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(a,k)-REGULARIZED FAMILIES AND SOME EVOLUTION EQUATIONS

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Thesis Introduction

This thesis corresponds to the requirement to obtain the doctoral degree in mathematics, given by the department of mathematics of the University of Chile. This work is composed for three investigations, in order to be published. The first and second research is given in a context about the theory of the (a,k)-regularized families. This type of families generalizes the theories of C_0-semigroups, r-times-integrated semigroups, k-convoluted semigroups, r-times integrated cosine families, and k-times resolvent families, among others. On the other hand some of this families can be associated to an evolution differential equation (or fractional differential equations). The theory and principal results about the (a,k)-regularized families are resumed in the chapter one.

The first research treated in the second chapter is related to the association of the one functional equation which characterizes an (a,k)-regularized family. This notion generalizes and unified the Abel's and the D'Alambert's equation associated to the families C_0- semigroup and cosine respectively and gives new functional equations to families which are not been associated, for example the resolvent families [53]. The functional equation associated to an (a,k)-regularized families $\{R(t)\}_{t\geq 0}$ is read as follows

$$R(s)(a*R)(t)-(a*R)(s)R(t)=k(s)(a*R)(t)-k(t)(a*R)(s)$$

where a,k denotes two complex functions on $[0,\infty]$ called the kernels of the family. It is important emphasize that this research are published in joint C. Lizama under the title *On a functional equation associated with* (a,k)*-regularized resolvent families* see [37].

In the chapter three we include a research about the uniform stability of an (a,k)-regularizes family in a Hilbert space. Basically we give sufficient conditions in terms of the kernels a,k to ensure the uniform stability of the an (a,k)-regularized family. Our main result can be seen as substantial generalization of the Gearhart-Greiner-Prüss characterization of exponential stability for strongly continuous semigroups, see for example [15, Theorem V.1.11]. On the other hand the applications are very interesting, for example if we fix a cosine family, never can be uniform stable, however we can ensure easily the follows result,

Corollary Let $b>0$. Suppose that A is the generator of a strongly continuous cosine families of operators $\{C(t)\}_{t\geq 0}$ satisfying the following conditions:

1. $(b+\lambda)^2\in\rho(A)$ for all $\Re\lambda\geq 0$.
2. $\sup_{\Re\lambda>0}\|(\lambda+b)((\lambda+b)^2-A)^{-1}\|<\infty.$

Then $\{e^{-bt}C(t)\}_{t\geq 0}$ is uniformly stable.

This paper was written and submitted in joint with the researchers P. J. Miana and C. Lizama, under the title *uniform stability of* $(a;k)$*-regularized families* [35].

Finally in the chapter four, we put aside the concept of (a,k)-regularized families to study the concept of the maximal regularity associated to the of first order differential equation with finite delay,

$$(I)\qquad u'(t)=Au(t)+Fu_t+f(t),\qquad t\in\mathbb{R},$$

where $(A,D(A))$ is a (unbounded) linear operator on a Banach space X, $u_t(\cdot)=u(t+\cdot)$ on $[-r,0]$, $r>0$, and the delay operator is supposed to belong to $\mathscr{B}(L^p([-r,0],X),X)$. The maximal regularity in this context is understood as, what conditions we need to be imposed over the operator A, F to ensure that given a function $f \in L^p(\mathbb{R}, X)$ exists a unique solution of the problem (I) in the same space as f. The main result of this work characterizes the maximal regularity on $L^p(\mathbb{R}, X)$ of the equation (I) in a U.M.D spaces and reads as follows.

Theorem. *Assume Let $1 < p < \infty$ and X an UMD-space. The following assertion are equivalent.*

1. *Problem (I) is maximal L^p-regular.*
2. *For all $\xi \in \mathbb{R}$, the operator $(i\xi - Fe^{i\xi\cdot} - A)$ is invertible, and $\|(i\xi - Fe^{i\xi\cdot} - A)^{-1}\| \leq \frac{c}{1+|\xi|}$, and the set $\{i\xi(i\xi - Fe^{i\xi\cdot} - A)^{-1} : \xi \in \mathbb{R} \setminus \{0\}\}$ is $\mathscr{R}$-bounded.*

Additionally we compare the connection between of the maximal regularity on $L^P(\mathbb{R}, X)$ of the equation

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R},$$

and the maximal regularity on $L^P(\mathbb{R}, X)$ of the equation (I). To finish we given an application in this context to the elliptic equation with finite delay. This work is a preprint in joint with the researcher V. Poblete and is titled *Maximal regularity on L^p of first order differential equations with finite delay* [51].

CHAPTER 1

Theory of (a,k)-regularized families

1.1. Introduction

Definition 1.1. Let $k \in C(\mathbb{R}_+)$, $k \neq 0$ and $a \in L^1_{loc}(\mathbb{R}_+)$, $a \neq 0$ be given. Assume that A is a linear operator with domain $D(A)$. A strongly continuous family $\{R(t)\}_{t\geq 0} \subset \mathscr{B}(X)$ is called an (a,k)-regularized resolvent family on X having A as a generator if the following properties hold:

(i) $R(0) = k(0)I$;

(ii) $R(t)x \in D(A)$ and $R(t)Ax = AR(t)x$ for all $x \in D(A)$ and $t \geq 0$;

(iii) $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)x ds,\ t \geq 0,\ x \in D(A).$

This notion generalizes the theories of C_0-semigroups, r-times-integrated semigroups, k-convoluted semigroups, r-times integrated cosine families, and k-times resolvent families, among others. We observe that existence as well as structural properties of (a,k)-regularized families have been studied by several authors in recent years (see [27, 31] and references therein).

Existence, uniqueness and qualitative properties of solutions for wide classes of linear evolution equations are associated to (a,k)-regularized families. For example, the abstract Cauchy problem of first and second order, Volterra equations of convolution type like

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t),$$

and fractional order differential equations, among others, see for example [23, 27] .

We note that a large number of results concerning C_0-semigroups, resolvent families, convoluted semigroups and cosine functions, can be presented in a new and unified look on the theory of general (a,k)-regularized families. However, the study of stability for this general structure remains an untreated topic in the literature.

1.2. Some results about $(a,k)-$regularized families

In this section we review some of the main results in the literature about the theory of (a,k)-regularized resolvent families. This notion was introduced in [31] and studied, as well as extended, in subsequent papers (see e.g. [56], [57], [27], [34], [39], [40] and [41]).

Let us fix some notations. From now on, we take X to be a complex Banach space with norm $\|\cdot\|$. We denote by $\mathscr{B}(X)$ the Banach algebra of all bounded linear operators on X endowed with the operator norm, which again is denoted by $\|\cdot\|$. The identity operator on X is denoted by $I \in \mathscr{B}(X)$, and $\mathbb{R}_+$ denotes the interval $[0,\infty)$. For a closed operator A, we denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_a(A)$ the spectrum, the point spectrum, the residual spectrum, and the approximate spectrum of A respectively.

We emphasize that the main properties of this theory admit very clear and simple proofs, and what is more interesting, it is easy to associate a suitable regularized resolvent family to a wide class of linear evolution equations, including e.g. fractional abstract differential equations. See [36].

Assume that a and k are both positive and one of them is non-decreasing. Let $\{R(t)\}_{t\geq 0}$ be an (a,k)-regularized resolvent family with generator A such that

$$\|R(t)\| \leq Mk(t), \quad t \geq 0, \tag{1.1}$$

for some constant $M > 0$. Then we have

$$Ax = \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{(a*k)(t)}, \quad x \in D(A). \tag{1.2}$$

Here we denote $(a*k)(t) := \int_0^t k(t-s)a(s)ds$ the finite convolution between a and k. The above representation of A in terms of $R(t)$ was established in [41] (see also [40]). We note that there is a one-to-one correspondence between (a,k)-regularized resolvent families and their generators. Moreover, we can prove that an (a,k)-regularized resolvent family is uniformly continuous if and only if its generator is a bounded linear operator [31].

We say that $\{R(t)\}_{t\geq 0}$ is of type (M,ω) if there exists constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|R(t)\| \leq Me^{\omega t} \qquad \text{for all } t \geq 0.$$

The next result corresponds to the generation theorem for the theory. We assume that the Laplace transform for $a(t)$ and $k(t)$ exists for all $\lambda > \omega$.

Theorem 1.1. *([31]) Let A be a closed linear densely defined operator on X. Then $\{R(t)\}_{t\geq 0}$ is (a,k)-regularized resolvent family of type (M,ω) if and only if the following conditions hold:*

1. *$\widehat{a}(\lambda) \neq 0$ and $\frac{1}{\widehat{a}(\lambda)} \in \rho(A)$ for all $\lambda > \omega$*
2. *$H(\lambda) := \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$ satisfies the estimates*

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}}, \qquad \lambda > \omega, \ n \in \mathbb{N}_0.$$

In the case where $k(t) \equiv 1$, Theorem 3.4 is well known. In fact, if a$(t) \equiv 1$, then it is just the Hille-Yosida theorem; if $a(t) \equiv t$, then it is the generation theorem for generators of cosine functions due to Sova and Fattorini; for arbitrary $a(t)$, it is the generation theorem due essentially to Da Prato and Iannelli [53]. In the case where $k(t) = \frac{t^n}{n!}$ and $a(t) \equiv 1$, it is the generation theorem for n-times integrated semigroups [24]; if $k(t) = \frac{t^n}{n!}$ and $a(t)$ is arbitrary, it corresponds to the generation theorem for integrated solutions of Volterra equations due to Arendt and Kellermann [6].

Although the generation theorem characterizes all generators of (a,k)-regularized resolvent families, it is difficult to verify the estimate of all derivatives of the operator $H(\lambda)$ in concrete applications. Thus, one tries to build up the operator from simpler ones using perturbation techniques. The following is the main result available until now. It corresponds to the extension of the Miyadera-Voigt perturbation theorem in the theory of C_0-semigroups. In this theorem, the perturbation B is bounded only from the domain of the generator $D(A)$, endowed with the graph norm $\|x\|_A := \|x\| + \|Ax\|$.

Theorem 1.2. *([41]) Let A be a closed operator on X. Assume that A generates an (a,k)-regularized resolvent family $\{R(t)\}_{t\geq 0}$ of type (M,ω) and suppose that*

(i) there exists $b \in L^1_{loc}(\mathbb{R}_+)$ such that $(k * b)(t) = a(t)$ for all $t \geq 0$.

