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Regularity and Qualitative Properties for Solutions of Some Evolution Equations

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Abstract

The main objective of this doctoral thesis is the study of existence and uniqueness, as well as qualitative properties of solutions for evolution equations defined in abstract Banach spaces.

Concerning to evolution equations, one of main subject of study is the study of existence of solutions. For this reason, we find sufficient conditions that guarantee existence of mild solutions for some evolution equations. Specifically, we study conditions which guarantee existence of *mild solutions* for an integro–differential equation with non–local initial conditions and a non–autonomous second order differential equation with non–local initial conditions. Our approach is based on resolvent operator theory.

On th other hand, it is well known that concerning to evolution equations, another important subject of interest is the study of qualitative properties of their solutions. Motivated by this, we study existence and uniqueness of *periodic strong solutions* for some interesting evolution equations having *maximal regularity* property. Specifically, we study maximal regularity property on periodic Lebesgue, Besov and Triebel–Lizorkin spaces for a third–order differential equation and a fractional order differential equation with finite delay. In the case of periodic Lebesgue spaces our results involve the notion of UMD–spaces and the concept of R–boundedness of some families of operators. In the cases of Besov and Triebel–Lizorkin spaces our results only involve boundedness conditions for some families of operators. Our approach is based on operator–valued Fourier multipliers theorems.

Resumen

El objetivo principal de esta tesis doctoral es el estudio de existencia y unicidad, así como también, propiedades cualitativas de soluciones de ecuaciones de evolución definidas en espacios de Banach abstractos.

En relación a una ecuación de evolución, uno de los principales temas de estudio es la existencia de soluciones. Por este motivo, nosotros encontramos condiciones suficientes que garantizan existencia de *soluciones mild* para algunas ecuaciones de evolución. Específicamente, estudiamos condiciones que aseguran existencia de *soluciones mild* para una ecuación integro–diferencial con condiciones iniciales no locales y una ecuación diferencial de segundo orden no autónoma con condiciones iniciales no locales. Nuestros métodos están basados en la teoría de operador resolvente.

Por otro lado, es un hecho conocido que en relación a una ecuación de evolución otro tópico de interés es el estudio de propiedades cualitativas de sus soluciones. Motivados por esto, estudiamos existencia y unicidad de *soluciones periódicas fuertes* para algunas interesantes ecuaciones de evolución teniendo la propiedad de regularidad maximal. Específicamente, estudiamos la propiedad de regularidad maximal en espacios periódicos de Lebesgue, Besov y Triebel–Lizorkin para una ecuación diferencial de tercer orden y una ecuación diferencial de orden fraccionario con retardo finito. En el caso de espacios periódicos de Lebesgue, nuestros resultados involucran la noción de espacios UMD y el concepto de R–acotamiento de algunas familias de operadores. En los casos de espacios periódicos de Besov y Triebel–Lizorkin, nuestros resultados involucran sólo condiciones de acotamiento de estas familias de operadores. Los métodos usados están basados en teoremas operador–valuados de multiplicadores de Fourier.

Introduction

Because many natural phenomena arising from applied fields can be described by partial differential equations and their generalizations, the study of properties of the solutions of these equations is a very important and active field of research. In many cases, partial differential equations can be transformed into an ordinary differential equation with values in an infinite dimensional space. This motivates the study of *evolution equations* in abstract spaces, especially in Banach spaces.

This thesis is concerned with the study of existence, uniqueness and qualitative properties of solutions for some classes of abstract evolution equations. This work is the outcome of the author's research during his Math Ph.D. study at *Universidad de Chile* (March 2009 – September 2012). The main results obtained in this research work are available through the following four articles made in this period

1) C. Lizama, J.C. Pozo, *Existence of mild solutions for semilinear integro–differential equations with non–local conditions.* Submitted

2) V. Poblete, J.C. Pozo, *Periodic solutions of an abstract third–order differential equation.* Submitted

3) V. Poblete, J.C. Pozo, *Periodic solutions for a fractional order abstract neutral differential equation with finite delay.* Preprint

4) H. R. Henríquez, V.Poblete, J.C. Pozo, *Existence of mild–solutions of a non–autonomous second order evolution equation with non–local initial conditions.* Preprint

Most of the results in the thesis is based on two methods of theory of evolution equations.

1. Theory of resolvent operators and variation of parameters formula.
2. Maximal regularity property and operator–valued Fourier multipliers.

It is well known that concerning to an evolution equation, one of the main subject of study is the existence of a solution. However, there exist several notions of solution of an evolution equation. *Strong solution* is the best, but more demanding notion. A weaker concept of solution is the concept of *mild solution.* In fact, a strong solution is a mild solution that satisfies additional differentiability properties. Many author prove existence of strong solutions proving existence of mild solutions and giving smoothness conditions in the initial value. Reader can see the works [83, 84, 125, 147] and references therein.

The theory of resolvent operator has been subject of increasing interest in past decades, because it is a central issue for the study of mild solutions of evolution equations. The resolvent operator is applied to inhomogeneous equations to derive various variation of parameters formula. In this direction, we refer the works made by de Andrade and Lizama [50], Arjunan, dos Santos and Cuevas [54], Lizama and N'Guérékata [111], Prüss [131,132]. Moreover, there exists several methods for proving existence theorems for the resolvent, for example operational calculus in Hilbert spaces, perturbation arguments, and Laplace–transform method, (for more information see the works made by Grimmer and Prüss [69] and Prüss [133, 134]). In Chapters 2 and 3, we study the existence of mild solutions for two very interesting evolution equations. Our approach is based in resolvent operator theory and variation of parameters formula.

Another important subject of research concerning to evolution equations is the study of qualitative properties of their solutions. In particular, the problem of existence of solutions having a periodicity property has been considered by several authors, the reader can see [13, 77, 79, 80, 114] and references therein. In the same manner, the study of regularity properties of solutions for evolution equations has been an active topic of research in last decades. In particular, *maximal regularity property* has received much attention in recent years due to its applications to evolution equations. Indeed, maximal regularity is an important tool in the study of the following problems:

- Existence and uniqueness of solutions of quasi-linear and non-linear partial differential equations.
- Existence and uniqueness of solutions of Volterra integral equations.
- Existence and uniqueness of solutions of neutral equations.
- Existence and uniqueness of solutions of non-autonomous evolution equations.

In these applications, maximal regularity is usually used to reduce, via a fixed-point argument, a non-linear (respectively a non-autonomous) problem to a linear (respectively an autonomous) problem. In some cases, maximal regularity is needed to apply an implicit function theorem. (See [46, p.72]).

Several techniques are used to study the problem of maximal regularity of evolution equations. One of these is the Fourier multipliers or symbols. There exists an extensive literature about vector–valued Fourier theorems and concrete applications. The reader can see the works made by, Amann [5], Arendt, Batty and Bu [8], Arendt and Bu [10, 11, 12], Bu [26], Bu and Fang [27, 28, 29, 30], Bu and Kim [25, 31, 32], Clément, de Pagter, Sukochev and Witvliet [43], Denk, Hieber and Prüss [52], Girardi and Weis [61, 62], Kalton and Lancien [90], Keyantuo and Lizama [93, 94, 95, 96], Lizama [110], Poblete [126, 127] and references therein.

As we have mentioned, this thesis consists in the study of **4 problems**, two of them are related with guarantee existence of mild solutions for some interesting evolution equations; the other two problems are related with guarantee existence of strong solution with periodicity and maximal regularity properties for evolution equations. In what follows, we will give a brief description of each Chapter of this thesis.

Chapter 1 contains notation and preliminary results. Furthermore, in this chapter, for reader's convenience, we have summarized some relevant concepts and Theorems, concerning to general evolution equations.

In Chapter 2, we study the following problem. Find conditions that guarantee existence of a mild solution of the semi–linear integro–differential equation with non–local initial conditions

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u(t)), \quad t \in [0,1] \\ u(0) &= g(u). \end{aligned}\right\} \tag{1}$$

where $A : D(A) \subseteq X \to X$ and $B(t) : D(B(t)) \subseteq X \to X$ for $t \in I = [0,1]$ are closed linear operators in a Banach space X. We assume that $D(A) \subseteq D(B(t))$ for every $t \in I$ and $f : I \times X \to X$, $g : \mathscr{C}(I;X) \to X$ are given X–valued functions.

Evolution equations with non–local initial conditions are more realistic to describe natural phenomena than classical initial value problem, because additional information is taken into account. For this reason, in recent decades there has been a lot of interest in this type of problems and applications. For importance of non–local initial conditions in different fields of applied sciences the reader can see [42, 155, 156] and the references cited therein.

First investigations in this area were made by Byszewski in the papers [36, 37, 38, 39]. Thenceforth, many authors have worked in evolution equations with non–local initial conditions, the reader can see [16, 57, 88, 115, 121, 158] for abstract results and concrete applications.

The initial–valued problem (1), that is $u(0) = u_0$ for some $u_0 \in X$, has been the subject of many research papers in recent years, because it has many applications in different fields such as thermodynamics, electrodynamics, continuum mechanics among others, see [134]. For this reason, the study of non–local initial for equation (1) is a very interesting problem.

The main tool that we use is resolvent operator theory. In fact, to achieve our goal we use a mixed method, combining existence of a family $\{R(t)\}_{t\in I}$ called resolvent operator for equation (1), a formula of variation of parameters and a fixed–point argument used in [158].

We prove existence of mild solution of equation (1), under conditions of compactness of g and norm continuity of $R(t)$ for $t > 0$. Moreover, in the special case $B(t) = b(t)A$, where the operator A is defined on a Hilbert space and the kernel b is a scalar map, we are able to give sufficient conditions for existence of mild solutions only in terms of spectral properties of the operator A and regularity properties of the kernel b. We remark that, this type of spectral conditions is a new feature that has not been observed even in the special case $B \equiv 0$. Finally, to prove the feasibility of the abstract results, we consider an example for a particular choice of $b(t)$ and A, which is defined by

$$(Ax)(t,z) = \sum_{i,j=1}^{n} a_{ij}(z)\frac{\partial x(t,z)}{\partial z_i \partial z_j} + \sum_{i=1}^{n} b_i(z)\frac{x(t,z)}{\partial z_i} + \overline{c}(z)x(t,z),$$

where the given coefficients $a_{ij}, b_j, \overline{c}$ $(i, j = 1,2,\ldots,n)$ satisfy the usual uniformly ellipticity conditions. We remark that the results of this Chapter can be found in the joint work made by Lizama–Pozo [115]

In Chapter 3, we study the following problem. Find conditions that guarantee existence of a mild solution of second order non–autonomous equation with non–local initial conditions.

$$\left.\begin{array}{rcl} u''(t) &=& A(t)u(t) + f(t,u(t)), \quad t \in [0,a] \\ u(0) &=& g(u), \\ u'(0) &=& h(u). \end{array}\right\} \tag{2}$$

where $A(t) : D(A(t)) \subseteq X \to X$ for $t \in J = [0,a]$ denote closed linear operators defined in a Banach space X. We assume that $D(A(t)) = D$ for all $t \in J$. The function $f : J \times \mathscr{C}(J;X) \to X$ satisfies Carathéodory type conditions, and the functions $g, h : \mathscr{C}(J;X) \to X$ are continuous maps.

The study of second order evolution equations is very interesting problem, because this type of equation arises in several natural phenomena. In the autonomous case, this is $A(t) = A$ for all t, there exists an extensive literature. The existence of solutions are closely related with the concept of cosine family. For abstract results and concrete applications we refer the reader to [17, 41, 58, 140, 144] and references therein. In the non–autonomous case, the study of solutions becomes much more complicated. However, we prove existence of mild– solutions for equation (2) imposing very general conditions for the operators $A(t)$ and the functions f, g and h.

The principal tool which we use is resolvent operator theory. In fact, we prove the existence of an evolution operator $\{S(t,s)\}_{t,s\in J}$, and then we derive a variation of parameters formula. Finally, we check the feasibility of our abstract results in the special case $A(t) = A + B(t)$ where the operator A is the generator of a cosine family and the operators $B(t)$ satisfy appropriate conditions.

We remark that the results of this Chapter can be found in the joint work made by Henríquez–Poblete–Pozo [82] .

In Chapter 4, we study the following problem. Find a characterization of maximal regularity for a linear abstract third–order differential equation.

Recent investigations have demonstrated that **third order differential equations** describe several models arising from very interesting natural phenomena, such as wave propagation in viscous thermally relaxing fluids, flexible space structure with internal damping, a thin uniform rectangular panel, like a solar cell array, or a spacecraft with flexible attachments (cf. e.g., [18, 19, 20, 21, 65, 66, 67]).

Motivated by this fact, many authors have worked in abstract third–order differential equations. In particular, the following equation has been widely studied

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Au'(t) + F(t, u(t)), \quad for \ t \in \mathbb{R}^+ \tag{3}$$

where A is a closed linear operator defined on a Banach space X, the function F is a given and X–valued, and $\alpha, \beta, \gamma \in \mathbb{R}^+$. We mention some aspects that equation (3) has been analyzed. In [45], a characterization of solutions for its linear version, i.e. $F(t, u(t)) = f(t)$, have been obtained in Hölder spaces $C^s(\mathbb{R}; X)$ by Cuevas and Lizama. In the same manner, Fernández, Lizama and Poblete in [59] characterize well-posedness in Lebesgue spaces, $L^p(\mathbb{R}; X)$. In addition, Fernández, Lizama and Poblete, in [60], study regularity of mild and strong solutions defined in $\mathbb{R}$ when the underlying spaces are Hilbert spaces and some qualitative properties of their solutions. On the other hand, existence of bounded mild solutions of the semi-linear equation (3) is studied by De Andrade and Lizama in [50].

However, the existence of periodic strong solutions of equation (3) has not been addressed in the existing literature. For this reason, Chapter 4 is devoted to study the existence of periodic strong solutions for the following abstract third–order equation

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t), \quad t \in [0, 2\pi], \tag{4}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$, where the operators A and B are closed linear operators defined on a Banach space X satisfying

$D(A)\cap D(B)\neq\{0\}$, the constants $\alpha,\beta,\gamma\in\mathbb{R}^+$, and f belongs to either periodic Lebesgue spaces , or periodic Besov spaces, or periodic Triebel–Lizorkin spaces. We remark, the study of existence of solutions for equation (4) in the particular case $A\equiv B$ is a manner to study periodic solutions of equation (3).

Our approach is based in maximal regularity property for evolution equations and operator–valued Fourier multiplier theorems. In case of periodic Lebesgue spaces, our results involve the key notions of UMD**–spaces** and R**–boundedness** (see Chapter 1 section 1.4 for definition and related results) of the families of operators

$$\left\{kB(i\alpha k^3+k^2+i\gamma kB+\beta A)^{-1})\right\}_{k\in\mathbb{Z}} \text{ and } \left\{ik^3(i\alpha k^3+k^2+i\gamma kB+\beta A)^{-1}\right\}_{k\in\mathbb{Z}}.$$

On the other hand, in the case of periodic Besov or Triebel–Lizorkin spaces, our results only involve boundedness condition of the preceding families.

In general, it is not easy to verify the R–boundedness or boundedness condition of a specific family of operators, especially when two different operators are involved. However, we verify our hypothesis in the special case $B=A^{1/2}$ where A is a sectorial operator; the scalar values α,β, and γ related with equation (4) play a crucial role in this proof.
We remark that the results of this Chapter can be found in the joint work made by Poblete–Pozo [129].

In Chapter 5, we study the following problem. Find sufficient conditions that guarantee the existence of a periodic strong solution for a fractional neutral equation with finite delay.

The fractional calculus which allows us to consider integration and differentiation of any order, not necessarily integer, has been the object of extensive study for analyzing not only anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [6, 118] and references therein), but also fractional phenomena in optimal control (see, e.g., [119, 130, 137]). As indicated in [68, 117, 137] and the related references given there, the advantages of fractional derivatives become apparent in modelling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields. One of the emerging branches of the study is the Cauchy problems of abstract differential equations involving fractional derivatives in time. In recent decades there has been a lot of interest in this type of problems, its applications and various generalizations (cf. e.g., [1, 3, 44, 85] and references therein). It is significant to study this class of problems, because, in this way, one is more realistic to describe the memory and hereditary properties of various materials and processes (cf. [86, 100, 119, 130]).

In the same manner, several systems of great interest in science and engineering are modeled by partial neutral functional differential equations. The reader can see [64, 74, 75, 149, 151, 152, 157]. Many of these equation can be written as an abstract neutral functional differential equation (ANFDE). Additionally, it is well known that one of the most interesting topics, both from a theoretical as practical point of view, of the qualitative theory of differential equations and functional differential equations is the existence of periodic solutions. In particular, the existence of periodic solutions of ANFDE has been considered in several works [71, 89, 116, 145, 146, 153].

Motivated by both practical and theoretical considerations, this Chapter is devoted to the study sufficient conditions that guarantee existence and uniqueness of strong solution for the following fractional neutral differential equation with finite delay

$$D^{\alpha}\big(u(t)-Bu(t-r)\big)=Au(t)+Fu_t+GD^{\beta}u_t+f(t),\quad t\in[0,2\pi], \tag{5}$$

with $0<\beta<\alpha\leqslant 2$, where $r>0$ is a fixed number and $A:D(A)\subseteq X\to X$ and $B:D(B)\subseteq X\to X$ are linear closed operators defined in a Banach space X such that $D(A)\subseteq D(B)$. Here the function u_t is given by $u_t(\theta)=u(t+\theta)$ for θ in an appropriate domain, denotes the history of the function $u(\cdot)$ at t and $D^{\beta}u_t(\cdot)$ is defined by $D^{\beta}u_t(\cdot)=\big(D^{\beta}u\big)_t(\cdot)$. The delay operators F and G are bounded linear map defined on an suitable space and f is a given function that belongs to either periodic Besov spaces, or periodic Triebel–Lizorkin spaces.

Our approach is based in a mixed method. We prove and use maximal regularity on periodic Besov spaces (respectively periodic Triebel–Lizorkin spaces) of an auxiliary equation and a fixed–point argument to proving existence of a strong $B^s_{p,q}$–solution (respectively $F^s_{p,q}$–solution) of equation (5). Here the auxiliary equation is given by

$$D^{\alpha}u(t)=Au(t)+Fu_t+GD^{\beta}u_t+f(t),\quad t\in[0,2\pi],\ \text{ and } 0<\beta<\alpha\leqslant 2. \tag{6}$$

with boundary periodic conditions. All terms in preceding equation are defined in the same manner as equation (5).

Our main results involve, among other considerations, boundedness of the family of operators,

$$\big\{(ik)^{\alpha}\big((ik)^{\alpha}-F_k-(ik)^{\beta}G_k-A\big)^{-1}\big\}_{k\in\mathbb{Z}}$$

and regularity of the families of bounded operators $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$. Here the families of operators $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are defined by

$$F_kx=F(e_kx)\ \text{ and }\ G_kx=G(e_kx),\ \text{ where } e_kx(\cdot)=e^{ik\cdot}x\ \text{ for }\ x\in X.$$

In last section of this chapter, to prove the feasibility of the abstract results, we consider two examples for particular choices of operators A, B, F and G.

We remark that the main results of this Chapter can be found in the joint work made by Poblete–Pozo [128].

CHAPTER 1

Preliminaries

In this thesis X and Y always are complex Banach spaces. We denote the space of all linear operators from X to Y by $\mathscr{L}(X,Y)$. In the case $X=Y$, we will write briefly $\mathscr{L}(X)$. Let A be an operator defined on X. We will denote its domain by $D(A)$, its domain endowed with the graph norm by $[D(A)]$, its resolvent set by $\rho(A)$, and its spectrum set by $\sigma(A)=\mathbb{C}\setminus\rho(A)$.

We denote by $\mathscr{C}([0,a];X)$ the space of continuous functions $f:[0,a]\mapsto X$.

Let $f\in L^1_{loc}(\mathbb{R};X)$, we adopt the following notation for Fourier transform and Laplace transform,

$$\widehat{f}(\xi)=\frac{1}{2\pi}\int_0^{2\pi}e^{i\xi t}f(t)dt,\quad \text{and}\quad \widetilde{f}(\lambda)=\int_0^{\infty}e^{-\omega t}f(t)dt,$$

respectively.

1.1 Families $\mathscr{M}$–bounded and n–regular sequences

In order to give certain conditions which we will need in Chapters 4 and 5, we establish the following notation. Let $\{L_k\}_{k\in\mathbb{Z}}\subset\mathscr{L}(X,Y)$ be a sequence of operators. Set

$$(\Delta^0 L_k)=L_k,\quad (\Delta L_k)=(\Delta^1 L_k)=L_{k+1}-L_k$$

and for $n=2,3,\ldots$, set

$$(\Delta^n L_k)=\Delta(\Delta^{n-1}L_k).$$

Definition 1.1. [96] *We will say that a family of operators* $\{L_k\}_{k\in\mathbb{Z}}\subset\mathscr{L}(X,Y)$ *is* ***$\mathscr{M}$–bounded of order n*** $(n\in\mathbb{N}\cup\{0\})$ *if*

$$\sup_{0\leqslant l\leqslant n}\sup_{k\in\mathbb{Z}}\|k^l(\Delta^l L_k)\|<\infty. \tag{1.1}$$

Note that, for $j \in \mathbb{Z}$ fixed, $\sup_{0\leqslant l\leqslant n}\sup_{k\in\mathbb{Z}} \|k^l(\Delta^l L_k)\| < \infty$ if and only if $\sup_{0\leqslant l\leqslant n}\sup_{k\in\mathbb{Z}} \|k^l(\Delta^l L_{k+j})\| < \infty$. The statement follows directly from the binomial formula.

In the preceding definition when $n = 0$, the $\mathcal{M}$–boundedness of order n for $\{L_k\}_{k\in\mathbb{Z}}$ simply means that $\{L_k\}_{k\in\mathbb{Z}}$ is bounded.

When $n = 1$, this is equivalent to

$$\sup_{k\in\mathbb{Z}} \|L_k\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \|k(L_{k+1} - L_k)\| < \infty. \tag{1.2}$$

When $n = 2$, in addition to (1.2), we must have

$$\sup_{k\in\mathbb{Z}} \|k^2(L_{k+2} - 2L_{k+1} + L_k)\| < \infty. \tag{1.3}$$

When $n = 3$, in addition to (1.2) and (1.3), we must have

$$\sup_{k\in\mathbb{Z}} \|k^3(L_{k+3} - 3L_{k+2} + 3L_{k+1} - L_k)\| < \infty. \tag{1.4}$$

In the scalar case, that is, $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$, we will write $\Delta^n a_k = \Delta(\Delta^{n-1} a_k)$.

Definition 1.2. [91] *A sequence* $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}\setminus\{0\}$ *is called*

a) ***1–regular*** *if the sequence* $\left\{k\dfrac{(\Delta^1 a_k)}{a_k}\right\}_{k\in\mathbb{Z}}$ *is bounded;*

b) ***2–regular*** *if it is 1–regular and the sequence* $\left\{k^2\dfrac{(\Delta^2 a_k)}{a_k}\right\}_{k\in\mathbb{Z}}$ *is bounded;*

c) ***3–regular*** *if it is 2–regular and the sequence* $\left\{k^3\dfrac{(\Delta^3 a_k)}{a_k}\right\}_{k\in\mathbb{Z}}$ *is bounded.*

For useful properties and further details about n–regularity, see [99].

***Remark* 1.1.** *Note that if* $\{a_k\}_{k\in\mathbb{Z}}$ *is an 1–regular sequence then, for all* $j \in \mathbb{Z}$ *fixed, the sequence* $\left\{k\dfrac{a_{k+j} - a_k}{a_{k+j}}\right\}_{k\in\mathbb{Z}}$ *is bounded. In the cases* $n = 2, 3$, *analogous properties hold.*

1.2 Vector–valued Besov and Triebel–Lizorkin spaces

Periodic Besov and Triebel–Lizorkin spaces form part of functions spaces which are of special interest. They behave in a similar manner to Sobolev spaces and the property of maximal regularity can be stated elegantly on them. Furthermore, they generalize many important spaces. For example, periodic Hölder continuous functions of index s with $0 < s < 1$, is a particular case of periodic Besov spaces, see [12] for more details.

However, the main reason to work in these spaces is that a certain form of Mikhlin's multiplier theorem holds for operator–valued symbols on arbitrary Banach spaces X, unlike Lebesgue spaces $L^p(\mathbb{T};X)$, where this property is valid if and only if $p=2$ (for more information [61]).

Let $\mathscr{S}(\mathbb{R};X)$ be the Schwartz space on $\mathbb{R}$, let $\mathscr{S}'(\mathbb{R};X)$ be the space of all the tempered distributions on $\mathbb{R}$ and let $\mathscr{D}'(\mathbb{T};X)$ be the space of the X–valued 2π–periodic distributions. Let $\Phi(\mathbb{R})$ be the set of all systems $\phi=\{\phi_j\}_{j\geqslant 0}\subseteq \mathscr{S}(\mathbb{R};X)$ satisfying $supp(\phi_0)\subseteq[-2,2]$, and

$$supp(\phi_j)\subseteq[-2^{j+1},-2^{j-1}]\cup[2^{j-1},2^{j+1}],\qquad \sum_{j\geqslant 0}\phi_j(t)=1,\ for\ \ t\in\mathbb{R}$$

and, for $\alpha\in\mathbb{N}\cup\{0\}$, there is a $C_\alpha>0$ such that $\sup\limits_{j\geqslant 0,x\in\mathbb{R}} 2^{\alpha j}\|\phi_j^{(\alpha)}(x)\|\leqslant C_\alpha$. It is a well known fact from Littlewood–Paley decomposition theory that this type of systems there exist. More information about this can be found in [4,5,9,12].

Definition 1.3. [12] *Let $1\leqslant p,q\leqslant\infty, s\in\mathbb{R}$ and $\phi=(\phi_j)_{j\geqslant 0}\in\Phi(\mathbb{R})$. The X–valued periodic Besov spaces are defined by*

$$B^{s,\phi}_{p,q}(\mathbb{T};X))=\left\{f\in\mathscr{D}'(\mathbb{T};X):\|f\|_{B^{s,\phi}_{p,q}}<\infty\right\}$$

where

$$\|f\|_{B^{s,\phi}_{p,q}}=\Big(\sum_{j\geqslant 0}2^{jsq}\Big\|\sum_{k\in\mathbb{Z}}e_k\otimes\phi_j(k)\widehat{f}(k)\Big\|_p^q\Big)^{\frac{1}{q}}$$

with usual modifications when $p=\infty$ or $q=\infty$.
The space $B^{s,\phi}_{p,q}$ is independent of $\phi\in\Phi(\mathbb{R})$ and different choices of $\phi\in\Phi(\mathbb{R})$ generate equivalent norms. As consequence, we will denote $\|\cdot\|_{B^{s,\phi}_{p,q}}$ simply by $\|\cdot\|_{B^s_{p,q}}$.

We recall some important properties of these spaces:

(a) Let $1\leqslant p,q\leqslant\infty, s\in\mathbb{R}$ be fixed. The X–valued periodic space $B^s_{p,q}(\mathbb{T};X)$ is a Banach space.

(b) Let $1\leqslant p,q\leqslant\infty$ be fixed. If $s>0$, the natural injection from $B^s_{p,q}(\mathbb{T};X)$ into $L^p(\mathbb{T};X)$ is a continuous linear operator.

(c) (Lifting property) Let $1\leqslant p,q\leqslant\infty, s\in\mathbb{R}$, $f\in\mathscr{D}'(\mathbb{T};X)$ and $\alpha\in\mathbb{R}$ then $f\in B^s_{p,q}(\mathbb{T};X)$ if and only if $\sum_{k\neq 0}e_k\otimes(ik)^\alpha\widehat{f}(k)\in B^{s-\alpha}_{p,q}(\mathbb{T};X)$.

To define the X–valued periodic Triebel–Lizorkin spaces, we use the same notation for $\mathscr{S}(\mathbb{R};X)$, $\mathscr{S}'(\mathbb{R};X)$, $\mathscr{D}'(\mathbb{T};X)$ and $\Phi(\mathbb{R})$ as those of definition of X–valued periodic Besov spaces.

