ON THE INVERSION OF THE LAPLACE TRANSFORM FOR RESOLVENT FAMILIES IN UMD SPACES

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Abstract. We analize the inversion of the Laplace transform in UMD - spaces for resolvent families associated to an integral Volterra equation of convolution type.

1. Introduction

Let X be a complex Banach space. We consider the following Volterra equation of convolution type:

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \geq 0, \tag{1}$$

where A is a closed linear unbounded operator with domain $D(A)$ densely defined on X , $a \in L^1_{loc}(\mathbb{R}_+)$ is a scalar kernel, and $f \in L^1(\mathbb{R}_+; X)$. We recall from [12] that a family $\{S(t)\}_{t\geq 0}$ of bounded and linear operators defined in X is said to be a resolvent family for (1) if the following conditions are satisfied:

(R1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = I$;

(R2) $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(R3) The resolvent equation holds

$$S(t)x = x + \int_0^t a(t-s)S(s)Axds \tag{2}$$

for all $x \in D(A)$ and $t \geq 0$.

The notion of resolvent family is a natural extension of the concepts of a C_0-semigroup and a cosine operator function (obtained for $a(t) \equiv 1$ and $a(t) \equiv t$ respectively). The existence of a resolvent family allows one to find the solution for the equation (1). Several properties of resolvent families has been discussed in [8], [9],[11],[12],[4],[3].

In this paper we examine the convergence of the inverse Laplace transform for a resolvent family in a Banach space X.

In the sequel we always assume the existence of a resolvent $\{S(t)\}_{t\geq 0}$ for (1) which is in addition of type (M, ω_0), i.e. there are constants $M > 0$ and $\omega_0 \in \mathbb{R}_+$ such that

$$||S(t)|| \leq Me^{\omega_0 t} \text{ for all } t \geq 0.$$

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Also, we assume the existence of the Laplace transform of $a(t)$, denoted $\hat{a}(\lambda)$, for all $Re\lambda > \omega_0$. Under these conditions, the generation theorem for resolvent families (see [12], Theorem 1.3) give us the following

(H1) $\hat{a}(\lambda) \neq 0$ and $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $Re\lambda > \omega_0$;

(H2) $H(\lambda) := (\lambda - \lambda\hat{a}(\lambda)A)^{-1}$ satisfies

$$H(\lambda) = \hat{S}(\lambda) = \int_0^\infty e^{-\lambda t} S(t)dt, \text{ for all } Re\lambda > \omega_0 \tag{3}$$

Conversely, one may express the resolvent $\{S(t)\}_{t\geq 0}$ in terms of the Laplace transform $H(\lambda)$ by different formulas. For instance, by means of the Post-Widder inversion formula (see [10], Theorem 2.1) or by means of the complex inversion formula of the Laplace transform obtaining

$$S(t)x = \frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} e^{\lambda t} H(\lambda)x d\lambda \tag{4}$$

for all $t > 0, \omega > \omega_0$ and $x \in D(A)$ (see Proposition 2).

Our main result in this paper establish that in UMD spaces the formula (4) holds for all $x \in X$. We remark that in the particular case of $a(t) \equiv 1$ (i.e. $S(t)$ is a C_0-semigroup) we recover a result of A. Driouich and O. El-Mennaoui [6] Theorem 1 (see also [2] Theorem 3.12.2), in fact, the proof of our main result is very much inspired by the proof of Driouich and El-Mannaoui for the semigroup case. However, due to the more complicated structure of the Laplace transform $H(\lambda)$ the argument involved are more delicate and differ from those employed in [6]. We also observe that in the case of $a(t) \equiv t$, we recover the inversion formula for cosine operator functions in UMD spaces due to I. Cioranescu and V. Keyantuo [5]. Let us recall that a Banach space X is called UMD if the Hilbert transform $\mathcal{H}$ defined in the Schwartz space $\mathcal{S}(\mathbb{R}, X)$ by

$$\mathcal{H}f(t) := \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{|t-s| \geq \epsilon} \frac{f(s)}{t-s} ds$$

extends to a bounded linear operator on $L^p(\mathbb{R}; X)$ for some $p \in (1, \infty)$ (or, equivalently, for all $p \in (1, \infty)$, see [1]). Note, that every UMD space is reflexive and its dual is also a UMD space.

