t) = \epsilon^{\alpha}(g_{2-\alpha} * u_{\epsilon}'')(t)$ by definition, comparing Systems (18) and (19) we obtain (15). The proof is finished. $\square$

Next, before to show our main result, we need to prove some technical preliminaries. Recall that by Proposition 2 and Remark 3, if A be a sectorial operator of angle $\pi/2$ then A generates an (α, ϵ)-resolvent family $\{S_{\alpha,\epsilon}(t)\}_{t\geq 0}$ for each $\epsilon > 0$ and a C_0-semigroup in case $(\alpha, \epsilon) = (1, 0)$. In particular, under such assumption, and whenever $u_0 \in D(A)$, a unique strong solution (in the sense that $u_0 \in C^1([0, T]; X)$, $u_0(t) \in D(A)$ and verifies (3) on $[0, T]$) of the Equation (3) exists and is given by

$$u_0(t) = S_{1,0}(t)u_0.$$

Proposition 4. *Let* $1 < \alpha < 2$, $\epsilon > 0$ *and* A *be a sectorial operator of angle* $\pi/2$. *For all* $x \in D(A^2)$ *we have*

$$S_{1,0}(t)x - S_{\alpha,\epsilon}(t)x = (E_{\epsilon} * S_{\alpha,\epsilon} * S_{1,0})(t)A^2x + (E_{\epsilon} * S_{\alpha,\epsilon})(t)Ax, \tag{20}$$

and

$$\|S_{1,0}(t)x - S_{\alpha,\epsilon}(t)x\| \leq M^2Ct^2m_{\epsilon}(t)\|A^2x\| + MCtm_{\epsilon}(t)\|Ax\|, \tag{21}$$

where the constants $C > 0$ *and* $M > 0$ *does not depend on* $\epsilon > 0$.

Proof. To verify (20) note that the Laplace transform of $S_{1,0}(t)$ gives the resolvent operator $R(\lambda, A)$. Using this, Definition 4, the convolution properties and the uniqueness theorem for the Laplace transform we obtain (20).

Let us verify (21). For any $y \in X$, we have the following inequalities

$$\|(E_{\epsilon} * S_{\alpha,\epsilon})(t)y\| \leq \int_0^t E_{\epsilon}(t-s)\|S_{\alpha,\epsilon}(t)y\|ds \leq M\left(\int_0^t E_{\epsilon}(s)ds\right)\|y\| \leq MtCm_{\epsilon}(t)\|y\|,$$

where in the last inequality we have used Proposition 1, part (f). Hence

$$\|(E_{\epsilon} * S_{\alpha,\epsilon})(t)y\| \leq MtCm_{\epsilon}(t)\|y\|, \quad \text{for all} \quad y \in X. \tag{22}$$

Now, since $\|S_{\alpha,\epsilon}(t)\| \leq M$ for all $1 < \alpha < 2$ and for all $\epsilon > 0$, and $\|S_{1,0}(t)\| \leq M$, we obtain

$$\|(S_{\alpha,\epsilon} * S_{1,0})(t)y\| \leq \int_0^t \|S_{\alpha,\epsilon}(t-s)S_{1,0}(s)y\|ds \leq M^2t\|y\|.$$

Therefore, again using Proposition 1, part (f), we obtain

$$\begin{aligned}\|(E_\epsilon * S_{\alpha,\epsilon} * S_{1,0})(t)y\| &= \|\int_0^t E_\epsilon(t-s)(S_{\alpha,\epsilon} * S_{1,0})(s)yds\| \\ &\le \int_0^t E_\epsilon(t-s)M^2 s\|y\|ds \\ &\le M^2 t\left(\int_0^t E_\epsilon(s)ds\right)\|y\| \le M^2t^2Cm_\epsilon(t)\|y\|.\end{aligned}$$

Hence (21) follows. □

Finally, we arrive at the main result of this paper.

Theorem 4. *Let $1 < \alpha < 2$ be fixed. Given $\epsilon > 0$ suppose that A generates a bounded analytic semigroup on a complex Banach space X and $u_0(0), u_\epsilon(0) \in D(A^2)$. Then the solutions of (3) and (4) exists and the following estimate holds:*

$$\begin{aligned}\|u_\epsilon(t) - u_0(t)\| \le{}& M^2t^2Cm_\epsilon(t)\|A^2u_0(0)\| + MtCm_\epsilon(t)\|Au_0(0)\| + M\|u_\epsilon(0) - u_0(0)\| \\ &+ MCtm_\epsilon(t)\|Au_\epsilon(0)\|.\end{aligned} \tag{23}$$

Moreover, if $u_\epsilon(0) \to u_0(0)$ in X and the set $\{Au_\epsilon(0)\}_{\epsilon>0}$ is bounded then the strong solution $u_\epsilon(t)$ of Problem (2) converges to the unique strong solution $u_0(t)$ of Problem (3) as $\epsilon \to 0$ on closed subintervals of $[0,T]$.