Definition 1.4. [25] *Let $\phi = (\phi)_{k\in\mathbb{N}_0} \in \Phi(\mathbb{R})$ be fixed, for $1 \leqslant p,\ q \leqslant \infty$, and $s \in \mathbb{R}$. The X–valued periodic Triebel–Lizorkin spaces are defined by*

$$F^{s,\phi}_{p,q}(\mathbb{T};X) = \left\{ f \in \mathscr{D}'(\mathbb{T};X) : \|f\|_{F^{s,\phi}_{p,q}} < \infty \right\}$$

where

$$\|f\|_{F^{s,\phi}_{p,q}} = \left\| \Big(\sum_{j\geqslant 0} 2^{jsq} \Big\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\widehat{f}(k) \Big\|^q_X \Big)^{\frac{1}{q}} \right\|_p$$

with the usual modification when $p = \infty$ or $q = \infty$.
The space $F^{s,\phi}_{p,q}$ is independent of $\phi \in \Phi(\mathbb{R})$ and different choices of $\phi \in \Phi(\mathbb{R})$ generate equivalent norms. Consequently, we simply denote $\|\cdot\|_{F^{s,\phi}_{p,q}}$ by $\|\cdot\|_{F^{s}_{p,q}}$.

Note that X–valued periodic Triebel–Lizorkin spaces have analogous properties as those of X–valued periodic Besov spaces, the reader can see [25, 32]. We summarize the most important properties as follows

(a) Let $1 \leqslant p, q \leqslant \infty, s \in \mathbb{R}$ be fixed. The X–valued periodic space $F^s_{p,q}(\mathbb{T};X)$ is a Banach space.

(b) Let $1 \leqslant p, q \leqslant \infty$ be fixed. If $s > 0$, then the natural injection from $F^s_{p,q}(\mathbb{T};X)$ into $L^p(\mathbb{T};X)$ is a continuous linear operator.

(c) (Lifting property) Let $1 \leqslant p, q \leqslant \infty, s \in \mathbb{R}$, $f \in \mathscr{D}'(\mathbb{T};X)$ and $\alpha \in \mathbb{R}$ then $f \in F^s_{p,q}(\mathbb{T};X)$ if and only if $\sum_{k\neq 0} e_k \otimes (ik)^\alpha \widehat{f}(k) \in F^{s-\alpha}_{p,q}(\mathbb{T};X)$.

1.3 Operator–valued Fourier multipliers

In this section, we recall some operator–valued Fourier multipliers theorems, that we shall use to characterize maximal regularity of problems with periodic boundary conditions in Chapters 4 and 5.

We denote the space consisting of all 2π–periodic, X–valued functions by $E(\mathbb{T};X)$. The following definitions will be used in Chapters 4 and 5 with Lebesgue, Besov and Triebel–Lizorkin spaces.

Definition 1.5. *We say that the sequence $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathscr{L}(X,Y)$ is an $(E(X),E(Y))$– multiplier if for each $f \in E(\mathbb{T};X)$, there exists a function $u \in E(\mathbb{T};Y)$ such that*

$$\widehat{u}(k) = L_k \widehat{f}(k)\,, \ \text{for all } k \in \mathbb{Z}.$$

*In the case $X = Y$, we will say that $\{L_k\}_{k\in\mathbb{Z}}$ is an E–**multiplier**.*

The next Theorem, proved by Arendt and Bu in [10], establishes a sufficient condition that guarantees when a family $\{L_k\}_{k\in\mathbb{Z}}$ is a L^p–multiplier. It is remarkable, in order to do this the key concepts of family of operator R–bounded and UMD–spaces are needed (see section 1.4).

Theorem 1.1. *Let $p \in (1,\infty)$, and let X be UMD–space. Assume that $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathscr{L}(X)$. If the families of operators $\{L_k\}_{k\in\mathbb{Z}}$ and $\{k(\Delta^1 L_k)\}_{k\in\mathbb{Z}}$ are R–bounded, then $\{L_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier.*

The following Theorem, proved by Arendt and Bu in [12], establishes a sufficient condition that guarantees when a family $\{L_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. We remark, this theorem impose stronger conditions than Theorem 1.1 for family of operators $\{L_k\}_{k\in\mathbb{Z}}$, however it is valid on an arbitrary Banach space X.

Theorem 1.2. *Let $1 \leqslant p,\ q \leqslant \infty$, and $s \in \mathbb{R}$. Let X be a Banach space. If the family $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathscr{L}(X)$ is $\mathscr{M}$–bounded of order 2, then $\{L_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.*

The following Theorem, proved by Bu and Kim in [32] , establishes a sufficient condition that guarantees when a family $\{L_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier. We remark, as well as in Theorem 1.2, this theorem is valid for arbitrary Banach space X, however more conditions are imposed for the family of operators $\{L_k\}_{k\in\mathbb{Z}}$.

Theorem 1.3. *Let $1 \leqslant p,\ q \leqslant \infty$, and $s \in \mathbb{R}$. Let X be a Banach space. If the family $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathscr{L}(X)$ is $\mathscr{M}$–bounded of order 3, then $\{L_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.*

1.4 L^p–maximal regularity of evolution equations

The L^p–maximal regularity property is a special topic of evolution equations because is a fundamental tool for the study of non–linear problems. It is remarkable that classical theorems on L^p–multipliers are no longer valid for operator–valued functions unless the underlying space is isomorphic to a Hilbert space. However, Weis in [148] gives a characterization of L^p–maximal regularity in UMD–spaces using the key notion of R–boundedness and Fourier multipliers techniques. Thenceforth, many authors have used this concept in the study of L^p–maximal regularity. The reader can see [7, 10, 29, 30, 43, 59, 63, 97, 127]. In **Chapter 4** we characterize L^p–maximal regularity of a third–order differential equation in UMD–spaces, using R–boundedness for some families of operators. We state in this section the necessary definitions.

Let $\mathscr{S}(\mathbb{R};X)$ be the Schwartz space, consisting of all the rapidly decreasing X–valued functions. A Banach space will be called a UMD–space if the Hilbert transform can be extended to a bounded linear operator in $L^p(\mathbb{R};X)$, for some (and hence for all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathscr{S}(\mathbb{R};X)$ is defined by

$$(Hf)(s) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\varepsilon<|t|<\frac{1}{\varepsilon}} \frac{f(t-s)}{t}\,dt.$$

Examples of UMD–spaces include Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$, with $1 < p < \infty$, the Schatten–von Neumann classes $C_p(H)$ of operators on Hilbert spaces for $1 < p < \infty$, the Lebesgue spaces $L^p(\Omega,\mu)$ and $L^p(\Omega,\mu;X)$, with $1 < p < \infty$ and X a UMD–space. Moreover, every closed subspace of a UMD–space is an UMD–space. On the other hand, every UMD–space is reflexive, and therefore, $L^1(\Omega,\mu)$, $L^\infty(\Omega,\mu)$ (if Ω is an unbounded set) and periodic Hölder spaces of index α with $0 < \alpha < 1$, $C^\alpha([0,2\pi];X)$ are not UMD–spaces. For further information about these spaces, see [23, 33, 34].

As we have mentioned, the notion R–boundedness has proved to be a significant tool in the study of abstract multiplier operators. Preliminary concepts for the definition and properties of R–boundedness that we will use may be found in [52, 87, 90].

For $j \in \mathbb{N}$, denote by r_j the $j-th$ Rademacher function on $[0,1]$, i.e. $r_j(t) = sgn(\sin(2^j\pi t))$. For $x \in X$ we write $r_j x$ for the vector–valued function $t \to r_j(t)x$. The definition of R–boundedness is given as follows.

Definition 1.6. *Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ is called R–**bounded** if there exist a constant $C \geqslant 0$ and $p \in [1,\infty)$ such that for each $n \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ such that the inequality*

$$\Big\| \sum_{j=1}^{n} r_j T_j x_j \Big\|_{L^p((0,1);Y)} \leqslant C \Big\| \sum_{j=1}^{n} r_j x_j \Big\|_{L^p((0,1);X)}$$

*holds. The smallest such $C \geqslant 0$ is called an R–**bound** of $\mathcal{T}$, denoted $R_p(\mathcal{T})$.*

***Remark* 1.2.** *We remark that large classes of operators are R–bounded, (the reader can see* [63, 87, 143] *and references therein). Several properties of R–bounded families can be founded in the monograph of Denk-Hieber-Prüss* [52]. *For the reader convenience we have summarized the most important of them.*

(i) If $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ is R–bounded, then it is uniformly bounded with

$$\sup\{\|T\| : T \in \mathcal{T}\} \leqslant \mathcal{R}_p(\mathcal{T}).$$

(ii) The definition of R–boundedness is independent of $p \in [1,\infty)$.

(iii) When X and Y are Hilbert spaces, $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ is R–bounded if and only if $\mathcal{T}$ is uniformly bounded.

(iv) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subseteq \mathcal{L}(X,Y)$ be R–bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R–bounded as well, and $\mathcal{R}_p(\mathcal{T} + \mathcal{S}) \leqslant \mathcal{R}_p(\mathcal{T}) + \mathcal{R}_p(\mathcal{S})$.

(v) *Let X, Y and Z be Banach spaces and $\mathcal{T} \subseteq \mathcal{L}(X,Y)$, and $\mathcal{S} \subseteq \mathcal{L}(Y,Z)$ be R–bounded. Then*
$$\mathcal{T}\mathcal{S} = \{TS : T \in \mathcal{T}, S \in \mathcal{S}\}$$
is R–bounded as well, and $\mathcal{R}_p(\mathcal{T}\mathcal{S}) \leqslant \mathcal{R}_p(\mathcal{T})\mathcal{R}_p(\mathcal{S})$.

(vi) *Let X, Y be Banach spaces and $\mathcal{T} \subseteq \mathcal{L}(X,Y)$ be R–bounded. If $\{\alpha_k\}_{k\in\mathbb{Z}}$ is a bounded sequence, then $\{\alpha_k T : k \in \mathbb{Z}\ ,\ T \in \mathcal{T}\}$ is R–bounded.*

The next Proposition proved in [10] relates L^p–multipliers and R–bounded families of operators.

Proposition 1.1. *Let $p \in (1,\infty)$, and let X and Y be UMD–spaces. Assume that $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X,Y)$. If the family $\{L_k\}_{k\in\mathbb{Z}}$ is an $(L^p(X), L^p(Y))$–multiplier, then $\{L_k\}_{k\in\mathbb{Z}}$ is R–bounded.*

In order to work with L^p–maximal regularity for evolution equations, various researchers introduce the following vector–valued spaces of functions. See [10, 93, 99].

Definition 1.7. *Let $p \in [1,\infty)$, and let $n \in \mathbb{N}$. Let X and Y be Banach spaces. We define the following vector–valued function spaces.*

$$H^{n,p}_{per}(X,Y) = \left\{u \in L^p(\mathbb{T};X) : \exists\ v \in L^p(\mathbb{T};Y) \text{ such that } \widehat{v}(k) = (ik)^n \widehat{u}(k), \text{ for all } k \in \mathbb{Z}\right\}.$$

In the case $X = Y$, we just write $H^{n,p}_{per}(X)$. We highlight two important properties of these spaces:

- Let $n, m \in \mathbb{N}$. If $n \leqslant m$, then $H^{m,p}_{per}(X,Y) \subseteq H^{n,p}_{per}(X,Y)$.
- If $u \in H^{n,p}_{per}(X)$, then for all $0 \leqslant k \leqslant n-1$, we have $u^{(k)}(0) = u^{(k)}(2\pi)$.

***Remark* 1.3.** *For $1 \leqslant p \leqslant \infty$, by* [10, Lemma 2.2], *for all $n \in \mathbb{N}$ the family of operators $\{k^n M_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier if and only if $\{L_k\}_{k\in\mathbb{Z}}$ is an $(L^p(X), H^{n,p}_{per}(X))$–multiplier.*

Lemma 1.1. [10] *Let $f, g \in L^p(\mathbb{T};X)$, with $p \in [1,\infty)$. If A is a closed operator in a Banach space X, then the following two assertions are equivalent.*

(i) *$f(t) \in D(A)$ and that $Af(t) = g(t)$ a.e.*

(ii) *$\widehat{f}(k) \in D(A)$ and that $A\widehat{f}(k) = \widehat{g}(k)$, for all $k \in \mathbb{Z}$.*

1.5 Measure of Non–compactness

The theory of measures of non–compactness has many applications in Topology, Functional analysis and Operator theory. For more information the reader can see [14]. In this thesis we use the notion of Hausdorff measure of non–compactness.

Definition 1.8. *Let B be a bounded subset of a semi–normed linear space Y. The Hausdorff measure of non–compactness is defined by*

$$\gamma(B) = \inf\{\varepsilon > 0 : B \text{ has a finite cover by balls of radius } \varepsilon\}$$

***Remark* 1.4.** *This measure of non–compactness satisfies important properties, we have summarized the most important for our work. For more details see* [14].

(a) *If $A \subseteq B$ then $\gamma(A) \leqslant \gamma(B)$.*

(b) $\gamma(A) = \gamma(\overline{A})$, *where $\overline{A}$ denotes the closure of A.*

(c) $\gamma(A) = 0$ *if and only if A is totally bounded.*

(d) $\gamma(\lambda A) = |\lambda|\gamma(A)$ *with $\lambda \in \mathbb{R}$.*

(e) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$

(f) $\gamma(A + B) \leqslant \gamma(A) + \gamma(B)$, *where $A + B = \{a + b : \ a \in A, \ b \in B\}$.*

(g) $\gamma(A) = \gamma(\overline{co}(A))$ *where $\overline{co}(A)$ is the closed convex hull of A.*

The following Lemmas will be necessary for the problems that we study in **Chapter 2** and **Chapter 3**. In what follows, we denote by ζ the Hausdorff measure of non–compactness on X and by γ the Hausdorff measure of non–compactness on $\mathscr{C}([0, a]; X)$.

Lemma 1.2. *Let $S \subseteq \mathscr{C}([0, a]; X)$. If S is bounded and equicontinuous, then the set of functions $\overline{co}(S) \subseteq \mathscr{C}([0, a]; X)$ is also bounded and equicontinuous. Here $\overline{co}(S)$ denotes the convex hull of S.*

Lemma 1.3. [14] *Let $W \subseteq C([0, a]; X)$. If W is bounded, then $\zeta(W(t)) \leqslant \gamma(W)$ for all $t \in [0, a]$, where $W(t) = \{x(t) \ : \ x \in W\} \subseteq X$. Furthermore, if W is equicontinuous on $[0, a]$, then $\zeta(W(t))$ is continuous on $[0, a]$, and*

$$\gamma(W) = \sup\{\zeta(W(t)) \ : \ t \in [0, a]\}.$$

Lemma 1.4. [22]. *If W is a bounded set, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W$ such that*

$$\gamma(W) \leqslant 2\gamma(\{u_n\}_{n\in\mathbb{N}}) + \varepsilon.$$

Lemma 1.5. [108]. *Suposse that* $0 < \epsilon < 1$ *and* $h > 0$ *and let*

$$S_n = \epsilon^n + C_1^n \epsilon^{n-1} h + C_2^n \epsilon^{n-2} \frac{h^2}{2!} + \cdots + \frac{h^n}{n!}, \quad n \in \mathbb{N}$$

then $\lim_{n\to\infty} S_n = 0$*, where for* $0 \leqslant m \leqslant n$ *and the constants are defined by* $C_m^n = \binom{n}{m}$.

The following results are the key in the proof of the Theorems of Chapter 2 and Chapter 3. The first one was proved by Sadovskii [136] in 1967. In 1955 Darbo [48] proved the same result for γ–k–set contractions, $k < 1$. The second one is a sharpening of the first one and it is due to Liu, Guo, C. Wu, Y. Wu [108].

Definition 1.9. *A mapping* $F : \mathscr{C}([0,a];X) \to \mathscr{C}([0,a];X)$ *is said to be a* γ*–*k*–set contraction,* $k \in (0,1)$*, if* F *is continuous and if for all bounded subsets* B *of* $\mathscr{C}([0,a];X)$*,* $\gamma(F(B)) \leqslant k\gamma(B)$*.* F *is said to be* γ*–condensing if* F *is continuous and* $\gamma(F(A)) < \gamma(A)$ *for every bounded subset* A *of* $\mathscr{C}([0,a];X)$ *with* $\gamma(A) > 0$.

Theorem 1.4. *Suppose* M *is a nonempty bounded closed and convex subset of a Banach space* X *and suppose* $F : M \to M$ *is* γ*–condensing. Then* F *has a fixed point in* M.

Theorem 1.5. *Let* B *be a closed and convex subset of a complex Banach space* X*, let* $F : B \to B$ *be a continuous operator such that* $F(B)$ *is bounded. For each bounded subset* $C \subseteq B$*, set*

$$F^1(C) = F(C) \ \text{ and } \ F^n(C) = F(\overline{co}(F^{n-1}(C))), \quad n = 2,3,\ldots$$

If there exist a constant $0 \leqslant r < 1$ *and* $n_0 \in \mathbb{N}$ *such that for each bounded subset* $C \subseteq B$

$$\gamma(F^{n_0}(C)) \leqslant r\gamma(C)$$

then F *has a fixed point in* B.

CHAPTER 2

Mild Solutions for an Integro–Differential Equation with Non–local Initial Conditions

As we have mention in *Introduction*, evolution equations with non–local initial conditions generalize evolution equations with classical initial conditions. This notion is more complete for describing nature phenomena than the classical one because additional information is taken into account. For the importance of nonlocal conditions in different fields of applied sciences see [42, 53, 155, 156] and the references cited therein. For example, in [51] the author describes the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula $g(u) = \sum_{i=0}^{p} c_i u(t_i)$, where c_i, $i = 0, 1, \ldots, p$, are given constants and $0 < t_0 < t_1 < \cdots < t_p < 1$.

The earliest works in this area were made by Byszewski in [36, 37, 38, 39]. In these works, using semigroup methods and Banach fixed point theorem the author prove existence and uniqueness of mild and strong solutions of the problem

$$\left.\begin{aligned} u'(t) &= Au(t) + f(t, u(t)) \quad t \in [0, 1] \\ x(0) &= g(u). \end{aligned}\right\} \tag{2.1}$$

when A is an operator defined in a Banach space X and generates a semigroup $\{T(t)\}_{t \geqslant 0}$, f and g are given X–valued functions.

Thenceforth, problem (2.1) has been extensively examined. In fact, in [40] Byszewski and Lakshmikantham have studied existence and uniqueness of mild solution when f and g satisfy Lipschitz-type conditions. In [121, 122] Ntouyas and Tsamatos have studied this problem under conditions of compactness of the function g and the semigroup generated by A. Recently, in [158], Zhu, Song and Li have treated this problem without conditions of compactness on the semigroup $\{T(t)\}_{t\geqslant 0}$ and the function f.

On the other hand, the study of integro–differential equations has been an active topic of research in recent years since it has many applications in different areas. In addition, there exists an extensive literature about integro–differential equations with non–local initial conditions, (cf. e.g., [16, 56, 57, 88, 106, 115, 121, 122, 123, 158]). Our work is a contribution to this theory. In fact, this Chapter is devoted to the study of the existence of mild solutions for semi–linear integro–differential evolution equations. More precisely, we consider the following problem on an abstract Banach space X

$$\left.\begin{array}{rcl} u'(t) &=& Au(t)+\displaystyle\int_0^t B(t-s)u(s)ds+f(t,u(t)), \quad t\in[0,1] \\ u(0) &=& g(u). \end{array}\right\} \tag{2.2}$$

where $A: D(A)\subseteq X\to X$ and $B(t): D(B(t))\subseteq X\to X$ for $t\in I=[0,1]$ are closed linear operators in a Banach space X. We assume that $D(A)\subseteq D(B(t))$ for every $t\in I$ and $f: I\times X\to X$, $g:\mathscr{C}(I;X)\to X$ are given X–valued functions.

The initial valued version of equation (2.2) has been extensively studied by many researchers because it is very important in different fields such as thermodynamics, electrodynamics, continuum mechanics and population biology, among others. For more information see [15, 47, 134]. For this reason the study of mild solutions for equation (2.2) is a very interesting problem.

2.1 Main Results

Most of authors obtain the existence, uniqueness of solutions and well–posedness for equation (2.2) establishing the existence of a resolvent operator $\{R(t)\}_{t\in I}$ (see Grimmer and Prüss [69, 134]).

Next, we include some preliminaries concerning to resolvent operator $\{R(t)\}_{t\in I}$ related with equation (2.2).

Definition 2.1. *A family $\{R(t)\}_{t\in I}$ of bounded linear operators on X is called a resolvent operator of equation* (2.2) *if the following conditions are fulfilled.*

(R1) *For each $x\in X$, $R(0)x=x$ and $R(\cdot)x\in\mathscr{C}([0,1];X)$.*

(R2) *The map $R:[0,1]\to\mathscr{L}([D(A)])$ is strongly continuous.*

(R3) *For each $y\in D(A)$, the function $t\to R(t)y$ is continuously differentiable and*

$$\begin{array}{rcl} \dfrac{d}{dt}R(t)y &=& AR(t)y+\displaystyle\int_0^t B(t-s)R(s)yds \;= \\ &=& R(t)Ay+\displaystyle\int_0^t R(t-s)B(s)yds, \quad t\in I. \end{array} \tag{2.3}$$

Furthermore, we will say that the resolvent operator of equation (2.2), $\{R(t)\}_{t\in I}$, has the property ***(EP)*** if the application from $\mathscr{C}(I;X)$ to $\mathscr{C}(I;X)$ defined by $u(\cdot)\to R(\cdot)g(u)$ takes bounded sets into equicontinuous sets.

***Remark* 2.1.** *There exists several situations where the resolvent operator of equation* (2.2)*,* $\{R(t)\}_{t\in I}$*, has property **(EP)**, for example*

a) *If the function* $t\to R(t)$ *is continuous from* $(0,+\infty)$ *to* $\mathscr{L}(X)$ *endowed with the uniform operator norm* $\|\cdot\|_{\mathscr{L}(X)}$*.*

b) *If function* g *takes values in* $D(A)$*.*

c) *If function* g *is a compact operator.*

As we have mentioned, the existence of mild solutions of the *linear* classical version of equation (2.2), this is

$$\left.\begin{array}{rcl} u'(t) &=& Au(t)+\displaystyle\int_0^t B(t-s)u(s)ds+f(t), \quad t\in I \\ u(0) &=& u_0\in X, \end{array}\right\} \tag{2.4}$$

has been studied by Grimmer and Prüss. Assuming that the function $f\in L^1(I;X)$ they prove that

$$u(t)=R(t)u_0+\int_0^t R(t-s)f(s)ds, \quad t\in I, \tag{2.5}$$

is a mild solution of the problem (2.4). Motivated by this result, we adopt the following concept of solution.

Definition 2.2. *A function* $u\in\mathscr{C}(I;X)$ *is called a **mild solution** of equation* (2.2) *if satisfies the equation*

$$u(t)=R(t)g(u)+\int_0^t R(t-s)f(s,u(s))ds \qquad t\in I. \tag{2.6}$$

Clearly, a manner to guarantee the existence of a mild solution of equation (2.2) is using a fixed–point argument. For this reason, we apply an adaptation of the method used in [158] where the authors prove that equation (2.1) has a mild solution. The crucial concept that is involved by this technique is Hausdorff measure of non–compactness. See Chapter (1) *Preliminaries* for definition, properties and results related.

Henceforth, we assume that the following assertions hold.

(H1) Equation (2.1) admits a resolvent operator $\{R(t)\}_{t\in I}$ satisfying the property ***(EP)***.

(H2) The function $g : \mathscr{C}(I;X) \to X$ is a compact map, and for each $M \geqslant 0$ the number g_M defined by $g_M = \sup\big\{\|g(u)\| \,:\, \|u\| \leqslant M\big\}$ is finite.

(H3) The function $f : I \times X \to X$ satisfies the Carathéodory type conditions, that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t,\cdot)$ is continuous for almost all $t \in I$.

(H4) There exist a function $m \in L^1(I;\mathbb{R}^+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leqslant m(t)\Phi(\|u\|)$$

for all $x \in X$ and almost all $t \in I$.

(H5) There exists a function $H \in L^1(I;\mathbb{R}^+)$ such that for any subset of functions $S \subseteq \mathscr{C}(I;X)$, we have

$$\zeta\{f(t,S(t))\} \leqslant H(t)\zeta\{S(t)\}$$

for almost all $t \in I$.

The main result in this Chapter is the following theorem.

Theorem 2.1. *If the hipothesis **(H1)–(H5)** are satisfied and there exists a constant $R \geqslant 0$ such that*

$$Kg_M + K\Phi(M)\int_0^1 m(s)\,ds \leqslant M,$$

where $K = \sup\big\{\|R(t)\| : t \in I\big\}$, then the problem (2.2) *has at least one mild solution.*

Proof. Define $F : \mathscr{C}(I;X) \to \mathscr{C}(I;X)$ by

$$(Fu)(t) = R(t)g(u) + \int_0^t R(t-s)f(s,u(s))\,ds, \quad t \in I$$

for all $u \in \mathscr{C}(I;X)$.

First we show that F is a continuous map. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{C}(I;X)$ such that $u_n \to u$ (in the norm of $\mathscr{C}(I;X)$) Note that

$$\|F(u_n) - F(u)\| \leqslant K\|g(u_n) - g(u)\| + K\int_0^1 \|f(s,u_n(s)) - f(s,u(s))\|\,ds,$$

by hypotheses ***(H2)*** and ***(H3)*** and Lebesgue dominated convergence theorem we have that $\|F(u_n) - F(u)\| \to 0$ as $n \to \infty$.

Now denote by $B_M = \{u \in \mathscr{C}(I;X) : \|u(t)\|_X \leqslant M, \text{ for all } t \in I\}$ and note that for any $u \in B_M$ we have

$$\begin{aligned}\|(Fu)(t)\| &\leqslant \|R(t)g(u)\| + \left\|\int_0^t R(t-s)f(s,u(s))ds\right\| \\ &\leqslant Kg_M + K\Phi(M)\int_0^1 m(s)ds \leqslant M.\end{aligned}$$

Therefore, $F : B_M \to B_M$ and $F(B_M)$ is a bounded set. Moreover, since the family of operators $\{R(t)\}_{t\in I}$ has the property ***(EP)***, we have that $F(B_M)$ is an equicontinuous set of functions.

Define $\mathfrak{B} = \overline{co}(F(B_M))$. It follows from Lemma 1.4 that the set $\mathfrak{B}$ is equicontinuous and the operator $F : \mathfrak{B} \to \mathfrak{B}$ is bounded and continuous. In addition, $F(\mathfrak{B})$ is a bounded set of functions.