In the following, we will suppose that $a \in L^1_{loc}(\mathbb{R}_+)$ is of subexponential growth, that is $\int_0^\infty e^{-\epsilon t}|a(t)|dt < \infty$, for all $\epsilon > 0$. Under this condition, the Laplace transform, $\hat{a}(\lambda)$, exists for all $Re\lambda > 0$. Also we recall [12] that $a(t)$ is called k - *regular* ($k \in \mathbb{N}$), if there is a constant $c > 0$ such that

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \leq c|\hat{a}(\lambda)|$$

for all $Re\lambda > 0$ and $1 \leq n \leq k$. Observe that if $a \in L^1_{loc}(\mathbb{R}_+)$ is of subexponential growth and 1-regular, then $\hat{a}(is) := \lim_{\lambda \to is} \hat{a}(\lambda)$ exists for all $s \neq 0$. Moreover, $\hat{a}(\lambda) \neq 0$ for all $Re\lambda \geq 0,\ \lambda \neq 0$ (see [12], Lemma 8.1). We will need the following result from [11].

Lemma 1. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and suppose that*

$$H(\lambda) = (\lambda - \lambda \hat{a}(\lambda) A)^{-1}$$

exists for all $Re\lambda > 0$. Then,

$$H'(\lambda) = f(\lambda) H(\lambda) + g(\lambda) H(\lambda)^2,$$

where $f(\lambda) = -(\frac{1}{\lambda} + \frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)})$ and $g(\lambda) = \lambda \frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)}$ for all $Re\lambda > 0$.

With the notations of Lemma 1, we give the following

Lemma 2. *Let $a(t)$ be 3-regular; then there is a bounded function $b \in C^1(\mathbb{R}_+)$ such that $\hat{b}(\lambda) = f(\lambda)$ for all $Re\lambda > 0$.*

Proof. Since a is 1-regular we have $|f(\lambda)| \leq \frac{M}{|\lambda|}$ for all $Re\lambda > 0$. On the other hand $f'(\lambda) = -\left(-\frac{1}{\lambda^2} + \frac{\hat{a}''(\lambda)\hat{a}(\lambda) - \hat{a}'(\lambda)^2}{\hat{a}(\lambda)^2}\right)$ implies, $|f'(\lambda)| \leq \frac{M}{|\lambda|^2}$ for all $Re\lambda > 0$, because a is 2-regular. Now $f''(\lambda) = -\frac{2}{\lambda^3} - \frac{\hat{a}'''(\lambda)}{\hat{a}(\lambda)} + \frac{\hat{a}''(\lambda)}{\hat{a}(\lambda)} \cdot \frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)} - 2\frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)} \cdot \left(\frac{\hat{a}''(\lambda)\hat{a}(\lambda) - \hat{a}'(\lambda)^2}{\hat{a}(\lambda)^2}\right)$ and as a is 3-regular we obtain $|f''(\lambda)| \leq \frac{M}{|\lambda|^3}$ for all $Re\lambda > 0$. In short we have that $f(\lambda)$ satisfies

$$|\lambda^{n+1} f^{(n)}(\lambda)| \leq M \text{ for all } Re\lambda > 0 \text{ and } n = 0, 1, 2.$$

Hence, by Theorem 0.4 in [12] we obtain that there is a bounded function $b \in C^1(\mathbb{R}_+)$ such that $\hat{b}(\lambda) = f(\lambda)$ for all $Re\lambda > 0$.

□

We also need the following lemma. For details we refer to the monograph of J. Prüss ([12], Lemma 10.1).

Lemma 3. *Suppose c is a locally analytic function on $\mathbb{C}_+^\infty$. Then there is a function $k \in L^1(\mathbb{R}_+)$ such that*

$$c(t) = c(\infty) + \hat{k}(\lambda),$$

for all $\lambda \in \mathbb{C}_+^\infty$.

2. L^p - ESTIMATES FOR THE INVERSION OF THE LAPLACE TRANSFORM IN UMD SPACES

For $j = 1, 2$, let $\{F_j(t)\}_{t \geq 0} \subseteqq \mathbf{B}(X)$ be strongly continuous and of type (M, ω_0). Let

$$\hat{F}_j(z) = \int_0^\infty e^{-zt} F_j(t) dt, \quad Rez > \omega_0$$

be the Laplace transform. Then, for $\omega > \omega_0$, we have $\hat{F}_j(\omega + i\lambda) \in \mathbf{B}(X)$ and $\hat{F}_j(\omega + i\lambda)$ are strongly continuous for $\lambda \in \mathbb{R}$; moreover

$$\sup_{\lambda \in \mathbb{R}} ||\hat{F}_j(\omega + i\lambda)|| < \infty \text{ and } \lim_{|\lambda| \to \infty} \hat{F}_j(\omega + i\lambda) x = 0 \ \ (x \in X).$$