(ii) there exists constants $\mu > \omega$ and $\gamma \in [0,1)$ such that

$$\int_0^\infty e^{-\mu r} \left\| B \int_0^r b(r-s)R(s)x ds \right\| dr \leq \gamma \|x\| \qquad \text{for all } x \in D(A).$$

Then $A+B$ generates an (a,k)-regularized resolvent family $\{R(t)\}_{t\geq 0}$ on X such that $\|R(t)\| \leq \frac{M}{1-\gamma} e^{\mu t}$. In addition

$$R(t)x = R(t)x + \int_0^t R(t-r)B \int_0^r b(r-s)R(s)x ds dr, \qquad x \in X.$$

The next result show, roughly speaking, the continuous dependence of an (a,k)-regularized resolvent family $R(t)$ on its generator A. More precisely, the theorem below show that the convergence - in an appropriate sense - of a sequence of generators is equivalent to the convergence of the corresponding (a,k)-regularized resolvent families.

Theorem 1.3. *([32]) Let $\{k_n\}_{n\geq 0} \in L^1_{loc}(\mathbb{R}^+)$ and $\{a_n\}_{n\geq 0} \in AC_{loc}(\mathbb{R}^+)$ be of type (M,ω), $\omega \geq 0$, such that $\widehat{a}_n(\lambda) \neq 0$ for $\lambda > \omega$ and $\int_0^\infty e^{-\omega s}|a_n'(s)|ds < \infty$. Let A_n be closed and linear operators in X such that A_0 is densely defined. For each fixed $n \in \mathbb{N}_0$, assume that $R_n(t)$ is an (a_n,k_n)-regularized resolvent family generated by A_n in X, and that there exists constants $M > 0$ and $\omega \in \mathbb{R}$, independent of n, such that*

$$\|R_n(t)\| \leq M e^{\omega t}, \quad \text{for all } t \geq 0.$$

Suppose also $a_n(t) \to a_0(t)$ and $k_n(t) \to k_0(t)$ as $n \to \infty$. Then the following statements are equivalent:

1. $\lim_{n\to\infty} k_n(\lambda)(I - a_n(\lambda)A_n)^{-1} = k_0(\lambda)(I - a_0(\lambda)A_0)^{-1} \qquad$ *for all* $\lambda \geq \omega$
2. $\lim_{n\to\infty} R_n(t)x = R_0(t)x$ *for all $x \in X$, $t \geq 0$. Moreover the convergence is uniform in t on every compact subset of* $\mathbb{R}^+$.

Note that the above theorem is the extension of the Trotter-Kato theorem for the theory of C_0-semigroups, which follows in case $a(t) \equiv k(t) \equiv 1$.

In our next result, of concern are ergodic type theorems. Here the contributions to the theory are contained in the references [39], [41]. We below cite only a simple, but typical, result.

Theorem 1.4. *Let A be the generator of an (a,k)- regularized resolvent family $\{R(t)\}_{t\geq 0}$ such that*

$$\|R(t)\| \leq Mk(t) \qquad for\ all\ t \geq 0.$$

Suppose that

(i) $a(t)$ is positive, and $k(t)$ is nondecreasing and positive as well.

(ii) $\displaystyle \lim_{t\to\infty} \frac{k(t)}{(k*a)(t)} = 0$

(iii) $\displaystyle \sup_{t>0} \frac{k(t)(1*k)(t)}{(k*a)(t)} < \infty$

(iv) $\lim_{t\to\infty} \frac{(a*a*k)(t)}{(a*k)(t)} = \infty$

Define

$$A_t x := \frac{1}{k*a(t)} \int_0^t a(t-s)R(s)x\,ds; \quad x \in X \quad t > 0.$$

Then the following holds:

1. *The mapping* $Px := \lim_{t\to\infty} A_t x$ *is a bounded linear projection with* $Ran(P) = Ker(A)$, $Ker(P) = \overline{Ran(A)}$, *and*
$$D(P) = Ker(A) \oplus \overline{Ran(A)}.$$
2. *For* $0 < \beta \le 1$ *and* $x \in Ker(A) \oplus \overline{Ran(A)}$, *we have*
$$\| A_t x - Px \| = O\left(\left[\frac{k(t)}{a*k(t)} \right]^{\beta} \right) \qquad as\, |t| \to \infty.$$
3. *If* X *is reflexive then* $Ker(A) \oplus \overline{Ran(A)} = X$.

Note that in the case $k(t) = \frac{t^{\beta}}{\Gamma(\beta+1)}$, $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ $\alpha > 0$, $\beta \ge 0$ the conditions (i)-(iv) are automatically satisfied.

In the next result, we are interested in the relation between the spectrum of A and the spectrum of each one of the operators $R(t)$, $t \ge 0$. We denote by $s(t,\lambda)$ the unique solution of the convolution equation

$$s(t,\lambda) := a(t) + \lambda \int_0^t a(t-\tau)s(\tau,\lambda)\,d\tau.$$

We also define

$$r(t,\lambda) := k(t) + \lambda \int_0^t s(t-\tau,\lambda)k(\tau)\,d\tau.$$

From a purely formal point of view one would expect the relation $\sigma(R(t)) = r(t,\sigma(A))$. This, however, is not true in general. The following result corresponds to the inclusion theorem.

Theorem 1.5. *([40]) Let* A *be a closed operator on* X *and let* $R(t)$ *be an* (a,k)*-regularized resolvent family with generator* A. *Then*

(i) $\sigma(R(t)) \supset r(t,\sigma(A)), \quad t \ge 0,$

(ii) $\sigma_p(R(t)) \supset r(t,\sigma_p(A)), \quad t \ge 0,$

(iii) If A *is densely defined then* $\sigma_r(R(t)) \supset r(t,\sigma_r(A)), \quad t \ge 0,$

(iv) $\sigma_a(R(t)) \supset r(t,\sigma_a(A)), \quad t \ge 0.$

Remark 1.6. In the case $k(t) = \frac{t^{\beta}}{\Gamma(\beta+1)}$, $\beta \ge 0$, $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ $\alpha > 0$, we have that

$$r_{\alpha,\beta}(t,\lambda) = t^{\beta} E_{\alpha,\beta+1}(\lambda t^{\alpha})$$

where $E_{\alpha,\beta+1}$ denotes the Mitag-Leffler function, defined as follows:

$$E_{\alpha,\beta+1}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta+1)}.$$

In particular: $\alpha = 1, \beta = 0$ gives $E_{1,1}(z) = e^z$ and then $r_{1,0}(t,\lambda) = e^{\lambda t}$. Here $R(t)$ is the C_0-semigroup generated by A and therefore we recover the well known inclusion

$$e^{\sigma(A)t} \subset \sigma(R(t)), \quad t > 0.$$

If $\alpha = 2, \beta = 0$ we have $E_{2,1}(z^2) = \cosh(z)$ and then $r_{1,0}(t,\lambda) = \cosh\sqrt{(\lambda)}t$. Here we recover the inclusion [47]:

$$\cosh\sqrt{\sigma(A)}t \subset \sigma(R(t)), \quad t > 0.$$

In general, let $\alpha > 0$ and suppose that the fractional Cauchy problem:

$$D_t^\alpha u(t) = Au(t), \quad t > 0$$

is well posed. Then A generates an $(\alpha, 0)$-regularized resolvent family $R_\alpha(t)$ and we conclude that

$$E_{\alpha,1}(\sigma(A)t^\alpha) \subset \sigma(R_\alpha(t)), \quad t > 0.$$

This result was first proved by Li and Zheng [30].

Recent results on the theory of (a,k)-regularized resolvent families include conditions under which the complex inversion formula for the Laplace transform holds for (a,k)-regularized regularized families [33] and Kallman-Rota type inequalities [34]. However, following the analogy with the theories of C_0-semigroups and cosine operator functions, many problems are still to be solved.

Theorem 1.7. *[27]. Let A be the generator of an exponentially bounded (a,k)-regularized family. Let $k, a, c \in L^1_{loc}(\mathbb{R}_+)$ be such that $\int_0^\infty |a(t)|e^{-\beta t}dt < \infty$ and $\int_0^\infty |k(t)|e^{-\beta t}dt < \infty$ for some $\beta \in \mathbb{R}$. Assume*

(i) $c(t)$ *is a completely positive function, i.e.* $\dfrac{1}{\lambda\hat{c}(\lambda)}$ *and* $-\dfrac{\hat{c}(\lambda)'}{\hat{c}(\lambda)^2}$ *are completely monotonic on* $(0,\infty)$.

(ii) $\hat{a}_1(\lambda) = \hat{a}(\dfrac{1}{\hat{c}(\lambda)})$

(iii) $\hat{k}_1(\lambda) = \dfrac{1}{\lambda\hat{c}(\lambda)}\hat{k}(\dfrac{1}{\hat{c}(\lambda)})$

Then A is the generator of a exponentially bounded (a_1, k_1)-regularized family.

The case $k(t) \equiv 1$ gives $k_1(t) \equiv 1$ and recover [53, Theorem 4.1]. A remarkable case is $a(t) = g_\beta(t)$ and $k(t) \equiv 1$, because we have an explicit representation. Note that in order to apply the above Theorem, is enough to take $c(t) = g_{\alpha/\beta}(t)$ which is completely positive whenever $\alpha < \beta$. We restate from [7] the result.

Corollary 1.8. [7, Theorem 3.1] Let $0 < \alpha < \beta \leq 2$, $\gamma = \alpha/\beta$. If A be the generator of an exponentially bounded $(g_\beta, 1)$-regularized family $S_\beta(t)$, then A generates an exponentially bounded $(g_\alpha, 1)$-regularized family $S_\alpha(t)$, and

$$S_\alpha(t) = \int_0^\infty \Phi_\gamma(s) S_\beta(st^\gamma)ds, \quad t > 0, \tag{1.3}$$

where

$$\Phi_\gamma(t) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1,$$

is the Wright function.

More results and concrete examples on the theory of (a,k) - regularized families can be obtained from the recent article [27]. There, the author introduce the more general class of (local) (a,k)-regularized C-resolvent families and discuss its basic structural properties. In particular, the analysis done in [27] covers subjects like regularity, perturbations, duality, spectral properties and subordination principles, applying they in the study of the backwards fractional diffusion-wave equation and providing several illustrative examples.

1.3. Examples

In this section we show as the solution of an evolution equation can be associated with an (a,k)-regularized family.

Example 1.9. We consider now the *composite fractional relaxation equation*

$$u'(t) - AD_t^{\alpha}u(t) + u(t) = f(t), \quad u(0) = x, \quad 0 < \alpha < 1, \tag{1.4}$$

where A is a closed linear operator and D_t^{α} denotes Caputo's fractional derivative.

In the scalar case, the fractional differential equation in (1.4) with $\alpha = 1/2$ corresponds to the Basset problem, a classical problem in fluid dynamics concerning the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity. The abstract version (1.4) has been studied in [40] and [23].