From properties of Hausdorff measure of non–compactness we have that

$$\begin{aligned}\zeta(F(\mathfrak{B}(t))) &\leqslant \zeta\left\{R(t)g(\mathfrak{B})\right\} + \zeta\left(\int_0^t R(t-s)f(s,\mathfrak{B}(s))ds\right) \\ &\leqslant K\zeta\{g(\mathfrak{B})\} + K\int_0^t \zeta\{f(s,\mathfrak{B}(s))\}ds \\ &\leqslant K\int_0^t H(s)\zeta\{\mathfrak{B}(s)\}ds \leqslant K\gamma(\mathfrak{B})\int_0^t H(s)ds\end{aligned}$$

Since $H \in L^1(I;X)$ for $\delta < 1/K$ there exists $\varphi \in \mathscr{C}(I;\mathbb{R}^+)$ satisfying $\int_0^1 |H(s)-\varphi(s)|ds < \delta$. Therefore,

$$\begin{aligned}\zeta(F(\mathfrak{B}(t))) &\leqslant K\gamma(\mathfrak{B})\left[\int_0^t |H(s)-\varphi(s)|ds + \int_0^t \varphi(s)ds\right] \\ &\leqslant K\gamma(\mathfrak{B})\left[\delta + Nt\right],\end{aligned}$$

where $N = \|\varphi\|_\infty$. Thus , we have

$$\zeta(F(\mathfrak{B}(t))) \leqslant (a+bt)\gamma(\mathfrak{B}), \quad \text{where } a = \delta K \text{ and } b = KN.$$

Since $(F(\mathfrak{B})$ is an equicontinuous set of functions we have that

$$\gamma(F(\mathfrak{B})) \leqslant (a+b)\gamma(\mathfrak{B}).$$

In the same manner, it follows from properties of Hausdorff measure of non–compactness that

$$\begin{aligned}
\zeta(F^2(\mathfrak{B}(t))) &\leqslant \zeta(R(t)g(F^1\mathfrak{B})) + \zeta\left(\int_0^t R(t-s)f(s,F^1\mathfrak{B}(s))ds\right)\\
&\leqslant K\int_0^t \zeta\{f(s,F^1\mathfrak{B}(s)))\}ds \leqslant K\int_0^t H(s)\zeta\{F^1\mathfrak{B}(s)\}ds\\
&\leqslant K\int_0^t \big|H(s)-\varphi(s)\big|\zeta\{F^1\mathfrak{B}(s)\}ds + K\int_0^t \varphi(s)\zeta\{F^1\mathfrak{B}(s)\}ds\\
&\leqslant K\delta(a+bt)\gamma(\mathfrak{B}) + KN\left(at+\frac{bt^2}{2}\right)\gamma(\mathfrak{B})\\
&\leqslant \left[a(a+bt)+b\left(at+\frac{bt^2}{2}\right)\right]\gamma(\mathfrak{B}).
\end{aligned}$$

Therefore, for all $t \in I$ we have

$$\zeta(F^2(\mathfrak{B}(t))) \leqslant \left(a^2+2abt+\frac{(bt)^2}{2}\right)\gamma(\mathfrak{B}).$$

Furthermore, since $F^2(\mathfrak{B})$ is an equicontinuous set of functions, we have that

$$\gamma(F^2\mathfrak{B}) \leqslant \left(a^2+2ab+\frac{b^2}{2}\right)\gamma(\mathfrak{B}).$$

It follows from an inductive process that for all $n \in \mathbb{N}$, it hold

$$\zeta(F^n(\mathfrak{B}(t))) \leqslant \left(a^n + C_1^n a^{n-1}bt + C_2^n a^{n-2}\frac{(bt)^2}{2!} + C_3^n a^{n-3}\frac{(bt)^3}{3!}\cdots + \frac{(bt)^n}{n!}\right)\gamma(\mathfrak{B}).$$

In fact, suppose that

$$\zeta(F^n(\mathfrak{B}(t))) \leqslant \left(a^n + C_1^n a^{n-1}bt + C_2^n a^{n-2}\frac{(bt)^2}{2!} + \cdots + \frac{(bt)^n}{n!}\right)\gamma(\mathfrak{B}).$$

By properties of Hausdorff measure of non–compactness we have that for $(n+1)$ it holds

$$\begin{aligned}
\zeta(F^{n+1}(\mathfrak{B}(t))) &\leqslant \zeta\{R(t)g(F^n(\mathfrak{B}))\} + \zeta\left(\int_0^t R(t-s)f(s, F^n(\mathfrak{B}(s)))ds\right) \\
&\leqslant K\int_0^t \zeta\{f(s, F^n\mathfrak{B}(s)))\}ds \leqslant K\int_0^t H(s)\zeta\{F^n\mathfrak{B}(s)\}ds \\
&\leqslant K\int_0^t \left|H(s)-\varphi(s)\right|\zeta\{F^n\mathfrak{B}(s)\}ds + K\int_0^t \varphi(s)\zeta\{F^n\mathfrak{B}(s)\}ds \\
&\leqslant K\delta\left(a^n + C_1^n a^{n-1}bt + C_2^n a^{n-2}\frac{(bt)^2}{2!} + \cdots + \frac{(bt)^n}{n!}\right)\gamma(\mathfrak{B}) \\
&+ KN\left(a^n t + C_1^n a^{n-1}\frac{bt^2}{2} + C_2^n a^{n-2}\frac{b^2t^3}{3!} + \cdots + \frac{b^n t^{n+1}}{(n+1)!}\right)\gamma(\mathfrak{B}) \\
&\leqslant a\left(a^n + C_1^n a^{n-1}bt + C_2^n a^{n-2}\frac{(bt)^2}{2!} + \cdots + \frac{(bt)^n}{n!}\right)\gamma(\mathfrak{B}) \\
&+ b\left(a^n t + C_1^n a^{n-1}\frac{bt^2}{2} + C_2^n a^{n-2}\frac{b^2t^3}{3!} + \cdots + \frac{b^n t^{n+1}}{(n+1)!}\right)\gamma(\mathfrak{B}).
\end{aligned}$$

From the fact that, for all $1 \leqslant m \leqslant n$, it holds $C_{m-1}^n + C_m^n = C_m^{n+1}$ we have have

$$\zeta(F^{n+1}(\mathfrak{B}(t))) \leqslant \left(a^{n+1} + C_1^{n+1}a^n(bt) + C_2^{n+1}a^{n-1}\frac{(bt)^2}{2!} + \cdots + \frac{(bt)^{n+1}}{(n+1)!}\right)\gamma(\mathfrak{B}).$$

Hence, for all $n \in \mathbb{N}$, it hold

$$\zeta(F^n(\mathfrak{B}(t))) \leqslant \left(a^n + C_1^n a^{n-1}bt + C_2^n a^{n-2}\frac{(bt)^2}{2!} + C_3^n a^{n-3}\frac{(bt)^3}{3!}\cdots + \frac{(bt)^n}{n!}\right)\gamma(\mathfrak{B}).$$

Since, for all $n \in \mathbb{N}$ the set of functions $F^n(\mathfrak{B})$ is equicontinuous we have that

$$\gamma(F^n(\mathfrak{B})) \leqslant \left(a^n + C_n^1 a^{n-1}b + C_n^2 a^{n-2}\frac{b^2}{2!} + C_n^3 a^{n-3}\frac{b^3}{3!}\cdots + \frac{b^n}{n!}\right)\gamma(\mathfrak{B}).$$

Furthermore, since $0 \leqslant a < 1$ and $b > 0$, it follows from Lemma 1.4 that there exists $n_0 \in \mathbb{N}$ such that

$$a^{n_0} + C_{n_0}^1 a^{n_0-1}b + C_{n_0}^2 a^{n_0-2}\frac{b^2}{2!} + \cdots + \frac{b^{n_0}}{n_0!} = r < 1$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that $\gamma(F^{n_0}(\mathfrak{B})) \leqslant r\gamma(\mathfrak{B})$, with $r < 1$. It follows from Theorem 1.5 that F has a fixed point in $\mathfrak{B}$. This fixed point is a mild solution of equation (2.2). ∎

Our next result is related with a special case of equation (2.2). Consider the following equation

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t b(t-s)Au(s)ds + f(t,u(t)), \quad t \in I \\ u(0) &= g(u). \end{aligned}\right\} \tag{2.7}$$

where A is a closed linear operator defined on a Hilbert space $\mathscr{H}$, the kernel $b \in L^1_{loc}(0,1;X)$, and f is a given function.

To prove existence of mild solutions of equation (**??**), we will need the following definitions introduced in [134].

Let $a \in L^1_{loc}(\mathbb{R}^+;X)$. We say that a is Laplace transformable if there exists a constant $\omega \in \mathbb{R}$ such that $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$. We denote $\widetilde{a}(\lambda) = \int_0^\infty e^{-\lambda t}a(t)dt$, with $Re\lambda > \omega$

Definition 2.3. *Let $a \in L^1_{loc}(\mathbb{R}^+;X)$ be Laplace transformable and $k \in \mathbb{N}$. We say that $a(t)$ is k–regular if there exists a constant $C > 0$ such that*

$$|\lambda^n \widetilde{a}^{(n)}(\lambda)| \leqslant C|\widetilde{a}(\lambda)|$$

for all $Re\lambda \geqslant \omega$, $0 < n \leqslant k$.

Convolutions of k–regular kernels are again k–regular. Moreover, integration and differentiation are operations which preserve k–regularity as well. See [134, pp. 70].
We recall the following concept introduced in [134].

Definition 2.4. *Let $f \in C^\infty(I;X)$. We will say that f is completely monotone if and only if $(-1)^n f^{(n)}(\lambda) \geqslant 0$ for all $\lambda > 0$ and $n \in \mathbb{N}$.*

Definition 2.5. *Let $a \in L^1_{loc}(I;X)$ such that a is Laplace transformable. We say that a is completely positive if and only if*

$$\frac{1}{\lambda\widetilde{a}(\lambda)} \quad \text{and} \quad \frac{-\widetilde{a}'(\lambda)}{(\widetilde{a}(\lambda))^2}$$

are completely monotone.

Finally, we recall that one-parameter family $\{S(t)\}_{t\geqslant 0}$ of operators is said to be exponentially bounded of type (M,ω) if there are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leqslant Me^{\omega t}, \text{ for all } t \geqslant 0.$$

The next Proposition guarantees existence of a resolvent operator of equation (2.7), furthermore this resolvent operator is continuous in norm for $t > 0$. For this purpose we will introduce the conditions ***(C1)*** and ***(C2)***.

(C1) For all $t \in I$, the kernel a defined by $a(t) = \int_0^t b(s)ds + 1$, for all $t \in I$, is 2–regular and completely positive.

(C2) There exists $\mu_0 > \omega$ such that

$$\lim_{|\mu|\to\infty} \left\| \frac{1}{\widetilde{b}(\mu_0 + i\mu) + 1} \left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1} - A \right)^{-1} \right\| = 0$$

Proposition 2.1. *Suppose that A is the generates a C_0–semigroup of type (M,ω) on $\mathscr{H}$ a Hilbert space. If the conditions **(C1)–(C2)** are satisfied then, there exists a resolvent operator $\{R(t)\}_{t\in I}$ for equation* (2.7) *which is continuous in norm for $t > 0$.*

Proof. Integrating on time equation (2.7) we get

$$u(t) = \int_0^t a(t-s)Au(s)ds + \int_0^t f(s,u(s)) + g(u). \tag{2.8}$$

Since the scalar kernel a is completely positive and A generates a C_0–semigroup, it follows from [134, Theorem 4.2] that there exists a family of operators $\{R(t)\}_{t\in I}$ strongly continuous, exponentially bounded that commutes with A, satisfying

$$R(t)x = x + \int_0^t a(t-s)AR(s)xds. \tag{2.9}$$

On the other hand, using hypothesis ***(C2)*** and since the scalar kernel a is 2–regular, it follows from [109, Theorem 2.2] that $\{R(t)\}_{t\in I}$ is continuous on $\mathscr{L}(\mathscr{H})$ for $t > 0$. Further, since $a \in C^1(\mathbb{R}^+)$, it follows from equation (2.9) that $R(t)$ is differentiable for all $t > 0$ and satisfies

$$\frac{d}{dt}R(t)x = AR(t)x + \int_0^t b(t-s)AR(s)xds, \quad t > 0. \tag{2.10}$$

From the preceding equality, we conclude that $\{R(t)\}_{t\in I}$ is a resolvent operator for equation (2.7) which is norm continuous. ∎

Corollary 2.1. *Suppose that A generates a C_0–semigroup of type (M,ω) on $\mathscr{H}$ a Hilbert space and conditions **(C1)–(C2)** are fulfilled. If the hypothesis **(H2)–(H5)** are satisfied and there exists $M \geqslant 0$ such that*

$$Kg_M + K\Phi(M)\int_0^1 m(s)ds \leqslant M, \;\; where \;\; K = \sup\{\|R(t)\| : t \in I\},$$

then equation (2.7) *has at least a mild solution.*

Proof. It follows from Proposition 2.1 that the equation (2.7) admits a resolvent operator $\{R(t)\}_{t\in I}$ satisfying the property ***(EP)***. Moreover, since hypothesis ***(H2)-(H5)*** are satisfied, we apply Theorem 2.1 and conclude that equation (2.7) has a mild solution. ∎

2.2 An Example

In this section we prove the feasibility of our abstract results applying them to a concrete partial differential equation with non–local initial condition. Let $X = L^2(\mathbb{R}^n)$ and consider the following integro-differential equation

$$\left.\begin{array}{rcl} \dfrac{\partial w(t,\xi)}{\partial t} & = & Aw(t,\xi) + \displaystyle\int_0^t \beta e^{-\alpha(t-s)} Aw(s,\xi)ds + t^{-1/3}\cos(w(t,\xi)), \quad t \in I. \\ w(0,\xi) & = & \displaystyle\sum_{i=1}^{N} \int_{\mathbb{R}^n} qk(\xi,y)w(t_i,y)dy, \;\; \xi \in \mathbb{R}^n. \end{array}\right\} \tag{2.11}$$

where N is a positive integer, $0 < t_1 < t_2 < \cdots < t_N < 1;\; k \in L^2(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^+),\; q \in \mathbb{R}^+$, the constants α, β satisfy $-\alpha \leqslant \beta \leqslant 0 \leqslant \alpha$. The operator A is defined by

$$(Aw)(t,\xi) = \sum_{i,j=1}^{n} a_{ij}(z)\frac{\partial w(t,\xi)}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{n} b_i(\xi)\frac{\partial w(t,\xi)}{\partial \xi_i} + \overline{c}(\xi)w(t,\xi),$$

with given coefficients a_{ij} , b_i , $\overline{c}$, $(i, j = 1,2,\dots,n)$ satisfying the usual uniformly ellipticity conditions, and $D(A) = \{v \in X : v \in H^2(\mathbb{R}^n)\}$.

We will prove that there exists $q > 0$ sufficiently small such that equation (2.11) has a mild solution on X.
In what follows, we identify $u(t) = w(t,\cdot)$. With this identification the preceding equation takes the abstract form

$$\left.\begin{array}{rcl} u'(t) & = & Au(t) + \displaystyle\int_0^t b(t-s)Au(s)ds + f(t,u(t)), \quad t \in I \\ u(0) & = & g(u), \end{array}\right\} \tag{2.12}$$

where the function $g : \mathscr{C}(I;X) \to X$ is given by $g(u) = \sum_{i=1}^{N} qk_g(u(t_i))$ with

$$(k_g v)(\xi) = \int_{\mathbb{R}^n} k(\xi,y)v(y)dy, \quad \text{for } v \in X, \xi \in \mathbb{R}^n,$$

the function $f : I \times X \to X$ is defined by $f(t,u(t)) = t^{-1/3}\cos(u(t)) = t^{-1/3}\cos(w(t,\cdot))$, and the kernel b is given by $b(t) = \beta e^{-\alpha t}$.

Note that $\|g(u)\| \leqslant qN\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k^2(z,y)dydz\Big)^{1/2}\|u\|$, and the function k_g is completely continuous.
Furthermore, the function f satisfies $\|f(t,u(t))\| \leqslant t^{-1/3}\Phi(\|u\|)$, with $\Phi(\|u\|) \equiv 1$ and $\|f(t,u_1) - f(t,u_2)\| \leqslant t^{-1/3}\|u_1 - u_2\|$.
Therefore, conditions ***(H2)–(H5)*** are fulfilled.

Define $a(t) = \displaystyle\int_0^t \beta e^{-\alpha s} ds + 1$, for all $t \in I$. Since the kernel b defined by $b(t) = \beta e^{-\alpha t}$ is 2–regular, it follows that a is 2–regular. Furthermore, we claim that a is completely positive. In fact, we have

$$\widetilde{a}(\lambda) = \frac{\lambda + \alpha + \beta}{\lambda(\lambda + \alpha)}.$$

Define the functions f_1 and f_2 by $f_1(\lambda) = \dfrac{1}{\lambda \widetilde{a}(\lambda)}$ and $f_2(\lambda) = \dfrac{-\widetilde{a}'(\lambda)}{[\widetilde{a}(\lambda)]^2}$ respectively. In another words

$$f_1(\lambda) = \frac{\lambda + \alpha}{\lambda + \alpha + \beta} \quad \text{and} \quad f_2(\lambda) = \frac{\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta + \alpha^2}{(\lambda + \alpha + \beta)^2}.$$

Direct calculation shows that

$$f_1^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(n+1)!}{(\lambda + \alpha + \beta)^{n+1}} \quad \text{and} \quad f_2^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(\alpha + \beta)(n+1)!}{(\lambda + \alpha + \beta)^{n+2}} \quad \text{for } n \in \mathbb{N}.$$

Since $-\alpha \leqslant \beta \leqslant 0 \leqslant \alpha$, we have that f_1 and f_2 are completely monotone. Thus, the kernel a is completely positive.

On the other hand, it follows from [55] that A generates an analytic, non compact semigroup $\{T(t)\}_{t \geqslant 0}$ on $L^2(\mathbb{R}^n)$. Furthermore, there exists a constant $M > 0$ such that

$$M = \sup\{\| T(t) \| : t \geqslant 0\} < +\infty.$$

It follows from this fact and Hille–Yosida theorem that $z \in \rho(A)$ for $Re(z) > 0$.
A direct calculation shows that,

$$Re\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1}\right) = \frac{\mu_0^3 + \mu_0^2\alpha + \mu_0^2(\alpha + \beta) + \mu_0\alpha(\alpha + \beta) + \mu_0\mu^2 - \mu^2\beta}{(\alpha + \beta)^2 + 2\mu_0(\alpha + \beta) + \mu_0^2 + \mu^2}.$$

Hence, for $\mu_0 > 0$, we have that, $Re\left(\dfrac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1}\right) > 0$. This implies that

$$\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1} - A\right)^{-1} \in \mathscr{L}(X).$$

Moreover,

$$\left\| \frac{1}{\widetilde{b}(\mu_0 + i\mu) + 1}\left(\frac{\mu_0 + i\mu}{\widetilde{b}(\mu_0 + i\mu) + 1} - A\right)^{-1} \right\| \leqslant \left\| \frac{M}{\mu_0 + i\mu} \right\|.$$

Therefore,

$$\lim_{|\mu|\to\infty}\left\|\frac{1}{\widetilde{b}(\mu_0+i\mu)+1}\left(\frac{\mu_0+i\mu}{\widetilde{b}(\mu_0+i\mu)+1}-A\right)^{-1}\right\|=0.$$

It follows from Proposition 2.1, equation (2.11) admits a resolvent operator $\{R(t)\}_{t\in I}$ satisfying property ***(EP)***. Let $K=\sup\{\|R(t)\|: t\in I\}$ and $c=N\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}k^2(z,y)dydz\Big)^{1/2}$. A direct computation shows that for each $M\geqslant 0$ the number g_M is equal to $g_M=qcM$.

Therefore the expression $g_M K+K\Phi(M)\int_0^1 m(s)ds$, is equivalent to $qcKM+\dfrac{3K}{2}$.

Since, there exists $q>0$ such that $qcK<1$ it follows that, there exists $M\geqslant 0$ satisfying

$$qcKM+\frac{3K}{2}\leqslant M.$$

From Corollary 2.1 we conclude that there exists a mild solution of equation (2.11).

CHAPTER 3

Mild Solutions of Second Order Non–autonomous Cauchy Problem with non–local initial conditions

As we have mentioned in preceding Chapter, the study of mild solutions of evolution equations is very important because it has a lot of applications. In particular, the study mild solutions for problems with non–local initial conditions have taken great interest in last decades.

This chapter is devoted to study the existence of mild solutions of non–local initial value problem described as a second order non–autonomous abstract differential problem

$$\left.\begin{array}{rcl} u''(t) & = & A(t)u(t) + f(t,u(t)) \quad t \in [0,a] \\ u(0) & = & g(u) \\ u'(0) & = & h(u) \end{array}\right\} \tag{3.1}$$

For $t \in J = [0,a]$, $A(t) : D(A(t)) \subseteq X \to X$ for $t \in J = [0,a]$ denote closed linear operators defined in a Banach space X. We assume that $D(A(t)) = D$ for all $t \in J$. The function $f : J \times \mathscr{C}(J;X) \to X$ satisfies Carathéodory type conditions, and the functions $g, h : \mathscr{C}(J;X) \to X$ are continuous maps.

There exists an extensive literature concerning second order problems. In the autonomous case, this is $A(t) \equiv A$, for all $t \in J$, the solutions of these equations are closely related with the concept of *cosine families*. We refer the reader to [58, 140, 141, 142, 144] for basic concepts about these families.

In the same manner, the study of mild solutions of equation (3.1) are closely related with the concept of evolution operator $\{S(t,s)\}_{t,s \in J}$. In the literature several techniques have been discussed to establish the existence of the evolution operator $\{S(t,s)\}_{t,s\in J}$. In particular, a very studied situation is that $A(t)$ is the perturbation of an operator A that generates a cosine operator function. For this reason, below we briefly review some essential properties of the theory of cosine functions. We will mention a few of properties and notations needed to establish our main results.

Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a strongly continuous cosine family $\{C_0(t)\}_{t\geqslant 0}$ of bounded linear operators in X and let $\{S_0(t)\}_{t\geqslant 0}$ be the sine family associated with $\{C_0(t)\}_{t\geqslant 0}$, which is defined by $S_0(t)x = \int_0^t C_0(s)x ds$, for $x \in X$ and $t \in \mathbb{R}$. It is immediate that

$$C_0(t)x - x = A\int_0^t S_0(s)x ds, \quad \text{for all } \ x \in X \ \text{ and } \ t \geqslant 0.$$

The notation E stands for the space consisting of vectors $x \in X$ such that the function $C_0(\cdot)x$ is of class C^1. Kisyński in [101] has proved that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\leqslant t\leqslant 1} \|AS_0(t)x\|, \quad x \in E,$$

is a Banach space. It is known that the operator valued function $G(t) = \begin{bmatrix} C_0(t) & S_0(t) \\ AS_0(t) & C_0(t) \end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space $E \times X$. generated by the operator $\mathscr{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from this property that $S_0(t) : X \to E$ is a bounded linear operator such that the operator valued map $S_0(\cdot)$ is strongly continuous and $AS_0(t) : E \to X$ is a bounded linear operator such that, for each $x \in E$, satisfies $AS_0(t)x \to 0$ as $t \to 0$. Furthermore, recall that, if $f : [0,\infty) \to X$ is a locally integrable function, then $u(t) = \int_0^t S_0(t-s)f(s)ds$ defines an E–valued continuous function.

We finally mention that the function $u(\cdot)$ given by

$$u(t) = C_0(t-s)x + S_0(t-s)y + \int_s^t S_0(t-\xi)f(\xi)d\xi, \quad t \in J, \tag{3.2}$$

is called mild solution of the problem

$$\left.\begin{array}{rcl} u''(t) &=& Au(t) + f(t), \quad t \in J, \\ u(s) &=& x, \\ u'(s) &=& y. \end{array}\right\} \tag{3.3}$$

If $x \in E$, the function $u(\cdot)$ given by (3.2) is continuously differentiable and

$$u'(t) = AS_0(t-s)x + C_0(t-s)y + \int_s^t C_0(t-\xi)f(\xi)d\xi, \quad t \in J. \tag{3.4}$$

If $x \in D(A)$, $y \in E$ and f is a continuously differentiable function, then the function $u(\cdot)$ is a classical solution of problem (3.3).

Non–autonomous second order problems have received much attention in recent years due their applications in different fields. Specially, many authors have studied the initial value Cauchy equation

$$\left.\begin{array}{rcl} u''(t) & = & A(t)u(t)+f(t) \quad t \in J \\ u(s) & = & x \\ u'(s) & = & y. \end{array}\right\} \tag{3.5}$$

We refer the reader to [17, 78, 107, 138, 150]. As we have mentioned, the existence of solutions of this equation is related with the existence of the evolution operator $\{S(t,s)\}_{t,s\in J}$ for homogeneous equation

$$\left.\begin{array}{rcl} u''(t) & = & A(t)u(t), \quad t \in J. \\ u(s) & = & x \\ u'(s) & = & y. \end{array}\right\} \tag{3.6}$$

In this thesis, we will use the concept of evolution operator $\{S(t,s)\}_{t,s\in J}$ associated with equation (3.6) introduced by Kozak in [103]. With this purpose, we assume that the domain of $A(t)$ is a subspace D dense in X and independent of $t \in J$, and for each $x \in D$ the function $t \to A(t)x$ is continuous.

Definition 3.1. *A map $S : J \times J \to \mathscr{L}(X)$ is said to be an evolution operator of equation* (3.6) *if the following conditions are fulfilled:*

(D1) *For each $x \in X$ the map $(t,s) \to S(t,s)x$ is continuously differentiable and*

- ***(a)*** *For each $t \in J$, $S(t,t) = 0$.*
- ***(b)*** *For all $t,s \in J$ and each $x \in X$, $\frac{\partial}{\partial t}S(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}S(t,s)x|_{t=s} = -x$.*

(D2) *For all $t,s \in J$, if $x \in D$, then $S(t,s)x \in D$, the map $(t,s) \to S(t,s)x$ is of class C^2 and*

- ***(a)*** $\frac{\partial^2}{\partial t^2}S(t,s)x = A(t)S(t,s)x$.
- ***(b)*** $\frac{\partial^2}{\partial s^2}S(t,s)x = S(t,s)A(s)x$.
- ***(c)*** $\frac{\partial^2}{\partial s \partial t}S(t,s)x|_{t=s} = 0$.

(D3) *For all $s,t \in J$, if $x \in D$ then $\frac{\partial}{\partial t}S(t,s)x \in D$. Further, there exist $\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x$ and*

- ***(a)*** $\frac{\partial^3}{\partial t^2 \partial s}S(t,s)x = A(t)\frac{\partial}{\partial s}S(t,s)x$. *Moreover, the map $(t,s) \to A(t)\frac{\partial}{\partial s}S(t,s)x$ is continuous.*
- ***(b)*** $\frac{\partial^3}{\partial s^2 \partial t}S(t,s)x = A(t)\frac{\partial}{\partial t}S(t,s)x$.

Assuming that $f : J \to X$ is an integrable function, Kozak in [103] has proved that the function $u : J \to X$ given by

$$u(t) = -\frac{\partial}{\partial s} S(t,s)x + S(t,s)y + \int_s^t S(t,\xi) f(\xi) d\xi, \tag{3.7}$$

is a mild solution of equation (3.6). Motivated by this result, we will say that a function $u \in \mathscr{C}(J;X)$ is a ***mild solution*** of problem (3.1) if verifies

$$u(t) = -\frac{\partial}{\partial s} S(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi) f(\xi, u(\xi)) d\xi. \tag{3.8}$$

Henceforth, we assume that there exists an evolution operator $\{S(t,s)\}_{t \leqslant s, t \leqslant a}$ associated with equation (3.6). To abbreviate the text, we introduce the operator $C(t,s) = -\frac{\partial S(t,s)}{\partial s}$. With this notation, the mild solution of problem (3.1) is

$$u(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi) f(\xi, u(\xi)) d\xi. \tag{3.9}$$

In addition, we set $K > 0$ such that

$$\sup_{t,s \in J} \|S(t,s)\| \leqslant K \quad \text{and} \quad \sup_{t,s \in J} \|C(t,s)\| \leqslant K. \tag{3.10}$$

We denote N_1 a positive constant such that

$$\|S(t+h,s) - S(t,s)\| \leqslant N_1 |h| \quad \text{for all} \quad t, s, t+h \in J. \tag{3.11}$$

3.1 Main Results

In this section we will present our main results. In the same manner as that of Chapter 2, an argument to prove existence of mild solution of non–autonomous problem (3.1) is using fixed–point Theorems. The Theorems that we will use are related with the concept of measure of non–compactness.

Similarly to Chapter 2, in order to give a condition that we will need in the proof of main results of this Chapter, we consider the following list of assertions.

(Cgh1) The functions g and h are continuous maps. Further, for each $M \geqslant 0$ the numbers g_M and h_M defined by

$$g_M = sup\{\|g(u)\| : \|u\| \leqslant M\} \quad \text{and} \quad h_M = sup\{\|h(u)\| : \|u\| \leqslant M\}$$

are finite.

(Cgh2) The functions f and g satisfy

$$\gamma(g(B)) + \gamma(h(B)) < \frac{1}{2K}\gamma(B)$$

for all bounded set of continuous functions B. Here the constant K is defined as in inequality (3.10).

(Cg1) The map from $\mathscr{C}(J;X)$ to $\mathscr{C}(J;X)$ given by $u(\cdot) \to C(\cdot,0)g(u)$ takes bounded sets into equicontinuous sets.

(Cg2) The function $g : \mathscr{C}(J;X) \to X$ satisfies $\gamma(g(B)) < \dfrac{1}{K}\gamma(B)$ for all bounded set of functions $B \subseteq \mathscr{C}(J;X)$. Here the constant K is defined as in equation 3.10.

(CS1) The evolution operator $\{S(t,s)\}_{t,s\in J}$ is a compact family of operators.

(Cf1) The map $f : J \times X \to X$ satisfies the Carathéodory type conditions, that is, $f(\cdot,x)$ is measurable for all $x \in X$ and $f(t,\cdot)$ is continuous for almost all $t \in J$.

(Cf2) There exist functions $m \in L^1(J;\mathbb{R}^+)$ and non–decreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x)\| \leqslant m(t)\Phi(\|x\|)$$

for all $x \in X$ and almost all $t \in J$.

(Cf3) There exists a function $H \in L^1(J;\mathbb{R}^+)$ such that for any subset of functions $S \subseteq \mathscr{C}(J;X)$, we have

$$\zeta(f(t,S(t))) \leqslant H(t)\zeta(S(t))$$

for almost all $t \in J$.

Condition **(Cg1)** play a crucial role in the proof of Theorem 3.1, for this reason we will show several situations where this condition is valid.