Denote for $t \in \mathbb{R}, r < r'$ and $x \in X$

$$I_1(t,r,r')x = \frac{1}{2\pi}\int_r^{r'} e^{i\lambda t}\hat{F}_1(\omega+i\lambda)xd\lambda,$$

$$I_2(t,r,r')x = \frac{1}{2\pi}\int_r^{r'} e^{i\lambda t}\hat{F}_1(\omega+i\lambda)\hat{F}_2(\omega+i\lambda)xd\lambda.$$

Lemma 4. *We have, for all $x \in X$:*

$$\begin{array}{llcl} (a) & I_2(t,r,r')x & = & \int_0^\infty I_1(t-s,r,r')e^{-\omega s}F_2(s)xds \\ (b) & I_1(t,r,r')x & = & \frac{1}{2i}e^{itr'}\mathcal{H}(e^{-ir'\cdot}e^{-\omega\cdot}\chi_{[0,\infty)}(\cdot)F_1(\cdot)x)(t) \\ & & - & \frac{1}{2i}e^{itr}\mathcal{H}(e^{-ir\cdot}e^{-\omega\cdot}\chi_{[0,\infty)}(\cdot)F_1(\cdot)x)(t) \end{array}$$

where $\mathcal{H}$ is the Hilbert transform and $\chi_{[0,\infty)}(\cdot)$ denotes the characteristic function.

Proof.

$$\begin{aligned} I_2(t,r,r')x &= \frac{1}{2\pi}\int_r^{r'} e^{i\lambda t}\hat{F}_1(\omega+i\lambda)[\int_0^\infty e^{-(\omega+i\lambda)s}F_2(s)xds]d\lambda \\ &= \frac{1}{2\pi}\int_0^\infty [\int_r^{r'} e^{i\lambda(t-s)}\hat{F}_1(\omega+i\lambda)d\lambda]e^{-\omega s}F_2(s)xds \\ &= \frac{1}{2\pi}\int_0^\infty I_1(t-s,r,r')e^{-\omega s}F_2(s)xds \end{aligned}$$

where we used the fact that the function $(s,\lambda) \to ||e^{i\lambda(t-s)}e^{-\omega s}\hat{F}_1(\omega + i\lambda)F_2(s)x||$ belongs to $L^1(\mathbb{R}_+ \times [r', r])$.

We shall prove further the second formula. Note first that the function $(s,\lambda) \to ||e^{i\lambda(t-s)}e^{-\omega s}F_1(s)x|| \in L^1(\mathbb{R}_+ \times [r', r])$, consequently we have:

$$\begin{aligned} I_1(t,r,r')x &= \frac{1}{2\pi}\int_r^{r'} e^{i\lambda t}[\int_0^\infty e^{-(\omega+i\lambda)s}F_1(s)xds]d\lambda \\ &= \frac{1}{2\pi}\int_0^\infty [\int_r^{r'} e^{i\lambda(t-s)}d\lambda]e^{-\omega s}F_1(s)xds \\ &= \frac{1}{2\pi i}p.v.\int_0^\infty \frac{e^{i(t-s)r'} - e^{i(t-s)r}}{t-s}e^{-\omega s}F_1(s)xds \\ &= \frac{e^{itr'}}{2\pi i}p.v.\int_0^\infty \frac{e^{-isr'}e^{-\omega s}}{t-s}F_1(s)xds - \frac{e^{itr}}{2\pi i}p.v.\int_0^\infty \frac{e^{-isr}e^{-\omega s}}{t-s}F_1(s)xds \\ &= \frac{e^{itr'}}{2i}\mathcal{H}(e^{-ir'\cdot}e^{-\omega\cdot}\chi_{[0,\infty)}(\cdot)F_1(\cdot)x)(t) \\ &- \frac{e^{itr}}{2i}\mathcal{H}(e^{-ir\cdot}e^{-\omega\cdot}\chi_{[0,\infty)}(\cdot)F_1(\cdot)x)(t) \end{aligned}$$

□

Proposition 1. *Let X be an UMD space and $p \in (1,\infty)$; then there exists a constant $C > 0$ such that*

$$(c) \qquad ||I_1(\cdot, r, r')x||_{L^p(\mathbb{R},X)} \leq C||x||,$$

$$(d) \qquad ||I_2(\cdot, r, r')x||_{L^\infty(\mathbb{R},X)} \leq C||x||,$$

for all $x \in X$ and $r < r'$.

