FRACTIONAL RELAXATION EQUATIONS ON BANACH SPACES

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Abstract. We study existence and qualitative properties of solutions for the abstract fractional relaxation equation

(0.1) $$u'(t) - AD_t^{\alpha}u(t) + u(t) = f(t), \quad 0 < \alpha < 1, \quad t \geq 0, \qquad u(0) = 0,$$

on a complex Banach space X, where A is a closed linear operator, D_t^{α} is the Caputo derivative of fractional order $\alpha \in (0,1)$, and f is an X-valued function. We also study conditions under which the solution operator has the properties of maximal regularity and L^p integrability. We characterize these properties in the Hilbert space case.

1. Introduction

The main strategy to study equation (0.1), will be the application of the theory of (a,k)-regularized resolvent. The notion of (a,k)-regularized resolvent families was first introduced in [8] as a generalization of solution operator families for linear Volterra integral equations of convolution type; see [9, 10, 12, 14]. We recall the resolvent equation from [8]

(1.2) $$R(t)x = k(t)x + A(a * R)(t)x, \quad t \geq 0, \quad x \in X.$$

In the above definition $(a * R)(t)x := \int_0^t a(t-s)R(s)xds$ stands for the finite convolution product.

2. preliminaries

We recall that the Caputo derivative of fractional order $\alpha \in (0,1)$ is defined as

(2.1) $$D_t^{\alpha}u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(\tau)}{(t-\tau)^{\alpha}} d\tau,$$

whenever $u \in C^1(\mathbb{R}_+, X)$. Then the Laplace transform of $D_t^{\alpha}u$ is given by

(2.2) $$\widehat{D_t^{\alpha}u}(\lambda) = \lambda^{\alpha}\hat{u}(\lambda) - \lambda^{\alpha-1}u(0), \quad Re(\lambda) > 0, \quad 0 < \alpha < 1;$$

for details and further properties see [1, 2, 4] and references therein.

Next we recall the generalized Mittag-Leffler function which is defined in the complex plane by the power series

(2.3) $$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad \alpha > 0, \quad \beta \in \mathbb{R}, \quad z \in \mathbb{C};$$

see [5, 11]. This is an entire trascendental function. An interesting property related with its Laplace transform is the following identity cf. [7, 11],

(2.4) $$\int_0^{\infty} e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^{\alpha}) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \omega}, \quad Re\lambda > \omega^{1/\alpha}, \quad \omega > 0.$$

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A function $a(t)$ is said to be of *positive type* if $Re[\widehat{a}(\lambda)] \geq 0$ for all $Re\lambda > 0$, and it is called 2-*regular*, cf. [13], if there is a constant $c > 0$ such that

$$|\lambda\widehat{a}^{'}(\lambda)| \leq c|\widehat{a}(\lambda)| \text{ and } |\lambda^2\widehat{a}^{''}(\lambda)| \leq c|\widehat{a}(\lambda)| \text{ for all } Re\lambda > 0. \tag{2.5}$$

The following lemma will be needed in the following sections.

Lemma 2.1. *Let* $a(t) := t^{-\alpha}E_{1,1-\alpha}(-t), \quad 0 < \alpha < 1$. *Then* $a(t)$ *is of positive type and* 2*-regular.*

Proof. Thanks to (2.4) we obtain that $\widehat{a}(\lambda) = \dfrac{\lambda^\alpha}{\lambda+1}$, and the assertion follows immediately. □

Henceforth we shall denote $a_\alpha(t) := t^{-\alpha}E_{1,1-\alpha}(-t), \quad 0 < \alpha < 1$ and $\quad k(t) := e^{-t}$.

