UNIFORM STABILITY OF RESOLVENT FAMILIES

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Abstract. In this article we study uniform stability of resolvent families associated to an integral equation of convolution type. We give sufficient conditions for the uniform stability of the resolvent family in Hilbert and Banach spaces. Our main result can be viewed as a substantial generalization of the Gearhart - Greiner - Prüss 's characterization of exponential stability for strongly continuous semigroups.

1. Introduction

Let X be a Banach space. In this paper we are concerned with the study of the asymptotic behavior for the following integral Volterra equation of scalar type

$$u(t)=\int_0^t a(t-s)Au(s)ds+f(t),\ t\geq 0, \tag{1}$$

where A is a closed and linear operator with domain $D(A)$ dense in X, $a\in L^1_{loc}(\mathbb{R}_+)$ is a scalar kernel and $f\in W^{1,1}(\mathbb{R}_+,X)$.
Equation (1) has been extensively studied in the last years, mainly because of its applications in the theory of linear viscoelasticity. See, for instance, the monograph [5].
It is well known that equation (1) is well-posed if, and only if (1) admits a *resolvent family*, that is, there is a strongly continuous family $S(t)$, $t\geq 0$ of bounded and linear operators defined in X, which commutes with A and satisfies the resolvent equation

$$S(t)x=x+\int_0^t a(t-s)AS(s)xds,\ t\geq 0,\ x\in D(A). \tag{2}$$

In particular, the resolvent family for (1) in the case $a(t)\equiv 1$ correspond to the C_0-semigroup generated by A.
Due to the special feature of a convolution in (1), it is appropriated to employ the Laplace transform for its study. Formally, the Laplace transform $H(\lambda)=\hat{S}(\lambda)$ of the resolvent family is represented by

$$H(\lambda)=(\lambda-\lambda\hat{a}(\lambda)A)^{-1}. \tag{3}$$

The problem to find necessary and sufficient conditions for the stability of resolvent families, in general, is difficult to handle, essentially due to the complex structure of (3). In 1992, Arendt and Prüss [1] studied the existence of $\lim_{t\to\infty}S(t)=P$ in various senses. In particular, they gave sufficient conditions for the strong stability

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of $S(t)$. This result extends to the well-known ABLP-theorem on stability for C_0-semigroups (see [4]).
To our knowledge, there is no further study on the stability of resolvent. Our purpose in this article is to give some advances in the analysis of sufficient conditions in terms of (3) for the uniform stability of resolvent families in Hilbert and Banach spaces. The arguments used in [5], Theorem 10.2 and Theorem 10.5 to prove integrability give tools to prove two general results on uniform stability of resolvents. The first one (Theorem 1) concerning the hyperbolic case and valid in Hilbert spaces, generalizes the Gearhart-Greiner-Prüss characterization of exponential stability for strongly continuous semigroups. The second one (Theorem 2), is concerned with the parabolic case and is true in general Banach spaces.

Our basic assumption to get the above mentioned results is only 1-regularity of the kernel $a(t)$. We remark that in general it is more difficult to prove integrability properties than stability for resolvents. However, the kernel $a(t)$ need not to have the property that $1/\lambda\hat{a}(\lambda)$ is locally analytic and therefore the results in [5], Theorems 10.2 and Theorem 10.5 cannot be applied. We show in this paper that this condition is not needed for stability of resolvents.

Some immediate consequences of our results in the study of the asymptotic behavior of (1) can be obtained; For instance, suppose $f \in W^{1,1}(\mathbb{R}_+; X)$. Then, by the variation of parameters formula, we can conclude that $\|u(t)\| \to 0$ as $t \to \infty$. Another application is the connection between the solutions of (1) and the solutions of the equation on the line

$$v(t) = \int_0^\infty a(s)Av(t-s)ds + g(s)\,,\, t \in \mathbb{R}, \tag{4}$$

where $g \in W^{1,1}(\mathbb{R}; X)$. If (1) admits a resolvent $S(t)$ which is uniformly stable and uniformly integrable (see Corollary 3). The solution of (4) is given by

$$v(t) = \int_0^\infty S(t)\dot{g}(t-s)ds\,,\, t \in \mathbb{R}.$$

Assume that $\|\dot{f}(t) - \dot{g}(t)\| \to 0$ as $t \to +\infty$. From the variation of parameters formula we have

$$u(t) - v(t) = S(t)f(0) + \int_0^t S(\tau)[\dot{f}(t-\tau) - \dot{g}(t-\tau)]d\tau - \int_t^\infty S(\tau)\dot{g}(t-\tau)d\tau.$$