Proof. Using (b) we obtain

$$\begin{aligned} ||2I_1(\cdot,r,r')x||_{L^p(\mathbb{R},X)} &\leq ||\mathcal{H}(e^{-ir'\cdot}e^{-\omega\cdot}\chi_{[0,\infty)}(\cdot)F_1(\cdot)x||_{L^p(\mathbb{R},X)} \\ &+ ||\mathcal{H}(e^{-ir\cdot}e^{-\omega\cdot}\chi_{[0,\infty)}(\cdot)F_1(\cdot)x||_{L^p(\mathbb{R},X)} \\ &\leq 2C_1||e^{-\omega\cdot}\chi_{[0,\infty)}(\cdot)F_1(\cdot)x||_{L^p(\mathbb{R},X)} \\ &\leq 2C_1M(\frac{1}{(\omega-\omega_0)p})^{1/p}||x||. \end{aligned}$$

where C_1 is the norm of the Hilbert transform in $L^p(\mathbb{R},X)$. Consider further $x \in X$ and $x^* \in X^*$; we have by (a)

$$\begin{aligned} |< x^*, I_2(t,r,r')x >| &= |\int_0^\infty < x^*, I_1(t-s,r,r')e^{-\omega s}F_2(s)x > ds| \\ &= |\int_0^\infty < I_1^*(t-s,r,r')x^*, e^{-\omega s}F_2(s)x > ds| \\ &\leq \int_0^\infty ||e^{-\omega s}F_2(s)x||||I_1^*(t-s,r,r')x^*||ds \\ &\leq ||I_1^*(\cdot,r,r')x^*||_{L^p(\mathbb{R},X)}||e^{-\omega\cdot}\chi_{[0,\infty)}(\cdot)F_2(\cdot)x||_{L^q(\mathbb{R},X)} \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Since X^* is also a UMD space, we can use (c) to estimate $||I_1^*(\cdot,r,r')x^*||$ and obtain that there exists a constant $C_2 > 0$ such that

$$|< x^*, I_2(t,r,r')x >| \leq C_2||x^*||||x||$$

and thus (d) holds.

□

3. Main result

We start our considerations with the following inversion result in general Banach spaces.

Lemma 5. *Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous family of type (M,ω_0) and let $b \in C^1(\mathbb{R}_+)$ be of type (K,ω_0); then*

$$(b*S)(t)x = \lim_{r'\to\infty}\lim_{r\to-\infty}\frac{1}{2\pi}\int_r^{r'} e^{(\omega+i\lambda)t}\widehat{(b*S)}(\omega+i\lambda)xd\lambda,$$

for each $\omega > \omega_0$ and all $x \in X$. Moreover, the convergence is uniform on t for any compact interval of $(0,\infty)$

Proof. By Theorem 6.3.1 in [7] we have for all $x \in X$ and $\omega > \omega_0$

$$(5) \qquad \int_0^t (b'*S)(s)xds = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\widehat{(b'*S)}(\lambda)x\frac{d\lambda}{\lambda}$$

and

$$\int_0^t S(s)x ds = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\hat{S}(\lambda)x\frac{d\lambda}{\lambda} \tag{6}$$

where the integrals are convergent uniformly with respect to t in any compact interval of $(0,\infty)$. We have:

$$(b' * S)(s)x = (b * S)'(s)x - b(0)S(s)x, \quad s \in \mathbb{R}_+$$

and

$$\widehat{(b' * S)}(\lambda)x = \lambda\widehat{(b * S)}(\lambda)x - b(0)\hat{S}(\lambda)x, \quad \lambda > \omega_0$$

so that (5) yields:

$$\begin{aligned}(b * S)(t)x - b(0)\int_0^t S(s)x ds &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\widehat{(b * S)}(\lambda)x d\lambda \\ &- b(0)\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\hat{S}(\lambda)x\frac{d\lambda}{\lambda}.\end{aligned}$$

Using (6) in the above equality we obtain

$$(b * S)(t)x = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\widehat{(b * S)}(\lambda)x d\lambda$$

where the convergence of the integral is uniform with respect to t in any compact interval of $(0,\infty)$.