3. Well posedness

First we recall the definition of an $(a, k)-$regularized resolvent; see [8]. Let A be a closed linear operator and let $a \in L^1_{loc}(\mathbb{R}_+)$, and $k \in C(\mathbb{R}_+)$. A strongly continuous function $R : \mathbb{R}_+ \to \mathcal{B}(X)$ is called $(a, k)-regularized$ resolvent family with generator A if it satisfies: $(i)\ R(0) = k(0)I$, $(ii)\ R(t)x \in D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t > 0$, $(iii)\ (a * R)(t)x \in D(A)$ and

$$R(t)x = k(t)x + A(a * R)(t)x, \quad t \geq 0, \quad x \in X. \tag{3.1}$$

Theorem 3.1. *Let* $0 < \alpha < 1$. *Assume that* A *is the generator of a bounded analytic* C_0*- semigroup. Then* A *is the generator of an* (a_α, k)*-regularized resolvent family* $R \in C^1((0,\infty); \mathcal{B}(X))$.

Proof. Since $a_\alpha(t)$ is of positive type, we obtain by [13, Corollary 3.1] that $\frac{1}{\widehat{a_\alpha}(\lambda)} \in \rho(A)$ for all $Re\lambda > 0$ and there is a constant $M \geq 1$ such that $H(\lambda) := (I - \hat{a}_\alpha(\lambda)A)^{-1}/\lambda$ satisfies

$$||H(\lambda)|| \leq \frac{M}{|\lambda|} \quad \text{for all } Re\lambda > 0. \tag{3.2}$$

From the above, and 2-regularity of $a_\alpha(t)$, we get by [13, Theorem 3.1] that A generates a $(1, a_\alpha)$-regularized resolvent family $S \in C^1((0,\infty); \mathcal{B}(X))$ i.e., a resolvent. Thus, there is a constant $C \geq 1$ such that the estimates

$$||S(t)|| \leq C, \qquad t \geq 0 \text{ and } ||S'(t)|| \leq C/t \tag{3.3}$$

holds. Next let $x \in X$ and define

$$R(t)x = S(t)x - \int_0^t e^{-(t-\tau)}S(\tau)xd\tau, \quad t > 0. \tag{3.4}$$

But then, $R(t)$ is a continuously differentiable (a_α, k)-regularized family generated by A. In fact, (i) follows from $R(0)x = S(0)x = x = k(0)x$. (ii) and the first part of (iii) follows from the corresponding properties for the resolvent $S(t)$. To prove (3.1) we notice that

$$\begin{aligned} A(a_\alpha * R)(t)x &= A(a_\alpha * [S(t)x - (k * S)(t)x]) \\ &= A(a_\alpha * S)(t)x - A(a_\alpha * k * S)(t)x \\ &= S(t)x - Ix - k * [S(t)x - Ix] \\ &= S(t)x - Ix - (k * S)(t)x + (k * I)(t)x, \end{aligned}$$

where $(k * I)(t)x = -k(t)x + Ix$, since $k(t) = e^{-t}$. Hence

$$A(a_\alpha * R)(t)x = S(t)x - (k * S)(t)x - Ix + (-k(t) + Ix) = R(t)x - k(t)x,$$

proving the claim and the theorem. □

Remark 3.2. We notice that integrating (0.1) we obtain

$$u(t) - A(t^{-\alpha} * u)(t) + (1 * u)(t) = (1 * f)(t), \tag{3.5}$$

since $D_t^\alpha u(t) = (t^{-\alpha} * u')(t)$ and $u(0) = 0$.
Now let $v = u + (1 * u)$. Hence $u = v - (e^{-t} * v)$. Then equation (3.5) is equivalent to

$$v(t) = A(a_\alpha * v)(t) + g(t) \qquad v(0) = 0 \qquad g = (1 * f) \tag{3.6}$$

where $a_\alpha(t) = t^{-\alpha} - (t^{-\alpha} * e^{-t}) = t^{-\alpha} E_{1,1-\alpha}(-t)$.

Now by applying the results of [8] we relate the solutions of (0.1) to those of the following integral equation of Volterra type, which includes (3.6),

$$u(t) = g(t) + A \int_0^t (t-s)^{-\alpha} E_{1,1-\alpha}(s-t) u(s) ds, \quad t \geq 0, \quad 0 < \alpha < 1. \tag{3.7}$$

Furthermore, (0.1) is equivalent to (3.7) when $g = (1 * f)$.