Taking Laplace transform, we obtain

$$\hat{u}(\lambda) = \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}u(0) - \frac{1}{\lambda}A(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}u(0) + \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}\hat{f}(\lambda), \tag{1.5}$$

whenever $\frac{\lambda+1}{\lambda^{\alpha}} \in \rho(A)$. Note the unpleasant fact that A is involved in the above formula. We then replace the identity $\frac{1}{\lambda}I = \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1} - \frac{1}{\lambda}A(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1} + \frac{1}{\lambda^{\alpha+1}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}$ in (1.5) to obtain the equivalent form

$$\hat{u}(\lambda) = \frac{1}{\lambda}u(0) - \frac{1}{\lambda}\frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}u(0) + \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}\hat{f}(\lambda). \tag{1.6}$$

As before, we look for an strongly continuous family $R_{\alpha}(t) \subset \mathscr{B}(X)$ such that

$$\hat{R}_{\alpha}(\lambda) = \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}}I - A)^{-1}. \tag{1.7}$$

Comparing equation (1.6) with theorem 1.1 we need to find two Laplace transformable functions, $a(t)$ and $k(t)$, such that $\hat{a}(\lambda) = \frac{\lambda^{\alpha}}{\lambda+1}$ and $\hat{k}(\lambda) = \frac{1}{\lambda+1}$. It happens when we choose $k(t) = e^{-t}$ and $a(t) = t^{\alpha}E_{1,1-\alpha}(-t)$, ($E_{\alpha,\beta}$ denotes the Mittag-Leffler function).

Example 1.10. Our next example consists in the *composite fractional oscillation equation*

$$u''(t) - AD_t^{\alpha}u(t) + u(t) = f(t), \quad u(0) = x, \; u'(0) = y \quad 0 < \alpha < 2, \tag{1.8}$$

where A is a closed linear operator with domain $D(A) \subset X$. The fractional differential equation in (1.8) with $0 < \alpha < 2$ models an oscillation process with fractional damping term. It was formerly treated by Caputo, who provided a preliminary analysis by the Laplace transform. The special cases $\alpha = 1/2$ and $\alpha = 3/2$ have been discussed by Bagley and Torvik [59]. We note that the Bagley- Torvik equation was originally derived to study the motion of a rigid plate in a Newtonian fluid (see, e.g., [52], [59], [54]). The abstract form have been studied in [23].

Taking Laplace transform, we obtain

$$\hat{u}(\lambda) = \frac{1}{\lambda^{\alpha-1}}(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1}u(0) - \frac{1}{\lambda}A(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1}u(0) + \frac{1}{\lambda^\alpha}(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1}u'(0) + \frac{1}{\lambda^\alpha}(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1}\hat{f}(\lambda), \tag{1.9}$$

whenever $\frac{\lambda^2+1}{\lambda^\alpha} \in \rho(A)$. Taking into account the identity

$$\frac{1}{\lambda}I = \frac{1}{\lambda^{\alpha-1}}(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1} - \frac{1}{\lambda}A(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1} + \frac{1}{\lambda^{\alpha+1}}(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1}$$

we obtain the equivalent form

$$\hat{u}(\lambda) = \frac{1}{\lambda}u(0) - \frac{1}{\lambda}\frac{1}{\lambda^\alpha}(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1}u(0) + \frac{1}{\lambda^\alpha}(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1}u'(0) + \frac{1}{\lambda^\alpha}(\frac{\lambda^2+1}{\lambda^\alpha} - A)^{-1}\hat{f}(\lambda). \tag{1.10}$$

We now look for an strongly continuous family $T_\alpha(t) \subset \mathscr{B}(X)$ such that

$$\hat{T}_\alpha(\lambda) = \frac{1}{\lambda^\alpha}(\frac{\lambda^2+1}{\lambda^\alpha}I - A)^{-1}. \tag{1.11}$$

Comparing equation (1.11) with theorem 1.1 we note that we need to find two Laplace transformable functions, $a(t)$ and $k(t)$, such that $\hat{a}(\lambda) = \frac{\lambda^\alpha}{\lambda^2+1}$ and $\hat{k}(\lambda) = \frac{1}{\lambda^2+1}$. It holds when we choose $k(t) = \sin(t)$ and $a(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2)$.

Putting now $\hat{T}_\alpha(\lambda)$ in equation (1.11) we have

$$\hat{u}(\lambda) = \frac{1}{\lambda}u(0) - \frac{1}{\lambda}\hat{T}_\alpha(\lambda)u(0) + \hat{T}_\alpha(\lambda)u'(0) + \hat{T}_\alpha(\lambda)\hat{f}(\lambda). \tag{1.12}$$

From inversion of the Laplace transform, we obtain the following variation of constants formula for the composite fractional oscillation equation (1.8):

$$u(t) = u(0) - \int_0^t T_\alpha(s)u(0)ds + T_\alpha(t)u'(0) + \int_0^t T_\alpha(t-s)f(s)ds. \tag{1.13}$$

CHAPTER 2
Functional equations

2.1. Introduction

Functional equations arise in most parts of mathematics. Well known examples are Cauchy's equation, the functional equations for the Riemann zeta function, the equation for entropy and numerous equations in combinatorics. Still other examples arise in probability theory, geometry and operator theory [1].

The theory of functional equations for bounded operators, emerged after the book of Hille and Phillips [20] in 1957. A strongly continuous semigroup $T(t)$ of bounded and linear operators on a Banach space X, is defined by means of Abel's functional equation:

$$\begin{cases} T(t)T(s) = T(t+s), \qquad t \geq 0, \\ T(0) = I, \end{cases}$$

which, in turn, characterizes the well posedness of the abstract Cauchy problem of first order:

$$\begin{cases} u'(t) = Au(t), \quad t \geq 0; \\ u(0) = u_0, \end{cases}$$

where $Ax = \lim\limits_{t \to 0^+} \dfrac{T(t)x - x}{t}$ for all $x \in D(A)$. In 1966, Sova [58] introduces the concept of strongly continuous cosine operator functions, $C(t)$, by means of D'Alembert's functional equation:

$$\begin{cases} C(t+s) + C(t-s) = 2C(t)C(s), \quad t, s \in \mathbb{R}; \\ C(0) = I. \end{cases}$$

which characterizes the well posedness of the abstract Cauchy problem of second order:

$$\begin{cases} u''(t) = Au(t), \quad t \geq 0; \\ u(0) = u_0; \\ u'(0) = u_1. \end{cases}$$

where now $Ax = 2 \lim\limits_{t \to 0^+} \dfrac{C(t)x - x}{t^2}$ for all $x \in D(A)$. Let A be a linear operator defined on a Banach space X. In [53] Prüss proved that the Volterra equation of scalar type:

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t), \tag{2.1}$$

is well posed if and only if it admits a resolvent family, i.e. a strongly continuous family $S(t)$ of bounded and linear operators which commutes with A and satisfies the so called resolvent equation [53, Definition 1.3]:

$$S(t)x = x + \int_0^t a(t-s)AS(s)xds, \quad t \geq 0, \quad x \in X.$$

Resolvent families of operators have been known for a long time. They have many applications in the study of abstract differential and integral equations. However, it seems that they have never been associated to a functional equation, except in very special cases of the scalar kernel $a(t)$ as for example $a(t) = 1$ or $a(t) = t$, which corresponds to the cases of strongly continuous semigroups and cosine operator functions, respectively. Very recently, the intermediate case $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $1 < \alpha < 2$ has brought the interest of some authors (see [49], [48]).

In this chapter, we shall be concerned with a commutative one parameter family of strongly continuous operators $R_{a,k}(t)$, depending on two scalar kernels $a(t)$ and $k(t)$, satisfying $R_{a,k}(0) = k(0)I$ and the functional equation

$$R_{a,k}(s)(a * R_{a,k})(t) - (a * R_{a,k})(s)R_{a,k}(t) = k(s)(a * R_{a,k})(t) - k(t)(a * R_{a,k})(s), \quad t, s \geq 0. \quad (2.2)$$

In case $k(t) \equiv 1$ and $a(t)$ positive, one of our main results in this chapter show that the functional equation (2.2) characterizes a resolvent family, and hence the well-posedness of the Volterra equation (2.1). Moreover, we have the representation

$$Ax = \lim_{t \to 0^+} \frac{R_{a,1}(t)x - x}{\int_0^t a(s)ds}, \quad x \in D(A), \quad (2.3)$$

which includes the case of semigroups and cosine operator functions for $a(t) \equiv 1$ and $a(t) \equiv t$ respectively. Our discussion will not be restricted to resolvent families; The more general case of (a, k)-regularized resolvent families [31] is included in our results. Indeed, in such case we will see that in addition to (2.2) the condition

$$\sup_{t>0} \frac{\int_0^t |a(s)|ds}{\left|\int_0^t k(t-s)a(s)ds\right|} < +\infty, \quad (2.4)$$

is necessary to characterize an (a, k)-regularized resolvent family. A remarkable consequence is that the domain of A must be necessarily dense on the Banach space X. In particular, setting $k(t) \equiv 1$ we obtain a new (but equivalent) functional equation for strongly continuous semigroups (i.e. the case $a(t) \equiv 1$) and strongly continuous cosine operator functions (i.e. $a(t) \equiv t$), respectively. On the other hand, we prove that the condition

$$\lim_{s \to 0+} \frac{(a * a * k)(s)}{(a * k)(s)} = 0,$$

that include e.g the theory of α-times integrated semigroups, is also necessary to characterize an (a, k)-regularized resolvent family. However, the immediate denseness of $D(A)$ is not automatically obtained in such case, in concordance with the theory of integrated semigroups [3].

2.2. Sufficient Conditions

Let X be a complex Banach space and $f \in L^1_{loc}(\mathbb{R}^+, X)$. The Laplace integral is defined by

$$\widehat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt := \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} f(t)dt.$$

Also define abs(f):=inf$\{Re\lambda : \widehat{f}(\lambda)$ exist$\}$. Recall that a function f is called Laplace transformable if abs$(f) < \infty$. Note that a locally Bochner integrable function f is Laplace transformable if and only if its antiderivative $F(t) = \int_0^t f(s)ds$ is exponentially bounded, see [3, Section 1.4]. The following theorem is the main result in this section.

Theorem 2.1. *Let $\{R(t)\}_{t\geq 0} \subset \mathscr{B}(X)$ be an (a,k)-regularized resolvent family generated by a closed operator A such that $\rho(A) \neq \emptyset$. Then, for all $t,s \geq 0$ we have $R(t)R(s) = R(s)R(t)$, and the functional equation*

$$(FE) \qquad R(s)(a*R)(t) - (a*R)(s)R(t) = k(s)(a*R)(t) - k(t)(a*R)(s) \tag{2.1}$$

holds.