Hence $\|u(t) - v(t)\| \to 0$ as $t \to \infty$. This shows that the solutions $u(t)$ of (1) and $v(t)$ of (4) are asymptotic to each other as $t \to \infty$.

2. Conditions For Uniform Stability

Recall that a resolvent family $\{S(t)\}_{t\geq 0}$ defined in a Banach space X is called *uniformly stable* if

$$\lim_{t\to+\infty} \|S(t)\| = 0.$$

In the following, we will suppose that $a \in L^1_{loc}(\mathbb{R}_+)$ is of subexponential growth, that is $\int_0^\infty e^{-\varepsilon t} \mid a(t) \mid dt < \infty$, for all $\varepsilon > 0$. In this situation, the Laplace transform, $\hat{a}(\lambda)$, exist for all $Re\lambda > 0$.
Also we recall that $a(t)$ is called k-regular ($k \in \mathbb{N}$), if there is a constant $c > 0$ such that

$$\mid \lambda^n \hat{a}^{(n)}(\lambda) \mid \leq c \mid \hat{a}(\lambda) \mid$$

for all $Re\lambda > 0$ and $1 \leq n \leq k$.
Observe that if $a \in L^1_{loc}(\mathbb{R}_+)$ is of subexponential growth and 1-regular, then $\hat{a}(is) := \lim_{\lambda\to is} \hat{a}(\lambda)$ exist for all $s \neq 0$. Moreover, $\hat{a}(\lambda) \neq 0$ for $Re\lambda \geq 0$, $\lambda \neq 0$ (see [5], Lemma 8.1).

Theorem 1. *Suppose $a(t) \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular; assume that (1) admits a resolvent $S(t)$ with finite growth bound $\omega_0(S) < \infty$ in a Hilbert space H, and the following conditions*

(H1) $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ *for all* $Re\lambda \geq 0$, $\lambda \neq 0$.
(H2) $\lambda\hat{a}(\lambda) \to a(\infty) \neq 0$ *as* $\lambda \to 0$ *and* $0 \in \rho(A)$.
(H3) $H(\lambda)$ *is uniformly bounded in* $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \,:\, Re\lambda > 0\}$.

Then $S(t)$ is uniformly stable.

Proof. By hypothesis there are constants $M > 0$ and $\omega_0 \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega_0 t}$. We may suppose that $\omega_0 \geq 0$. Let $\omega > \omega_0 + 1$ be given and define $R(t) := e^{-\omega t}S(t)$, then $\|R(t)\| \leq Me^{-(\omega-\omega_0)t}$. Let $x \in H$ be fixed, and observe that $\chi_{[0,+\infty)}(\cdot)R(\cdot)x$ is in $L^2(\mathbb{R}; H)$, where $\chi_{[0,+\infty)}(\cdot)$ denotes the characteristic function. In fact,

$$\begin{aligned}\|\chi_{[0,+\infty)}(\cdot)R(\cdot)x\|_2^2 &= \int_0^\infty \|R(t)x\|^2 dt \leq M^2 \int_0^\infty (e^{-(\omega-\omega_0)t}\|x\|)^2 dt \\ &\leq \frac{M^2 \cdot \|x\|^2}{2(\omega-\omega_0)}\end{aligned}$$

hence, $\|\chi_{[0,+\infty)}(\cdot)R(\cdot)x\|_2 \leq \dfrac{M \cdot \|x\|}{\sqrt{2}\sqrt{\omega-\omega_0}}$.