We recall next that a function $u \in C(\mathbb{R}_+; X)$ is called a strong solution of (3.7) if $a_\alpha * u \in C(\mathbb{R}_+, D(A))$ and (3.7) is satisfied. Now the following is a direct consequence of Theorem 3.1 and [8, Corollary 2.13].

Corollary 3.3. *Assume that A is the generator of a bounded analytic C_0- semigroup and $g \in C^1(\mathbb{R}_+, X)$. Then there exist a unique solution of (3.7).*

We say that $u \in C^1(\mathbb{R}_+; X)$ is a strong solution of (0.1) if $D_t^\alpha u(t) \in D(A)$ for all $t \geq 0$ and u satisfies (0.1).

Proposition 3.4. *Let $f \in C(\mathbb{R}_+, X)$. Assume that u is a solution of (0.1). Then u is a solution of (3.7) with $g = k * f$.*

Proof. By hypothesis u satisfies

$$u'(t) - AD_t^\alpha u(t) + u(t) = f(t), \quad 0 < \alpha < 1, \quad t > 0, \tag{3.8}$$

and convolving this last equation against k, yields

$$(k * u')(t) - (k * AD_t^\alpha u)(t) + (k * u)(t) = (k * f)(t), \quad 0 < \alpha < 1, \quad t > 0.$$

Since A is a closed linear operator and $D_t^\alpha u(t) \in D(A)$, it follows that $k * D_t^\alpha u(t) \in D(A)$ and $k * AD_t^\alpha u = Ak * D_t^\alpha u$. But then,

$$(k * u')(t) - A(k * D_t^\alpha u)(t) + (k * u)(t) = (k * f)(t), \quad 0 < \alpha < 1, \quad t > 0. \tag{3.9}$$

Next we note the following two identities

$$(k * u')(t) = (k' * u)(t) + u(t)k(0) = -(k * u)(t) + u(t), \quad t \geq 0 \tag{3.10}$$

and

$$(k * D_t^\alpha u)(t) = D_t^\alpha k * u + k(0)\ t^{-\alpha} * u\ , \quad t \geq 0. \tag{3.11}$$

Since $k(t) = e^{-t}$ it follows that

$$(D_t^\alpha k)(t) = t^{-\alpha} E_{1,1-\alpha}(-t) - t^{-\alpha} = a_\alpha(t) - t^{-\alpha}, \quad t \geq 0. \tag{3.12}$$

Then by (3.12) it follows that identity (3.11) equals

$$(k * D_t^{\alpha} u)(t) = (a_\alpha * u)(t), \quad t \geq 0. \tag{3.13}$$

In particular $a_\alpha * u \in C(\mathbb{R}_+, D(A))$. Hence replacing (3.13) and (3.10) into (3.9) we get

$$u(t) - A(a_\alpha * u)(t) = (k * f)(t), \quad t \geq 0. \tag{3.14}$$

□

In what follows, we set for $\beta \geq 0$, $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$ if $t > 0$ and $g_\beta(t) := 0$ if $t \leq 0$.

Theorem 3.5. *Assume that A is the generator of a bounded analytic semigroup. Then for all $f \in L^1_{loc}(\mathbb{R}_+; D(A))$,*

$$u(t) = \int_0^t R(t-s) f(s) ds \tag{3.15}$$

is a strong solution of (0.1), with initial condition $u(0) = 0$.

Proof. First we note that by Theorem 3.1, u' exists and

$$u'(t) = \int_0^t R'(t-s) f(s) ds + f(t).$$

Since $f(t) \in D(A)$ and A is closed it then follows that $u'(t) = (R' * f)(t) + f(t) \in D(A)$. Now by the closedness of A, we obtain $D_t^{\alpha} u(t) = (g_{1-\alpha} * u')(t) = (g_{1-\alpha} * R' * f)(t) + (g_{1-\alpha} * f)(t) \in D(A)$ for all $t \geq 0$. Next we show that u satisfies (0.1). Since

$$(g_{1-\alpha} * u)(t) = (1 * D_t^{\alpha} u)(t), \quad 0 < \alpha \leq 1, \tag{3.16}$$

by applying the definition of the Caputo's derivative and $u(0) = 0$.