Lemma 3.1. *Let $g : \mathscr{C}(J;X) \to X$ be a continuous map. If g is a compact function then the function $u(\cdot) \to C(\cdot,0)g(u)$ takes bounded sets into equicontinuous sets.*

Proof. Let $S \subseteq \mathscr{C}(J;X)$ be a bounded set of continuous functions, this is, there exists $M > 0$ such that $\|u\| \leqslant M$ for all $u \in S$. Let $\varepsilon > 0$ be an arbitrary positive number. Since the function g is compact we have that there exist $x_1, x_2, \ldots, x_p \in X$ such that $g(S) \subseteq \bigcup_{i=1}^{p} B\left(x_i, \dfrac{\varepsilon}{4K}\right)$.
For $i \in \{1,2\ldots,p\}$, define the functions $f_i : J \to X$ given by $f_i(t) = C(t,0)x_i$. Clearly these functions are continuous. Furthermore, since all functions f_i are defined in a

compact set it follows that for all $i \in \{1,2,\dots,p\}$ the functions f_i are uniformly continuous. Hence, for $i \in \{1,2,\dots,p\}$ there exists $h_i > 0$ such that for all $t \in J$ we have $\|(C(t+h_i,0)-C(t,0))x_i\| \leqslant \frac{\varepsilon}{2}$.

Denote $\delta = \min\{h_i : i = 1,2,\dots,p\}$. Thus, we have that $\|(C(t+\delta,0)-C(t,0))x_i\| \leqslant \frac{\varepsilon}{2}$ for all $t \in J$ and $i \in \{1,2\dots,p\}$.

Let $u \in S$, we know that there exists $i \in \{1,2,\dots,p\}$ such that $\|g(u)-x_i\| \leqslant \frac{\varepsilon}{2}$. Therefore

$$\begin{aligned}\|C(t+\delta,0)g(u)-C(t,0)g(u)\| &\leqslant \|(C(t+\delta,0)-C(t,0))(g(u)-x_i)\| \\ &\quad + \|(C(t+\delta,0)-C(t,0))x_i\| \\ &\leqslant 2K\|g(u)-x_i\| + \frac{\varepsilon}{2} \\ &\leqslant \varepsilon.\end{aligned}$$

This argument is valid for any $u \in S$. Since, δ is independent of u and t we have that the set

$$\{C(\cdot,0)g(u) \,:\, u \in S\}$$

is an equicontinuous set of continuous functions. ■

Lemma 3.2. *Assume that $A(t) = A + B(t)$ for all $t \in J$, here the operator A is the generator of a cosine family $\{C_0(t)\}_{t\in J}$ defined on X, and $B(t) \in \mathscr{L}(E;X)$ for all $t \in J$, and $B(t)z \in C^1(J;X)$ for $z \in E$. Let $g : \mathscr{C}(J;X) \to X$ be a continuous map such that $g(u) \in E$ for all $u \in \mathscr{C}(J;X)$, then the function $u(\cdot) \to C(\cdot,0)g(u)$ takes bounded sets into equicontinuous sets.*

Proof. Under the hypothesis, it has been proved by Serizawa and Watanabe in [139] that $\{C(t,s)\}_{t,s\in J}$ is differentiable in E. ■

In the same manner as that of Chapter 2, henceforth we assume that the following assertions hold:

Theorem 3.1. *Suppose that the function g and h are compact maps and the conditions* ***(Cgh1), (Cf1), (Cf2)*** *and* ***(Cf3)*** *are fulfilled. If there exists a constant $R \geqslant 0$ such that*

$$K(g_M + h_M) + K\Phi(M)\int_0^a m(s)ds \leqslant M,$$

then the problem (3.1) *has at least one mild solution.*

Proof. Define $F : \mathscr{C}(J;X) \to \mathscr{C}(J;X)$ by

$$(Fu)(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,s)f(s,u(s))ds, \quad t \in J.$$

First, we show that F is a continuous map. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathscr{C}(J;X)$ such that $u_n \to u$ (in the norm of $\mathscr{C}(J;X)$) Note that

$$\begin{aligned}\|F(u_n) - F(u)\| &\leqslant K\|g(u_n) - g(u)\| + K\|h(u_n) - h(u)\| \\ &\quad + K\int_0^a \|f(s,u_n(s)) - f(s,u(s))\|ds.\end{aligned}$$

Since, the functions g and h are continuous maps, it follows from Lebesgue dominated theorem that $\|F(u_n) \to F(u)\| \to 0$ as $n \to \infty$.
Now denote by $B_M = \{u \in \mathscr{C}(J;X) : \|u(t)\| \leqslant M \text{ for all } t \in J\}$ and note that for any $u \in B_M$ we have

$$\begin{aligned}\|(Fu)(t)\| &\leqslant \|C(t,0)g(u)\| + \leqslant \|S(t,0)h(u)\| + \left\|\int_0^t S(t,s)f(s,u(s))ds\right\| \\ &\leqslant K(g_M + h_M) + K\Phi(M)\int_0^a m(s)ds \leqslant M.\end{aligned}$$

Therefore, $F : B_M \to B_M$ and $F(B_M)$ is a bounded set. Moreover, $F(B_M)$ is an equicontinuous set of functions. In fact, let $\varepsilon > 0$ be an arbitrary positive number. By Lemma 3.1 there exists $\delta_1 > 0$ such that $\|(C(t+\delta_1,0) - C(t,0))g(u)\| \leqslant \frac{\varepsilon}{4}$. Choose $\delta_2 > 0$ such that

$$\delta_2 < \min\left\{\delta_1, \frac{\varepsilon}{4h_M N_1}, \frac{\varepsilon}{4K\Phi(M)M}, \frac{\varepsilon}{4\Phi(M)\|m\|_1}\right\},$$

where m_∞ is defined by $m_\infty = \sup\{m(t) \ : \ t \in J\}$ and $\|m\|_1$ denotes the integral norm of function m defined in condition ***(Cf2)***, the function Φ is defined in condition ***(Cf2)***, and K and N_1 have been chosen as inequalities (3.10) and (3.11). The constant h_M is defined in condition ***(Cgh1)***
For all $|t_2 - t_1| < \delta_2$ and for all $u \in B_M$ we have

$$\begin{aligned}\|(Fu)(t_2) - (Fu)(t_1)\| &\leqslant \|C(t_2)g(u) - C(t_1)g(u)\| + \|S(t_2)h(u) - S(t_1)h(u)\| \\ &\quad + K\Phi(M)\int_{t_1}^{t_2} m(s)ds + \Phi(M)\int_0^{t_1}\|S(t_2,s) - S(t_1,s)\|m(s)ds \\ &\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

Define $\mathfrak{B} = \overline{co}(F(B_M))$. It follows from Lemma 1.2 that the set $\mathfrak{B}$ is equicontinuous and the operator $F : \mathfrak{B} \to \mathfrak{B}$ is bounded and continuous. In addition, $F(\mathfrak{B})$ is a bounded set of functions. Same argument made to show that $F(B_M)$ is an equicontinuous set of functions show that $F(\mathfrak{B})$ is an equicontinuous set of functions.

From properties of Hausdorff measure of non–compactness we have that

$$
\begin{aligned}
\zeta(F(\mathfrak{B}(t))) &\leqslant \zeta\big\{C(t,0)g(\mathfrak{B})\big\} + \zeta\big\{S(t,0)h(\mathfrak{B})\big\} + \zeta\left(\int_0^t S(t,s)f(s,\mathfrak{B}(\cdot))ds\right)\\
&\leqslant K\zeta\{g(\mathfrak{B})\} + K\zeta\{h(\mathfrak{B})\} + K\int_0^t \zeta\{f(s,\mathfrak{B}(s))\}ds\\
&\leqslant K\gamma(\mathfrak{B})\int_0^t H(s)ds
\end{aligned}
$$

Since $H \in L^1(J;\mathbb{R}^+)$ for $\delta < 1/K$ there is $\varphi \in \mathscr{C}(J;\mathbb{R}^+)$ such that $\int_0^a |H(s)-\varphi(s)|ds < \delta$. Therefore,

$$
\begin{aligned}
\zeta(F(\mathfrak{B}(t))) &\leqslant K\gamma(\mathfrak{B})\left[\int_0^t |H(s)-\varphi(s)|ds + \int_0^t |\varphi(s)|ds\right]\\
&\leqslant K\delta\gamma(\mathfrak{B}) + KNt,
\end{aligned}
$$

where $N = \|\varphi\|_\infty$. Thus, we have

$$
\gamma(F(\mathfrak{B}(t))) \leqslant (A+Bt)\gamma(\mathfrak{B}) \text{ where } A = K\delta \text{ and } B = KN.
$$

Following the same arguments as those of proof of Theorem 2.1 in Chapter 2, we have that

$$
\gamma(F^n(\mathfrak{B})) \leqslant \left(A^n + C_1^n A^{n-1}Ba + C_2^n A^{n-2}\frac{(Ba)^2}{2!} + \cdots + \frac{(Ba)^n}{n!}\right)\gamma(\mathfrak{B}).
$$

Furthermore, since $0 \leqslant A < 1$ and $Ba > 0$, it follows from Lemma 1.4 that there exists $n_0 \in \mathbb{N}$ such that

$$
A^{n_0} + C_1^{n_0}A^{n_0-1}B + C_2^{n_0}A^{n_0-2}\frac{B^2}{2!} + \cdots + \frac{B^{n_0}}{n_0!} = r < 1
$$

Therefore, $\gamma(F^{n_0}(\mathfrak{B})) \leqslant r\gamma(\mathfrak{B})$, with $r < 1$. It follows from Theorem 1.5 that F has a fixed point in $\mathfrak{B}$. This fixed point is a mild solution of equation (3.1). ■

Theorem 3.1 imposes some restrictive conditions about functions g and h. In fact, non–local initial conditions that arise in specific applications, are very often condensing maps. Motivated by this, the next result imposes much more weak restrictions for g and h.

Theorem 3.2. *Suppose that the functions g and h are condensing maps and the conditions* ***(Cgh1), (Cgh2), (Cf1)*** *and* ***(Cf2)*** *are fulfilled. If there exists a constant $R \geqslant 0$ such that*

$$
K(g_M + h_M) + K\Phi(M)\int_0^a m(s)ds \leqslant M,
$$

then the problem (3.1) *has at least one mild solution.*

Proof. Define $F : \mathscr{C}(J;X) \to \mathscr{C}(J;X)$ by

$$(Fu)(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,s)f(s,u(s))ds, \ \ t \in J$$

for all $u \in \mathscr{C}(J;X)$.
Following the same argument as that of proof of Theorem 3.1, we prove that F is a continuous map. We introduce the mappings

$$F_1 : \mathscr{C}(J;X) \to \mathscr{C}(J;X) \ \ \text{defined by} \ \ (F_1 u)(t) = C(t,0)g(u) + S(t,0)h(u), \ \ t \in J.$$

and

$$F_2 : \mathscr{C}(J;X) \to \mathscr{C}(J;X) \ \ \text{defined by} \ \ (F_2 u)(t) = \int_0^t S(t,s)f(s,u(s))ds, \ \ t \in J$$

Since $\{S(t,s)\}_{t,s\in J}$ is a family of operators such that

$$\|S(t+h,s) - S(t,s)\| \leqslant N_1|h| \ \ \text{for all} \ \ t,s,t+h \in J,$$

it follows from Arzela–Ascoli theorem that F_2 is a compact map. On the other hand F_1 is a γ–condensing map. In fact, $\gamma(F_1B) \leqslant \gamma(C(\cdot,0)g(B)) + \gamma(S(\cdot,0)g(B)) < \gamma(B)$ for all bounded set of functions B.
By hypothesis, there exists a constant $R \geqslant 0$ such that

$$K(g_M + h_M) + K\Phi(M)\int_0^a m(s)ds \leqslant R.$$

Therefore, $F : B_M \to B_M$. It follows from Theorem 1.4 that F has a fixed point in B_M and this fixed point is a mild solution of equation (3.1). ∎

Assume now that $A(t) = A + B(t)$ for all $t \in J$, where $B(\cdot) : J \to \mathscr{L}(E;X)$ is a map such that the function $t \to B(t)x$ is continuously differentiable in X for each $x \in E$. It has been established by Serizawa and Watanabe [139] that for each $(y,z) \in D(A) \times E$ the non–autonomous abstract Cauchy problem

$$\left.\begin{array}{rcl} u''(t) &=& (A + B(t))u(t), \quad t \in J \\ u(0) &=& x, \\ u'(0) &=& y, \end{array}\right\} \tag{3.12}$$

has a unique solution $u(\cdot)$ such that the function $t \to u(t)$ is continuously differentiable in E. It is clear that the same argument allows us to conclude that equation

$$\left.\begin{array}{rcl} u''(t) &=& (A + B(t))u(t), \quad t \in J \\ u(s) &=& x, \\ u'(s) &=& y, \end{array}\right\} \tag{3.13}$$

has a unique solution $u(\cdot, s)$ such that the function $t \to u(t,s)$ is continuously differentiable in E. It follows from (3.7) that

$$u(t,s) = C_0(t-s)x + S_0(t-s)y + \int_s^t S_0(t-\xi)B(\xi)u(\xi,s)d\xi$$

In particular, for $x = 0$ we have

$$u(t,s) = S_0(t-s)y + \int_s^t S_0(t-\xi)B(\xi)u(\xi,s)d\xi$$

Consequently,

$$\|u(t,s)\|_1 \leqslant \|S_0(t-s)\|_{\mathscr{L}(X,E)}\|z\| + \int_s^t \|S_0(t-\xi)\|_{\mathscr{L}(X,E)}\|B(\xi)\|_{\mathscr{L}(X,E)}\|u(\xi,s)\|_1 d\xi.$$

Applying the Gronwall–Bellman lemma, there exists a constant $\widetilde{N} \geqslant 0$ such that $\|u(t,s)\|_1 \leqslant \widetilde{N}\|z\|, \quad s,t \in J.$

We define the operator $S(t,s)y = u(t,s)$. It follows from the previous estimate that $S(t,s)$ is a bounded linear map on E. Since E is dense in X , we can extend $S(t,s)$ to X. We keep the notation $S(t,s)$ for this extension.

It is well known that, except in the case $\dim(X) < \infty$, the cosine function $C_0(t)$ cannot be compact for all $t \in \mathbb{R}$. By contrast, for the cosine functions that arise in specific applications, the sine function $S_0(t)$ is very often a compact operator for all $t \in \mathbb{R}$. This motivates the following result proved by Henríquez in [78]. We remark, this result shows a situation where the condition ***(CS1)*** is valid.

Lemma 3.3. *Under the preceding conditions, $S(\cdot,\cdot)$ is an evolution operator for equation* (3.14). *Moreover, if $S_0(t)$ is compact for all $t \in \mathbb{R}$, then $S(t,s)$ is also compact for all $s \leqslant t$.*

Now consider the particular case of equation (3.1).

$$\left.\begin{array}{rcl} u''(t) &=& (A+B(t))u(t) + f(t,u(t)), \quad t \in J \\ u(s) &=& g(u), \\ u'(s) &=& h(u). \end{array}\right\} \qquad (3.14)$$

Next Theorem gives sufficient conditions to guarantee existence of a mild solution of equation (3.14). We remark, this results imposes much more weak conditions for function h. In same manner as those of Theorems 3.1 and 3.2, we enumerate following conditions.

Theorem 3.3. *Suppose that the functions g and h are condensing maps and the conditions **(Cgh1), (Cg2), (Cf1)** and **(Cf2)** are fulfilled. If the evolution operator $\{S_0(t)\}_{t\in J}$ is a compact family of operators and if there exists a constant $M \geqslant 0$ such that*

$$K(g_M + h_M) + K\Phi(M)\int_0^a m(s)ds \leqslant M,$$

then the problem (3.14) *has at least one mild solution.*

Proof. Define $F:\mathscr{C}(J;X)\to\mathscr{C}(J;X)$ by

$$(Fu)(t)=C(t,0)g(u)+S(t,0)h(u)+\int_0^t S(t,s)f(s,u(s))ds,\quad t\in J.$$

Following the same argument as that of proof of Theorem 3.1, we obtain that F is a continuous map. We introduce the mappings

$$F_1:\mathscr{C}(J;X)\to\mathscr{C}(J;X)\quad\text{defined by}\quad (F_1u)(t)=C(t,0)g(u),\quad t\in J,$$

and

$$F_2:\mathscr{C}(J;X)\to\mathscr{C}(J;X)\quad\text{defined by}\quad (F_2u)(t)=S(t,0)h(u)+\int_0^t S(t,s)f(s,u(s))ds,\quad t\in J.$$

It follows from Lemma 3.2 that $\{S(t,s)\}_{t,s\in J}$ is a compact evolution operator, following same argument of proof of Theorem 3.2 we have that F_2 is a compact map. Thus, for all $B\subseteq\mathscr{C}(J;X)$ we have

$$\gamma(F(B))\leqslant\gamma(F_1B)<\gamma(B).$$

Therefore, F is a γ–condensing operator. By Hypothesis, there exists $M\geqslant 0$ such that $F:B_M\to B_M$. It follows from Theorem 1.4 that F has a fixed point in B_M, and this fixed–point is a mild solution of equation (3.14). ∎

***Remark* 3.1.** *Note that proof of Theorem 3.3 still is valid for general $A(\cdot)$ such that the corresponding evolution operator $\{S(t,s)\}_{t,s\in J}$ of equation* (3.1) *is a compact evolution operator.*

3.2 Examples

The one–dimensional wave equation modeled as an abstract Cauchy problem has been studied extensievolutionvely. See for example [154]. In this section, we apply the abstract results established in preceding section to study the existence of solutions of the non–autonomous wave equation with non–local initial conditions. Specifically, we will study the following problem

$$\left.\begin{array}{rcl}\dfrac{\partial^2 w(t,\xi)}{\partial t^2} &=& \dfrac{\partial^2 w(t,\xi)}{\partial\xi^2}+b(t)\dfrac{\partial w(t,\xi)}{\partial\xi}+F(t,w(t,\xi)),\quad t\in J,\ \text{and}\ 0\leqslant\xi\leqslant 2\pi\\ w(t,0) &=& w(t,2\pi)\quad t\in J\\ w(0,\xi) &=& \displaystyle\sum_{i=0}^m g_i w(t_i,\xi)\\ \dfrac{\partial w(0,\xi)}{\partial t} &=& \displaystyle\sum_{i=0}^m h_i w(t_i,\xi)\end{array}\right\}$$

We model this problem in the space $X = L^2(\mathbb{T};\mathbb{R})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{T}$ and 2π–periodic functions on $\mathbb{R}$. Specifically, in what follows we denote by $L^2(\mathbb{T};\mathbb{R})$ the space of 2π–periodic 2–integrable functions from $\mathbb{R}$ into $\mathbb{R}$. Similarly, $H^2(\mathbb{T};\mathbb{R})$ denotes the Sobolev space of 2π–periodic functions $u:\mathbb{R}\mapsto\mathbb{R}$ such that $u'' \in L^2(\mathbb{T};\mathbb{R})$.

In what follows, we use the identification $u(t) = w(t,\cdot)$ for all $t \in J$. In another words, for all $t \in J$, we have the the function $u(t) : [0,2\pi] \to \mathbb{R}$ defined by $u(t)(\xi) = w(t,\xi)$. Furthermore, consider the operator A defined by

$$Az = \frac{d^2 z(\xi)}{d\xi^2} \text{ with domain } D(A) = \{z \in L^2(\mathbb{T};\mathbb{R}) : z \in H^2(\mathbb{T};\mathbb{R})\}.$$

For $t \in J$ the operators $B(t)$ are defined by

$$B(t)z = b(t)\frac{dz(\xi)}{d\xi} \text{ with domain } D = \{z \in L^2(\mathbb{T};\mathbb{C}) : z \in H^1(\mathbb{T};\mathbb{C})\},$$

and the functions g and h defined by

$$g(u) = \sum_{i=0}^{p} g_i u(t_i) \text{ and } h(u) = \sum_{i=0}^{p} h_i u(t_i),$$

for $i = 0,1,\dots,p$, the numbers g_i and h_i, are given constants and $0 < t_0 < \cdots < t_p < a$. Assume that $\sum_{i=0}^{p}(gi + h_i) < 1$.

With this considerations preceding equation can be written in the abstract form

$$\left.\begin{array}{rcl} u''(t) &=& (A+B(t))u(t) + f(t,u(t)), \quad t \in J \\ u(0) &=& g(u), \\ u'(0) &=& h(u). \end{array}\right\} \qquad (3.15)$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $\{C_0(t)\}_{t\in J}$ in X. Moreover, A has discrete spectrum, the spectrum of A consists of eigenvalues $-n^2$ for $n \in \mathbb{Z}$ with associated eigenvectors

$$z_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi} \text{ for } n \in \mathbb{Z}.$$

Furthermore, the set $\{z_n : n \in \mathbb{Z}\}$ is an orthonormal basis of X. In particular,

$$Az = \sum_{n\in\mathbb{Z}} -n^2 \langle z, z_n\rangle z_n$$

for $z \in D(A)$. The cosine function $C_0(t)$ is given by

$$C_0(t)z = \sum_{n\in\mathbb{Z}} \cos(nt)\langle z, z_n\rangle z_n, \quad t \in \mathbb{R},$$

with associated sine function

$$S_0(t)z = t\langle z, z_0\rangle z_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\sin(nt)}{n}\langle z, z_n\rangle z_n \quad t \in \mathbb{R}.$$

It is clear that $\|C_0(t)\| \leqslant 1$ for all $t \in \mathbb{R}$. Thus, $C_0(\cdot)$ is uniformly bounded on $\mathbb{R}$. Moreover, the family $\{S_0(t)\}_{t\in\mathbb{R}}$ is compact family of operators such that $\|S_0(t)\| \leqslant 1$ for all $t \in \mathbb{R}$. It has been proved by Henríquez in [78] that the equation (3.2) admits an evolution operator $\{S(t,s)\}_{0\leqslant t,s\leqslant a}$. From Lemma 3.1 we have that the evolution operator $\{S(t,s)\}_{t,s\in J}$ is compact.

Consider the function $f(t,u(t)) = \alpha(t)\beta(u(t))$ for all $t \in J$, where α is an integrable function and β is a bounded map.

By direct computation, the conditions ***(Cgh1), (Cg2), (Cf1)*** and ***(Cf2)*** are satisfied. In addition, since

$$\sum_{i=0}^{p}(g_i + h_i) < 1,$$

there exists $M > 0$ such that

$$M\sum_{i=0}^{p}(g_i + h_i) + L\|\alpha\|_1 \leqslant M,$$

where L is the bound of function β and $\|\alpha\|_1 = \displaystyle\int_0^a \|\alpha(t)\|dt$.
It follows from Theorem 3.3 that there exists a mild solution for equation (3.15) in $L^2(\mathbb{R}^n;\mathbb{R})$.

CHAPTER 4

Periodic Solutions of an Abstract Third–Order Differential Equation

Recent investigations have demonstrated that third–order differential equations describe several models arising from natural phenomena, such as wave propagation in viscous thermally relaxing fluids or flexible space structures with internal damping, for example, a thin uniform rectangular panel, like a solar cell array, and a spacecraft with flexible attachments. For more information see [18, 19, 20, 21, 65, 66, 67] and references therein.

Considering the influence of an external force, many of these equations take the abstract form

$$\alpha u'''(t) + u''(t) = \beta A u(t) + \gamma A u'(t) + F(t, u(t)), \quad for \ t \in \mathbb{R}^+, \tag{4.1}$$

where A is a closed linear operator defined on a Banach space X, the function F is a given and X–valued map, and the constants $\alpha, \beta, \gamma \in \mathbb{R}^+$.

Equation (4.1) has been studied in many aspects. Next, we just mention a few of them. A characterization of solutions for its linear version, i.e. $F(t, u(t)) = f(t)$, have been obtained in Hölder spaces $C^s(\mathbb{R}; X)$ by Cuevas and Lizama in [45]. In the same manner, Fernández, Lizama and Poblete in [59] characterize well-posedness in Lebesgue spaces, $L^p(\mathbb{R}; X)$. Further, Fernández, Lizama and Poblete, in [60], study regularity and qualitative properties of mild and strong solutions defined in $\mathbb{R}^+$ where the underlying space is a Hilbert space. On the other hand, existence of bounded mild solutions of the semi–linear equation (4.1) is studied in [50] by De Andrade and Lizama.

As we have said in *Introduction*, concerning to abstract evolution equations, the study of solutions having periodicity property is a very important subject of research. However, for abstract third–order differential equation (4.1), this aspect has not been addressed in the existing literature. For this reason, this Chapter is dedicated to study existence and uniqueness of periodic strong solutions for abstract third–order equation

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t), \quad t \in [0,2\pi], \tag{4.2}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$. Here A and B are closed linear operators defined on a Banach space X with $D(A) \cap D(B) \neq \{0\}$, the constants $\alpha, \beta, \gamma \in \mathbb{R}^+$, and f belongs to either periodic Lebesgue spaces , or periodic Besov spaces, or periodic Triebel–Lizorkin spaces. Our approach is based in vector–valued Fourier theorems and maximal regularity property. We remark, the study of existence of solutions for equation (4.2) in the particular case $A \equiv B$ is a manner to study periodic solutions of equation (4.1).

With a specific norm, we will denote the space consisting of all 2π–periodic, X–valued functions by $E(\mathbb{T};X)$, and denote the set consisting of all functions in $E(\mathbb{T};X)$ which are n times differentiable by $E^n(\mathbb{T};X)$. The following definitions will be used in subsequent sections with either periodic Lebesgue spaces, periodic Besov spaces or periodic Triebel–Lizorkin spaces.

Definition 4.1. *A function u is called a **strong** E**–solution** of equation* (4.2) *if $u \in E^3(\mathbb{T};X) \cap E^1(\mathbb{T};[D(B)]) \cap E(\mathbb{T};X)$ and equation* (4.2) *holds a.e. in* $[0,2\pi]$.

Definition 4.2. *We say that solutions of equation* (4.2) *has E**–maximal regularity** if for each $f \in E(\mathbb{T};X)$, equation* (4.2) *has a unique strong E–solution.*

For the rest of this chapter we introduce the following notation. Given $\alpha, \beta, \gamma > 0$, and closed linear operators A and B defined on a Banach space X, with $D(A) \cap D(B) \neq \{0\}$. For $k \in \mathbb{Z}$, we will write

$$a_k = ik^3 \quad \text{and} \quad b_k = i\alpha k^3 + k^2, \tag{4.3}$$

and the operators

$$N_k = (b_k + i\gamma kB + \beta A)^{-1} \quad \text{and} \quad M_k = a_k N_k. \tag{4.4}$$

Furthermore, we denote

$$\rho(A,B) = \{k \in \mathbb{Z} : N_k \text{ exists and is bounded}\} \quad \text{and} \quad \sigma(A,B) = \mathbb{Z} \setminus \rho(A,B).$$

4.1 Maximal regularity for a third–order differential equation in periodic Lebesgue spaces

In this section, we give a characterization of L^p–maximal regularity for equation (4.2). For this reason we prove Theorem 4.1 and to carry out its proof, we need the following results.

Lemma 4.1. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined on a Banach space X. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded families of operators, then*

$$\{ka_k(\Delta^1 N_k)\}_{k\in\mathbb{Z}} \text{ and } \{k^2 B(\Delta^1 N_k)\}_{k\in\mathbb{Z}}$$

are R–bounded families of operators.

Proof. First note that $\{a_k N_k\}_{k\in\mathbb{Z}}$ is R–bounded if and only if $\{b_k N_k\}_{k\in\mathbb{Z}}$ is R–bounded. Furthermore, for all $j \in \mathbb{Z}$ fixed, we have $\{a_k N_{k+j}\}_{k\in\mathbb{Z}}$ and that $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are R–bounded families. For $k \in \mathbb{Z}$, we have

$$(\Delta^1 N_k) = N_{k+1}(b_k - b_{k+1} - i\gamma B)N_k = -(\Delta^1 b_k)N_{k+1}N_k - i\gamma N_{k+1}BN_k. \tag{4.5}$$

Hence, for all $k \in \mathbb{Z} \setminus \{0\}$ we have

$$ka_k(\Delta^1 N_k) = -k\frac{(\Delta^1 b_k)}{b_k}\frac{b_k}{a_k}a_k N_{k+1}M_k + \gamma a_k N_{k+1}kBN_k$$

and,

$$\begin{aligned} k^2 B(\Delta^1 N_k) &= -k(\Delta^1 b_k)kBN_{k+1}N_k - i\gamma kBN_{k+1}kBN_k \\ &= -k\frac{(\Delta^1 b_k)}{b_k}\frac{b_k}{a_k}kBN_{k+1}M_k - i\gamma kBN_{k+1}kBN_k. \end{aligned}$$

Direct computation shows that if $k = 0$ the operators $k^1 a_k(\Delta^1 N_k)$ and $k^2 B(\Delta^1 N_k)$ are bounded. In addition, $\{b_k\}_{k\in\mathbb{Z}}$ is a 1–regular sequence and $\sup_{k\in\mathbb{Z}\setminus\{0\}} |b_k/a_k| < \infty$. The Lemma results from the properties of R–bounded families. ■

Lemma 4.2. *Let $p \in (1,\infty)$, and let X be a UMD–space. If $\alpha, \beta, \gamma > 0$, and A and B are closed linear operators defined on X, then the following two assertions are equivalent.*

(i) The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are R–bounded.