Proof. Note that by [31, Lemma 2.2] we have that for all $x \in X$, $(a*R)(t)x \in D(A)$ and

$$R(t)x = k(t)x + A(a*R)(t)x. \tag{2.2}$$

Hence

$$\begin{aligned}(a*R)(s)R(t)x &= (a*R)(s)k(t)x + (a*R)(s)A(a*R)(t)x \\ &= (a*R)(s)k(t)x + A(a*R)(s)(a*R)(t)x \\ &= (a*R)(s)k(t)x + R(s)(a*R)(t)x - k(s)(a*R)(t)x\end{aligned}$$

where we used the item (ii) in Definition 1.2 and equation (2.2). This show that (FE) holds for all $x \in X$ and $t,s \geq 0$.

Now we show that $R(t)R(s) = R(s)R(t)$ for all $t,s \geq 0$. Let $F(t) := R(t)R(s)$ and $H(t) := R(s)R(t)$ then for all $x \in D(A)$

$$F(t)x = k(t)R(s)x + (a*F)(t)Ax$$

$$H(t)x = k(t)R(s)x + (a*H)(t)Ax$$

where we used Definition 1.2, the fact that A is closed and $R(t)Ax = AR(t)x$ for all $x \in D(A)$, $t \geq 0$. It then follows that $W(t) := F(t) - H(t)$ satisfies

$$W(t)x = a*W(t)Ax \qquad \textit{for all } x \in D(A).$$

Note that by (ii) in the Definition 1.2, $W(t)x \in D(A)$ for all $x \in D(A)$ and hence by (iii) we obtain

$$k*W(t)x = (R - a*RA)*W(t)x = R*(W - a*WA)(t)x = 0.$$

Now let $\lambda \in \rho(A)$, $y \in X$ and define $x = (\lambda - A)^{-1}y$. Then $(\lambda - A)k*W(t)x = 0$ implies that $k*W(t)y = 0$ for each $y \in X$. Therefore, by Titchmarsh's Theorem, we obtain that $W(t)x = 0$ for each $x \in X$ which ends the proof.

□

Remark 2.2. Assume that $R(t)$ is Laplace transformable. We note that an application of the double Laplace transform to (FE) gives the following identity

$$\hat{R}(\lambda)\hat{R}(\mu) = \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}\frac{1}{\frac{1}{\hat{a}(\lambda)} - \frac{1}{\hat{a}(\mu)}}\hat{R}(\mu) - \frac{\hat{k}(\mu)}{\hat{a}(\mu)}\frac{1}{\frac{1}{\hat{a}(\lambda)} - \frac{1}{\hat{a}(\mu)}}\hat{R}(\lambda).$$

Let $S,\ T:\mathbb{R}^+\to\mathscr{B}(X)$ be strongly continuous functions satisfying $\|S(t)\|\le Me^{\omega t}$ and $\|T(t)\|\le Me^{\omega t}\ \ (t\ge 0)$ for some $\omega\in\mathbb{R},\ M\ge 0$ (for simplicity we may assume the same constants). For $h\in\mathbb{R}$ we shall denote S^h the translation $S^h(u):=S(u+h)\chi_{[-h,+\infty)}(u)$ for $u\in\mathbb{R}$ and

$$(T*S)(t):=\int_0^t T(t-s)S(s)ds,\quad t>0,$$

the convolution product between T and S. We will need the following result.

Lemma 2.3. ([25, Lemma 4.1]) Let $S,\ T:[0,\infty)\to\mathscr{B}(X)$ be strongly continuous functions satisfying the assumptions above. For $\lambda>\mu>\omega$, the following identities are valid:

$$\hat{S}(\lambda)\hat{T}(\mu)=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}S(t)T(s)dsdt \tag{2.3}$$

$$\frac{1}{\mu-\lambda}(\hat{S}(\lambda)-\hat{S}(\mu))=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}S(t+s)dsdt \tag{2.4}$$

$$\frac{1}{\mu-\lambda}\hat{T}(\mu)[\hat{S}(\lambda)-\hat{S}(\mu)]=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}(T*S^t)(s)dsdt \tag{2.5}$$

Defining $S(t)=S(-t)$ for $t<0$ we have,

$$\frac{1}{\mu+\lambda}(\hat{S}(\lambda)+\hat{S}(\mu))=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}S(s-t)dsdt,\quad \lambda+\mu>0. \tag{2.6}$$

and

$$\frac{1}{\mu+\lambda}\hat{T}(\mu)(\hat{S}(\lambda)+\hat{S}(\mu))=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}(T*S^{-t})(s)dsdt,\quad \lambda+\mu>0. \tag{2.7}$$

and defining $S(t):=-S(-t)$ for $t<0$ we obtain

$$\frac{-1}{\mu+\lambda}\hat{T}(\mu)(\hat{S}(\lambda)-\hat{S}(\mu))=\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-\mu s}(T*S^{-t})(s)dsdt,\quad \lambda+\mu>0. \tag{2.8}$$

In what follows, we restate and analyze consequences of Theorem 2.1 in several particular cases. They are important because includes different theories of strongly continuous operators and, as consequence, involves the well posedness of wide classes of abstract evolution equations.

Example 2.4. (Semigroups) $k(t)\equiv a(t)\equiv 1$. In this case we have that $R(t)$ corresponds to a C_0-semigroup and the associated functional equation (FE) reads:

$$R(s)\int_0^t R(\tau)d\tau-R(t)\int_0^s R(\tau)d\tau=\int_0^t R(\tau)d\tau-\int_0^s R(\tau)d\tau,\quad t,s\ge 0, \tag{2.9}$$

Corollary 2.5. Assume that $R(t)$ is Laplace transformable. Then equation (2.9) is equivalent to Abel's functional equation.

Proof. By (2.4) we have the identity:

$$\int_0^\infty\int_0^\infty e^{-\lambda t-\mu s}R(t+s)dsdt=\frac{\hat{R}(\lambda)-\hat{R}(\mu)}{\mu-\lambda}.$$

Hence, applying the double Laplace transform to the Abel's functional equation

$$R(t+s)=R(t)R(s),\quad t,s\ge 0,$$

we obtain

$$\hat{R}(\lambda)\hat{R}(\mu) = \frac{\hat{R}(\lambda) - \hat{R}(\mu)}{\mu - \lambda}$$

which is equivalent to

$$\frac{1}{\mu}\hat{R}(\lambda)\hat{R}(\mu) - \frac{1}{\lambda}\hat{R}(\mu)\hat{R}(\lambda) = \frac{1}{\lambda\mu}\hat{R}(\mu) - \frac{1}{\lambda\mu}\hat{R}(\lambda).$$

Hence, inversion of the double Laplace transform to the above identity gives (2.9). The converse is analogue. □

Example 2.6. (Cosine operator families) $k(t) \equiv 1$, $a(t) = t$. In this case we have that $R(t)$ corresponds to a cosine operator family [3, Section3.14] and the associated functional equation (FE) reads:

$$R(s)\int_0^t (t-\tau)R(\tau)d\tau - R(t)\int_0^s (s-\tau)R(\tau)d\tau = \int_0^t (t-\tau)R(\tau)d\tau - \int_0^s (s-\tau)R(\tau)d\tau. \qquad (2.10)$$

Corollary 2.7. Assume that $R(t)$ is Laplace transformable. Then equation (2.10) is equivalent to the D'Alembert's functional equation.

Proof. By (2.6) we have the identity:

$$\int_0^\infty \int_0^\infty e^{-\lambda t - \mu s} R(t-s)\,ds\,dt = \frac{\hat{R}(\lambda) + \hat{R}(\mu)}{\mu + \lambda},$$

valid whenever $R(t)$ is extended as an even function to $\mathbb{R}$, which is indeed the case of cosine operator families.

We apply the double Laplace transform to the D'Alembert's functional equation:

$$R(t+s) + R(t-s) = 2R(t)R(s), \quad t, s \geq 0,$$

and we obtain

$$\frac{\hat{R}(\lambda) - \hat{R}(\mu)}{\mu - \lambda} + \frac{\hat{R}(\lambda) + \hat{R}(\mu)}{\mu + \lambda} = 2\hat{R}(\lambda)\hat{R}(\mu)$$

which, after an algebraic manipulation, is equivalent to

$$\frac{\lambda}{\lambda^2 - \mu^2}\hat{R}(\mu) - \frac{\mu}{\lambda^2 - \mu^2}\hat{R}(\lambda) = \hat{R}(\lambda)\hat{R}(\mu)$$

(compare with Remark 2.2 in case $a(t) \equiv t$ and $k(t) \equiv 1$). The above identity is equivalent to:

$$\hat{R}(\lambda)\frac{1}{\mu^2}\hat{R}(\mu) - \hat{R}(\mu)\frac{1}{\lambda^2}\hat{R}(\lambda) = \frac{1}{\mu^2\lambda}\hat{R}(\mu) - \frac{1}{\lambda^2\mu}\hat{R}(\lambda).$$

Hence, inversion of the double Laplace transform to the above identity gives (2.10). The converse is analogue.

□

Example 2.8. (Sine operator family) $k(t) \equiv t$, $a(t) = t$. In this case we have that $R(t)$ corresponds to a Laplace transformable sine family [3, Section3.15] and the associated functional equation (FE) reads:

$$R(s)\int_0^t (t-\tau)R(\tau)d\tau - R(t)\int_0^s (s-\tau)R(\tau)d\tau = s\int_0^t (t-\tau)R(\tau)d\tau - t\int_0^s (s-\tau)R(\tau)d\tau. \qquad (2.11)$$

Remark 2.9. The functional equation (2.11) is equivalent to the following

$$2R(t)R(s) = \int_0^s R(s-\tau)d\tau - \int_0^t R(t+\tau)d\tau + \int_0^t R(\tau)d\tau - \int_0^s R(\tau)d\tau, \quad t,s \geq 0. \tag{2.12}$$

To show this, we apply the double Laplace transform to the equation (2.12) and then Lemma 2.3, to conclude that:

$$\begin{aligned} 2\hat{R}(t)\hat{R}(s) &= -\frac{\hat{R}(\lambda)-\hat{R}(\mu)}{\mu(\mu+\lambda)} - \frac{\hat{R}(\lambda)-\hat{R}(\mu)}{\lambda(\mu-\lambda)} + \frac{\hat{R}(\lambda)}{\lambda\mu} - \frac{\hat{R}(\mu)}{\lambda\mu} \\ &= 2\frac{\hat{R}(\lambda)-\hat{R}(\mu)}{\mu^2-\lambda^2}. \end{aligned}$$

Finally, using Remark 2.2 in case $a(t) \equiv t$ and $k(t) \equiv t$ we see from the last identity that (2.11) is equivalent to the equation (2.12). We notice that the functional equations (2.11) and (2.12) seems to be new for the theory of sine operator functions [3].