But then by (3.16) equation (0.1) is equivalent to

$$u(t) - (g_{1-\alpha} * Au)(t) + (1 * u)(t) = (1 * f)(t). \tag{3.17}$$

On the other hand $R(t)$ is an $(a_\alpha, k)-$regularized resolvent where $a_\alpha(t) = t^{-\alpha} E_{1,1-\alpha}(-t)$, and $k(t) = e^{-t}$ it then follows by the resolvent equation (3.1) that

$$(R * f)(t) = (k * f)(t) + (a_\alpha * AR * f)(t).$$

Hence

$$\begin{aligned} u(t) - (g_{1-\alpha} * Au)(t) &= (R * f)(t) - (g_{1-\alpha} * AR * f)(t) \\ &= (k * f)(t) + ((a_\alpha - g_{1-\alpha}) * AR * f)(t). \end{aligned} \tag{3.18}$$

Now by a direct application of the basic properties of the Mittag Leffler function [**?**], follows the identity

$$g_{1-\alpha}(t) - a_\alpha(t) = (1 * a_\alpha)(t). \tag{3.19}$$

Hence by replacing (3.19) into (3.18) and by equation (3.1) follows that

$$\begin{aligned} u(t) - (g_{1-\alpha} * Au)(t) &= (k * f)(t) - (1 * a * AR * f)(t) \\ &= (k * f)(t) - (1 * (R - k) * f)(t). \end{aligned}$$

Moreover, $(1 * (R - k) * f)(t) = (1 * R * f)(t) - ((1 - k) * f)(t)$, since $(1 * k)(t) = 1 - k(t)$. But then, $(k * f)(t) - (1 * (R - k) * f)(t) = (1 * f)(t) - (1 * u)(t)$, and hence $u(t)$ satisfies equation (3.17). □

Corollary 3.6. *Assume that A is the generator of a bounded analytic semigroup. Then for all $f \in L^1_{loc}(\mathbb{R}_+; D(A))$ and $x \in D(A)$,*

$$v(t) = R(t)x + \int_0^t R(t-s)f(s)ds \tag{3.20}$$

is a strong solution of (0.1), with initial condition $u(0) = x$.

Proof. We first note that $w(t) = R(t)x$ solves (0.1) with $f \equiv 0$. In fact, by (3.19) and equation (3.1) we have

$$R(t)x = k(t)x + A(a_\alpha * R)(t)x = k(t)x + A(g_{1-\alpha} * R)(t)x - A(1 * a_\alpha * R)(t)x.$$

Since A is closed and $x \in D(A)$ it follows, that $R'(t)x = k'(t)x + A(g_{1-\alpha} * R')(t)x - A(a_\alpha * R)(t)x$. Furthermore, $w'(t) = R'(t)x$. Then a direct computation shows that $w'(t) = -R(t)x + AD_t^\alpha R(t)x = -w(t) + AD_t^\alpha w(t)$. Finally, we set $v(t) = w(t) + u(t)$. Then we have $v'(t) = w'(t) + u'(t) = -v(t) + AD_t^\alpha v(t) + f(t)$, proving the result.

□

4. Maximal regularity

We introduce the concept of maximal regularity for equation (0.1) in analogy with the first order Cauchy problem; see e.g. [3].

Definition 4.1. *We say that (0.1) has L^p-maximal regularity if there is a strong solution of (0.1) satisfying $AD_t^\alpha u \in L^p(\mathbb{R}_+; X)$ for each $f \in L^p(\mathbb{R}_+; X)$.*

Trough this section we shall need the notion of R-bounded sets, and $UMD-$spaces; see [3] and the references therein. We recall

Remark 4.2. We remark some of the elementary properties of R-bounded sets: (i) If $\mathbf{S}$ and $\mathbf{T}$ are both R-bounded, then $\mathbf{S} + \mathbf{T}$ and $\mathbf{ST}$ are R-bounded, (ii) Let I be the identity operator on X. Then each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R- bounded whenever $\Omega \subset \mathbb{C}$ is bounded.