(ii) The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are L^p–multipliers.

Proof. $(i) \Rightarrow (ii)$. By hypothesis, $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded families of operators. According to Theorem 1.1, it suffices to show that the families

$$\{k(\Delta^1 M_k)\}_{k\in\mathbb{Z}} \text{ and } \{k(\Delta^1 kBN_k)\}_{k\in\mathbb{Z}}$$

are R–bounded. For this, note that

$$k(\Delta^1 M_k) = k\frac{(\Delta^1 a_k)}{a_k}a_k N_{k+1} + ka_k(\Delta^1 N_k).$$

Similarly, we write

$$k(\Delta^1 kBN_k) = k^2 B(\Delta^1 N_k) + kBN_{k+1}.$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is a 1–regular sequence, statement (ii) results from Lemma 4.1 and the properties of R–bounded families.
$(ii) \Rightarrow (i)$. Statement (i) follows from Proposition 1.1. ■

Theorem 4.1. *Let $p \in (1,\infty)$, and let X be a UMD–space. The following two assertions are equivalent.*

(i) Equation (4.2) *has L^p–maximal regularity.*

(ii) $\sigma(A,B) = \emptyset$. The families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded.

Proof. $(i) \Rightarrow (ii)$. Let $k \in \mathbb{Z}$ be fixed, and let $x \in X$. Define $h(t) = e^{ikt}x$. A simple computation shows that $\widehat{h}(k) = x$. By hypothesis, there exists a function $u \in H^{3,p}_{per}(X) \cap H^{1,p}_{per}(X;[D(B)]) \cap L^p(\mathbb{T};[D(A)])$ such that, for almost all $t \in [0,2\pi]$, we have

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + h(t).$$

Applying Fourier transform to both sides of the preceding equality, we obtain

$$(-i\alpha k^3 - k^2 - i\gamma kB - \beta A)\widehat{u}(k) = x.$$

Since x is arbitrary, we have that $(-b_k - i\gamma kB - \beta A)$ is surjective.
On the other hand, let $z \in D(A)\cap D(B)$, and assume $(-b_k - i\gamma kB - \beta A)z = 0$. Substituting $u(t) = e^{ikt}z$ in equation (4.2), we see that u is a periodic solution of this equation when $f \equiv 0$. The uniqueness of the solution implies that $z = 0$.
Now suppose $(b_k + i\gamma kB + \beta A)$ has no bounded inverse. Then for each $k \in \mathbb{Z}$, there exists a sequence $\{y_{k,n}\}_{n\in\mathbb{Z}} \subseteq X$ such that

$$\|y_{n,k}\| \leqslant 1 \quad \text{and} \quad \|N_k y_{k,n}\| \geqslant n^2, \quad \text{for all } \ n \in \mathbb{Z}.$$

Write $x_k = y_{k,k}$. We obtain $\|N_k x_k\| \geqslant k^2$, for all $k \in \mathbb{Z}$.
Let $g(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{x_k}{k^2} e^{ikt}$. Note that $g \in L^p(\mathbb{T};X)$. By hypothesis, there exists a unique strong L^p–solution $u \in L^p(\mathbb{T};X)$. Applying Fourier transform to equation (4.2), we have $\widehat{u}(k) = -N_k \widehat{g}(k)$, for all $k \in \mathbb{Z}$. We know

$$u(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} -\frac{x_k}{k^2} e^{ikt} N_k, \quad \text{for almost } t \in [0,2\pi].$$

For all $k \in \mathbb{Z}$, we have $\left\|\frac{x_k}{k^2} N_k\right\| \geqslant 1$ and conclude that $u \notin L^p(\mathbb{T};X)$. Since u is a strong L^p–solution of equation (4.2), this is a contradiction. Hence $N_k \in \mathscr{L}(X)$, for all $k \in \mathbb{Z}$. Therefore, $\sigma(A,B) = \emptyset$.

Next let $f \in L^p(\mathbb{T};X)$. By hypothesis, there exits a unique function $u \in H^{3,p}_{per}(X) \cap H^{1,p}_{per}(X;[D(B)]) \cap L^p(\mathbb{T};[D(A)])$ such that

$$\alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t)$$

for almost all $t \in [0,2\pi]$. Applying Fourier transform to both sides of the preceding equation, we have

$$(-b_k - i\gamma kB - \beta A)\widehat{u}(k) = \widehat{f}(k)$$

for all $k \in \mathbb{Z}$. Since $\sigma(A,B) = \emptyset$, we have

$$\widehat{u}(k) = (-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k), \text{ for all k} \in \mathbb{Z}.$$

Multiplying by $i\gamma k$ on both sides of the preceding equality, we obtain

$$i\gamma k\widehat{u}(k) = -i\gamma k(b_k + i\gamma kB + \beta A)^{-1}\widehat{f}(k).$$

Since $u \in H^{1,p}_{per}(X;[D(B)])$, there is a function $v \in L^p(\mathbb{T};[D(B)])$ satisfying $\widehat{v}(k) = i\gamma k\widehat{u}(k)$, for all $k \in \mathbb{Z}$. Therefore,

$$\widehat{v}(k) = -i\gamma k(b_k + i\gamma kB + \beta A)^{-1}\widehat{f}(k), \text{ for all } k \in \mathbb{Z}.$$

Define $w = Bv$. Since $v \in L^p(\mathbb{T};[D(B)])$, we conclude $w \in L^p(\mathbb{T};X)$. Since B is a closed linear operator, it follows from Lemma 1.1 that,

$$\widehat{w}(k) = -i\gamma kBN_k\widehat{f}(k), \quad \text{for all } k \in \mathbb{Z}.$$

This implies that the family $\{kBN_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier.

On the other hand, since $u \in L^p(\mathbb{T};[D(A)])$, we define $r = -\beta Au$, and we have $r \in L^p(\mathbb{T};X)$. Since A is linear and closed, it follows from Lemma 1.1 that

$$\widehat{r}(k) = -\beta AN_k\widehat{f}(k), \text{ for all } k \in \mathbb{Z}.$$

Hence, the family $\{-\beta AN_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier. Now for all $k \in \mathbb{Z}$, we have

$$b_kN_k = I - i\gamma kBN_k - \beta AN_k.$$

Since the sum of L^p–multipliers is also an L^p–multiplier, we conclude $\{b_kN_k\}_{k\in\mathbb{Z}}$ is an L^p–multiplier. Since, the sequence $\{a_k/b_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded and $\frac{a_k}{b_k}b_kN_k = M_k$, we have that $\{M_k\}_{k\in\mathbb{Z}}$ an L^p–multiplier. It now follows from Proposition 1.1 that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded families.

$(ii) \Rightarrow (i)$. By hypothesis, the conditions of Lemma 4.2 are satisfied. Therefore, $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are L^p–multipliers. From Remark 1.1 we conclude that $\{(-b_k - i\gamma kB - \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an $(L^p(X), H^{3,p}_{per}(X))$–multiplier. Thus, given a function $f \in L^p(\mathbb{T};X)$, there exists a function $u \in H^{3,p}_{per}(X)$ such that, for all $k \in \mathbb{Z}$,

$$\widehat{u}(k) = (-b_k - \beta A - i\gamma kB)^{-1}\widehat{f}(k). \tag{4.6}$$

Moreover, from Lemma 1.1 we have that $u(t) \in D(A) \cap D(B)$ for almost all $t \in [0,2\pi]$.

As we have shown, the family $\{ikB(-b_k - i\gamma kB - \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p–multiplier. Then there exists a function $v \in L^p(\mathbb{T}; X)$ satisfying

$$\widehat{v}(k) = ikB(-b_k - i\gamma kB - \beta A)^{-1}\widehat{f}(k)$$

for all k. According to equality (4.6), we have $\widehat{v}(k) = ikB\widehat{u}(k)$, for all $k \in \mathbb{Z}$.
On the another hand, since $H^{3,p}_{per}(X) \subseteq H^{1,p}_{per}(X)$, there exists a function $w \in L^p(\mathbb{T}; X)$ such that $\widehat{w}(k) = ik\widehat{u}(k)$, for all $k \in \mathbb{Z}$. Since B is closed linear operator, we have

$$\widehat{v}(k) = B(ik\widehat{u}(k)) = B\widehat{w}(k) = \widehat{Bw}(k)$$

for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $v = Bw$. This implies that $w \in L^p(\mathbb{T}; [D(B)])$. Therefore, $u \in H^{1,p}_{per}(X; [D(B)])$. We claim that $u \in L^p(\mathbb{T}; [D(A)])$. In fact, using the identity

$$\beta A(b_k + i\gamma kB + \beta A)^{-1} = I - b_k(b_k + i\gamma kB + \beta A)^{-1} - i\gamma kB(b_k + i\gamma kB + \beta A)^{-1}$$

we see that $\{\beta A(b_k + i\gamma kB + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p–multiplier. Thus, there exists a function $h \in L^p(\mathbb{T}; X)$ satisfying

$$\widehat{h}(k) = A(b_k + i\gamma B + \beta A)^{-1}\widehat{f}(k)$$

for all k. It follows from equality (4.2) that $\widehat{h}(k) = A\widehat{u}(k)$, for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, we have $h = Au$. This implies that $u \in L^p(\mathbb{T}; [D(A)])$ as asserted. Therefore, $u \in H^{3,p}_{per}(X) \cap H^{1,p}_{per}(X; [D(B)]) \cap L^p(\mathbb{T}; [D(A)])$.

We have shown that $u \in H^{3,p}_{per}(X)$. Thus, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$, and $u''(0) = u''(2\pi)$. Since A and B are closed linear operators, it now follows from equality (4.3) that

$$\alpha\widehat{u'''}(k) + \widehat{u''}(k) = \beta\widehat{Au}(k) + \gamma\widehat{Bu}(k) + \widehat{f}(k), \text{ for all } k \in \mathbb{Z}.$$

From the uniqueness of the Fourier coefficients we conclude that equation (4.2) holds a.e. in $[0, 2\pi]$. Therefore, u is a strong L^p–solution of equation (4.2). It remains to show that this solution is unique. Indeed, let $f \in L^p(\mathbb{T}; X)$. Suppose equation (4.2) has two strong L^p–solutions, u_1 and u_2. A direct computation shows that

$$(-b_k - i\gamma kB - \beta A)\left[\widehat{u_1}(k) - \widehat{u_2}(k)\right] = 0$$

for all $k \in \mathbb{Z}$. Since $(-b_k - i\gamma kB - \beta A)$ is invertible, we have $\widehat{u_1}(k) = \widehat{u_2}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $u_1 \equiv u_2$. Therefore, equation (4.2) has L^p–maximal regularity. ∎

It is not easy to verify the R–boundedness condition of a specific family of operators, especially when two different operators are involved. Our next Corollary require additional conditions about the operators A and B, however is a more practical result to check that families $\{a_k N_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are R–bounded. With this purpose, for $k \in \mathbb{Z}$ define the operators $S_k = \left(-\frac{b_k}{\beta} - A\right)^{-1}$.

Corollary 4.1. *Let* $1 < p < \infty$*, and let* X *be a* UMD*–space. Supose that for all* $k \in \mathbb{Z}$ *we have* $-\dfrac{b_k}{\beta} \in \rho(A)$*. Assume that the families of operators* $\mathscr{F}_1 = \{a_k S_k : k \in \mathbb{Z}\}$ *and* $\mathscr{F}_2 = \left\{ ik\dfrac{\gamma}{\beta} BS_k \,:\, k \in \mathbb{Z} \right\}$ *are* R*–bounded. If* $\mathscr{R}_p(\mathscr{F}_2) < 1$ *then equation* (4.2) *has* L^p*–maximal regularity.*

Proof. According to [81, Lemma 3.17], the family

$$\left\{ \left(I - \frac{ik\gamma}{\beta} BS_k \right)^{-1} \right\}_{k \in \mathbb{Z}}$$

is R–bounded. For $k \in \mathbb{Z}$ the operators $M_k = a_k S_k \left(I - \dfrac{ik\gamma}{\beta} BS_k \right)^{-1}$ $kBN_k = kBS_k \left(I - \dfrac{ik\gamma}{\beta} BS_k \right)^{-1}$.
By properties of R–boundedness, we conclude the families $\{M_k\}_{k \in \mathbb{Z}}$ and $\{kBN_k\}_{k \in \mathbb{Z}}$ are R–bounded. The Corollary results from Theorem 4.1. ■

Following corollary is given as an answer to the study of existence of periodic solutions for equation (4.1). With this purpose, we denote the complex sequence $\{d_k\}_{k \in \mathbb{Z}}$ given by

$$d_k = -\frac{i\alpha k^3 + k^2}{i\gamma k + \beta} \quad \text{for } k \in \mathbb{Z}.$$

Corollary 4.2. *Let* $p \in (1, \infty)$*, and let* X *be a* $UMD-$*space. The following two assertions are equivalent.*

(i) Equation (4.2)*, with* $B \equiv A$*, has* L^p*–maximal regularity.*

(ii) $\{d_k\}_{k \in \mathbb{Z}} \subseteq \rho(A)$ *and that* $\{d_k(d_k - A)^{-1}\}_{k \in \mathbb{Z}}$ *is* R*–bounded.*

Proof. $(i) \Rightarrow (ii)$. According to Theorem 4.1, we have that $\sigma(A, A) = \emptyset$ and that the operators $(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} \in \mathscr{L}(X)$, for all $k \in \mathbb{Z}$. In addition, $\{ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1}\}_{k \in \mathbb{Z}}$ is $R-$bounded. Hence, this family of operators is bounded. Then there exists a constant $C > 0$ such that $\sup\limits_{k \in \mathbb{Z}} \| ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} \| \leqslant C$.
This implies

$$\|(d_k - A)^{-1}\| \leqslant \frac{|i\gamma k + \beta|}{|ik^3|} C, \quad \text{for all } k \in \mathbb{Z} \setminus \{0\}.$$

Since $0 \in \rho(A, A)$ if and only if $0 \in \rho(A)$, we have $\{d_k\}_{k \in \mathbb{Z}} \subseteq \rho(A)$. Properties of $R-$bounded families and the equality

$$d_k(d_k - A)^{-1} = \frac{i\alpha k^3 + k^2}{ik^3} ik^3(i\alpha k^3 + k^2 + (i\gamma k + \beta)A)^{-1}$$

show that $\{d_k(d_k - A)^{-1}\}_{k \in \mathbb{Z}}$ is $R-$bounded.

$(ii) \Rightarrow (i)$. For this, note that assertion (ii) guarantees that condition (ii) of Theorem 4.1 is satisfied. In fact, let $k \in \mathbb{Z}$, by hypothesis $d_k \in \rho(A)$, this implies that $(d_k - A)^{-1}$ is well defined in $\mathscr{L}(X)$. Since $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is $R-$bounded, there exists a constant $C \geqslant 0$ such that

$$\sup_{k\in\mathbb{Z}} \|d_k(d_k - A)^{-1}\| = \sup_{k\in\mathbb{Z}} |i\alpha k^3 + k^2| \, \|(i\alpha k^3 + k^2 + (i\gamma k + \beta)A)^{-1}\| \leqslant C.$$

Then, for all $k \in \mathbb{Z} \setminus \{0\}$, we obtain

$$\|(-i\alpha k^3 - k^2 - (i\gamma k + \beta)A)^{-1}\| \leqslant \frac{C}{|i\alpha k^3 + k^2|}.$$

Since $0 \in \rho(A)$ if and only if $0 \in \rho(A, A)$, we have $\sigma(A, A) = \emptyset$.
We combine properties of $R-$bounded families with the identities

$$ik^3(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} = \frac{ik^3}{i\alpha k^3 + k^2} d_k(d_k - A)^{-1}$$

and

$$kA(i\alpha k^3 + k^2 + i\gamma kA + \beta A)^{-1} = \frac{-k}{i\gamma k + \beta}(d_k(d_k - A)^{-1} - I)$$

to obtain that the families $\{ik^3(b_k + i\gamma kA + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ and $\{kA(b_k + i\gamma kA + \beta A)^{-1}\}_{k\in\mathbb{Z}}$ are $R-$bounded. ∎

4.2 Maximal regularity for a third–order differential equation in periodic Besov spaces

In this section, we give a characterization of $B^s_{p,q}$–maximal regularity for equation (4.2). For this reason we prove Theorem 4.2 and to carry out its proof, we need the following results.

Lemma 4.3. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined on X. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded families of operators, then*

$$\{k^2 a_k(\Delta^2 N_k)\}_{k\in\mathbb{Z}} \quad \textit{and} \quad \{k^3 B(\Delta^2 N_k)\}_{k\in\mathbb{Z}}$$

are bounded families of operators.

Proof. We make the same considerations as that of Lemma 4.1. We have that $\{a_k N_k\}_{k\in\mathbb{Z}}$ is bounded if and only if $\{b_k N_k\}_{k\in\mathbb{Z}}$ is bounded. Further, for all $j \in \mathbb{Z}$ fixed, we have that $\left\{a_k N_{k+j}\right\}_{k\in\mathbb{Z}}$ and $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are bounded families. For $k \in \mathbb{Z} \setminus \{0\}$, we have

$$k^2 a_k(\Delta^2 N_k) = i\gamma k a_k(N_k - N_{k+2})kBN_{k+1} - M_k\, k^2 \frac{(\Delta^2 b_k)}{b_k}\frac{b_k}{a_k} a_k N_{k+1}$$
$$+ k a_k(N_{k+2} - N_k)k\frac{(\Delta^1 b_{k+1})}{b_k}\frac{b_k}{a_k} a_k N_{k+1}$$

and

$$k^3 B(\Delta^2 N_k) = k^2 B(N_k - N_{k+2})kBN_{k+1} - kBN_k\, k^2 \frac{(\Delta^2 b_k)}{b_k}\frac{b_k}{a_k} a_k N_{k+1}$$
$$- k^2 B(N_{k+2} - N_k)\, k\frac{(\Delta^1 b_{k+1})}{b_k}\frac{b_k}{a_k} a_k N_{k+1}.$$

A direct computation shows that if $k = 0$ the operators $k^2 a_k(\Delta^2 N_k)$ and $k^3 B(\Delta^2 N_k)$ are bounded. Since $\{b_k\}_{k\in\mathbb{Z}}$ is a 2–regular sequence, same calculation made in proof of Lemma 4.1 shows that the families of operators $\{k^2 a_k(\Delta^2 N_k)\}_{k\in\mathbb{Z}}$ and $\{k^3 B(\Delta^2 N_k)\}_{k\in\mathbb{Z}}$ are bounded. ■

Lemma 4.4. *Let $1 \leqslant p, q \leqslant \infty$, and $s > 0$. Let $\alpha, \beta, \gamma \in \mathbb{R}_+$, and let A and B be closed linear operators defined on a Banach space X. The following two assertions are equivalent.*

(i) The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are bounded.

(ii) The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$–multiplier.

Proof. $(i) \Rightarrow (ii)$. According to Theorem 1.2 , we need to show that the families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are $\mathscr{M}$–bounded of order 2. Exactly, the same calculation made in proof of Lemma 4.2 displays that $\{k(\Delta^1 M_k)\}_{k\in\mathbb{Z}}$ and $\{k(\Delta^1 kBN_k)\}_{k\in\mathbb{Z}}$ are bounded. Now note that

$$k^2(\Delta^2 M_k) = k^2 a_k(\Delta^2 N_k) + k^2\frac{(\Delta^2 a_k)}{a_k} a_k N_{k+1} - k\frac{(\Delta^1 a_k)}{a_k} k a_k(N_k - N_{k+2}).$$

Also

$$k^2(\Delta^2 kBN_k) = k^3 B(\Delta^2 N_k) + k^2 B(N_{k+2} - N_k).$$

A simple verification shows that if $k = 0$ the operator $k^2(\Delta^2 M_k)$ is bounded. Furthermore, it follows from Lemma 4.1 and Lemma 4.3 that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are $\mathscr{M}$–bounded of order 2.

$(ii) \Rightarrow (i)$. It follows from the Closed Graph theorem that there exists a $C \geqslant 0$ (independent of f) such that, for $f \in B^s_{p,q}(\mathbb{T}; X)$, we have

$$\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \widehat{f}(k) \Big\|_{B^s_{p,q}} \leqslant C\|f\|_{B^s_{p,q}}.$$

Let $x \in X$, and define $f(t) = e^{ikt}x$ for $k \in \mathbb{Z}$ fixed. The preceding inequality implies

$$\|e_k\|_{B^s_{p,q}} \|M_k x\|_{B^s_{p,q}} = \|e_k M_k x\|_{B^s_{p,q}} \leqslant C \|e_k\|_{B^s_{p,q}} \|x\|_{B^s_{p,q}}.$$

Hence for all $k \in \mathbb{Z}$ we have $\|M_k\| \leqslant C$. Consequently, $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. Similarly, $\sup_{k\in\mathbb{Z}} \|kBN_k\| < \infty$. ■

The following Theorem gives a characterization of maximal regularity on periodic Besov spaces for equation (4.2). Its proof is very similar to as that of Theorem (4.1), so we will pass over it.

Theorem 4.2. *Let $1 \leqslant p,\ q \leqslant \infty$, and $s > 0$. Let X be a Banach space. The following two assertions are equivalent.*

(i) Equation (4.2) has $B^s_{p,q}$–maximal regularity.

(ii) $\sigma(A,B) = \emptyset$. The families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded.

In the same manner as preceding section, we give a more practical criteria to checking boundedness condition about families $\{a_k N_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$. We use the same notation introduced in Corollary 4.1 for the family $\{S_k\}_{k\in\mathbb{Z}}$. We do not give its proof because it follows the same lines as those of proof of Corollary 4.1.

Corollary 4.3. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$ and X a Banach space. Suppose that for all $k \in \mathbb{Z}$ we have $-\dfrac{b_k}{\beta} \in \rho(A)$. Assume that the families $\{a_k S_k\}_{k\in\mathbb{Z}}$ and $\left\{\dfrac{i\gamma k}{\beta} BS_k\right\}_{k\in\mathbb{Z}}$ are bounded. If $\sup_{k\in\mathbb{Z}} \|a_k S_k\| < 1$, then the equation (4.2) has $B^s_{p,q}$–maximal regularity.*

As Corollary 4.2 in preceding section, the following result is an answer to the study of existence of periodic solutions for equation (4.1). Its proof follows the same lines as those of proof of Corollary 4.2, so we pass over it.

Corollary 4.4. *Let X a Banach space and $1 \leqslant p,q \leqslant \infty$ and $s > 0$ The following assertions are equivalent,*

(i) The equation (4.1) with $B \equiv A$, has $B^s_{p,q}$–maximal regularity.

(ii) $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded.

4.3 Maximal regularity for a third–order differential equation in periodic Triebel–Lizorkin spaces

In this section, we give a characterization of $F^s_{p,q}$–maximal regularity for Equation (4.2). For this reason we prove Theorem 4.3, its proof will depend on our next results.

Lemma 4.5. *Let $\alpha, \beta, \gamma > 0$, and let A and B be closed linear operators defined on X. If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded families, then*

$$\{k^3 a_k(\Delta^3 N_k)\}_{k\in\mathbb{Z}} \quad \{k^4 B(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$$

are bounded families.

Proof. We make the same considerations as those of Lemmas 4.1 and 4.3. We have that $\{a_k N_k\}_{k\in\mathbb{Z}}$ is bounded if and only if $\{b_k N_k\}_{k\in\mathbb{Z}}$ is bounded. Further, for all $j \in \mathbb{Z}$ fixed, we have that $\left\{a_k N_{k+j}\right\}_{k\in\mathbb{Z}}$ and $\{kBN_{k+j}\}_{k\in\mathbb{Z}}$ are bounded families. Using the calculations as those of Lemma 4.3, for all $k \in \mathbb{Z}$ we see that

$$\Delta^2 N_k \quad = (N_{k+2} - N_k)(-(\Delta^1 b_{k+2}) - i\gamma B)N_{k+1} - N_k(\Delta^2 b_k)N_{k+1}.$$

Therefore, for $k \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned} k^3 a_k(\Delta^3 N_k) = {} & k^2 a_k\big((\Delta^2 N_{k+1}) + (\Delta^2 N_k)\big)\, k\,(-(\Delta b_{k+2}) - i\gamma B)N_{k+2} \\ & - k a_k(N_{k+2} - N_k)\; k^2 \frac{(\Delta^2 b_{k+1})}{b_k} \frac{b_k}{a_k} a_k N_{k+2} \\ & + k a_k(N_{k+2} - N_k)\; k^2(-(\Delta b_{k+1}) - i\gamma B)(\Delta^1 N_{k+1}) \\ & - k^3 \frac{(\Delta^3 b_k)}{b_k}\, a_k N_{k+1}\, b_k N_{k+2} - k^2 \frac{(\Delta^2 b_k)}{b_k}\, k a_k(\Delta^1 N_k)\, b_k N_{k+2} \\ & - k^2 \frac{(\Delta^2 b_k)}{b_k} b_k N_k\; k a_k(N_{k+2} - N_k), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} k^4 B(\Delta^3 N_k) = {} & k^3 B(\Delta^2 N_{k+1})\; k(-(\Delta^1 b_{k+2}) - i\gamma B)N_{k+2} \\ & + k^3 B(\Delta^2 N_k)\; k(-(\Delta^2 b_{k+2}) - i\gamma B)N_{k+2} \\ & + k^2 B(N_{k+2} - N_k)\; k^2 \frac{(\Delta^2 b_{k+1})}{b_{k+2}}\; b_{k+2} N_{k+2} \\ & - k^2 B(N_{k+2} - N_k)\; k^2(-(\Delta^2 b_{k+2}) - i\gamma B)(N_{k+2} - N_k) \\ & + \frac{k^3(\Delta^3 b_k)}{b_k}\; a_k N_{k+1} b_k N_{k+2} - \frac{k^2(\Delta^2 b_k)}{b_k}\; k^2 B(\Delta^1 N_k) b_k N_{k+2} \\ & - \frac{k^2(\Delta^2 b_k)}{b_k} b_k B N_k\; k^2(N_{k+2} - N_k). \end{aligned} \tag{4.8}$$

A direct computation shows that if $k = 0$ the operators $k^3 a_k(\Delta^3 N_k)$ and $k^4 B(\Delta^3 N_k)$ are bounded. Since $\{b_k\}_{k\in\mathbb{Z}}$ is a 3–regular sequence, it follows from Lemmas 4.1 and 4.3 that all of the terms on the right side of identities (4.7) and (4.8) are uniformly bounded. Therefore, $\{k^3 a_k(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$ and $\{k^4 B(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$ are bounded families. ∎

Lemma 4.6. *Let $1 \leqslant p,\ q \leqslant \infty$, and $s > 0$, and let A and B be closed linear operators defined on a Banach space X. The following two assertions are equivalent.*

(i) The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are bounded.

(ii) The families $\{kBN_k\}_{k\in\mathbb{Z}}$ and $\{M_k\}_{k\in\mathbb{Z}}$ are $F^s_{p,q}$–multiplier.

Proof. $(i) \Rightarrow (ii)$. The proof of Lemma 4.4 shows that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$, are $\mathcal{M}$–bounded of order 2. Moreover, we have

$$\begin{aligned} k^3(\Delta^3 M_k) = {} & k^3 a_k(\Delta^3 N_k) + k^3(a_{k+3} - a_k)(\Delta^2 N_{k+1}) + k^3(\Delta^2 a_{k+1})(\Delta^1 N_{k+1}) \\ & - 2k^3(\Delta^2 a_k)(\Delta^1 N_{k+1}) + (\Delta^3 a_k)N_{k+2}, \end{aligned}$$

and

$$k^3(\Delta^3 kBN_k) = k^4 B(\Delta^3 N_k) + 3k^3 B(\Delta^2 N_{k+1}).$$

It follows from Lemmas 4.1, 4.3 and 4.5 that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}$ are $\mathcal{M}$–bounded of order 3. Statement (ii) now follows from Theorem 1.3.