Example 2.10. (Laplace transformable (α,β)-resolvent operators). This example recover Theorem 3.11 in [10]. Let $k(t) = \frac{t^\beta}{\Gamma(\beta+1)}$; $a(t) \equiv g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$; $\alpha > 0, \beta \geq 0$. In this case we have that $R(t)$ corresponds to a Laplace transformable $(g_\alpha, g_{\beta+1})$-regularized resolvent family [10, Theorem 3.12] and the associated functional equation (FE) reads:

$$R(s)\int_0^t g_\alpha(t-\tau)R(\tau)d\tau - R(t)\int_0^s g_\alpha(s-\tau)R(\tau)d\tau = g_{\beta+1}(s)\int_0^t g_\alpha(t-\tau)R(\tau)d\tau - g_{\beta+1}(t)\int_0^s g_\alpha(s-\tau)R(\tau)d\tau.$$

Note that, taking $1 \leq \alpha = \beta+1$, this example also includes the concept of α-resolvent families introduced in [2], which characterizes the well posedness of the fractional Cauchy problem

$$D_t^\alpha u(t) = Au(t), \quad u(0) = x, \quad u^{(k)}(0) = 0, \quad k = 1,...,[\alpha]-1 \tag{2.13}$$

where D_t^α denotes the Riemann-Liouville fractional derivative. On the other hand, the special case $\beta = 0$ gives the theory of solution operators introduced by Bazhlekova [7] for the fractional Cauchy problem (2.13) where now D_t^α denotes Caputo's fractional derivative. In this case, the functional equation has the following form:

$$R(s)\int_0^t (t-\tau)^{\alpha-1}R(\tau)d\tau - R(t)\int_0^s (s-\tau)^{\alpha-1}R(\tau)d\tau = \int_0^t (t-\tau)^{\alpha-1}R(\tau)d\tau - \int_0^s (s-\tau)^{\alpha-1}R(\tau)d\tau \tag{2.14}$$

for $\alpha \geq 1$.

Observe the remarkable fact that in the scalar case, i.e. $X = \mathbb{R}$ and $A = \rho \in \mathbb{R}$, we have that the Mittag-Leffler function $E_\alpha(\rho t^\alpha)$ satisfies the functional equation (2.14), because it is the unique solution of (2.13) with the fractional derivative considered in the sense of Caputo. In other words, we have

$$\begin{aligned} E_\alpha(\rho s^\alpha)\int_0^t (t-\tau)^{\alpha-1}E_\alpha(\rho\tau^\alpha)d\tau \quad &- \quad E_\alpha(\rho t^\alpha)\int_0^s (s-\tau)^{\alpha-1}E_\alpha(\rho\tau^\alpha)d\tau \\ &= \int_0^t (t-\tau)^{\alpha-1}E_\alpha(\rho\tau^\alpha)d\tau - \int_0^s (s-\tau)^{\alpha-1}E_\alpha(\rho\tau^\alpha)d\tau, t,s \geq 0, \end{aligned}$$

for $\alpha \geq 1$. In particular, it shows that the functional equation (2.14) is a proper generalization of Abel's functional equation (corresponding to the case $\alpha = 1$) and D'Alembert functional equation (corresponding to the case $\alpha = 2$) since in case $\alpha = 1$ we have $E_1(\rho t) = e^{\rho t}$ and in case $\alpha = 2$ we have $E_2(\rho t^2) = \cosh(\sqrt{\rho}t)$.

Example 2.11. (Laplace transformable k-times integrated semigroups). We take in this example $k(t)=\frac{t^k}{\Gamma(k+1)}$, $k=0,1,...$ and $a(t)\equiv 1$. We have that $R(t)$ is an k-times integrated semigroup and the functional equation has the form

$$R(s)\int_0^t R(\tau)d\tau - R(t)\int_0^s R(\tau)d\tau = \frac{s^k}{k!}\int_0^t R(\tau)d\tau - \frac{t^k}{k!}\int_0^s R(\tau)d\tau$$

Following the same type of arguments as in Corollaries 2.5 and 2.7 (see also the following example), we note that the above equation is equivalent to the following well known formula that originally define k-times integrated semigroups (see [3, Section 3.2, Proposition 2.3.4]:

$$R(t)R(s) = \frac{1}{(k-1)!}[\int_t^{t+s}(s+t-r)^{k-1}R(r)dr - \int_0^s (s+t-r)^{k-1}R(r)dr].$$

Example 2.12. (K-convoluted semigroups) Let K be a complex-valued, locally integrable function on $[0,\infty)$. We take in this example $k(t)=\int_0^t K(\sigma)d\sigma$ and $a(t)\equiv 1$. We have that $R(t)$ is an K-convoluted semigroup (see [28, Definition 2.1] and references therein) and the functional equation has the form

$$R(s)\int_0^t R(\tau)d\tau - R(t)\int_0^s R(\tau)d\tau = \int_0^s K(\sigma)d\sigma\int_0^t R(\tau)d\tau - \int_0^t K(\sigma)d\sigma\int_0^s R(\tau)d\tau, \tag{2.15}$$

which is equivalent with the standard definition:

$$R(t)R(s)x = [\int_0^{t+s} - \int_0^t - \int_0^s]K(t+s-r)R(r)xdr. \tag{2.16}$$

Indeed, if R is Laplace transformable then we can apply the double Laplace transform to the equation (2.16) and use Lemma 2.3 to conclude that:

$$\begin{aligned} 2\hat{R}(\lambda)\hat{R}(\mu) &= \frac{\hat{K}(\lambda)\hat{R}(\lambda)-\hat{K}(\mu)\hat{R}(\mu)}{\mu-\lambda} - \hat{R}(\lambda)\frac{\hat{K}(\mu)-\hat{K}(\lambda)}{\lambda-\mu} - \hat{R}(\mu)\frac{\hat{K}(\lambda)-\hat{R}(\mu)}{\mu-\lambda} \\ &= \frac{\hat{K}(\lambda)\hat{R}(\mu)-\hat{K}(\mu)\hat{R}(\lambda)}{\lambda-\mu} \end{aligned}$$

Then observing the formula in Remark 2.2 we note that (2.15) is equivalent to the functional equation (2.16).

Example 2.13. (K-convoluted cosine functions). Let K be a complex-valued, locally integrable function on $[0,\infty)$. We take in this example $k(t)=\int_0^t K(\sigma)d\sigma$ and $a(t)=t$. We have that $R(t)$ is an K-convoluted semigroup and the functional equation has the form.

$$R(s)\int_0^t (t-\tau)R(\tau)d\tau - R(t)\int_0^s (s-\tau)R(\tau)d\tau = \int_0^s K(\sigma)d\sigma\int_0^t (t-\tau)R(\tau)d\tau - \int_0^t K(\sigma)d\sigma\int_0^s (s-\tau)R(\tau)d\tau$$

in contrast with the (equivalent) recently discovered expression [25]:

$$2R(t)R(s) = \int_t^{s+t} K(s+t-r)R(r)dr - \int_0^s K(s+t-r)R(r)dr +$$

$$+\int_{t-s}^t K(r-t+s)R(r)dr + \int_0^s K(r+t-s)R(r)dr,$$

where $t\geq s\geq 0$.

Example 2.14. (Resolvent families). Taking $k(t) \equiv 1, a \in L^1_{loc}(\mathbb{R}_+)$ we obtain the following functional equation, which seems to be the first, for the theory of integral equations of convolution type [53]:

$$R(s)\int_0^t a(t-\tau)R(\tau)d\tau - R(t)\int_0^s a(s-\tau)R(\tau)d\tau = \int_0^t a(t-\tau)R(\tau)d\tau - \int_0^s a(s-\tau)R(\tau)d\tau.$$

Example 2.15. (Integral resolvents). Taking $a(t) = k(t)$ we obtain, to our knowledge, the first functional equation for the theory of integral resolvents. Of course, it includes the scalar case [18]:

$$R(s)\int_0^t a(t-\tau)R(\tau)d\tau - R(t)\int_0^s a(s-\tau)R(\tau)d\tau = a(s)\int_0^t a(t-\tau)R(\tau)d\tau - a(t)\int_0^s a(s-\tau)R(\tau)d\tau.$$

Example 2.16. (An special case). Taking $k(t) = 1$ and $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\gamma e^{-\beta t}, \gamma \neq 0$, we obtain

$$\begin{aligned} R(s)\int_0^t (t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}R(\tau)d\tau - R(t)\int_0^s (s-\tau)^{\alpha-1}e^{-\beta(s-\tau)}R(\tau)d\tau \\ = \int_0^t (t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}R(\tau)d\tau - \int_0^s (s-\tau)^{\alpha-1}e^{-\beta(s-\tau)}R(\tau)d\tau. \end{aligned}$$

In this example, the kernel $a(t)$ is important in viscoelasticity theory [53].

2.3. Necessary conditions

Let $k, a \in L^1_{loc}(\mathbb{R}_+)$ be given and $R : \mathbb{R}_+ \to B(X)$ be a strongly continuous family such that

$$(FE) \qquad R(s)(a*R)(t) - (a*R)(s)R(t) = k(s)(a*R)(t) - k(t)(a*R)(s) \tag{2.17}$$

holds for all $s, t \geq 0$. In this section, we study in what extent the functional equation (FE) is sufficient to imply that $R(t)$ is an (a, k) resolvent family. In passing, we are going to unify and clarify from a general perspective a basic property of the theories of semigroups and cosine operator families: Automatic denseness of the domain of the generator.

Theorem 2.17. *Let $k \in C(\mathbb{R}_+)$ $a \in L^1_{loc}(\mathbb{R}_+)$ be given and let $R(t) \subset \mathscr{B}(X)$ be a commutative and strongly continuous family such that $R(0) = k(0)I$ and satisfies (FE). Define*

$$D(B) := \left\{ x \in X : \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{(a*k)(t)} \; exist \right\}$$

and

$$Bx := \lim_{t\to 0^+} \frac{R(t)x - k(t)x}{(a*k)(t)} \qquad x \in D(B), \tag{2.18}$$

and suppose the following condition

$$\sup_{t>0} \frac{\int_0^t |a(s)|ds}{|a*k(t)|} < +\infty. \tag{2.19}$$

Then $R(t)$ is an (a, k)-regularized resolvent family with generator B. Moreover, B is closed and $D(B)$ is dense in X.