Proof. Since A generates an analytic semigroup then Theorem 2 implies that A generates an (α,ϵ)-regularized family $\{S_{\alpha,\epsilon}(t)\}_{t\ge 0}$ such that $\|S_{\alpha,\epsilon}(t)\| \le M$ for all $t > 0$ and for all $\epsilon > 0$. Then, by the representation of $u_\epsilon(t)$ (see Theorem 3) and $u_0(t)$, we obtain the following identity

$$u_\epsilon(t) - u_0(t) = (S_{\alpha,\epsilon}(t) - S_{1,0}(t))u_0(0) + S_{\alpha,\epsilon}(t)(u_\epsilon(0) - u_0(0)) + (E_\epsilon * S_{\alpha,\epsilon})(t)Au_\epsilon(0). \tag{24}$$

In view of Proposition 4, the following estimate

$$\|S_{1,0}(t)u_0(0) - S_{\alpha,\epsilon}(t)u_0(0)\| \le M^2Ct^2m_\epsilon(t)\|A^2u_0(0)\| + MCtm_\epsilon(t)\|Au_0(0)\|, \tag{25}$$

holds. On the other hand, is clear that

$$\|S_{\alpha,\epsilon}(t)(u_\epsilon(0) - u_0(0))\| \le M\|u_\epsilon(0) - u_0(0)\|.$$

Moreover, by (22) we get that

$$\|(E_\epsilon * S_{\alpha,\epsilon})(t)Au_\epsilon(0)\| \le MCtm_\epsilon(t)\|Au_\epsilon(0)\|. \tag{26}$$

Combining the above inequalities, we obtain (23). Finally, in order to prove the convergence, it is enough to take into account that the set $\{Au_\epsilon(0)\}_{\epsilon>0}$ is bounded, say, by a constant $K > 0$, and $m_\epsilon(t) \to 0$ as $\epsilon \to 0$ for any $t > 0$. Hence, for any $I := [a,b] \subset [0,T]$ with $0 < a < b$, we have

$$\begin{aligned}\sup_{t\in I}\|u_\epsilon(t) - u_0(t)\| \le{}& M^2b^2Cm_\epsilon(a)\|A^2u_0(0)\| + MbCm_\epsilon(a)\|Au_0(0)\| + M\|u_\epsilon(0) - u_0(0)\| \\ &+ MCbm_\epsilon(a)K,\end{aligned}$$

and then $\lim_{\epsilon\to 0}\sup_{t\in I}\|u_\epsilon(t) - u_0(t)\| = 0$. Therefore, for each $\epsilon > 0$ the solution $u_\epsilon(t)$ of Problem (2) converges to the unique solution $u_0(t)$ of the Equation (3) as $\epsilon \to 0$ on closed subintervals of $[0,T]$. □

Next, we prove the convergence of integrals.

Corollary 1. *Under the same conditions of Theorem 4 we have*

$$\int_a^b \|u_\epsilon(t) - u_0(t)\| dt \to 0 \quad \text{as } \epsilon \to 0,$$

for all $a, b \in \mathbb{R}$ with $0 < a < b$.

Proof. For any $a, b > 0$ with $0 < a < b$ we have that

$$\int_a^b \|u_\epsilon(t) - u_0(t)\| dt \leq (b-a) \sup_{t\in[a,b]} \|u_\epsilon(t) - u_0(t)\|,$$

and the result follows from the main theorem.

□

Finally, we prove the convergence of derivatives.

Corollary 2. *Let $1 < \alpha < 2$, $\epsilon > 0$ and suppose that A generates a bounded analytic semigroup on a complex Banach space X such that $0 \in \sigma_p(A)$. Assume that $u_0, u'_\epsilon(0) \in D(A^2)$ and $0 \neq u'_\epsilon(0) \in ker(A)$. If $u'_\epsilon(0) \to Au_0$, then the solutions of (3) and (4) exists and $u'_\epsilon(t) \to u'_0(t)$ as $\epsilon \to 0$ on closed subintervals of $[0, T]$.*

Proof. Let us define $v_\epsilon(t) := u'_\epsilon(t)$. Observe that by definition, (15) is equivalent to write

$$\epsilon^\alpha (g_{2-\alpha} * u''_\epsilon)(t) + u'_\epsilon(t) = Au_\epsilon(t). \tag{27}$$

Using that A is closed we obtain that

$$\epsilon^\alpha \frac{d}{dt}(g_{2-\alpha} * u''_\epsilon)(t) + u''_\epsilon(t) = Au'_\epsilon(t),$$

or, equivalently

$$\epsilon^\alpha \frac{d}{dt}(g_{2-\alpha} * v'_\epsilon)(t) + v'_\epsilon(t) = Av_\epsilon(t).$$

Applying the rule (10) for the derivative of the convolution we get

$$\epsilon^\alpha [v'_\epsilon(0) g_{2-\alpha}(t) + (g_{2-\alpha} * v''_\epsilon)(t)] + v'_\epsilon(t) = Av_\epsilon(t).$$

Observe now that from (27) and the fact that $1 < \alpha < 2$, we have $u'_\epsilon(0) = Au_\epsilon(0)$ and hence from the identity (17) we have

$$u''_\epsilon(t) = -(E'_\epsilon * S_{\alpha,\epsilon})(t) A^2 u_\epsilon(0), \quad t > 0.$$

We conclude that $v'_\epsilon(0) = u''_\epsilon(0) = 0$. Then, we have that v_ϵ satisfies

$$\begin{cases} \epsilon^\alpha D_t^\alpha v_\epsilon(t) + v'_\epsilon(t) = Av_\epsilon(t), & t \in [0, T], \quad 1 < \alpha < 2, \quad \epsilon > 0, \\ v_\epsilon(0) = u'_\epsilon(0), \end{cases} \tag{28}$$

The result is now the consequence of Theorem 4.

□

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