□

Proposition 2. *Let* $\{S(t)\}_{t\geq 0}$ *be a resolvent family of type* (M,ω_0) *for the equation (1) and let* $a \in C^1(\mathbb{R}_+)$ *be of type* (K,ω_0)*; then for each* $x \in D(A)$ *and* $\omega > \omega_0$ *we have*

$$S(t)x = \lim_{r\to-\infty\ r'\to\infty}\frac{1}{2\pi}\int_r^{r'} e^{(\omega+i\lambda)t}H(\omega+i\lambda)x d\lambda.$$

where the convergence is uniform on t *for any compact interval of* $(0,\infty)$.

Proof. For each $x \in D(A)$ we have by Lemma 5 and (2)

$$\begin{aligned}(a * S)(t)Ax &= \lim_{r\to-\infty\ r'\to\infty}\frac{1}{2\pi}\int_r^{r'} e^{(\omega+i\lambda)t}\widehat{(a * S)}(\omega+i\lambda)Ax d\lambda \\ &= \lim_{r\to-\infty\ r'\to\infty}\frac{1}{2\pi}\int_r^{r'} e^{(\omega+i\lambda)t}(\hat{S}(\omega+i\lambda)x - \frac{x}{\omega+i\lambda})d\lambda \\ &= \lim_{r\to-\infty\ r'\to\infty}\frac{1}{2\pi}\int_r^{r'} e^{(\omega+i\lambda)t}H(\omega+i\lambda)x d\lambda \\ &- \lim_{r\to-\infty\ r'\to\infty}\frac{1}{2\pi}\int_r^{r'} e^{(\omega+i\lambda)t}\frac{x}{\omega+i\lambda}d\lambda \\ &= \lim_{r\to-\infty\ r'\to\infty}\frac{1}{2\pi}\int_r^{r'} e^{(\omega+i\lambda)t}H(\omega+i\lambda)x d\lambda - x\end{aligned}$$

Hence, the resolvent equation (2) implies the assertion.

□

We can give now our main result.

Theorem 1. *Let $\{S(t)\}_{t\geq 0}$ be a resolvent family of type (M,ω_0) for the equation (1), defined in a UMD space X; suppose that $a(t)$ is 3-regular, $\lambda\frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)}$ is locally analytic, and $|\hat{a}(\lambda)|\leq \frac{C}{|\lambda|}$ for all $|\lambda|>1$, then we have for all $x\in X$,*

$$S(t)x=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(\omega+i\lambda)t}H(\omega+i\lambda)xd\lambda,\quad t>0,\ \ \omega>\omega_0$$

where the convergence is uniform for t in compact intervals of $(0,\infty)$.

Proof. Let $r<0<r', t>0, x\in X$ and consider

$$I(t,r,r')x=\frac{1}{2\pi}\int_r^{r'}e^{i\lambda t}\hat{S}(\omega+i\lambda)xd\lambda.$$

An integration by parts yields:

$$\begin{aligned}I(t,r,r')x &= \frac{1}{2i\pi t}(e^{itr'}\hat{S}(\omega+ir')x-e^{itr}\hat{S}(\omega+ir)x)\\ &- \frac{1}{2i\pi t}\int_r^{r'}e^{i\lambda t}\hat{S}(\omega+i\lambda)'xd\lambda.\end{aligned}$$

Since $\lim\limits_{r\to-\infty}\hat{S}(\omega+ir)x=\lim\limits_{r'\to\infty}\hat{S}(\omega+ir')x=0$, in order to prove that $\lim I(t,r,r')$ exists as $r'\to\infty$ and $r\to-\infty$, we only have to prove that $\lim\int_r^{r'}e^{i\lambda t}\hat{S}(\omega+i\lambda)'xd\lambda$ exists as $r'\to\infty$ and $r\to-\infty$. Since a is 3-regular we have by Lemma 1

$$\hat{S}(\omega+i\lambda)'=f(\omega+i\lambda)\hat{S}(\omega+i\lambda)+g(\omega+i\lambda)\hat{S}(\omega+i\lambda)^2$$

with $\sup\limits_{Rez>0}|zf(z)|<\infty$ and $\sup\limits_{Rez>0}|g(z)|<\infty$. Hence we have

$$\begin{aligned}\int_r^{r'}e^{i\lambda t}\hat{S}(\omega+i\lambda)'xd\lambda &= \int_r^{r'}e^{i\lambda t}f(\omega+i\lambda)\hat{S}(\omega+i\lambda)xd\lambda\\ &+ \int_r^{r'}e^{i\lambda t}g(\omega+i\lambda)\hat{S}(\omega+i\lambda)^2xd\lambda.\end{aligned}$$