Because H is a Hilbert space, the Plancherel theorem show us that the Fourier transform $\mathcal{F}$ satisfies $\|\mathcal{F}f\|_2 = \sqrt{2\pi}\|f\|_2$ for all $f \in L^2(\mathbb{R}, H)$. On the other hand, because $S(t)$ is an exponentially bounded resolvent for (1); its Laplace transform $\hat{S}(\lambda)$ is then well-defined, holomorphic and satisfies $H(\lambda) = \hat{S}(\lambda)$ for all $Re\lambda > 0$. Hence, we have for all $x \in H$ and $s \in \mathbb{R}$:

$$\begin{aligned}H(\omega+is)x &= \hat{S}(\omega+is)x = \int_0^\infty e^{-(\omega+is)t}S(t)x dt = \int_0^\infty e^{-ist}e^{-\omega t}S(t)x dt \\ &= \int_0^\infty e^{-ist}R(t)x dt = \int_{-\infty}^{+\infty} e^{-ist}\chi_{[0,+\infty)}(t)R(t)x dt \\ &= \mathcal{F}(\chi_{[0,+\infty)}(\cdot)R(\cdot)x)(s).\end{aligned}$$

It follows from Plancherel theorem that $H(\omega + i\cdot)x \in L^2(\mathbb{R}; H)$ and

$$\|H(\omega+i\cdot)x\|_2 = \sqrt{2\pi} \cdot \|\chi_{[0,+\infty)}(\cdot)R(\cdot)x\|_2 \leq M \cdot \sqrt{\frac{\pi}{\omega-\omega_0}} \cdot \|x\|. \tag{5}$$

Writing $H(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1} = \frac{1}{\lambda\hat{a}(\lambda)}(\frac{\lambda}{\lambda\hat{a}(\lambda)} - A)^{-1}$ we observe that $\lim_{\lambda\to 0} H(\lambda)$ exist in $B(H)$ due to $(H1)$ and $(H2)$. Hence, from 1-regularity of $a(t)$, conditions (H1) and (H3) and the Banach-Steinhaus theorem we obtain that $H(i\rho)$ is bounded, for each $\rho \in \mathbb{R}$. It follows from the uniform boundedness principle that $H(\lambda)$ is also uniformly bounded in the imaginary axis $i\mathbb{R}$.

On the other hand, from the identity

$$\lambda H(\lambda) - \lambda\hat{a}(\lambda)AH(\lambda) = I$$

valid for all $Re\lambda \geq 0$, $\lambda \neq 0$, we obtain

$$H(i\rho)x = H(\omega+i\rho)x + \frac{\omega}{\omega+i\rho}H(i\rho)x + h(\rho)H(i\rho)(H(\omega+i\rho)x - \frac{x}{\omega+i\rho}) \tag{6}$$

for all $\rho \neq 0$, where $h(\rho) = i\rho(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)} - 1)$.
It follows from the proof of Theorem 10.5 in [5] that $h(\rho)$ is bounded for $|\rho| \geq 1$.

Choose a function $\varphi(\rho)$ in $C_0^\infty(\mathbb{R})$, defined by $\varphi(\rho) = 1$ for $|\rho| < 1$ and, $\varphi(\rho) = 0$ for $|\rho| \geq 2$. Define $\psi(\rho) = 1 - \varphi(\rho)$ for all $\rho \in \mathbb{R}$, then using the uniform boundedness of $H(i\cdot)$ in $\mathbb{R}$ and (5) in (6), we conclude that $\psi(\cdot)H(i\cdot)x \in L^2(\mathbb{R}; H)$ and,

$$\begin{aligned}
\|\psi(\cdot)H(i\cdot)x\|_2^2 &= \int_{-\infty}^{+\infty} \|\psi(\rho)H(i\rho)x\|^2 d\rho \\
&= \int_{|\rho|\geq 2} \|H(i\rho)x\|^2 d\rho + \int_{1\leq|\rho|\leq 2} \|\psi(\rho)H(i\rho)x\|^2 d\rho \\
&\leq M_0 \cdot \|x\|^2.
\end{aligned}$$