Proof. Since $R(0) = k(0)I$, we have to prove conditions (ii) and (iii) from Definition 1.1. Fix $x \in D(B)$ and $t \geq 0$. For $s \geq 0$ we have from the commutativity of the family $R(t)$ that

$$\frac{R(s)R(t)x - k(s)R(t)x}{a * k(s)} = \frac{R(t)(R(s)x - k(s)x)}{a * k(s)}. \tag{2.20}$$

Since $R(t)$ is bounded, by definition of B we have that

$$\lim_{s \to 0^+} \frac{R(t)(R(s)x - k(s)x)}{a * k(s)}$$

exist and equals to $R(t)Bx$. Then (2.20) implies that $R(t)x \in D(B)$ and $BR(t)x = R(t)Bx$ for all $t \geq 0$. It proves condition (ii) in Definition 1.1. In order to show condition (iii), let $x \in X$ be given and note that by (2.19) we have that

$$\begin{aligned} \left\| \frac{a * R(s)x}{a * k(s)} - x \right\| &\leq \frac{1}{|a * k(s)|} \int_0^s |a(s-\mu)| \left\| R(\mu)x - k(\mu)x \right\| d\mu \\ &\leq \frac{1}{|a * k(s)|} \int_0^s |a(s-\mu)| d\mu \sup_{\mu \in [0,s]} \left\| R(\mu)x - k(\mu)x \right\| \to 0 \qquad as\, s \to 0^+ \end{aligned} \tag{2.21}$$

It follows from (2.17) and (2.21) that

$$\frac{R(s)(a * R)(t)x - k(s)(a * R)(t)x}{a * k(s)} = \frac{(a * R(s))(R(t)x - k(t)x)}{a * k(s)} \to R(t)x - k(t)x \tag{2.22}$$

as $s \to 0^+$ for all $x \in X$. Then, for all $x \in X$ and $t \geq 0$, $(a * R)(t)x \in D(B)$ and

$$B(a * R)(t)x = R(t)x - k(t)x. \tag{2.23}$$

Now let $x \in D(B)$. Note that $(a * R)(t)x \in D(B)$ and hence, by (2.18) and the commutativity of $R(t)$, we have

$$\begin{aligned} B(a * R)(t)x &= \lim_{s \to 0^+} \frac{R(s)(a * R)(t)x - k(s)(a * R)(t)x}{a * k(s)} \\ &= \lim_{s \to 0^+} (a * R)(t) \frac{[R(s)x - k(s)x]}{a * k(s)} \\ &= (a * R)(t)Bx. \end{aligned} \tag{2.24}$$

It then follows from (2.23) and (2.24) that, for all $x \in D(B)$

$$R(t)x = k(t)x + B(a * R)(t)x = k(t)x + (a * R)(t)Bx. \tag{2.25}$$

It shows (iii) in Definition 1.1 and hence that $R(t)$ is an (a, k)-regularized resolvent family generated by B.
We now show closedness. Let $(x_n) \in D(B)$ be a sequence such that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. It follows from the second equality in (2.25) that

$$R(t)x = k(t)x + (a * R)(t)y.$$

Then by (2.21) we have

$$\frac{R(t)x - k(t)x}{a * k(t)} = \frac{(a * R)(t)y}{a * k(t)} \to y$$

as $t \to 0^+$. It shows $x \in D(B)$ and $Bx = y$.

Finally, we show that $D(B)$ is dense in X. Let $x \in X$ be given. Then it was proved that $a * R(t)x \in D(B)$ for all $t \geq 0$. Hence, defining

$$x_n = \frac{a * R(1/n)}{a * k(1/n)} x, \quad n \in \mathbb{N},$$

it follows from (2.21) that $x_n \in D(B)$ and $\lim_{n\to\infty} x_n = x$, proving the claim and the theorem.

□

Remark 2.18. We notice that the denseness of $D(A)$ is always present under the conditions of the above theorem. In particular, we recover a well known result in the theories of semigroups and cosine operator functions and, more important, gives new results for other theories of strongly continuous functions of operators. An immediate application of this result is that the Laplacian operator with Dirichlet boundary conditions cannot be the generator on L^p spaces for $p \neq 2$, of those classes of (a,k)-regularized resolvent families that have $a(t)$ and $k(t)$ satisfying (2.19).

Our next result, studies a complementary case of Theorem 2.17. Here we deal with integrated versions of the operator families where, as we known, the density of the domain is not present, in general.

Theorem 2.19. *Let $k \in C(\mathbb{R}_+)$ and $a \in L^1_{loc}(\mathbb{R}_+)$. Let $R(t) \subset \mathscr{B}(X)$ be a commutative strongly continuous family such that $R(0) = k(0)I$ and satisfies (FE). Define $D(B)$ and B as in the above theorem. Suppose the following condition*

$$\lim_{s\to 0+} \frac{(a*(a*k))(s)}{a*k(s)} = 0. \tag{2.26}$$

Then $R(t)$ is an (a,k)-regularized resolvent family with generator B.

Proof. Since $R(0) = k(0)$, we have to prove conditions (ii)-(iii) in Definition 1.1. The proof of (ii) is the same as in Theorem 2.17. To show (iii), fix $x \in D(B)$ and note that

$$\left\| \frac{a*R(s)x}{a*k(s)} - x \right\| \leq \frac{1}{|a*k(s)|} \int_0^s |a(s-\mu)| \|(R(\mu)x - k(\mu)x)\| d\mu \tag{2.27}$$

Since $x \in D(B)$, there exists $\delta > 0$ such that for all $|\mu| \leq \delta$

$$\left\| \frac{R(\mu)x - k(\mu)x}{(a*k)(\mu)} - Bx \right\| < 1.$$

Then for all $|\mu| \leq \delta$,

$$\left\| R(\mu)x - k(\mu)x \right\| \leq (1 + \|Bx\|)|(a*k)(\mu)| =: C_x|(a*k)(\mu)|. \tag{2.28}$$

It follows from (2.27), (2.28) and the condition (2.26) that

$$\begin{aligned} \left\| \frac{a*R(s)x}{a*k(s)} - x \right\| &\leq \frac{1}{|a*k(s)|} \int_0^s |a(s-\mu)C_x(a*k)(\mu)| d\mu \\ &= \frac{|a*(a*k)(s)|}{|a*k(s)|} C_x \to 0 \qquad as\ s \to 0^+. \end{aligned} \tag{2.29}$$

Since $R(t)x - k(t)x \in D(B)$, it follows from (2.17) and (2.29) that

$$\frac{R(s)(a*R)(t)x - k(s)(a*R)(t)x}{a*k(s)} = \frac{(a*R(s))(R(t)x - k(t)x)}{a*k(s)} \to R(t)x - k(t)x \tag{2.30}$$

as $s \to 0^+$. Then for all $x \in D(B)$, we have $(a * R)(t)x \in D(B)$ and

$$B(a * R)(t)x = R(t)x - k(t)x. \tag{2.31}$$

Finally, note that $(a * R)(t)x \in D(B)$ and

$$B(a * R)(t)x = (a * R)(t)Bx. \tag{2.32}$$

Hence

$$R(t)x = k(t)x + B(a * R)(t)x = k(t)x + (a * R)(t)Bx,$$

proving the theorem.

□

Example 2.20. Suppose that $a(t)$ and $k(t)$ are positive kernels, then the condition (2.26) holds. In fact, denote $c(t) = (a * k)(t)$. Then $c(t)$ is positive non-decreasing and therefore

$$(a * c)(t) = \int_0^t a(t-s)c(s)ds \le \int_0^t a(t-s)c(t)ds = c(t)\int_0^t a(s)ds, \quad t \ge 0.$$

Hence

$$\frac{(a * a * k)(t)}{(a * k)(t)} \le \int_0^t a(s)ds \to 0$$

as $t \to 0$.

Remark 2.21. In view of Theorems 2.1, 2.17 and 2.19, the Definition 1.1 is equivalent, under certain conditions on the kernels $a(t)$ and $k(t)$, to the functional equation (FE). This fact can be used to define (a,k)-regularized families in a local way, and avoid the use of Laplace transform in the development of the theory. We note that more general classes of families of bounded operators like, e.g. (a,k)-regularized C-regularized families can be understood using (FE) (see e.g. [26, Definition 1.2]). We left the details to the interested reader.

2.4. Applications

In this section, we give one example to show how this class of regularized families defined by functional equations appears in new concrete problems, while other methods not apply directly.

Consider the following nonlinear third order differential equation

$$u'''(t) + \alpha u''(t) + c^2 Au(t) + bAu'(t) = f(t, u(t), u'(t), u''(t)) \tag{2.33}$$

with given initial conditions $u(0), u'(0), u''(0)$, and where α, b, c are positive real numbers and f is a vector-valued function.

Equation (2.1) has recently attracted the attention of a number of authors because their applications in different fields as, for example, high intensity ultrasound and vibrations of flexible materials. See [22], [17], [14] and references therein. In such cases, usually the operator A is the negative Laplacian and $f(t, u, u_t, u_{tt}) = (K(u_t)^2 + |\nabla u|^2)_t$ for a suitable constant $K > 0$. For example, in high intensity ultrasound, u is the velocity potential of the acoustic phenomenon described on some bounded $\mathbb{R}^3$-domain. In the abstract case, A is a non-negative, self-adjoint operator (possibly with compact resolvent) defined on a Hilbert space H.

Mathematical understanding of the linearized equation

$$u'''(t) + \alpha u''(t) + c^2 Au(t) + bAu'(t) = 0 \tag{2.34}$$

is meant as a preliminary critical step for the subsequent analysis of the full nonlinear model (2.33).