The purpose of this section is to show the following result on the L^p-maximal regularity for equation (0.1).

Theorem 4.3. *Let X be a UMD-space and $1 < p < \infty$. Assume that A generates a bounded analytic semigroup. Suppose that $\{\frac{i\rho+1}{(i\rho)^\alpha}\}_{\rho \in \mathbb{R}\setminus\{0\}} \subset \rho(A)$ and the set*

$$\Big\{ \frac{i\rho+1}{(i\rho)^\alpha} \Big(\frac{i\rho+1}{(i\rho)^\alpha} - A \Big)^{-1} \Big\}_{\rho \in \mathbb{R}\setminus\{0\}} \tag{4.1}$$

is R-bounded. Then equation (0.1) has L^p-maximal regularity.

Proof. First we set $M(\rho) := \frac{i\rho+1}{(i\rho)^\alpha} \big(\frac{i\rho+1}{(i\rho)^\alpha} - A \big)^{-1}$ and let T_M be defined by

$$T_M f := \mathcal{F}^{-1} M(\cdot) \mathcal{F} f, \quad t \in \mathbb{R} \tag{4.2}$$

for all $\mathcal{F}f \in \mathcal{D}(\mathbb{R}; X)$, where $\mathcal{F}$ denotes the Fourier transform.

Since the set $\{M(\rho)\}_{\rho \in \mathbb{R}\setminus\{0\}}$ is R-bounded by hypothesis, then we claim that the set $\{\rho M'(\rho)\}_{\rho \in \mathbb{R}\setminus\{0\}}$ is R-bounded. In fact, we have

$$\rho M'(\rho) = -\rho \frac{\hat{a}'(i\rho)}{\hat{a}(i\rho)} M(\rho) + \rho \frac{\hat{a}'(i\rho)}{\hat{a}(i\rho)} M(\rho)^2 \tag{4.3}$$

where

$$g(i\rho) := i\rho\frac{\hat{a}'(i\rho)}{\hat{a}(i\rho)} = \frac{(\alpha-1)i\rho}{i\rho+1} - \frac{\alpha}{i\rho+1} \tag{4.4}$$

and hence the claim follows by Remark 4.2. Then [3, Theorem 3.19] implies that $T_M \in \mathcal{B}(L^p(\mathbb{R};X))$. Now for $f \in L^p(\mathbb{R};X)$ we define

$$(Lf)(t) := u'(t) + u(t) = \int_0^t R'(t-s)f(s)ds + \int_0^t R(t-s)f(s)ds + f(t), \tag{4.5}$$

and note that

$$\widehat{(Lf)}(\rho) = i\rho\hat{R}(i\rho)\hat{f}(\rho) + \hat{R}(i\rho)\hat{f}(\rho) = (i\rho+1)\hat{R}(i\rho)\hat{f}(\rho) = M(\rho)\hat{f}(\rho) \tag{4.6}$$

Hence, by uniqueness of the Fourier transform, we have $L = T_M \in \mathcal{B}(L^p(\mathbb{R};X))$. By Theorem 3.5 we conclude from (4.5) that $AD_t^\alpha u = u' + u - f = Lf - f \in L^p(\mathbb{R};X)$ for each $f \in L^p(\mathbb{R}_+;X)$.

□

It is well known that Hilbert spaces are UMD spaces. Furthermore, in Hilbert spaces the notion of R-boundedness and boundedness are equivalent. With these remarks under consideration we obtain the following.