Analogously, we can prove that $H(\omega + i\cdot)^*x \in L^2(\mathbb{R}; H)$ and following the same argument as above we conclude that $\psi(\cdot)H(i\cdot)^*x \in L^2(\mathbb{R}, H)$.
By Parseval's theorem, there exist a function $u \in L^2(\mathbb{R}; H)$ such that

$$\mathcal{F}(u(\cdot))(\rho) = \psi(\rho)H(i\rho)x \text{ for a.a. } \rho \in \mathbb{R}.$$

It follows that

$$\begin{aligned}
\mathcal{F}(u(\cdot))'(\rho) &= \psi'(\rho)H(i\rho)x + i\psi(\rho)H'(i\rho)x \\
&= \psi'(\rho)H(i\rho)x + i\psi(\rho)(-\tfrac{H(i\rho)}{i\rho}x + H(i\rho)i\rho\tfrac{\hat{a}'(i\rho)}{\hat{a}(i\rho)}(H(i\rho)x - \tfrac{x}{i\rho}))
\end{aligned}$$

hence, by 1-regularity of $a(t)$ and the fact that $\psi(\cdot)H(i\cdot)x$, $\psi(\cdot)H(i\cdot)^*x^*$ are in $L^2(\mathbb{R}; H)$ for each $x, x^* \in H$, we get

$$\int_{-\infty}^{+\infty} |< \mathcal{F}(u(\cdot))'(\rho), x^* >| \, d\rho \leq M_0 \cdot \|x\| \cdot \|x^*\|. \tag{7}$$

On the other hand, again from the uniform boundedness of $H(i\cdot)$ in $\mathbb{R}$ we have that for each $t > 0$

$$\begin{aligned}
S_0(t) :&= \int_{-\infty}^{+\infty} \varphi(\rho)H(i\rho)e^{i\rho t}d\rho \\
&= \int_{-2}^{2} \varphi(\rho)H(i\rho)e^{i\rho t}d\rho \in B(H).
\end{aligned} \tag{8}$$

Hence, by Riemman-Lebesgue lemma follows that $S_0(t) \to 0$ in $B(H)$ as $t \to +\infty$.

Finally, for $x,\ x^* \in H$ we have that

$$
\begin{aligned}
< S(t)x, x^* > &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} < H(i\rho)x, x^* > e^{i\rho t} d\rho \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} < \varphi(\rho)H(i\rho)x, x^* > e^{i\rho t} d\rho \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty} < \psi(\rho)H(i\rho)x, x^* > e^{i\rho t} d\rho.
\end{aligned}
$$

Integrating by parts in the second integral, we get

$$
\begin{aligned}
< S(t)x, x^* > &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} < \varphi(\rho)H(i\rho)x, x^* > e^{i\rho t} d\rho \\
&\quad + \frac{1}{2\pi it}\int_{-\infty}^{+\infty} < (\psi(\rho)H(i\rho))' x, x^* > e^{i\rho t} d\rho \\
&= \frac{1}{2\pi} < S_0(t)x, x^* > + \frac{1}{2\pi it}\int_{-\infty}^{+\infty} < \mathcal{F}(u(\cdot))'(\rho), x^* > e^{i\rho t} d\rho.
\end{aligned}
$$

It follows from (7) and (8) that

$$
|< S(t)x, x^* >| \leq \frac{1}{2\pi}\|S_0(t)\| \cdot \|x\| \cdot \|x^*\| + \frac{1}{2\pi t} M_0 \cdot \|x\| \cdot \|x^*\|.
$$

Therefore, $\|S(t)\| \leq \frac{1}{2\pi}\|S_0(t)\| + \frac{1}{2\pi t}M_0$, from which we obtain the result.