$(ii) \Rightarrow (i)$. The proof follows the same lines as that of Theorem 4.4. ∎

The following Theorem gives a characterization of maximal regularity for equation (4.2) on periodic Triebel–Lizorkin spaces. Its proof is similar to proof of Theorems 4.1 and 4.3, so we pass over it.

Theorem 4.3. *Let $1 \leqslant p$ and $q \leqslant \infty$. If $s > 0$ and X is a Banach space, then the following two assertions are equivalent.*

(i) Equation (4.2) has $F^s_{p,q}$–maximal regularity.

(ii) $\sigma(A,B) = \emptyset$. The families $\{M_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$ are bounded.

In the same manner as preceding sections, we give a more practical criteria to checking boundedness condition about families $\{a_kN_k\}_{k\in\mathbb{Z}}$ and $\{kBN_k\}_{k\in\mathbb{Z}}$. We use the same notation introduced in Corollary 4.1 for the family $\{S_k\}_{k\in\mathbb{Z}}$. We do not give its proof because it follows the same lines as those of proof of Corollary 4.1.

Corollary 4.5. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$ and X a Banach space. Supose that for all $k \in \mathbb{Z}$ we have $-\dfrac{b_k}{\beta} \in \rho(A)$. Assume that the families $\{a_kS_k\}_{k\in\mathbb{Z}}$ and $\left\{\dfrac{i\gamma k}{\beta}BS_k\right\}_{k\in\mathbb{Z}}$ are bounded. If $\sup_{k\in\mathbb{Z}} \|a_kS_k\| < 1$, then the equation (4.2) has $F^s_{p,q}$–maximal regularity.*

As Corollary 4.1 and 4.3 in preceding sections, the following result is an answer to the study of existence of periodic solutions for equation (4.1). Its proof follows the same lines as those of proof of Corollary 4.1, so we pass over it.

Corollary 4.6. *Let X a Banach space and $1 \leqslant p,q \leqslant \infty$ and $s > 0$ The following assertions are equivalent,*

(i) The equation (4.1) with $B \equiv A$, has $F^s_{p,q}$–maximal regularity.

(ii) $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded.

4.4 Examples.

To finish this chapter, in this section, we apply our results to some interesting examples.

***Example* 4.1.** *Let $\alpha,\beta,\gamma\in\mathbb{R}^+$. Let $1\leqslant p,\ q\leqslant\infty$, and $s>0$. Consider the abstract equation*

$$\alpha u'''(t)+u''(t)=\beta Au(t)+\gamma Au'(t)+f(t),\quad \textit{for } t\in[0,2\pi] \tag{4.9}$$

with boundary conditions $u(0)=u(2\pi)$, $u'(0)=u'(2\pi)$ and $u''(0)=u''(2\pi)$, and A a positive selfadjoint operator defined on a Hilbert space H such that $\inf_{\lambda\in\sigma(A)}\{\lambda\}\neq 0$. If $f\in B^s_{p,q}(\mathbb{T};H)$ (resp. $F^s_{p,q}(\mathbb{T};H)$), then equation (4.3) has $B^s_{p,q}$–maximal regularity (resp. $F^s_{p,q}$–maximal regularity).

Proof. *We have $d_k=\dfrac{-(\alpha\gamma k^4+\beta k^2)}{(\gamma k)^2+\beta^2}+i\dfrac{(\gamma-\alpha\beta)k^3}{(\gamma k)^2+\beta^2}$. Since A is positive selfadjoint such that $\inf_{\lambda\in\sigma(A)}\|\lambda\|\neq 0$, we know that $\sigma(A)\subseteq[\varepsilon,+\infty)$, with $\varepsilon>0$. This implies that $d_k\in\rho(A)$, for all $k\in\mathbb{Z}$. Moreover, by* [92, Chapter 5, Section 3.5], *we know that for $k\in\mathbb{Z}$, $\|(d_k-A)^{-1}\|=\dfrac{1}{dist(d_k,\sigma(A))}$. Therefore, $\sup_{k\in\mathbb{Z}}\|d_k(d_k-A)^{-1}\|<\infty$. According to Corollaries 4.4 and 4.6, equation* (4.9) *has, respectively, $B^s_{p,q}$–maximal regularity and $F^s_{p,q}$–maximal regularity.* ∎

For the next example we need to introduce some preliminaries on sectorial operators. Denote by $\Sigma_\phi\subseteq\mathbb{C}$ the open sector

$$\Sigma_\phi=\{\lambda\in\mathbb{C}\setminus\{0\}\ /\ |\arg\lambda|<\phi\}.$$

We denote

$$\mathcal{H}(\Sigma_\phi)=\{f:\Sigma_\phi\to\mathbb{C}\ \text{ holomorphic}\}$$

and

$$\mathcal{H}^\infty(\Sigma_\phi)=\{f:\Sigma_\phi\to\mathbb{C}\ \text{ holomorphic and bounded}\}.$$

$\mathcal{H}^\infty(\Sigma_\phi)$ is a Banach space endowed with the norm

$$\|f\|^\phi_\infty=\sup_{|\arg(\lambda)|<\phi}|f(\lambda)|.$$

We further define the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows:

$$\mathcal{H}_0(\Sigma_\phi)=\bigcup_{\alpha,\beta<0}\{f\in\mathcal{H}(\Sigma_\phi):\|f\|^\infty_{\alpha,\beta}<\infty\}$$

where

$$\|f\|^\infty_{\alpha,\beta}=\sup_{|\lambda|\leqslant 1}|\lambda^\alpha f(\lambda)|+\sup_{|\lambda|\geqslant 1}|\lambda^{-\beta}f(\lambda)|.$$

Definition 4.3. [91] *A closed linear operator A in X is called* ***sectorial*** *if the following two conditions hold.*

(i) We have $\overline{D(A)} = X$, $\overline{R(A)} = X$, *and* $(-\infty, 0) \subseteq \rho(A)$.

(ii) We have $\sup_{t>0} \| t(t+A)^{-1} \| \leqslant M$, *for some* $M > 0$.

*A is called R–**sectorial** if the family $\{t(t+A)^{-1}\}_{t>0}$ is R–bounded. We denote the class of sectorial operators (resp. R–sectorial operators) in X by $\mathbf{S}(X)$ (resp. $\mathbf{RS}(X)$).*

If $A \in \mathbf{S}(X)$, then $\Sigma_\phi \subseteq \rho(-A)$, for some $\phi > 0$ and $\sup\limits_{|\arg(\lambda)|<\phi} \|\lambda(\lambda + A)^{-1}\| < \infty$.

We denote the *spectral angle* of $A \in \mathbf{S}(X)$ by

$$\phi_A = \inf\{\phi \, : \, \Sigma_{\pi-\phi} \subseteq \rho(-A) \, , \, \sup_{\lambda \in \Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\| < \infty\}$$

Definition 4.4. *Let A be a sectorial operator. If there exist $\phi > \phi_A$ and a constant $K_\phi > 0$ such that*

$$\|f(A)\| \leqslant K_\phi \|f\|_\infty^\phi, \quad \textit{for all } f \in \mathcal{H}_0(\Sigma_\phi)$$

then we say that a sectorial operator A ***admits a bounded $\mathcal{H}^\infty$–calculus****.*

We denote the class of sectorial operators A which admit a bounded $\mathcal{H}^\infty$–calculus by $\mathcal{H}^\infty(X)$. Moreover, the $\mathcal{H}^\infty$–angle is defined by

$$\phi_A^\infty = \inf\{\phi > \phi_A : \text{ Definition (4.4) holds}\}.$$

***Remark* 4.1.** *Let A be a sectorial operator which admits a bounded $\mathcal{H}^\infty$–calculus. If the set*

$$\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\theta) \ , \ \|h\|_\infty^\theta\} \leqslant 1\}$$

is R–bounded for some $\theta > 0$, we say that A ***admits an R–bounded $\mathcal{H}^\infty$–calculus****. We denote the class of such operators by $\mathcal{RH}^\infty(X)$. The $\mathcal{RH}^\infty$–angle is analogous to the $\mathcal{H}^\infty$–angle, and is denoted $\theta_A^{R_\infty}$. For further information about sectorial and R–sectorial operators, see* [91].

We state the following Proposition from functional calculus theory without proof (compare [52]). The proof of Lemma of 4.7 depends on this result.

Proposition 4.1. *Let $A \in \mathcal{RH}^\infty(X)$ and suppose that $\{h_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta_A^{R_\infty}$, where Λ is an arbitrary index set. Then the set $\{h_\lambda(A)\}_{\lambda\in\Lambda}$ is R–bounded.*

Lemma 4.7. *Let $\alpha, \beta \in \mathbb{R}^+$. Assume that X is a UMD–space. Suppose that $A \in \mathcal{RH}^\infty(X)$, with $\mathcal{RH}^\infty$–angle $\theta_A^{R_\infty} < \frac{\pi}{2}$, then the families of operators*

$$\left\{ ik^3 \left(-\frac{i\alpha k^3 + k^2}{\beta} - A \right)^{-1} \right\}_{k\in\mathbb{Z}} \quad \textit{and} \quad \left\{ ikA^{1/2} \left(-\frac{i\alpha k^3 + k^2}{\beta} - A \right)^{-1} \right\}_{k\in\mathbb{Z}}$$

are R–bounded.

Proof. For every $\lambda \in \mathbb{C}$ and $z \in \Sigma_{\pi/2}$, we define the functions

$$F^1(\lambda, z) = \frac{\beta\lambda^3}{\alpha\lambda^3 + \lambda^2 - \beta z} \quad \text{and} \quad F^2(\lambda, z) = \frac{\beta\lambda z^{1/2}}{\alpha\lambda^3 + \lambda^2 - \beta z}.$$

We claim that there is a constant $M \geqslant 0$ such that $\sup_{k\in\mathbb{Z}} \| F^j(ik, \cdot)\|_{\mathscr{H}^\infty(\Sigma_{\pi/2})} \leqslant M$, for $j = 1, 2$. Indeed, note that

$$(\alpha\lambda^3 + \lambda^2 - \beta z) = (\alpha\lambda^3 + \lambda^2)\Big(1 - \frac{\beta z}{\alpha\lambda^3 + \lambda^2}\Big).$$

Hence, there exists a constant $M_1 \geqslant 0$ such that

$$|F^1(ik, z)| = \frac{|i\beta k^3|}{|i\alpha k^3 + k^2|\Big|1 + \frac{\beta z}{i\alpha k^3 + k^2}\Big|} \leqslant \frac{M_1}{\Big|1 + \frac{\beta z}{i\alpha k^3 + k^2}\Big|}.$$

On the other hand, note also that

$$(\alpha\lambda^3 + \lambda^2 - \beta z) = \sqrt{\alpha\lambda^3 + \lambda^2}\, z^{1/2}\Big(1 - \frac{\beta^{1/2} z^{1/2}}{\sqrt{\alpha\lambda^3 + \lambda^2}}\Big)\Big(\beta^{1/2} + \frac{\sqrt{\alpha\lambda^3 + \lambda^2}}{z^{1/2}}\Big).$$

Thus, there exists $M_2 \geqslant 0$ such that

$$|F^2(ik, z)| \leqslant \frac{M_2}{\Big|1 - \frac{\beta^{1/2} z^{1/2}}{\sqrt{-i\alpha k^3 - k^2}}\Big|\Big|\beta^{1/2} + \frac{\sqrt{-i\alpha k^3 - k^2}}{z^{1/2}}\Big|}.$$

Since $z \in \Sigma_{\pi/2}$, we have that the denominators never vanish. In addition, $F^1(0, z) = 0 = F^2(0, z)$. Therefore, there exists $M \geqslant 0$ such that $\sup_{k\in\mathbb{Z}} \| F^j(ik, \cdot)\|_{\mathscr{H}^\infty(\Sigma_{\pi/2})} \leqslant M$, for $j = 1, 2$. It follows from Proposition 4.1 that the sets $\{F^j(ik, A)\}_{k\in\mathbb{Z}}$ with $j = 1, 2$ are R–bounded. In particular, for all $k \in \mathbb{Z}$ the operators $(-i\alpha k^3 - k^2 - \beta A)^{-1}$ exist. Furthermore, for all $k \in \mathbb{Z}$ the operators $\Big(-\dfrac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1}$ exist in $\mathscr{L}(X)$, and

$$\Big\{ik^3\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1}\Big\}_{k\in\mathbb{Z}} \quad \text{and} \quad \Big\{ikA^{1/2}\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1}\Big\}_{k\in\mathbb{Z}}$$

are R–bounded families of operators. ∎

***Example* 4.2.** *Let X be a UMD–space, and let $p \in (1, \infty)$. Suppose $A \in \mathscr{R}\mathscr{H}^\infty(X)$, with $\mathscr{R}\mathscr{H}^\infty$–angle $\theta_A^{R_\infty} < \frac{\pi}{2}$. Consider the family of operators*

$$\mathscr{F} = \Big\{ikA^{1/2}\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A\Big)^{-1} : k \in \mathbb{Z}\Big\}$$

with $\alpha, \beta > 0$. If $\gamma > 0$ is such that $\frac{\gamma}{\beta}\mathscr{R}_p(\mathscr{F}) < 1$, then the equation

$$\alpha u'''(t) + u''(t) = \beta A u(t) + \gamma A^{1/2} u'(t) + f(t), \quad \textit{for } t \in [0, 2\pi] \tag{4.10}$$

with boundary conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$, has L^p–maximal regularity.

Proof. *According to Lemma 4.7, the families of operators*

$$\left\{ ikA^{1/2}\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A \Big)^{-1} \right\}_{k\in\mathbb{Z}} \quad \textit{and} \quad \left\{ ik^3\Big(-\frac{i\alpha k^3 + k^2}{\beta} - A \Big)^{-1} \right\}_{k\in\mathbb{Z}}$$

are R–bounded. Since $\dfrac{\gamma}{\beta}\mathscr{R}_p(\mathscr{F}) < 1$, it follows from Corollary 4.1 that equation (4.10) *has L^p–maximal regularity.* ■

CHAPTER 5

Periodic Solutions of a Fractional Neutral Differential Equation with Finite Delay

It is well known that neutral functional differential equations are used to represent important physical systems. We refer to [72, 76, 102] for a discussion about this aspect of the theory. Similarly, motivated by the fact that abstract neutral functional differential equations (abbreviated, ANFDE) arise in many areas of applied mathematics, this type of equations has received much attention in recent years ([49, 73, 76]). On the other hand, because several important physical phenomena are modeled by **abstract fractional differential equations**, this type of equations have been studied extensively last time for many authors. We refer the reader to the works [2, 104, 120, 124, 135] and the references listed therein for recent information on this subject.

There exists several notions of fractional differentiation. In this thesis we use the fractional differentiation in sense of Liouville–Grünwald–Letnikov. This concept was introduced in [70, 105] and has been widely studied by several authors. In these works the fractional derivative is defined directly as a limit of a fractional difference quotient. In [35], the authors apply this approach based on fractional differences to study fractional differentiation of periodic scalar functions. This idea has been used to extend the definition of fractional differentiation to vector–valued functions, (see [98]). In the case of periodic functions this concept enables one to set up a fractional calculus in the L^p setting with the usual rules, as well as provides a connection with the classical Weyl fractional derivative (see [137]).

Let $\alpha > 0$. Given $f \in L^p(\mathbb{T}; X)$ for $1 \leqslant p < \infty$ the Riemann difference

$$\Delta_t^{\alpha} f(x) := \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj),$$

$\left(\text{where } \binom{\alpha}{j} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-j-1)}{j!}\right)$ exists almost everywhere and

$$\|\Delta_t^\alpha f\|_{L^p(\mathbb{T};X)} \leqslant \sum_{j=0}^{\infty} \left|\binom{\alpha}{j}\right| \|f\|_{L^p(\mathbb{T};X)}$$

since $\sum_{j=0}^{\infty} |\binom{\alpha}{j}| < \infty$ (see [35]).

The following definition is a direct extension of Definition 2.1 in [35] to vector–valued case. See [98] for its connection with differential equations.

Definition 5.1. *Let X be a complex Banach space, $\alpha > 0$ and $1 \leqslant p < \infty$. Let $f \in L^p(\mathbb{T};X)$. If there exists $g \in L^p(\mathbb{T};X)$ such that $\lim_{t\to 0^+} t^{-\alpha}\Delta_t^\alpha f = g$ in the $L^p(\mathbb{T};X)$ norm, then g is called the α^{th}–Liouville–Grünwald–Letnikov derivative of f in the mean of order p.*

We abbreviate this terminology by α^{th}-derivative and we denote it by $D^\alpha f = g$.

***Example* 5.1.** *The α^{th}-derivative of e^{iax} for any real a is given by $(ia)e^{iax}$. In particular, $D^\alpha \sin x = \sin\big(x + \frac{\pi}{2}\alpha\big)$ and $D^\alpha \cos x = \cos\big(x + \frac{\pi}{2}\alpha\big)$.*

We also mention here a few properties of this fractional derivative. The proof follows the same steps as in the scalar case given in Proposition 4.1 in [35].

Proposition 5.1. *Let $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{T};X)$. For $\alpha, \beta > 0$ the following properties hold:*

(a) If $D^\alpha f \in L^p(\mathbb{T};X)$, then $D^\beta f \in L^p(\mathbb{T};X)$ for all $0 < \beta < \alpha$.

(b) $D^\alpha D^\beta f = D^{\alpha+\beta} f$ whenever one of the two sides is well defined.

Let $f \in L^p(\mathbb{T};X)$ and $\alpha > 0$. It has been proved by Butzer and Westphal [35] that $D^\alpha f \in L^p(\mathbb{T};X)$ if and only if there exists $g \in L^p(\mathbb{T};X)$ such that $(ik)^\alpha \widehat{f}(k) = \widehat{g}(k)$, where $(ik)^\alpha = |k|^\alpha e^{\frac{\pi i \alpha}{2}\operatorname{sgn}(k)}$. In this case $D^\alpha f = g$.

In this chapter, we study a characterization of maximal regularity on Besov and Triebel–Lizorkin spaces of fractional differential equation with finite delay

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad t \in [0,2\pi], \text{ and } 0 < \beta < \alpha \leqslant 2. \tag{5.1}$$

with different periodic boundary conditions depending on the values of α and β. They are

$$\left.\begin{array}{ll} u(0) = u(2\pi) & \text{if } \ 0 < \beta < \alpha \leqslant 1 \\[2ex] D^{\alpha-1}u(0) = D^{\alpha-1}(2\pi) \text{ and } \ u(0) = u(2\pi) & \text{if } \ 0 < \beta < 1 < \alpha \leqslant 2 \\[2ex] D^{\alpha-1}u(0) = D^{\alpha-1}(2\pi) \ , \ \ u(0) = u(2\pi) \text{ and} & \\ D^{\beta-1}u(0) = D^{\beta-1}(2\pi) & \text{if } \ 1 < \beta < \alpha \leqslant 2 \end{array}\right\}$$

Here the function u_t is given by $u_t(\theta) = u(t+\theta)$ for θ in an appropriate domain, denotes the history of the function $u(\cdot)$ at t and $D^\beta u_t(\cdot)$ is definded by $D^\beta u_t(\cdot) = \big(D^\beta u\big)_t(\cdot)$. The delay operators F and G are bounded linear map defined on an suitable space and f is a given function that belongs to a Besov or Triebel–Lizorkin spaces. The operator $A : D(A) \subseteq X \to X$ is a linear closed operator defined in a complex Banach space X.

In recent years, several particular cases of equation (5.1) have been studied by many authors. If $\alpha = 1$ and $F \equiv G \equiv 0$, Arendt and Bu in [10, 12] have studied L^p–maximal regularity and $B^s_{p,q}$–maximal regularity, and Bu and Kim in [32], have studied $F^s_{p,q}$–maximal regularity. On the other hand, Lizama in [110] has obtained a characterization of existence and uniqueness of strong L^p–solutions, and Lizama and Poblete in [112] study C^s–maximal regularity of the corresponding equation on the real line. In the same manner, if $\alpha = 2$ and $\beta = 1$, Bu in [24] characterizes C^s–maximal regularity on $\mathbb{R}$. Furthermore, if $\alpha = 2$ and $\beta = 1$, Bu and Fang in [30] have studied this equation simultaneosly in cathegories of periodic Lebesgue, Besov and Triebel–Lizorkin spaces. Moreover, if $1 < \alpha < 2$ and $G \equiv 0$, Lizama and Poblete in [113] study L^p–maximal regularity of this equation.

We use maximal regularity on Besov spaces (respectively Triebel–Lizorkin spaces) of equation (5.1) and a fixed point argument to proving existence of a strong $B^s_{p,q}$–solution (respectively $F^s_{p,q}$–solution) of neutral fractional differential equation with finite delay

$$D^\alpha\big(u(t) - Bu(t-r)\big) = Au(t) + Fu_t + GD^\beta u_t + f(t) \quad t \in [0, 2\pi], \tag{5.2}$$

with $0 < \beta < \alpha \leqslant 2$, where $r > 0$ is a fixed number and $B : D(B) \subseteq X \to X$ is a linear closed operator defined in a Banach space X such that $D(A) \subseteq D(B)$. All of the rest of terms of this equation are defined like in equation (5.1). This method is an adaptation of the technique that Henriquez and Poblete use successfully in [81] to prove that equation (5.2) in the particular case $G \equiv 0$ has a unique strong L^p-solution for some $1 < p < \infty$.

With a specific norm, we denote the space consisting of all 2π–periodic, X–valued functions by $E(\mathbb{T}; X)$. Let $\alpha > 0$ and denote the set consisting of all functions in $E(\mathbb{T}; X)$ which are α times differentiable in sense of Liouville–Grünwald–Letnikov (if it is well defined) by $E^\alpha(\mathbb{T}; X)$. The following definitions will be used in subsequent sections with periodic Besov and Triebel–Lizorkin periodic spaces.

Definition 5.2. *A function u is called a **strong** E–**solution** of equation* (5.1) *if $u \in E^\alpha(\mathbb{T}; X) \cap E(\mathbb{T}; [D(A)]) \cap E(\mathbb{T}; X)$ and equation* (5.1) *holds a.e. in* $[0, 2\pi]$.

Definition 5.3. *We say that solutions of equation* (5.1) *has E–**maximal regularity** if for each $f \in E(\mathbb{T}; X)$, equation* (5.1) *has a unique strong E–solution.*

For the rest of this chapter we introduce the following notation. Given $0 < \beta < \alpha \leqslant 2$, and a closed linear operator A defined on a Banach space X. For $k \in \mathbb{Z}$, we will write

$$a_k = (ik)^\alpha \quad \text{and} \quad b_k = (ik)^\beta \tag{5.3}$$

and the bounded operators F_k and G_k defined by

$$F_k x = F(e_k x) \text{ and } G_k x = G(e_k x), \text{ where } e_k x(t) = e^{ikt}x. \tag{5.4}$$

Moreover, we will denote

$$N_k = (a_k I - F_k - b_k G_k - A)^{-1}. \tag{5.5}$$

and

$$M_k = a_k\big(a_k I - b_k G_k - F_k - A\big)^{-1} = a_k N_k \tag{5.6}$$

In order to give conditions which we will need later, we say that $\{F_k\}_{k\in\mathbb{Z}}$ satisfies the condition

(F2) If the family of operators $\left\{\frac{k^2}{a_k}\left(\Delta^2 F_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ is a bounded.

(F3) If $\{F_k\}_{k\in\mathbb{Z}}$ satisfies ***(F2)*** and the family $\left\{\frac{k^3}{a_k}\left(\Delta^3 F_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded.

In the same manner, we say that the family $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition

(G2) If the families of operators $\left\{\frac{kb_k}{a_k}\left(\Delta^1 G_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ and $\left\{\frac{k^2 b_k}{a_k}\left(\Delta^2 G_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ are bounded.

(G3) If $\{G_k\}_{k\in\mathbb{Z}}$ satisfies ***(G2)*** and the family $\left\{\frac{k^3 b_k}{a_k}\left(\Delta^3 G_k\right)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ is bounded.

5.1 Periodic Strong $B^s_{p,q}$–solution.

The first objective of this section is the study $B^s_{p,q}$–maximal regularity of equation (5.1). For this purpose, given $\alpha > 0$ we present a characterization of the Besov space $B^{s+\alpha}_{p,q}(\mathbb{T};X)$ in terms of Liouville–Grünwald–Letnikov fractional derivative.

Proposition 5.2. *Let $\alpha > 0$ and a function $u \in B^s_{p,q}(\mathbb{T};X)$ with $1 \leqslant p,q \leqslant \infty$ and $s > 0$. $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$ if and only if there exists $g \in B^s_{p,q}(\mathbb{T};X)$ such that $(ik)^\alpha \widehat{u}(k) = \widehat{g}(k)$, and in this case $D^\alpha u = g$. In fact we have*

$$B^{s+\alpha}_{p,q}(\mathbb{T};X) = \{u \in B^s_{p,q}(\mathbb{T};X) : D^\alpha u \in B^s_{p,q}(\mathbb{T};X)\}.$$

Proof. Let $u \in B^s_{p,q}(\mathbb{T};X)$ such that $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$. We claim that $u \in B^{s+\alpha}_{p,q}(\mathbb{T};X)$. In fact, since $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$, it follows from *lifting property* of Besov spaces, that

$$\sum_{k\neq 0} e_k \otimes \widehat{D^\alpha u}(k) \in B^s_{p,q}(\mathbb{T};X).$$

Since $\widehat{D^\alpha u}(k) = (ik)^\alpha \hat{u}(k)$ for all $k \in \mathbb{Z}$, we have that

$$\sum_{k\neq 0} e_k \otimes (ik)^\alpha \hat{u}(k) \in B^s_{p,q}(\mathbb{T};X).$$

By *lifting property* this is equivalent to $u \in B^{s+\alpha}_{p,q}(\mathbb{T};X)$.
Reciprocally, suppose that $u \in B^{s+\alpha}_{p,q}(\mathbb{T};X)$. by *lifting property* this is equivalent that

$$\sum_{k\neq 0} e_k \otimes (ik)^\alpha \hat{u}(k) \in B^s_{p,q}(\mathbb{T};X) \tag{5.7}$$

Since $s > 0$, we have that

$$\sum_{k\neq 0} e_k \otimes (ik)^\alpha \hat{u}(k) \in L^p(\mathbb{T};X)$$

It follows from [35, Theorem 4.1] that there exists $g \in L^p(\mathbb{T};X)$ such that $\hat{g}(k) = (ik)^\alpha \hat{u}(k)$ for all $k \in \mathbb{Z}$. Furthermore, it follows from the affirmation (5.7) that $g \in B^s_{p,q}(\mathbb{T};X)$. Therefore $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$. ■

With this characterization we redefine elegantly $B^s_{p,q}$–maximal regularity of solutions of equation (5.1) in particular case $s > 0$.

Definition 5.4. *Let* $1 \leqslant p,q \leqslant \infty$, $s > 0$ *and let* $f \in B^s_{p,q}(\mathbb{T};X)$. *A function u is called strong* $B^s_{p,q}$*–solution of equation* (5.1) *if* $u \in B^{s+\alpha}(\mathbb{T};X) \cap B^s_{p,q}(\mathbb{T};[D(A)])$ *and u satisfies the equation* (5.1) *for almost* $t \in [0,2\pi]$. *We say that solutions of equation* (5.1) *has* $B^s_{p,q}$*–maximal regularity if, for each* $f \in B^s_{p,q}(\mathbb{T};X)$ *equation* (5.1) *has an unique strong* $B^s_{p,q}$*–solution.*

The proof of Theorem 5.1 will depend of our next results.

Lemma 5.1. *Let* $1 \leqslant p,q \leqslant \infty, s > 0$ *and* $0 < \beta < \alpha \leqslant 2$. *Let* $G \in \mathscr{L}(B^s_{p,q}(\mathbb{T};X);X)$. *If the family* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy the condition* ***(G2)*** *then*

$$\left\{\frac{k}{a_k}\big(\Delta^1 b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}} \quad \textit{and} \quad \left\{\frac{k^2}{a_k}\big(\Delta^2 b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

are bounded families of operators.