The usual operator-theoretic method to solve (2.34) is to rewrite it as a first order abstract Cauchy problem of the form

$$U'(t) = \mathscr{A}U(t), \quad t \geq 0, \; U(0) = U_0,$$

on the Banach space $\mathscr{X} := X \times X \times X$ with the usual norm, and where

$$\mathscr{A} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2 A & -bA & -\alpha I \end{pmatrix}$$

is defined on $D(\mathscr{A}) := D(A) \times D(A) \times X$. It is known that if A is unbounded then, for $b = 0$, the matrix of operators $\mathscr{A}$ cannot be the generator of a C_0-semigroup on $\mathscr{X}$, or even in some subspaces of it (see [22, Theorem 1.1]). It can be seen directly observing the entries of the resolvent operator

$$(\lambda\mathscr{I} - \mathscr{A})^{-1} = \begin{pmatrix} (\lambda^2 + \alpha\lambda + bA)H(\lambda) & (\lambda + \alpha)H(\lambda) & H(\lambda) \\ -c^2 AH(\lambda) & (\lambda^2 + \alpha\lambda)H(\lambda) & \lambda H(\lambda) \\ -\lambda c^2 AH(\lambda) & -(\lambda b + c^2)H(\lambda) & \lambda^2 H(\lambda) \end{pmatrix}$$

where $H(\lambda) = (\lambda^3 + \alpha\lambda^2 + b\lambda A + c^2 A)^{-1}$. Indeed, for $b = 0$ we replace the identity

$$\lambda^3 H(\lambda) + \alpha\lambda^2 H(\lambda) + c^2 AH(\lambda) = I$$

and obtain

$$(\lambda\mathscr{I} - \mathscr{A})^{-1} = \begin{pmatrix} (\lambda^2 + \alpha\lambda)H(\lambda) & (\lambda + \alpha)H(\lambda) & H(\lambda) \\ \lambda^3 H(\lambda) + \alpha\lambda^2 H(\lambda) - I & (\lambda^2 + \alpha\lambda)H(\lambda) & \lambda H(\lambda) \\ \lambda^4 H(\lambda) + \alpha\lambda^3 H(\lambda) - \lambda I & -(\lambda b + c^2)H(\lambda) & \lambda^2 H(\lambda) \end{pmatrix}$$

and now we observe that the the entries $\lambda^3 H(\lambda) + \alpha\lambda^2 H(\lambda) - I$ and $\lambda^4 H(\lambda) + \alpha\lambda^3 H(\lambda) - \lambda I$ cannot be a Laplace transform. Therefore, $\mathscr{A}$ cannot generate a C_0-semigroup on $\mathscr{X}$.

However, for $\alpha \neq 0$, we can directly associate to the equation (2.34) an (a, k)-regularized family on X, with $k(t) = e^{-\alpha t}$ and $a(t) = \frac{b\alpha - c^2}{\alpha^2}(1 - e^{-\alpha t}) + \frac{c^2}{\alpha} t$. Indeed, for such choice of the pair (a, k) we have

$$\hat{k}(\lambda) = \frac{1}{\lambda + \alpha} \text{ and } \hat{a}(\lambda) = \frac{c^2 + \lambda b}{\lambda^3 + \alpha\lambda^2}.$$

Hence

$$\lambda^2 H(\lambda) = \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)} - A\right)^{-1}.$$

Now take the (formal) Laplace transform of the left hand side of (2.34) with initial conditions $u(0) = x, u'(0) = y, u''(0) = z$. We obtain in case $b = 0$

$$\hat{u}(\lambda) = (\lambda^2 + \alpha\lambda)H(\lambda)x + (\lambda + \alpha)H(\lambda)y + H(\lambda)z.$$

Therefore, in case $b = 0$ we can still have that (2.34) is well posed in the sense that A is the generator of an (a,k)-regularized family $\{S(t)\}_{t\geq 0}$ and, in this case, a (mild) solution of the problem can be explicitly given by the formula

$$u(t) = [S(t) + \alpha(1 * S)(t)]x + [(1 * S)(t) + \alpha(t * S)(t)]y + (t * S)(t)z,$$

where $x, y, z \in X$. Finally, note that

$$k(t) = e^{-\alpha t} \text{ and } a(t) = c^2 \int_0^t (t-s)k(s)ds + b\int_0^t k(s)ds$$

are positive kernels and therefore, by Example 2.20, the condition (2.26) in Theorem 2.19 is satisfied. In this way, a method based on a direct approach - defined by an specific functional equation - is now available. Note that in case $b \neq 0$ the matrix operator $\mathscr{A}$ is the generator of a C_0-semigroup $\mathscr{T}(t)$ on $\mathscr{X}$, but it can never be compact. It has nothing to do with the corresponding properties of $S(t)$. Therefore maximal regularity results for the corresponding non-homogeneous version of equation (2.34) cannot be proved by reduction to a first order problem. The same remark applies to stability questions; even if $S(t)$ is integrable, the type of $\mathscr{T}(t)$ cannot be negative unless the type of $S(t)$ already is; this implies that it is not possible to obtain sharp integrability results for the solution $u(t)$ of (2.34) by means of an indirect approach.

Remark 2.22. The argument given above for the justification of the introduction of (a,k)-regularized families for a third order abstract differential equation is analog to those given in the origins of the theory of cosine families, see e.g. Fattorini [16], which has proved along the years to be very efficient to handle directly incomplete second order abstract Cauchy problems.

CHAPTER 3

Stability of (a,k)-regularized families in a Hilbert space

3.1. Introduction

In this chapter, we study uniform stability of (a,k)-regularized families. Note that the theory of stability is important since stable (a,k)-regularized families correspond one-to-one to stable well-posed abstract linear equations. It is also important since stability plays a central role in the structural theory of operators.

We give sufficient conditions for the uniform stability of the (a,k)-regularized family in Hilbert spaces. Our main result can be seen as substantial generalization of the Gearhart-Greiner-Prüss characterization of exponential stability for strongly continuous semigroups, see for example [15, Theorem V.1.11]. Our results also allow to study the identification of the kernels a and k. More precisely, we prove the following main result in the third section.

Suppose a,k are 1-regular kernels and that k satisfies the **(H)**-condition (see below). Assume that A generates an (a,k)-resolvent family $\{S(t)\}_{t\geq 0}$ with finite growth bound in a Hilbert space $\mathscr{H}$, and the following conditions hold.

(**H1**) $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\Re(\lambda) \geq 0$, $\lambda \neq 0$.

(**H2**) For all $x \in \mathscr{H}$, $\lim\limits_{\lambda\to 0} \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1}x = Bx$ exists.

(**H3**) $\sup\limits_{\Re\lambda>0} \| \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1} \| < \infty$.

Then $\{S(t)\}_{t\geq 0}$ is uniformly stable (Theorem 3.6).

In the second section we present some technical results about Laplace transforms and its estimates. We finish this chapter with some examples and comments concerning stability of strongly continuous cosine families and α-resolvent families associated to fractional differential equations, see Section 4. Finally, an example concerning stability of the solutions of the Basset equation is also considered.

3.2. Estimates of Laplace transform

We say that $k \in L^1_{loc}(\mathbb{R}_+)$ is of subexponential growth if for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|k(t)| \le C_\varepsilon e^{\varepsilon t}$ *a.e.* $t \ge 0$. In this case the Laplace transform, $\hat{k}(\lambda)$, given by

$$\hat{k}(\lambda) := \int_0^\infty e^{-\lambda t} k(t) dt,$$

exists for all $\Re\lambda > 0$.

Definition 3.1. Let $k \in L^1_{loc}(\mathbb{R}_+)$ be of subexponential growth such that there exists $\lim_{\lambda \to i\rho} \hat{k}(\lambda) = \hat{k}(i\rho)$ for all $|\rho| \ge 1$. We say that k satisfies the **(H)**-condition if there exists a constant $M > 0$ such that

$$\frac{1}{|\rho \hat{k}(i\rho)|} \le M$$

for all $|\rho| \ge 1$.

In what follows, we denote:

$$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} \text{ for } \alpha > 0 \text{ and } e_{-b}(t) := e^{-bt} \text{ for } b \ge 0.$$

Example. The functions $e_{-b}\cos$, for $b > 0$; g_α, for $0 < \alpha \le 1$; $\chi_{(1,\infty)} g_{n+1}$, for $n \in \mathbb{N}$ and e_{-b} for $b \ge 0$ satisfy the **(H)**-condition. However the characteristic function $\chi_{(0,1)}$, sin, cos and g_α for $\alpha > 1$ does not satisfy this **(H)**-condition.

We recall that $b \in L^1_{loc}(\mathbb{R}_+)$ of subexponential growth is called n-regular ($n \in \mathbb{N}$) if there exist a constant $c > 0$ such that

$$|\lambda^k \frac{d^k \hat{b}}{d\lambda^k}(\lambda)| \le c|\hat{b}(\lambda)|$$

for all $\Re\lambda > 0$ and $0 \le k \le n$, see [53, Definition 3.3]. In this work, we will use only the definition of 1-regular.

Example 3.2. We give some examples of 1-regular functions.

- Let $a = e_{-b} g_\alpha$ where $\Re b \ge 0$, $\alpha > 0$. We note that

  $$\sup_{\Re\lambda > 0} \left| \frac{\lambda \hat{a}'(\lambda)}{\hat{a}(\lambda)} \right| = \sup_{\Re\lambda > 0} \frac{\alpha|\lambda|}{|\lambda + b|} < \infty.$$

  Then a is an 1-regular function.

- Let $0 < \alpha < 1$ and $a = g_{1-\alpha} - \left(g_{1-\alpha} * e_{-1}\right)$. We have that $\hat{a}(\lambda) = \frac{\lambda^\alpha}{\lambda+1}$, then

  $$\left| \frac{\lambda \hat{a}'(\lambda)}{\hat{a}(\lambda)} \right| = \left| \frac{\alpha + (\alpha - 1)\lambda}{\lambda + 1} \right|,$$

  which is bounded for all $\Re\lambda > 0$.

Remark 3.3. Note that if $b \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular, then

(i) $\hat{b}(i\rho) = \lim_{\lambda \to i\rho} \hat{b}(\lambda)$ exist for each $\rho \neq 0$.

(ii) $\hat{b}(\lambda) \neq 0$ for all $\Re\lambda \geq 0$, $\lambda \neq 0$.

(iii) $|\rho \hat{b}'(i\rho)| \leq c|\hat{b}(i\rho)|$ for a.a. $\rho \in \mathbb{R}$.

see [53, Lemma 8.1].

Lemma 3.4. Let $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ be of subexponential growth and 1-regular; Assume that k satisfies the **(H)**-condition and let $\omega > 0$ be fixed. Then

(i)
$$\sup_{|\rho|\geq 1} \left| \frac{1}{\hat{k}(i\rho)} \left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega + i\rho)} - 1 \right) \right| < \infty.$$

(ii) If $\hat{k}(\omega + i(\cdot)) \in L^2(\mathbb{R} \setminus [-1,1])$ then $\left(1 - \frac{\hat{k}(\omega+i(\cdot))}{\hat{k}(i(\cdot))}\right) \in L^2(\mathbb{R} \setminus [-1,1])$.

(iii) If $\lim\limits_{\lambda\to\infty} \lambda \hat{k}(\lambda)$ exists, then $\hat{k}(\omega + i(\cdot)) \in L^2(\mathbb{R} \setminus [-1,1])$.