Corollary 1. *Suppose that the equation $u = a * Au + f$, where $a(t)$ is 1-regular admits a resolvent $S(t)$ with finite growth bound $\omega_0(S) < \infty$ in a Hilbert space H. If $S(t)$ is strongly integrable then $S(t)$ is uniformly stable.*

Proof. Follows from Theorem 1 and Theorem 10.5 in [5].

The special case $a(t) \equiv 1$ give us with the following result on stability of C_0-semigroups due to Gearhart, Greiner and Prüss (see [2], Theorem 1.11).

Corollary 2. *Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ with finite growth bound $\omega_0(S) < \infty$ defined in a Hilbert space H. The following conditions are equivalent.*

(a) *The semigroup $T(t)$ is uniformly stable.*
(b) $\{\lambda \in \mathbb{C} : Re\lambda \geq 0\} \subset \rho(A)$ *and* $\sup_{Re\lambda>0}\|R(\lambda; A)\| < \infty$.

Proof. Assume (b). Is easy to see that $a(t) \equiv 1$ is 1-regular and the hypothesis (H1)-(H3) are clearly satisfied. Hence (a) follows by Theorem 1.

Assume (a). Since the semigroup is uniformly stable, we conclude from ([2], Proposition 1.2) that it is also exponentially stable, that is, there is a constant $\omega_0 > 0$ such that $\|S(t)\| \leq Me^{-\omega_0 t}$. Hence $\{\lambda \in \mathbb{C} : Re\lambda \geq 0\} \subseteq \rho(A)$ and $\|R(\tau + is; A)\| \leq \frac{M}{\tau+\omega_0} < \frac{M}{\omega_0}$ for all $\tau \geq 0$ and $s \in \mathbb{R}$ from which we obtain (b).

We recall the following:

Definition 1. *Equation (1) is called* parabolic *if the following conditions are satisfied.*

(P1) $\hat{a}(\lambda) \neq 0$, $1/\hat{a}(\lambda) \in \rho(A)$ *for all* $Re\lambda > 0$.
(P2) *There exists a constant* $M \geq 1$ *such that*

$$\|H(\lambda)\| \leq \frac{M}{|\,\lambda\,|} \text{ for all } Re\lambda > 0. \tag{9}$$

Theorem 2. *Let* X *be a Banach space. Suppose that* $a(t)$ *is 1-regular and (H2) of Theorem 1 holds. If (1) is parabolic then (1) admits a resolvent* $S(t)$ *uniformly stable.*

Proof. From parabolicity and 1-regularity of $a(t)$ we have by Theorem 3.1 in [5] the existence of a resolvent family $S(t)$, continuous and bounded for $t \geq 0$. Hence $S(\cdot) \in L^1_{loc}(\mathbb{R}_+, B(X))$ and moreover by the identity $\lambda H(\lambda) - \lambda\hat{a}(\lambda)AH(\lambda) = I$, for $Re\lambda > 0$, we have that

$$\begin{aligned} H'(\lambda) &= -\tfrac{H(\lambda)}{\lambda} + \hat{a}'(\lambda)A(I - \hat{a}(\lambda)A)^{-1}H(\lambda) \\ &= -\tfrac{H(\lambda)}{\lambda} + \tfrac{\hat{a}'(\lambda)}{\hat{a}(\lambda)} \cdot \hat{a}(\lambda)A \cdot \lambda H(\lambda) \cdot H(\lambda) \\ &= -\tfrac{H(\lambda)}{\lambda} + \tfrac{\hat{a}'(\lambda)}{\hat{a}(\lambda)} \cdot (\lambda H(\lambda) - I) \cdot H(\lambda). \end{aligned}$$