Corollary 4.4. *Suppose that A generates a bounded analytic semigroup on a Hilbert space H. Assume that $\{\frac{i\rho+1}{(i\rho)^\alpha}\}_{\rho\in\mathbb{R}\setminus\{0\}} \subset \rho(A)$ and*

$$\sup_{|\rho|>0} ||\frac{i\rho+1}{(i\rho)^\alpha}\Big(\frac{i\rho+1}{(i\rho)^\alpha} - A\Big)^{-1}|| < \infty. \tag{4.7}$$

Then equation (0.1) has L^p-maximal regularity.

Remark 4.5. Since $Re\Big(\frac{i\rho+1}{(i\rho)^\alpha}\Big) = \rho^{1-\alpha}cos((1-\alpha)\pi/2) + \rho^{-\alpha}cos(\alpha\pi/2) > 0$ for $\rho > 0$ and $0 < \alpha < 1$. Hence for $\rho < 0$ the complex number $\frac{i\rho+1}{(i\rho)^\alpha}$, has negative real part. Consequently, in spite of the fact that A generates a bounded analytic semigroup this is not sufficient to imply condition (4.7), of the above corollary.

5. Integrability

The integrability property for families of bounded linear operators is directly related with stability properties of linear evolution equations, in particular Volterra integral equations.

Next, as a natural extension of [13, Definition 10.2 (i)] we introduce the following definition of strongly L^p- integrable family.

Definition 5.1. *Let $1 \le p < \infty$ and $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ be a strongly measurable family of operators. Then $\{S(t)\}_{t\ge 0}$ is called strongly L^p- integrable if $S(\cdot)x \in L^p(\mathbb{R}_+;X)$ for each $x \in X$.*

Theorem 5.2. *Let X be a UMD-space and let A be the generator of a bounded analytic semigroup. Assume that $\{\frac{i\rho+1}{(i\rho)^\alpha}\}_{\rho\in\mathbb{R}\setminus\{0\}} \subset \rho(A)$ and*

$$\Big\{\frac{i\rho+1}{(i\rho)^\alpha}\Big(\frac{i\rho+1}{(i\rho)^\alpha} - A\Big)^{-1}\Big\}_{\rho\in\mathbb{R}\setminus\{0\}} \tag{5.1}$$

is R-bounded. Then $\{R(t)\}_{t\ge 0}$ is strongly L^p- integrable for all $1 < p < \infty$.

Proof. As in the proof of Theorem 4.3 we set $M(\rho) := \frac{i\rho+1}{(i\rho)^\alpha}\big(\frac{i\rho+1}{(i\rho)^\alpha} - A\big)^{-1}$ and let T_M be defined by

$$T_M f := \mathcal{F}^{-1} M(\cdot)\mathcal{F}f, \quad t \in \mathbb{R} \tag{5.2}$$

for all $\mathcal{F}f \in \mathcal{D}(\mathbb{R}; X)$, where $\mathcal{F}$ denotes the Fourier transform. Then we know that $T_M \in \mathcal{B}(L^p(\mathbb{R}; X))$. Define for $x \in X$ the function $f_x(t) = e^{-t}x$. Now, by (3.1) we notice that

$$\mathcal{F}(R)(\rho) = \frac{1}{(i\rho)^\alpha}\big(\frac{i\rho+1}{(i\rho)^\alpha} - A\big)^{-1}x = \frac{i\rho+1}{(i\rho)^\alpha}\big(\frac{i\rho+1}{(i\rho)^\alpha} - A\big)^{-1}(\mathcal{F}f_x)(\rho). \tag{5.3}$$

Therefore, by uniqueness $(T_M f_x)(t) = R(t)x$ and since $f_x \in L^p(\mathbb{R}_+; X)$, we get

$$\int_0^\infty ||R(t)x||^p dt = ||T_M f_x||_p^p \leq C||f_x||_p^p.$$

Hence $R(\cdot)x \in L^p(\mathbb{R}_+; X)$ for all $x \in X$. □

Remark 5.3. *The above result should be compared with* [13, Theorem 10.5] *where the strong integrability for resolvent families was obtained only in the setting of Hilbert spaces.*

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