Concerning the first integral in the second part of the above equality, we have by Lemma 2 that there exist a bounded function $b\in C^1(\mathbb{R}_+)$ such that $\hat{b}(\lambda)=\hat{f}(\lambda)$. Hence, by Lemma 5 the integral

$$\int_r^{r'}e^{(\omega+i\lambda)t}f(\omega+i\lambda)\hat{S}(\omega+i\lambda)xd\lambda = \int_r^{r'}e^{(\omega+i\lambda)t}\widehat{(b*S)}(\omega+i\lambda)xd\lambda$$

converges to $2\pi(b*S)(t)x$ as $r\to-\infty$ and $r'\to\infty$, uniformly for t in compact intervals of $(0,\infty)$. Next, we will prove the convergence of the integral

$$\int_r^{r'}e^{i\lambda t}g(\omega+i\lambda)\hat{S}(\omega+i\lambda)^2xd\lambda.$$

We shall consider first the case $x\in D(A^2)$. We have:

$$\hat{S}^2(\omega+i\lambda)x=\frac{x}{(\omega+i\lambda)^2}+2\frac{\hat{a}(\omega+i\lambda)}{\omega+i\lambda}\hat{S}(\omega+i\lambda)Ax+\hat{a}^2(\omega+i\lambda)\hat{S}(\omega+i\lambda)^2A^2x.$$

Using the hypothesis, it follows that there is a constant $C > 0$ such that

$$||g(\omega+i\lambda)\hat{S}^2(\omega+i\lambda)x|| \leq \frac{C}{|\omega+i\lambda|^2}(||x||+||Ax||+||A^2x||).$$

Consequently, the limit

$$\int_r^{r'} e^{i\lambda t} g(\omega+i\lambda)\hat{S}(\omega+i\lambda)^2 x d\lambda$$

exists as $r' \to \infty$ and $r \to -\infty$, for all $x \in D(A^2)$, uniformly for t in any compact interval of $(0,\infty)$.

Observe further that according the hypothesis and Lemma 3, there exists a function $k \in L^1(\mathbb{R}_+)$ such that

$$g(\omega+i\lambda)\hat{S}(\omega+i\lambda)^2 x = \hat{S}(\omega+i\lambda)[(\widehat{(k*S)}(\omega+i\lambda)x + g(\infty)\hat{S}(\omega+i\lambda)]$$

for all $x \in X$.

Let $F_1(t) = S(t)$ and $F_2(t) = (k*S)(t) + g(\infty)S(t)$. Note that F_2 is of exponential type since $k \in L^1(\mathbb{R}_+)$ and hence

$$g(\omega+i\lambda)\hat{S}(\omega+i\lambda)^2 x = \hat{F}_1(\omega+i\lambda)\hat{F}_2(\omega+i\lambda)x$$

for all $x \in X$.

We can now apply the estimate (d) in Proposition 1 to obtain

$$|| \int_r^{r'} e^{i\lambda t} g(\omega+i\lambda)\hat{S}(\omega+i\lambda)^2 x d\lambda||_{L^\infty(\mathbb{R}_+;X)} \leq C||x||,$$

for all $x \in X$.

Since $D(A^2)$ is dense in X the above integral converges for all $x \in X$. We conclude that

$$\frac{1}{2\pi} \lim_{r\to-\infty\ r'\to\infty} \int_r^{r'} e^{(\omega+i\lambda)t}\hat{S}(\omega+i\lambda)x d\lambda = R(t)x$$

exists for all $x \in X$, uniformly for t in any compact interval of $(0,\infty)$.

On the other hand, by equation (6), we have for all $x \in X$

$$\int_0^t S(s)xds = \frac{1}{2\pi} \lim_{r\to-\infty\ r'\to\infty} \int_r^{r'} e^{(\omega+i\lambda)t}\hat{S}(\omega+i\lambda)x \frac{d\lambda}{\omega+i\lambda}$$

uniformly for t in any compact interval of $(0,\infty)$. By differentiation we obtain $R(t)x = S(t)x$ for all $x \in X$.

□

Remark The functions $a(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ $(\alpha \geq 0)$ satisfy the conditions of the above Theorem. In particular, for $\alpha = 0$ and $\alpha = 1$ we recover the results in [6], Theorem 1 and [5], Proposition 2.12 respectively.

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