It follows from (9) and 1-regularity of $a(t)$, that $\|\lambda H'(\lambda)\| \leq M_1\|H(\lambda)\|$, for $Re\lambda > 0$. So, we are in the conditions of Lemma 8.1 in [5]. It follows that $H(\lambda)$ admits a $B(X)$-continuous extension to $\overline{\mathbb{C}}_+ \setminus \{0\}$. We will prove that $H(\lambda)$ is, in addition, continuous at $\lambda = 0$. Indeed, from (H2) and $0 \in \rho(A)$ we have that

$$H(0) = \lim_{\lambda\to 0} H(\lambda) = \lim_{\lambda\to 0} \frac{1}{\lambda\hat{a}(\lambda)}\left(\frac{\lambda}{\lambda\hat{a}(\lambda)} - A\right)^{-1} = \frac{-A^{-1}}{a(\infty)}$$

exist in $B(X)$. Hence $\|H(\lambda)\| \leq \frac{M_2}{1+|\lambda|}$ for $Re\lambda \geq 0$. It follows that $H(\lambda)$ is uniformly bounded on $\overline{\mathbb{C}}_+$. Let $x \in X$ and $x^* \in X^*$ be fixed, we observe that $< H(\omega + i\cdot)x, x^* > \in L^2(\mathbb{R})$ for all $\omega \geq 0$. It follows from Lemma 6.1 in [3] that $< S(t)x, x^* > \in L^2(\mathbb{R}_+)$, therefore by L^2-theory we have the next representation

$$< S(t)x, x^* > = \frac{1}{2\pi}\int_{-\infty}^{\infty} < H(i\rho)x, x^* > e^{i\rho t}d\rho\ , \quad \text{a.a. } t \in \mathbb{R}_+. \tag{10}$$

Now we consider $N \in \mathbb{N}$ fixed and $\varphi \in C_0^\infty(\mathbb{R})$ defined by $\varphi(\rho) = 1$ if $|\,\rho\,| \leq N$, 0 if $|\,\rho\,| \geq N+2$ and $0 \leq \varphi \leq 1$ in another case. Then by (10) we have that

$$\begin{aligned} < S(t)x, x^* > &= \frac{1}{2\pi}\int_{-\infty}^{\infty} < \varphi(\rho)H(i\rho)x, x^* > e^{i\rho t}d\rho \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} < (1-\varphi(\rho))H(i\rho)x, x^* > e^{i\rho t}d\rho. \end{aligned} \tag{11}$$

Hence integrating by parts in the second integral in (11) becomes

$$\begin{aligned} < S(t)x, x^* > &= \frac{1}{2\pi}\int_{-\infty}^{\infty} < \varphi(\rho)H(i\rho)x, x^* > e^{i\rho t}d\rho \\ &\quad + \frac{1}{2\pi it}\int_{-\infty}^{\infty} < [(1-\varphi(\rho))H(i\rho)]'x, x^* > e^{i\rho t}d\rho. \end{aligned} \tag{12}$$

It follows from boundedness of $H(\lambda)$ on $\overline{\mathbb{C}}_+$ that the first integral is absolutely integrable. For the second integral, using the estimate $\|\rho H'(i\rho)\| \leq M_2\|H(i\rho)\|$ a.a. $\rho \in \mathbb{R}$ and, the Cauchy-Schwarz and Hölder inequalities, it follows that it is also absolutely integrable. Therefore, x and x^* can be dropped in (12) and, $S(t)$ can be written as

$$\begin{aligned} S(t) = & \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\rho)H(i\rho)e^{i\rho t}d\rho + \frac{1}{2\pi it}\int_{-\infty}^{\infty}[(1-\varphi(\rho))H(i\rho)]'e^{i\rho t}d\rho \\ := & S_1(t)+S_2(t). \end{aligned} \tag{13}$$

It follows from the Riemman-Lebesgue lemma that $S_1(t) \to 0$ as $t \to +\infty$. On the other hand, $S_2(t)$ satisfies an estimate of the form $\|S_2(t)\| \leq \frac{M_2}{t}$. It follows that $S(t) \to 0$ as $t \to +\infty$.

As immediate consequence of Theorem 2 and Theorem 10.2 in [5], we obtain the following.

Corollary 3. *Suppose that $u = a * Au + f$ is parabolic in Banach space X, where $a(t)$ is 1-regular. If $S(t)$ is uniformly integrable then $S(t)$ is uniformly stable.*

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