Proof. We note that for all $k \in \mathbb{Z}$, it holds

$$\big(\Delta^1 b_k G_k\big) = \big(\Delta^1 b_k\big)G_{k+1} + b_k\big(\Delta^1 G_k\big).$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$,

$$\frac{k}{a_k}(\Delta^1 b_k G_k) = \frac{k\big(\Delta^1 b_k\big)}{b_k}\frac{b_k}{a_k}G_{k+1} + \frac{k b_k}{a_k}\big(\Delta^1 G_k\big).$$

On the other hand, for all $k \in \mathbb{Z}$ we have

$$\big(\Delta^2 b_k G_k\big) = \big(\Delta^1 b_{k+1}\big)\big[(\Delta^1 G_{k+1}) + (\Delta^1 G_k)\big] + \big(\Delta^2 b_k\big)G_k + b_{k+1}\big(\Delta^2 G_k\big).$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$ we have,

$$\frac{k^2}{a_k}(\Delta^2 b_k G_k) = \frac{k(\Delta^1 b_{k+1})}{b_k}\frac{k b_k}{a_k}[(\Delta^1 G_{k+1}) + (\Delta^1 G_k)] + \frac{k^2(\Delta^2 b_k)}{b_k}\frac{b_k}{a_k}G_k + \frac{k^2 b_{k+1}}{a_k}(\Delta^2 G_k).$$

Since the sequence $\{b_k\}_{k\in\mathbb{Z}}$ is 2–regular and $\sup_{k\in\mathbb{Z}} \|G_k\| \leqslant C\|G\|$ for some $C \geqslant 0$, it follows from the hypothesis that

$$\left\{\frac{k}{a_k}\big(\Delta^1 b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}} \text{ and } \left\{\frac{k^2}{a_k}\big(\Delta^2 b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

are bounded families. ∎

Lemma 5.2. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a linear closed operator defined in a Banach space X and $F, G \in \mathscr{L}(B^s_{p,q}(\mathbb{T}; X); X)$. Assume that for all $k \in \mathbb{Z}$, the operators N_k are well defined in $\mathscr{L}(X)$. If the family $\{F_k\}_{k\in\mathbb{Z}}$ satisfies the condition **(F2)** and $\{G_k\}_{k\in\mathbb{Z}}$ satisfies **(G2)** and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded then*

$$\big\{k a_k\big(\Delta^1 N_k\big)\big\}_{k\in\mathbb{Z}} \text{ and } \big\{k^2 a_k\big(\Delta^2 N_k\big)\big\}_{k\in\mathbb{Z}}$$

are bounded families of operators.

Proof. We note that for all $k \in \mathbb{Z}$ it holds

$$\begin{aligned}\big(\Delta^1 N_k\big) &= N_{k+1}\big(a_k - F_k - b_k G_k - a_{k+1} + F_{k+1} + b_{k+1}G_{k+1}\big)N_k \\ &= \big(-\Delta a_k\big)N_{k+1}N_k + N_{k+1}\big(\Delta^1 F_k\big)N_k + N_{k+1}\big(\Delta^1 b_k G_k\big)N_k.\end{aligned} \tag{5.8}$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$

$$k a_k\big(\Delta^1 N_k\big) = k\frac{\big(-\Delta^1 a_k\big)}{a_k} a_k N_{k+1} M_k + \frac{k}{a_k} a_k N_{k+1}\big(\Delta^1 F_k\big)M_k + a_k N_{k+1}\frac{k}{a_k}\big(\Delta^1 b_k G_k\big)M_k.$$

A direct computation, shows that $k a_k\big(\Delta^1 N_k\big)$ is bounded if $k = 0$. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2–regular and the families of operators $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are bounded, it follows from Lemma 5.1 that $\big\{k a_k\big(\Delta^1 N_k\big)\big\}_{k\in\mathbb{Z}}$ is a bounded family of operators.

On the other hand, for all $k \in \mathbb{Z}$ we have

$$\begin{aligned}(\Delta^2 N_k) &= [(\Delta^1 N_{k+1}) + (\Delta^1 N_k)][(-\Delta^1 a_{k+1}) + (\Delta^1 F_{k+1}) + (\Delta^1 b_{k+1} G_{k+1})]N_{k+1} \\ &+ N_k[(-\Delta^2 a_k) + (\Delta^2 F_k) + (\Delta^2 b_k G_k)]N_{k+1}.\end{aligned} \tag{5.9}$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$

$$\begin{aligned}k^2 a_k(\Delta^2 N_k) = k a_k[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)]\frac{k}{a_k}\left[-(\Delta^1 a_{k+1}) + (\Delta^1 F_{k+1}) + (\Delta^1 b_{k+1} G_{k+1})\right] a_k N_{k+1} \\ + M_k\left[k^2\frac{(-\Delta^2 a_k)}{a_k} + \frac{k^2}{a_k}(\Delta^2 F_k) + \frac{k^2}{a_k}(\Delta^2 b_k G_k)\right] a_k N_{k+1}.\end{aligned}$$

A direct computation shows that if $k = 0$ the operator $k^2 a_k(\Delta^2 N_k)$ is bounded. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2–regular, the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ are bounded, and the family $\{k a_k(\Delta^1 N_k)\}_{k\in\mathbb{Z}}$ is bounded, it follows from Lemma 5.1 that the family of operators $\{k^2 a_k(\Delta^2 N_k)\}_{k\in\mathbb{Z}}$ is bounded. ■

Lemma 5.3. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a linear closed operator defined in a Banach space X and $F, G \in \mathscr{L}(B^s_{p,q}(\mathbb{T}; X); X)$. Assume that for all $k \in \mathbb{Z}$, the operators N_k are well defined in $\mathscr{L}(X)$. If $\{F_k\}_{k\in\mathbb{Z}}$ satisfies the condition **(F2)** and $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition **(G2)** and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded then the family $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.*

Proof. According to Theorem 1.2, it suffices to show that the family of operators $\{F_k N_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$–bounded of order 2. With this purpose, note that $\sup_{k\in\mathbb{Z}} \|F_k\| \leqslant C\|F\|$ for some $C \geqslant 0$ and $\sup_{k\in\mathbb{Z}} \|N_k\| < \infty$. Therefore the family of operators $\{F_k N_k\}_{k\in\mathbb{Z}}$ is bounded.

On the other hand, we have

$$k(\Delta^1 F_k N_k) = \frac{k}{a_{k+1}}(\Delta^1 F_k) M_{k+1} + \frac{1}{a_k} F_k k a_k(\Delta^1 N_k)$$

and

$$k^2(\Delta^2 F_k N_k) = \frac{1}{a_k} F_{k+1} k^2 a_k(\Delta^2 N_k) + \frac{k^2}{a_k}(\Delta^2 F_k) M_k + \frac{k}{a_k}(\Delta^1 F_{k+1}) k a_k\big((\Delta^1 N_{k+1}) + (\Delta^1 N_k)\big)$$

It follows from Lemmas 5.1 and 5.2 that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$-bounded of order 2. ■

Lemma 5.4. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a linear closed operator defined in a Banach space X and $F, G \in \mathscr{L}(B^s_{p,q}(\mathbb{T}; X); X)$. Assume that for all $k \in \mathbb{Z}$, the operators N_k are well defined in $\mathscr{L}(X)$. If $\{F_k\}_{k\in\mathbb{Z}}$ satisfies **(F2)** and $\{G_k\}_{k\in\mathbb{Z}}$ satisfies **(G2)** and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded then $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.*

Proof. According to Theorem 1.2 it suffices to show that the family of operators $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$–bounded of order 2.
For this, note that $\sup_{k\in\mathbb{Z}} \|G_k\| \leqslant C\|G\|$ for some $C \geqslant 0$ and $\sup_{k\in\mathbb{Z}} \|b_k N_k\| < \infty$.
Therefore $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is bounded.
On the other hand, we have

$$k\big(\Delta^1 b_k G_k N_k\big) = \frac{k}{a_k}\big(\Delta^1 b_k G_k\big) a_k N_{k+1} + \frac{b_k}{a_k} G_k k a_k \big(\Delta^1 N_k\big)$$

and

$$\begin{aligned} k^2\big(\Delta^2 b_k G_k N_k\big) &= \frac{k}{a_k}\big(\Delta^1 b_{k+1} G_{k+1}\big) k a_k \big[\big(\Delta^1 N_{k+1}\big) + \big(\Delta^1 N_k\big)\big] + \frac{k^2}{a_k}(\Delta^2 b_k G_k) M_k \\ &+ \frac{b_{k+1}}{a_k} G_{k+1} k^2 a_k \big(\Delta^2 N_k\big). \end{aligned}$$

It follows from Lemmas 5.1 and 5.2 that $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$-bounded of order 2. ■

Lemma 5.5. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a linear closed operator defined in a Banach space X and $F, G \in \mathcal{L}(B^s_{p,q}(\mathbb{T}; X); X)$. Assume that for all $k \in \mathbb{Z}$, the operators N_k are well defined in $\mathcal{L}(X)$. If $\{F_k\}_{k\in\mathbb{Z}}$ satisfies **(F2)** and $\{G_k\}_{k\in\mathbb{Z}}$ satisfies **(G2)** then the following assertions are equivalent*

(i) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a bounded.

(ii) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.

Proof. $(i) \Rightarrow (ii)$. According Theorem 1.2 it suffices to show that that $\{M_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$–bounded of order 2. By hypothesis, $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. We note that

$$k(\Delta^1 M_k) = \frac{k(\Delta^1 a_k)}{a_{k+1}} M_{k+1} + k a_k (\Delta^1 N_k).$$

On the other hand, we have

$$k^2\big(\Delta^2 M_k\big) = \frac{k\big(\Delta^1 a_{k+1}\big)}{a_k} k a_k \big[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)\big] + \frac{k^2\big(\Delta^2 a_k\big)}{a_k} M_k + k^2 a_{k+1}\big(\Delta^2 N_k\big).$$

Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2–regular, it follows from Lemmas 5.1 and 5.2 that $\{M_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$–bounded of order 2.

$(ii) \Rightarrow (i)$. It follows from Closed Graph Theorem that there exists $C \geqslant 0$ (independent of f) such that for $f \in B^s_{p,q}(\mathbb{T}; X)$ we have,

$$\Big\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \widehat{f}(k) \Big\|_{B^s_{p,q}} \leqslant C \|f\|_{B^s_{p,q}}.$$

Let $x \in X$ and define $f(t) = e^{ikt}x$ for $k \in \mathbb{Z}$ fixed. Then the above inequality implies

$$\|e_k\|_{B^s_{p,q}}\|M_k x\|_{B^s_{p,q}} = \|e_k M_k x\|_{B^s_{p,q}} \leqslant C\|e_k\|_{B^s_{p,q}}\|x\|_{B^s_{p,q}}.$$

Hence for all $k \in \mathbb{Z}$ we have $\|M_k\| \leqslant C$. Thus $\sup_{k\in\mathbb{Z}} \|M_k\| < \infty$. ■

Next Theorem establishes a characterization of $B^s_{p,q}$–maximal regularity of solution of equation (5.1).

Theorem 5.1. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$. Let be X a Banach space. If the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$, defined by the operators F and G of equation* (5.1) *satisfy the conditions **(F2)** and **(G2)** respectively, then the following assertions are equivalent*

(i) The solution of equation (5.1) *has $B^s_{p,q}$–maximal regularity.*

(ii) $\{N_k\}_{k\in\mathbb{Z}} \subseteq \mathscr{L}(X)$ and the family $\{M_k\}_{k\in\mathbb{Z}}$ is bounded.

Proof. $(i) \Rightarrow (ii)$. We show that for $k \in \mathbb{Z}$ the operators $((ik)^\alpha I - (ik)^\beta G_k - F_k - A)$ are invertible. For this, let $k \in \mathbb{Z}$ fixed and $x \in X$, and define $h(t) = e^{ikt}x$, a direct calculation shows that $\widehat{h}(k) = x$. By hypothesis there exists a function $u \in B^{s+\alpha}_{p,q}(\mathbb{T}; X) \cap B^s_{p,q}(\mathbb{T}; [D(A)])$ such that , for almost all $t \in [0, 2\pi]$, we have

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + h(t).$$

Applying Fourier transform to both sides of the preceding equality, we obtain

$$\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)\widehat{u}(k) = \widehat{h}(k) = x,$$

since x and k are arbitrary, we have that for $k \in \mathbb{Z}$ the operators $\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)$ are surjective.

On the other hand, let $z \in D(A)$, and assume that $((ik)^\alpha - F_k - (ik)^\beta G_k - A)z = 0$. Substituting $u(t) = e^{ikt}z$ in the equation (5.1) we see that u is a periodic solution of this equation when $f \equiv 0$. The uniqueness of solution implies that $z = 0$.

Since for all $k \in \mathbb{Z}$ the linear operators N_k are closed defined in whole space X, it follows from Closed Graph Theorem that $N_k \in \mathscr{L}(X)$. Thus $\{N_k\}_{k\in\mathbb{Z}} \subseteq \mathscr{L}(X)$.

Let $f \in B^s_{p,q}(\mathbb{T}; X)$. By hypothesis, there exists a function $u \in B^{s+\alpha}_{p,q}(\mathbb{T}, X) \cap B^s_{p,q}(\mathbb{T}, [D(A)])$ such that u is the unique strong solution of the equation

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \quad t \in [0, 2\pi].$$

Applying Fourier transform to both sides of the preceding equation, we have

$$\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)\widehat{u}(k) = \widehat{f}(k)$$

for all $k \in \mathbb{Z}$. Since for all $k \in \mathbb{Z}$ the operator $\big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)$ is invertible, we have

$$\widehat{u}(k) = \big((ik)^\alpha - F_k - (ik)^\beta G_k - A\big)^{-1} \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

Hence, $(ik)^\alpha \widehat{u}(k) = \widehat{D^\alpha u}(k) = (ik)^\alpha N_k \widehat{f}(k) = M_k \widehat{f}(k)$ for all $k \in \mathbb{Z}$.

Since $u \in B^{s+\alpha}_{p,q}(\mathbb{T};X)$, by Proposition 5.2 we have that $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$. Therefore the family $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. It follows from Lemma 5.5 that $\{M_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators.

$(ii) \Rightarrow (i)$. By hypothesis, the conditions of Lemma 5.5 are satisfied. Therefore, $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Define the family of operator $\{I_k\}_{k\in\mathbb{Z}}$, by $I_k = \frac{1}{(ik)^\alpha} I$ when $k \neq 0$ and $I_0 = I$. It follows from Theorem 1.2 that $\{I_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Since $N_k = I_k L_k$ for all $k \in \mathbb{Z} \setminus \{0\}$ we have $\{N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Accordingly, given a arbitrary function $f \in B^s_{p,q}(\mathbb{T};X)$ there exist functions $u, w \in B^s_{p,q}(\mathbb{T},X)$ such that

$$\widehat{u}(k) = N_k \widehat{f}(k) \quad \text{and} \quad \widehat{w}(k) = (ik)^\alpha N_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{5.10}$$

Therefore,

$$\widehat{w}(k) = (ik)^\alpha \widehat{u}(k) = \widehat{D^\alpha u}(k) \quad \text{for all } k \in \mathbb{Z}.$$

By the uniqueness of the Fourier coefficients, $D^\alpha u = w$. This implies that that $D^\alpha u \in B^s_{p,q}(\mathbb{T};X)$. It follows from Proposition 5.2 that $u \in B^{s+\alpha}_{p,q}(\mathbb{T};X)$.

On the other hand, it follows from Lemma 5.3 that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Consequently, there exists a function $g \in B^s_{p,q}(\mathbb{T};X)$ such that

$$\widehat{g}(k) = F_k N_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

By equality (5.10) we have

$$\widehat{g}(k) = F_k \widehat{u}(k) \quad \text{for all } k \in \mathbb{Z}.$$

A direct computation shows that $\widehat{Fu_t}(k) = F_k \widehat{u}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $Fu_t = g$. This implies that that $Fu_t \in B^s_{p,q}(\mathbb{T};X)$.

In the same manner, it follows from Lemma 5.4 that $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Hence there exists a function $h \in B^s_{p,q}(\mathbb{T};X)$ such that

$$\widehat{h}(k) = (ik)^\beta G_k N_k \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

By equality (5.10) we have

$$\widehat{h}(k) = (ik)^\beta G_k \widehat{u}(k) \quad \text{for all } k \in \mathbb{Z}.$$

A direct computation shows that $(ik)^\beta G_k \widehat{u}(k) = \widehat{GD^\beta u_t}(k)$ for all $k \in \mathbb{Z}$. By the uniqueness of the Fourier coefficients, $GD^\beta u_t = h$. This implies that that $GD^\beta u_t \in B^s_{p,q}(\mathbb{T};X)$.

It follows from equality (5.10) that

$$\widehat{u}(k) = \big((ik)^{\alpha} - F_k - (ik)^{\beta} G_k - A\big)^{-1} \widehat{f}(k).$$

Thus,

$$\big((ik)^{\alpha} - F_k - (ik)^{\beta} G_k - A\big)\widehat{u}(k) = \widehat{f}(k)$$

for all $k \in \mathbb{Z}$. Using the fact that A is a closed operator, we have that $u(t) \in D(A)$ for almost $t \in [0,2\pi]$. Moreover, by uniqueness of Fourier coefficients we have

$$D_t^{\alpha} u(t) = Au(t) + Fu_t + GD^{\beta} u_t + f(t)$$

for almost $t \in [0,2\pi]$. Since $f, Fu_t, GD^{\beta}u_t$ and $D^{\alpha}u \in B^s_{p,q}(\mathbb{T};X)$, we conclude that $Au \in B^s_{p,q}(\mathbb{T};X)$. This implies that $u \in B^s_{p,q}(\mathbb{T};[D(A)])$. Therefore, u is a strong $B^s_{p,q}$–solution of equation (5.1).

Since $\big((ik)^{\alpha} I - (ik)^{\beta} G_k - F_k - A\big)^{-1}$ is invertible for all $k \in \mathbb{Z}$, this strong $B^s_{p,q}$–solution is unique. Therefore the solution of equation (5.1) has $B^s_{p,q}$–maximal regularity. ∎

As we have mentioned, the verification of assumptions concerning the family $\{N_k\}_{k\in\mathbb{Z}}$ is not an easy work. Our next Corollary require additional conditions about the operators A, F and G, however is a more practical result to check that $\{(ik)^{\alpha} N_k\}$ is bounded. Let $\alpha \geqslant 1$, define the operators $S_k = ((ik)^{\alpha} - A)^{-1}$, for all $k \in \mathbb{Z}$.

Corollary 5.1. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$. Let be X a Banach space. Assume that $\{(ik)^{\alpha}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$, defined by the operators F and G of equation (5.1) satisfy the conditions **(F2)** and **(G2)** respectively. If the family of operators $\{(ik)^{\alpha}\big((ik)^{\alpha} - A\big)^{-1}\}_{k\in\mathbb{Z}}$ is bounded, and $\sup_{k\in\mathbb{Z}} \Big\|\big((ik)^{\beta} G_k + F_k\big) S_k\Big\| < 1$ then equation (5.1) has $B^s_{p,q}$–maximal regularity.*

Proof. Since $\sup_{k\in\mathbb{Z}} \big\|\big((ik)^{\beta} G_k + F_k\big) S_k\big\| < 1$, we have that the family

$$\left\{\Big(I - \big((ik)^{\beta} G_k + F_k\big) S_k\Big)^{-1}\right\}_{k\in\mathbb{Z}}$$

is bounded. In addition

$$\begin{aligned} N_k &= \Big[\big((ik)^{\alpha} - A\big)\big(I - \big((ik)^{\beta} G_k + F_k\big) S_k\big)\Big]^{-1} \\ &= \big(I - \big((ik)^{\beta} G_k + F_k\big) S_k\big)^{-1}\big((ik)^{\alpha} - A\big)^{-1}. \end{aligned}$$

Therefore the family $\{(ik)^{\alpha} N_k\}_{k\in\mathbb{Z}}$ is bounded. Since the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions ***(F2)*** and ***(G2)*** respectively, it follows from Theorem 5.1 that equation (5.1) has $B^s_{p,q}$–maximal regularity. ∎

5.2 Periodic Strong $B^s_{p,q}$–solutions of a Neutral Fractional Differential Equation with Finite Delay

In this section, as we have said, we use the results about $B^s_{p,q}$–maximal regularity of solution of equation (5.1) to prove that fractional neutral differential equation

$$D^{\alpha}\big(u(t)-Bu(t-r)\big)=Au(t)+Fu_t+GD^{\beta}u_t+f(t), \tag{5.11}$$

has a periodic strong $B^s_{p,q}$–solution.

Note that, if equation (5.1) has $B^s_{p,q}$–maximal regularity for some $1\leqslant p,q\leqslant\infty$ and $s>0$, the linear map $\Psi: B^s_{p,q}(\mathbb{T};X)\to B^s_{p,q}(\mathbb{T};X)$ given by $\Psi(g)=D^{\alpha}v$, is well defined. Here v is the unique strong $B^s_{p,q}$–solution of the equation

$$D^{\alpha}v=Au+Fu_t+GD^{\beta}u_t+g(t). \tag{5.12}$$

Lemma 5.6. *Let $1\leqslant p,q\leqslant\infty$, $s>0$, and $1\leqslant\beta\leqslant\alpha$. Let be X a Banach space. Assume that B is a bounded operator, for all $k\in\mathbb{Z}$ the operators N_k is well defined in $\mathscr{L}(X)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ defined by the operators F and G of equation* (5.11) *satisfy the conditions **(F2)** and **(G2)** respectively. If $\{(ik)^{\alpha}N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}}|k|^{\alpha}\|B\|\|N_k\|<1$ and $\|B\|\|\Psi\|<1$, then the family $\{(I-e^{-ikr}(ik)^{\alpha}BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier.*

Proof. Denote $R_k=(I-e^{-ikr}(ik)^{\alpha}BN_k)^{-1}$ for all $k\in\mathbb{Z}$. Since $\sup\limits_{k\in\mathbb{Z}}|k|^{\alpha}\|B\|\|N_k\|<1$, the family $\{R_k\}_{k\in\mathbb{Z}}\subseteq\mathscr{L}(X)$. Let $f\in B^s_{p,q}(\mathbb{T};X)$. Define the map $\mathscr{P}: B^s_{p,q}(\mathbb{T};X)\to B^s_{p,q}(\mathbb{T};X)$ by

$$\mathscr{P}\varphi(t)=B\Psi(\varphi)(t-r)+f(t).$$

By Theorem 5.1 the map $\mathscr{P}$ is well defined. Moreover $\mathscr{P}$ is a contraction, thus there exists a function $g\in B^s_{p,q}(\mathbb{T};X)$ such that

$$g(t)\;=\;B\Psi(g)(t-r)+f(t)\;=\;BD^{\alpha}u(t-r)+f(t), \tag{5.13}$$

where u is the unique strong $B^s_{p,q}$–solution of the equation

$$D^{\alpha}u(t)=Au(t)+Fu_t+GD^{\beta}u_t+g(t)\quad t\in[0,2\pi],\quad 0<\beta<\alpha\leqslant 2. \tag{5.14}$$

Applying Fourier transform to both side of equation (5.13) we have

$$\widehat{g}(k)=e^{-ikr}(ik)^{\alpha}B\widehat{u}(k)+\widehat{f}(k)\quad\text{for all } k\in\mathbb{Z} \tag{5.15}$$

On the other hand, applying Fourier transform to both sides of equation (5.14) we have

$$\widehat{u}(k)=N_k\widehat{g}(k)\quad\text{for all } k\in\mathbb{Z} \tag{5.16}$$

Therefore

$$\widehat{g}(k) = e^{-ikr}(ik)^{\alpha}BN_k\widehat{g}(k) + \widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

This implies that $\widehat{g}(k) = R_k\widehat{f}(k)$ for all $k \in \mathbb{Z}$.
Hence, the family of operators $\{(I - e^{-ikr}(ik)^{\alpha}BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. ∎

Next Theorem gives sufficient condition that guarantee existence of a strong $B^s_{p,q}$–solution of equation (5.11). We use the notation introduced in preceding Lemma.

Theorem 5.2. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded operator, for all $k \in \mathbb{Z}$ the operators N_k is well defined in $\mathscr{L}(X)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ defined by the operators F and G of equation* (5.11) *satisfy the conditions **(F2)** and **(G2)** respectively. If $\{(ik)^{\alpha}N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}}|k|^{\alpha}\|B\|\|N_k\| < 1$ and $\|B\|\|\Psi\| < 1$, then for every $f \in B^s_{p,q}(\mathbb{T};X)$ there exists an unique strong $B^s_{p,q}$–solution of Equation* (5.11).

Proof. It follows from Lemma 5.6 that the family of operators $\{(I-e^{-ikr}(ik)^{\alpha}BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$–multiplier. Denote $R_k = (I - e^{-ikr}(ik)^{\alpha}BN_k)^{-1}$. Let $f \in B^s_{p,q}(\mathbb{T};X)$. Since $\{R_k\}_{k\in\mathbb{Z}}$ is $B^s_{p,q}$–multiplier, there exists $g \in B^s_{p,q}(\mathbb{T};X)$ such that

$$\widehat{g}(k) = R_k\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{5.17}$$

On the other hand, by Theorem 5.1, there exists a function $u \in B^s_{p,q}(\mathbb{T};X)$ such that u is the unique strong $B^s_{p,q}$–solution of equation

$$D^{\alpha}u(t) = Au(t) + Fu_t + GD^{\beta}u_t + g(t), \quad t \in [0,2\pi], \quad 0 < \beta < \alpha \leqslant 2. \tag{5.18}$$

Applying Fourier transform to both side of the preceding equality we have

$$\widehat{u}(k) = N_k\widehat{g}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{5.19}$$

It follows from equality (5.17) that

$$\widehat{u}(k) = N_kR_k\widehat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \tag{5.20}$$

Note that

$$N_kR_k = \big((ik)^{\alpha} - e^{-ikr}(ik)^{\alpha}B - (ik)^{\beta}G_k - F_k - A\big)^{-1} \text{ for all } k \in \mathbb{Z}.$$

Thus,

$$\big((ik)^{\alpha} - e^{-ikr}(ik)^{\alpha}B - (ik)^{\beta}G_k - F_k - A\big)\widehat{u}(k) = \widehat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

Since A is a closed linear operator, it follows from uniqueness of Fourier coefficients that u satisfies the equation

$$D^{\alpha}\big(u(t) - Bu(t-r)\big) = Au(t) + Fu_t + GD^{\beta}u_t + f(t) \quad \text{for almost } t \in [0,2\pi].$$

Therefore u is a strong $B^s_{p,q}$–solution of equation (5.11). It only remains to show that the strong $B^s_{p,q}$–solution is unique. Indeed let $f \in B^s_{p,q}(\mathbb{T};X)$. Suppose equation (5.11) has two strong $B^s_{p,q}$–solutions, u_1 and u_2. A direct computation shows that

$$\big((ik)^\alpha - e^{-ikr}(ik)^\alpha B - (ik)^\beta G_k - F_k - A\big)(\widehat{u_1}(k) - \widehat{u_2}(k)) = 0$$

for all $k \in \mathbb{Z}$. Since $\big((ik)^\alpha - e^{-ikr}(ik)^\alpha B - (ik)^\beta G_k - F_k - A\big)$ is invertible, for all $k \in \mathbb{Z}$ we have that $\widehat{u_1}(k) = \widehat{u_2}(k)$. By the uniqueness of the Fourier coefficients, $u_1 \equiv u_2$. ■

5.3 Periodic Strong $F^s_{p,q}$–solution.

Recall, we are studying the equations

$$D^\alpha u(t) = Au(t) + Fu_t + GD^\beta u_t + f(t), \tag{5.21}$$

and

$$D^\alpha\big(u(t) - Bu(t-r)\big) = Au(t) + Fu_t + GD^\beta u_t + f(t), \tag{5.22}$$

where all terms of these equations are defined in (5.1) and (5.2). Our approach is similar as that of preceding section. For this reason, the first objective of this section is the study $F^s_{p,q}$–maximal regularity of equation (5.21). For this purpose, in the same manner of preceding section, given $\alpha > 0$ we present a characterization of the Triebel–Lizorkin space $F^{s+\alpha}_{p,q}(\mathbb{T};X)$ in terms of Liouville–Grünwald–Letnikov fractional derivative.

Proposition 5.3. *Let $\alpha > 0$ and a function $u \in F^s_{p,q}(\mathbb{T};X)$ with $1 \leqslant p,q \leqslant \infty$ and $s > 0$. $D^\alpha u \in F^s_{p,q}(\mathbb{T};X)$ if and only if there exists $g \in F^s_{p,q}(\mathbb{T};X)$ such that $(ik)^\alpha \widehat{u}(k) = \widehat{g}(k)$, and in this case $D^\alpha u = g$. In fact we have*

$$F^{s+\alpha}_{p,q}(\mathbb{T};X) = \{u \in F^s_{p,q}(\mathbb{T};X) : D^\alpha u \in F^s_{p,q}(\mathbb{T};X)\}.$$

Proof. The proof follows the same lines as those of proof Proposition 5.2 ■

With this characterization we redefine elegantly $F^s_{p,q}$–maximal regularity of solutions of equation (5.21) when $s > 0$.

Definition 5.5. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$ and let $f \in F^s_{p,q}(\mathbb{T};X)$. A function u is called strong $F^s_{p,q}$–solution of equation (5.21) if $u \in F^{s+\alpha}(\mathbb{T};X) \cap F^s_{p,q}(\mathbb{T};[D(A)])$ and u satisfies the equation (5.21) for almost $t \in [0,2\pi]$. We say that equation (5.21) has $F^s_{p,q}$–maximal regularity if, for each $f \in F^s_{p,q}(\mathbb{T};X)$ equation (5.21) has a unique strong $F^s_{p,q}$–solution.*

The proof of Theorem 5.3 will depend of our next results.

Lemma 5.7. *Let* $1 \leqslant p,q \leqslant \infty, s>0$ *and* $0<\beta<\alpha \leqslant 2$. *Let* $G \in \mathscr{L}(F^s_{p,q}(\mathbb{T};X);X)$. *If* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfy **(G3)** then*

$$\left\{\frac{k^3}{a_k}\big(\Delta^3 b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

is a bounded family of operators.

Proof. We note that for all $k\in\mathbb{Z}$ it holds

$$\begin{aligned}(\Delta^3 b_k G_k) &= b_k(\Delta^3 G_k) + (b_{k+3}-b_k)(\Delta^2 G_{k+1}) + (\Delta^2 b_{k+1})(\Delta^1 G_{k+1})\\ &\quad + (\Delta^3 b_k)G_{k+2} - 2(\Delta^2 b_k)(\Delta^1 G_{k+1}).\end{aligned}$$

Therefore, for all $k\in\mathbb{Z}\setminus\{0\}$ we have

$$\begin{aligned}\frac{k^3}{a_k}(\Delta^3 b_k G_k) &= \frac{kb_k}{a_k}(\Delta^3 G_k) + \frac{k(b_{k+3}-b_k)}{b_k}\frac{k^2 b_k}{a_k}(\Delta^2 G_{k+1}) + \frac{k^2(\Delta^2 b_{k+1})}{b_k}\frac{kb_k}{a_k}(\Delta^1 G_{k+1})\\ &\quad + \frac{k^3(\Delta^3 b_k)}{b_k}\frac{b_k}{a_k}G_{k+2} - 2\frac{k^2(\Delta^2 b_k)}{b_k}\frac{kb_k}{a_k}(\Delta^1 G_{k+1}).\end{aligned}$$

Since the sequence $\{b_k\}_{k\in\mathbb{Z}}$ is 3–regular and $\{G_k\}_{k\in\mathbb{Z}}$ is a bounded family satisfying condition ***(G3)*** it follows from Lemma 5.1 that

$$\left\{\frac{k^3}{a_k}\big(\Delta^3 b_k G_k\big)\right\}_{k\in\mathbb{Z}\setminus\{0\}}$$

is a bounded family of operators. ∎

Lemma 5.8. *Let* $1 \leqslant p,q \leqslant \infty, s>0$ *and* $0<\beta<\alpha \leqslant 2$. *Let A be a linear closed operator defined in a Banach space X and* $F,G \in \mathscr{L}(F^s_{p,q}(\mathbb{T};X);X)$. *Assume that for all* $k\in\mathbb{Z}$, *the operators* N_k *are well defined in* $\mathscr{L}(X)$. *If the family* $\{F_k\}_{k\in\mathbb{Z}}$ *satisfies the condition **(F3)** and* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfies **(G3)** and the family of operators* $\{M_k\}_{k\in\mathbb{Z}}$ *is bounded then*

$$\big\{k^3 a_k\big(\Delta^3 N_k\big)\big\}_{k\in\mathbb{Z}}$$

is a bounded family of operators.

Proof. We note that for all $k\in\mathbb{Z}$ it holds

$$\begin{aligned}(\Delta^3 N_k) \;&=\; \big[(\Delta^2 N_{k+1}) + (\Delta^2 N_k)\big]\big[(-\Delta^1 a_{k+2}) + (\Delta^1 F_{k+2}) + (\Delta^1 b_{k+2}G_{k+2})\big]N_{k+1}\\ &+\; \big[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)\big]\big[(-\Delta^2 a_{k+1}) + (\Delta^2 F_k) + (\Delta^2 b_{k+1}G_{k+1})\big]N_{k+1}\\ &+\; \big[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)\big]\big[(-\Delta^1 a_{k+1}) + (\Delta^1 F_k) + (\Delta^1 b_{k+1}G_{k+1})\big](\Delta^1 N_k)\\ &+\; (\Delta^1 N_k)\big[(-\Delta^2 a_{k+1}) + (\Delta^2 F_{k+1}) + (\Delta^2 b_{k+1}G_{k+1})\big]N_{k+2}\\ &+\; N_k\big[(-\Delta^3 a_k) + (\Delta^3 F_k) + (\Delta^3 b_k G_k)\big]N_{k+2}\\ &+\; N_k\big[(-\Delta^2 a_k) + (\Delta^2 F_k) + (\Delta^2 b_k G_k)\big](\Delta^1 N_{k+1}).\end{aligned}$$

Therefore,

$$
\begin{aligned}
k^3 a_k(\Delta^3 N_k) = k^2 a_k\big[(\Delta^2 N_{k+1}) + (\Delta^2 N_k)\big]\frac{k}{a_{k+1}}\big[(-\Delta^1 a_{k+2}) + (\Delta^1 F_{k+2}) + (\Delta^1 b_{k+2}G_{k+2})\big]M_{k+1}\\
+ k a_k\big[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)\big]\frac{k^2}{a_k}\big[(-\Delta^2 a_{k+1}) + (\Delta^2 F_k) + (\Delta^2 b_{k+1}G_{k+1})\big]M_{k+1}\\
+ k a_k\big[(\Delta^1 N_{k+1}) + (\Delta^1 N_k)\big]\frac{k}{a_k}\big[(-\Delta^1 a_{k+1}) + (\Delta^1 F_k) + (\Delta^1 b_{k+1}G_{k+1})\big]k a_k(\Delta^1 N_k)\\
+ k a_k(\Delta^1 N_k)\frac{k^2}{a_{k+1}}\big[(-\Delta^2 a_{k+1}) + (\Delta^2 F_{k+1}) + (\Delta^2 b_{k+1}G_{k+1})\big]a_k N_{k+2}\\
+ M_k\frac{k^3}{a_k}\big[(-\Delta^3 a_k) + (\Delta^3 F_k) + (\Delta^3 b_k G_k)\big]a_k N_{k+2}\\
+ M_k\frac{k^2}{a_k}\big[(-\Delta^2 a_k) + (\Delta^2 F_k) + (\Delta^2 b_k G_k)\big]k a_k(\Delta^1 N_{k+1}).
\end{aligned}
$$

Direct computation shows that $k^3 a_k(\Delta^3 N_k)$ is a bounded operator in the particular case $k = 0$. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 3–regular, it follows from Lemmas 5.1 and 5.2, and hypothesis that the family $\{k^3 a_k(\Delta^3 N_k)\}_{k\in\mathbb{Z}}$ is bounded. ■

Lemma 5.9. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a linear closed operator defined in a Banach space X and $F, G \in \mathscr{L}(F^s_{p,q}(\mathbb{T}; X); X)$. Assume that for all $k \in \mathbb{Z}$, the operators N_k are well defined. If $\{F_k\}_{k\in\mathbb{Z}}$ satisfy **(F3)** and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy **(G3)** and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded then the family $\{F_k N_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.*

Proof. According to Theorem 1.3, it suffices to show that the family of operators $\{F_k N_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$–bounded of order 3. It follows from Lemma 5.3 that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$–bounded of order 2. It remains to show that $\{k^3(\Delta^3 F_k N_k)\}_{k\in\mathbb{Z}}$ is bounded. For this we note that

$$
\begin{aligned}
(\Delta^3 F_k N_k) = F_k(\Delta^3 N_k) + (F_{k+3} - F_k)(\Delta^2 N_{k+1}) + (\Delta^2 F_{k+1})(\Delta^1 N_{k+1})\\
+ (\Delta^3 F_k)N_{k+2} - 2(\Delta^2 F_k)(\Delta^1 N_{k+1}).
\end{aligned}
$$

Therefore, for all $k \in \mathbb{Z}\setminus\{0\}$ we have

$$
\begin{aligned}
k^3(\Delta^3 F_k N_k) = \frac{1}{a_k}F_k k^3 a_k(\Delta^3 N_k) + \frac{k}{a_k}(F_{k+3} - F_k)k^2 a_k(\Delta^2 N_{k+1}) + \frac{k^2}{a_k}(\Delta^2 F_{k+1})k a_k(\Delta^1 N_{k+1})\\
+ \frac{k^3}{a_k}(\Delta^3 F_k)a_k N_{k+2} - 2\frac{k^2}{a_k}(\Delta^2 F_k)k a_k(\Delta^1 N_{k+1}).
\end{aligned}
$$

Clearly, if $k = 0$ the operator $k^3(\Delta^3 F_k N_k)$ is bounded. Since $\{F_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(F3)***, it follows that $\{F_k N_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$–bounded of order 3. ■

Lemma 5.10. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $0 < \beta < \alpha \leqslant 2$. Let A be a linear closed operator defined in a Banach space X and $F, G \in \mathscr{L}(F^s_{p,q}(\mathbb{T}; X); X)$. Assume that for all $k \in \mathbb{Z}$, the operators N_k are well defined. If $\{F_k\}_{k\in\mathbb{Z}}$ satisfy **(F3)** and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy **(G3)** and the family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is bounded then the family $\{(ik)^\beta G_k N_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.*

Proof. According to Theorem 1.3, it suffices to show that the family of operators $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$–bounded of order 3. It follows from Lemma 5.4 that $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$–bounded of order 2. It remains to show that $\{k^3(\Delta^3 b_k G_k N_k)\}_{k\in\mathbb{Z}}$ is bounded. For this we note that

$$\begin{aligned}(\Delta^3 b_k G_k N_k) &= b_k G_k(\Delta^3 N_k) + (b_{k+3}G_{k+3} - b_k G_k)(\Delta^2 N_{k+1}) + (\Delta^2 b_{k+1}G_{k+1})(\Delta^1 N_{k+1})\\ &\quad + (\Delta^3 b_k G_k)N_{k+2} - 2(\Delta^2 b_k G_k)(\Delta^1 N_{k+1})\end{aligned}$$

Therefore, for all $k \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned}k^3(\Delta^3 b_k G_k N_k) &= \frac{b_k}{a_k} G_k k^3 a_k(\Delta^3 N_k) + \frac{k}{a_k}(b_{k+3}G_{k+3} - b_k G_k)k^2 a_k(\Delta^2 N_{k+1})\\ &\quad + \frac{k^2}{a_k}(\Delta^2 b_{k+1}G_{k+1})k a_k(\Delta^1 N_{k+1}) + \frac{k^3}{a_k}(\Delta^3 b_k G_k)a_k N_{k+2}\\ &\quad - 2\frac{k^2}{a_k}(\Delta^2 b_k G_k)k a_k(\Delta^1 N_{k+1})\end{aligned}$$

A direct computation shows that $k^3(\Delta^3 b_k G_k N_k)$ is a bounded operator in the particular case $k = 0$. Since $\{G_k\}_{k\in\mathbb{Z}}$ satisfies the condition ***(G3)***, it follows from Lemmas 5.1, 5.2, 5.7 and 5.8 that $\{b_k G_k N_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$–bounded of order 3. ∎

Lemma 5.11. *Let $1 \leqslant p, q \leqslant \infty$, $s > 0$ and $1 \leqslant \beta \leqslant \alpha$. Let A be a linear closed operator defined in a Banach space X and $F, G \in \mathscr{L}(F^s_{p,q}(\mathbb{T}; X); X)$. Assume that for all $k \in \mathbb{Z}$, the operators N_k are well defined. If $\{F_k\}_{k\in\mathbb{Z}}$ satisfies **(F3)** and $\{G_k\}_{k\in\mathbb{Z}}$ satisfies **(G3)** then the following assertions are equivalent*

(i) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a bounded.

(ii) The family of operators $\{M_k\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.

Proof. $(i) \Rightarrow (ii)$. According Theorem 1.3 it suffices to show that $\{M_k\}_{k\in\mathbb{Z}}$ is $\mathscr{M}$–bounded of order 3. It follows from 5.5 that $\{M_k\}_{k\in\mathbb{Z}}$ is a family of operators $\mathscr{M}$–bounded of order 2. It remains to show that $\{k^3(\Delta^3 M_k)\}_{k\in\mathbb{Z}}$ is a bounded family of operators. For this we note

$$\begin{aligned}\Delta^3 M_k &= a_k(\Delta^3 N_k) + (a_{k+3} - a_k)(\Delta^2 N_{k+1}) + (\Delta^2 a_{k+1})(\Delta^1 N_{k+1})\\ &\quad + (\Delta^3 a_k)N_{k+2} - 2(\Delta^2 a_k)(\Delta^1 N_{k+1}).\end{aligned}$$

Therefore,

$$k^3(\Delta^3 M_k) = k^3 a_k(\Delta^3 N_k) + \frac{k(a_{k+3}-a_k)}{a_k} k a_k(\Delta^2 N_{k+1}) + \frac{k^2}{a_k}(\Delta^2 a_{k+1}) k a_k(\Delta^1 N_{k+1})$$
$$+ \frac{k^3(\Delta^3 a_k)}{a_k} a_k N_{k+2} - \frac{2k^2(\Delta^2 a_k)}{a_k} k a_k(\Delta^1 N_{k+1}).$$

Clearly, if $k=0$ the operator $k^3(\Delta^3 F_k N_k)$ is bounded. Since the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 3–regular, it follows from Lemma 5.2 and Lemma 5.8 that the family $\{M_k\}_{k\in\mathbb{Z}}$ is $\mathcal{M}$–bounded of order 3.
$(ii) \Rightarrow (i)$ The proof follows the same lines as those of Lemma 5.5. ∎

Theorem 5.3. *Let $1 \leqslant p,q \leqslant \infty$, $s>0$ and $0<\beta<\alpha\leqslant 2$. Let A be a linear closed operator defined in a Banach space X and $F,G \in \mathcal{L}(F^s_{p,q}(\mathbb{T};X);X)$. If $\{F_k\}_{k\in\mathbb{Z}}$ satisfies **(F3)** and $\{G_k\}_{k\in\mathbb{Z}}$ satisfies **(G3)** then the following assertions are equivalent,*

(i) The solution of equation (5.21) *has $F^s_{p,q}$–maximal regularity.*

(ii) $\{N_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X)$ and the family $\{M_k\}_{k\in\mathbb{Z}}$ is bounded.

Proof. The proof follows the same lines as that of Theorem 5.1. ∎

As we have mentioned, the verification of assumptions concerning the family $\{N_k\}_{k\in\mathbb{Z}}$ is not an easy work. Our next Corollary require additional conditions about the operators A, F and G, however is a more practical result to check that these families are bounded. Define the operators $S_k = (a_k - A)^{-1}$, for all $k\in\mathbb{Z}$.

Corollary 5.2. *Let $1 \leqslant p,q \leqslant \infty$, $s>0$. Let be X a Banach space. Assume that $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$, defined by the operators F and G of equation* (5.21) *satisfy the conditions **(F3)** and **(G3)** respectively. If the family of operators $\{(ik)^\alpha((ik)^\alpha - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded, and $\sup_{k\in\mathbb{Z}}\left\| \left((ik)^\beta G_k + F_k\right)\left((ik)^\alpha - A\right)^{-1}\right\| < 1$ then equation* (5.21) *has $F^s_{p,q}$–maximal regularity.*

Proof. The proof follows the same lines as those of Corollary 5.1. ∎

5.4 Periodic Strong $F^s_{p,q}$–solutions of a Neutral Fractional Differential Equation with Finite Delay

Now, we use the results about maximal regularity on periodic Triebel–Lizorkin spaces to prove that equation (5.22) has a strong $F^s_{p,q}$–solution.
Note that, if equation (5.21) has $F^s_{p,q}$–maximal regularity for some $1\leqslant p,q\leqslant\infty$ and $s\in\mathbb{R}$, the linear map $\Psi : F^s_{p,q}(\mathbb{T};X) \to F^s_{p,q}(\mathbb{T};X)$ given by $\Psi(g) = D^\alpha v$, where v is the unique strong $F^s_{p,q}$–solution of the equation (5.21), is well defined.

Lemma 5.12. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded operator, for all $k \in \mathbb{Z}$ the operators N_k is well defined in $\mathscr{L}(X)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ defined by the operators F and G of equation* (5.22) *satisfy the conditions **(F3)** and **(G3)** respectively. If $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$ and $\|B\|\|\Psi\| < 1$, then the family $\{(I - e^{-ikr}(ik)^\alpha BN_k)^{-1}\}_{k\in\mathbb{Z}}$ is a $F^s_{p,q}$–multiplier.*

Next Theorem gives sufficient condition that guarantee existence of a strong $B^s_{p,q}$–solution of equation (5.22). We use the notation introduced in preceding Lemma.

Theorem 5.4. *Let $1 \leqslant p,q \leqslant \infty$, $s > 0$, and $0 < \beta < \alpha \leqslant 2$. Let be X a Banach space. Assume that B is a bounded operator, for all $k \in \mathbb{Z}$ the operators N_k is well defined in $\mathscr{L}(X)$ and the families $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ defined by the operators F and G of equation* (5.22) *satisfy the conditions **(F3)** and **(G3)** respectively. If $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators, such that $\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\|\|N_k\| < 1$ and $\|B\|\|\Psi\| < 1$, then for every $f \in B^s_{p,q}(\mathbb{T};X)$ there exists an unique strong $F^s_{p,q}$–solution of equation* (5.22).

5.5 Examples.

In this section we apply the results that we have obtained in the previous sections to concrete equations.

***Example* 5.2.** *Let $X = \mathbb{C}$, $1 \leqslant p,q \leqslant \infty$, $s > 0$ and $1 \leqslant \beta < \alpha \leqslant 2$. Suppose that $\rho \in \mathbb{R}\setminus\{-1\}$. Consider the fractional differential equation with finite delay*

$$D^\alpha u(t) = \rho u(t) + u(t-2\pi) + D^\beta u(t-2\pi) + f(t) \quad t \in [0,2\pi] \tag{5.23}$$

where $f \in B^s_{p,q}(\mathbb{T};\mathbb{C})$. Writing $A = \rho I$ and $Fu_t = Gu_t = u(t-2\pi)$ we have an abstract differential equation of the form (5.1).

*Let $1 \leqslant p,q \leqslant \infty$ and $s > 0$, we claim that this equation has $B^s_{p,q}$-maximal regularity. In fact, for all $k \in \mathbb{Z}$ the operators F_k and G_k are given by $F_k = G_k = e^{2i\pi k} = 1$. Therefore the family $\{F_k\}_{k\in\mathbb{Z}}$ and $\{G_k\}_{k\in\mathbb{Z}}$ satisfy the conditions **(F2)** and **(G2)**. Moreover, the operators N_k take the form*

$$N_k = \frac{1}{(ik)^\alpha - (ik)^\beta - 1 - \rho}, \quad \textit{for all } k \in \mathbb{Z}.$$

Since $Im\big((ik)^\alpha - (ik)^\beta\big) \neq 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ and $1 + \rho \neq 0$, we have $\{(ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is a bounded family of operators. This means that there exists a constant $C_1 > 0$ such that $\sup_{k\in\mathbb{Z}} \|(ik)^\alpha N_k\| \leqslant C_1$.

On the other hand, it follows from Theorem 5.1 that equation (5.23) *has $B^s_{p,q}$-maximal regularity and $\Psi : B^s_{p,q}(\mathbb{T};\mathbb{C}) \to B^s_{p,q}(\mathbb{T};\mathbb{C})$, defined by $\Psi f = D^\alpha u$, where u is the unique strong $B^s_{p,q}$–solution of equation* (5.23)*, is a bounded linear operator. Thus there exists $C_2 > 0$ such that $\|\Psi\| \leqslant C_2$.*

Suppose that $\|B\| < \min\left\{\frac{1}{C_1}, \frac{1}{C_2}\right\}$, *it follows from Theorem 5.2 that equation*

$$D^{\alpha}\big(u(t) - Bu(t-r)\big) = \rho u(t) + u(t-2\pi) + D^{\beta}u(t-2\pi) + f(t) \quad t \in [0, 2\pi] \tag{5.24}$$

has a unique strong $B^s_{p,q}$*–solution.*

***Example* 5.3.** *Let* $1 \leqslant p, q \leqslant \infty$ *and* $s > 0$. *Consider the following neutral fractional differential equation with finite delay*

$$\begin{aligned} \frac{\partial^{\alpha}}{\partial t^{\alpha}}\big(u(t,x) - bu(t-2\pi, x)\big) &= \frac{\partial^2 u(t,x)}{\partial x^2} + \int_{-2\pi}^{0} q_1\gamma(s)u(t+s,x)ds \\ &+ \int_{-2\pi}^{0} q_2\gamma(s)D^{\beta}u(t+s,x)ds + f(t,x), \end{aligned} \tag{5.25}$$

with $1 < \beta < \alpha < 2$. *We will show that there exists* $b > 0$ *sufficiently small such that the equation* (5.25) *has* $B^s_{p,q}$*–maximal regularity. For this purpose, we assume that* $\gamma : [-2\pi, 0] \to \mathscr{L}(X)$ *is a function twice strongly continuously differentiable. Furthermore,* q_1 *and* q_2 *are positive numbers such that*

$$\left|q_1 + q_2\cos\big(\frac{\beta\pi}{2}\big)\right| \leqslant \left|q_2\cos\big(\frac{\beta\pi}{2}\big)\right|$$

and

$$q_2 C_{\gamma} < \sin\big(\frac{\alpha\pi}{2}\big),$$

where $C_{\gamma} = \left(\int_{-2\pi}^{0}\gamma^2(s)ds\right)^{1/2}$. *The function* f *satisfies Carathéodory type conditions.*

Considering the space $X = L^2([0,\pi])$ *and the operators* A, B, F *and* G *given by*

$$Av = \frac{\partial^2 v(x)}{\partial x^2} \quad \text{with domain} \quad D(A) = \{v \in X : v \in H^2([0,\pi]),\ v(0) = v(\pi) = 0\}$$

$$Bv = bv \quad \text{for all } v \in X \text{ with } b \in \mathbb{R}^+$$

$$Fv = \int_{-2\pi}^{0} q_1\gamma(s)v(s)ds \qquad Gv = \int_{-2\pi}^{0} q_2\gamma(s)v(s)ds \quad \text{for all } v \in X.$$

equation (5.25) *takes the abstract form of equation* (5.1).

Clearly $\|F\| \leqslant q_1 C_{\gamma}$ *and* $\|G\| \leqslant q_2 C_{\gamma}$. *Define, for all* $k \in \mathbb{Z}$ *the operators*

$$F_k = \int_{-2\pi}^{0} q_1 e^{iks}\gamma(s)ds \quad \text{and} \quad G_k = \int_{-2\pi}^{0} q_2 e^{iks}\gamma(s)ds$$

A direct computation shows that for all $k \in \mathbb{Z}\setminus\{0\}$

$$F_k = \frac{iq_1[\gamma(-2\pi) - \gamma(0)]}{k} - \frac{q_1[\gamma'(-2\pi) - \gamma'(0)]}{k^2} - \frac{iq_1}{k^2}\int_{-2\pi}^{0} e^{iks}\gamma''(s)ds$$

and

$$G_k = \frac{iq_2[\gamma(-2\pi)-\gamma(0)]}{k} - \frac{q_2[\gamma'(-2\pi)-\gamma'(0)]}{k^2} - \frac{iq_2}{k^2}\int_{-2\pi}^{0} e^{iks}\gamma''(s)\,ds.$$

Denote $P_k = \frac{iq_1}{k^2}\int_{-2\pi}^{0} e^{iks}\gamma''(s)\,ds$. *Since*

$$\sup_{k\in\mathbb{Z}\setminus\{0\}} \left\| \frac{k^2}{a_k}(\Delta^2 P_k) \right\| \leqslant 4q_1\|\gamma''\|_2 \left| \frac{1}{(k+2)^2} + \frac{2}{(k+1)^2} + \frac{1}{k^2} \right| < \infty,$$

the family $\{F_k\}_{k\in\mathbb{Z}}$ *satisfies the condition* ***(F2)***.
Following the same argument to prove that $\{F_k\}_{k\in\mathbb{Z}}$ *satisfies* ***(F2)***, *we note that the family* $\{G_k\}_{k\in\mathbb{Z}}$ *satisfies the condition* ***(G2)***.
On another hand, the spectrum of A *consists of eigenvalues* $-n^2$ *for* $n \in \mathbb{N}$. *Their associated eigenvectors are given by*

$$y_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx).$$

Morever, the set $\{x_n : n \in \mathbb{N}\}$ *is an orthonormal basis of* X. *In particular*

$$Ax = \sum_{n\in\mathbb{N}} -n^2\langle x, x_n\rangle x_n \qquad \textit{for all } x \in D(A). \tag{5.26}$$

Therefore $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and* $\big((ik)^\alpha I - A\big)^{-1} = \sum_{n\in\mathbb{N}} \frac{1}{(ik)^\alpha + n^2}\langle x, x_n\rangle x_n$.
Since $|(ik)^\alpha + n^2| \geqslant \big|Im\big((ik)^\alpha\big)\big| = |k|^\alpha \sin(\frac{\alpha\pi}{2})$ *we have*

$$\left\| \big((ik)^\alpha I - A\big)^{-1} \right\| \leqslant \frac{1}{|k|^\alpha \sin(\frac{\alpha\pi}{2})}. \tag{5.27}$$

On the other hand,

$$\|(ik)^\beta G_k + F_k\| \leqslant |(ik)^\beta q_2 + q_1| C_\gamma \leqslant q_2 \left|(ik)^\beta\right| C_\gamma = q_2 \left|k\right|^\beta C_\gamma. \tag{5.28}$$

It follows from (5.27) *that*

$$\sup_{k\in\mathbb{Z}} \|(ik)^\alpha\big((ik)^\alpha I - A\big)^{-1}\| < \infty$$

Moreover, from (5.28) *we have that*

$$\left\| \big((ik)^\beta G_k + F_k\big)\big((ik)^\alpha I - A\big)^{-1} \right\| \leqslant \frac{q_2\,|k|^\beta\, C_\gamma}{|k|^\alpha \sin(\frac{\alpha\pi}{2})}.$$

Since $q_2 C_\gamma < \sin(\frac{\alpha\pi}{2})$ *we have* $\sup_{k\in\mathbb{Z}} \left\| \big((ik)^\beta G_k + F_k\big) S_k \right\| < 1$. *From Corollary 5.1, it follows that fractional delay equation*

$$\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha} u(t,x) &= \frac{\partial^2 u(t,x)}{\partial x^2} + \int_{-2\pi}^{0} q_1 \gamma(s) u(t+s,x)\, ds \\ &+ \int_{-2\pi}^{0} q_2 \gamma(s) D^\beta u(t+s,x)\, ds + f(t,x) \end{aligned} \tag{5.29}$$

has $B^s_{p,q}$*–maximal regularity, for* $1 \leqslant p,q \leqslant \infty$ *and* $s > 0$.
Thus the map $\Psi : B^s_{p,q}(\mathbb{T};X) \to B^s_{p,q}(\mathbb{T};X)$, *defined by* $\Psi(f) = D^\alpha u$ *where* u *is the unique strong* $B^s_{p,q}$*–solution of equation* (5.29), *is a bounded linear operator. Therefore there exists* $C_2 \geqslant 0$ *such that* $\|\Psi\| \leqslant C_2$.
Moreover, there exists $C_1 \geqslant 0$ *such that* $\sup_{k\in\mathbb{Z}} |k|^\alpha \|N_k\| \leqslant C_1$. *If the constant* $b > 0$ *satisfies the condition* $b < \min\left\{\frac{1}{C_1}, \frac{1}{C_2}\right\}$ *we have*

$$\sup_{k\in\mathbb{Z}} |k|^\alpha \|B\| \|N_k\| < 1 \quad \textit{and} \quad \sup_{k\in\mathbb{Z}} \|B\| \|\Psi\| < 1$$

It follows from Theorem 5.2 that equation (5.25) *has a unique strong* $B^s_{p,q}$*-solution.*

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