Proof. We note that 1-regularity of a implies that $i\rho \left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)} - 1 \right)$ is bounded for $|\rho| \geq 1$. Hence, from the identity
$$\frac{1}{\hat{k}(i\rho)} \left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega + i\rho)} - 1 \right) = \frac{i\rho}{i\rho \hat{k}(i\rho)} \left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega + i\rho)} - 1 \right)$$
and the **(H)**-condition, the conclusion (i) follows. To show (ii), note that
$$\left(1 - \frac{\hat{k}(\omega + i\rho)}{\hat{k}(i\rho)}\right) = i\rho \left(\frac{\hat{k}(i\rho)}{\hat{k}(\omega + i\rho)} - 1 \right) \frac{1}{i\rho \hat{k}(i\rho)} \hat{k}(\omega + i\rho)$$
where $i\rho \left(\frac{\hat{k}(i\rho)}{\hat{k}(\omega+i\rho)} - 1 \right)$ and $\frac{1}{i\rho\hat{k}(i\rho)}$ are bounded on $\mathbb{R}\setminus[-1,1]$ by 1-regularity of k and **(H)**-condition, respectively.

To show (iii) note that by hypothesis there exists $M > 0$ such that
$$|\hat{k}(\omega + i\rho)| = \frac{1}{|\omega + i\rho|} |(\omega + i\rho)\hat{k}(\omega + i\rho)| \leq \frac{M}{|\omega + i\rho|},$$
which yields the claim. □

Remark 3.5. Recall that in case that $\lim\limits_{t\to 0} k(t)$ exists, then
$$\lim_{t\to 0} k(t) = \lim_{\lambda\to\infty} \lambda \hat{k}(\lambda),$$
(see for example [3, Proposition 4.1.3]) and hence we may replace the condition $\lim\limits_{\lambda\to\infty} \lambda \hat{k}(\lambda)$ by $\lim\limits_{t\to 0} k(t)$ in (iii) of Lemma 3.4.

3.3. Main result

Recall that a family $\{S(t)\}_{t\geq 0} \subset \mathscr{B}(X)$ satisfying
$$\lim_{t\to\infty} \|S(t)\| = 0,$$

is called *uniformly stable*. The next theorem gives sufficient conditions about uniformly stability of (a,k)-regularized family $\{S(t)\}_{t\geq 0}$. The case of C_0-semigroups is known as Gearhart-Greiner-Prüss theorem and may be found in [15, 60]. For resolvent families of operators, see [42, Theorem 1]. In this section, we modify the proof of [42] to consider the much more general case of (a,k)-regularized families. We write

$$H(\lambda) := \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1},$$

whenever is well defined. In view of [31, Proposition 3.1], if A generates an (a,k)-regularized family $\{S(t)\}_{t\geq 0}$, exponentially bounded of type (M,ω) (i.e. $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ with $M > 0$ and $\omega \in \mathbb{R}$) then $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\Re\lambda > \omega$ and

$$\hat{S}(\lambda) = H(\lambda), \text{ for all } \Re\lambda > \omega.$$

Theorem 3.6. *Suppose $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ are 1-regular and k satisfies the **(H)**-condition. Assume that A generates an (a,k)-resolvent family $\{S(t)\}_{t\geq 0}$ with finite growth bound in a Hilbert space $\mathscr{H}$, and the following conditions:*

(H1) $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ *for all* $\Re(\lambda) \geq 0$, $\lambda \neq 0$.

(H2) *For all* $x \in \mathscr{H}$, $\lim_{\lambda\to 0} H(\lambda)x = Bx$ *defines a bounded operator.*

(H3) $\sup_{\lambda\in\mathbb{C}_+} \|H(\lambda)\| < \infty$ *where* $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$.

Then $\{S(t)\}_{t\geq 0}$ is uniformly stable.

Proof. By hypothesis there are constants $M > 0$ and $\omega_0 \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{t\omega_0}$ for all $t \geq 0$. We may suppose that $\omega_0 \geq 0$. Let $\omega > \omega_0 + 1$ be given and define $R(t) := e^{-t\omega}S(t)$ and $\|R(t)\| \leq Me^{-(\omega-\omega_0)t}$ for $t \geq 0$. Let $x \in \mathscr{H}$ be fixed, and observe that $\chi_{[0,\infty)}(\cdot)R(\cdot)x$ is in $L^2(\mathbb{R},\mathscr{H})$ where $\chi_{[0,\infty)}$ denotes the characteristic function. In fact,

$$\begin{aligned}\|\chi_{[0,\infty)}(\cdot)R(\cdot)x\|_2^2 &\leq \int_0^\infty \|R(t)x\|^2 dt \leq M^2 \int_0^\infty \left(e^{-(\omega-\omega_0)t}\|x\|\right)^2 dt \\ &\leq \frac{M^2\|x\|^2}{2(\omega-\omega_0)};\end{aligned}$$

Hence

$$\|\chi_{[0,\infty)}(\cdot)R(\cdot)x\|_2 \leq \frac{M\|x\|}{2\sqrt{(\omega-\omega_0)}}.$$

Because $\mathscr{H}$ is a Hilbert space, the Plancherel theorem shows us that the Fourier transform $\mathscr{F}$ satisfies $\|\mathscr{F}f\|_2 = \sqrt{2\pi}\|f\|_2$ for all $f \in L^2(\mathbb{R},\mathscr{H})$. On the other hand, because $\{S(t)\}_{t\geq 0}$ is an exponentially bounded family, its Laplace transform $\hat{S}(\lambda)$ is well-defined, holomorphic and satisfies $H(\lambda) = \hat{S}(\lambda)$ for all $\Re\lambda > 0$. Hence, we have for all $x \in \mathscr{H}$ and $s \in \mathbb{R}$,

$$\begin{aligned}H(\omega+is)x &= \hat{S}(\omega+is)x = \int_0^\infty e^{-(\omega+is)t}S(t)x\,dt = \int_0^\infty e^{-\omega t}e^{-ist}S(t)x\,dt \\ &= \int_0^\infty e^{-ist}R(t)x\,dt = \int_{-\infty}^\infty e^{-ist}\chi_{[0,\infty)}(t)R(t)x\,dt \\ &= \mathscr{F}(\chi_{[0,\infty)}(\cdot)R(\cdot)x)(s)\end{aligned}$$

It follows from the Plancherel theorem that $H(\omega+i(\cdot))x \in L^2(\mathbb{R},\mathscr{H})$ and

$$\|H(\omega+i\cdot)x\|_2 = \sqrt{2\pi}\|\chi_{[0,\infty)}(\cdot)R(\cdot)x\|_2 \leq M\sqrt{\frac{\pi}{\omega-\omega_0}}\|x\|. \tag{3.1}$$

We observe that by (**H2**) $\lim_{\lambda\to 0} H(\lambda)x = Bx$ exist for all $x \in \mathscr{H}$ and $B \in B(\mathscr{H})$. Also, from 1-regularity of a, k, Remark 3.3 and (**H1**), we obtain that $H(i\rho)x := \lim_{\lambda\to i\rho} H(\lambda)x$ is well defined for all $x \in \mathscr{H}$ and, by (**H3**) and the Banach-Steinhaus theorem, that $H(i\rho)$ is bounded for each $\rho \in \mathbb{R}$. It follows from the uniform boundedness principle that H is also uniformly bounded in the imaginary axis $i\mathbb{R}$.
On the other hand, from the identity

$$\hat{k}^{-1}(\lambda)H(\lambda) - \hat{k}^{-1}(\lambda)\hat{a}(\lambda)AH(\lambda) = I$$

valid for all $\Re\lambda \geq 0$, $\lambda \neq 0$, we obtain

$$\begin{aligned} H(i\rho)x &= H(\omega+i\rho)x - \left(1 - \frac{\hat{k}(\omega+i\rho)}{\hat{k}(i\rho)}\right)H(i\rho)x \\ &+ \frac{1}{\hat{k}(i\rho)}\left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)} - 1\right)H(i\rho)[H(\omega+i\rho) - \hat{k}(\omega+i\rho)]x. \end{aligned} \tag{3.2}$$

Choose a function ϕ in $C_0^\infty(\mathbb{R})$, defined by $\phi(\rho) = 1$ for $|\rho| < 1$ and $\phi(\rho) = 0$ for $|\rho| \geq 2$. Define $\psi(\rho) = 1 - \phi(\rho)$, $\rho \in \mathbb{R}$. Then using the uniform boundedness of $H(i\cdot)$ in $\mathbb{R}$ and (3.1) in (3.2) together with Lemma 3.4 implies that $\psi(\cdot)H(i\cdot)x \in L^2(\mathbb{R}, \mathscr{H})$ and

$$\begin{aligned} \|\psi(\cdot)H(i\cdot)x\|_2^2 &= \int_\infty^\infty \|\psi(\rho)H(i\rho)x\|^2 d\rho \\ &= \int_{|\rho|\geq 2} \|H(i\rho)x\|^2 d\rho + \int_{1<|\rho|<2} \|H(i\rho)x\|^2 d\rho \\ &= M_0\|x\|^2. \end{aligned}$$

Analogously, we can prove that $H(\omega + i(\cdot))^* x \in L^2(\mathbb{R}, \mathscr{H})$ and following the same argument as above we conclude that $\psi(\cdot)H(i(\cdot))^* x \in L^2(\mathbb{R}, \mathscr{H})$. By Parseval's theorem, there exists a function $u \in L^2(\mathbb{R}, \mathscr{H})$ such that

$$\mathscr{F}(u(\cdot)x)(\rho) = \psi(\rho)H(i\rho)x \text{ for a.a } \rho \in \mathbb{R}.$$

It follows that

$$\begin{aligned} \mathscr{F}(u(\cdot)x)'(\rho) &= \psi'(\rho)H(i\rho)x + i\psi(\rho)H'(i\rho)x \\ &= \psi'(\rho)H(i\rho)x + i\psi(\rho)\left(\frac{-\hat{k}'(i\rho)}{\hat{k}(i\rho)}H(i\rho)x + H(i\rho)\frac{\hat{a}'(i\rho)}{\hat{k}(i\rho)\hat{a}(i\rho)}\left[H(i\rho)x - \hat{k}(i\rho)x\right]\right) \end{aligned}$$

hence, by (**H**) and 1-regularity of a, k there exists M_0 such that

$$\left|\frac{\hat{a}'(i\rho)}{\hat{k}(i\rho)\hat{a}(i\rho)}\right| = \left|\frac{1}{i\rho\hat{k}(i\rho)}\right|\left|\frac{i\rho\hat{a}'(i\rho)}{\hat{a}(i\rho)}\right| \leq M_0$$

for all $\rho \in \mathbb{R}$, $|\rho| > 1$. Moreover by the 1-regularity of k, we have that $\frac{\hat{k}'(i(\cdot))}{\hat{k}(i(\cdot))}$ is bounded in $\mathbb{R}\setminus[-1,1]$. This and the fact that $\psi(\cdot)H(i(\cdot))x$, $\psi(\cdot)H(i(\cdot))^* x^*$ are in $L^2(\mathbb{R}, \mathscr{H})$ for each $x, x^* \in \mathscr{H}$ gives,

$$\int_\infty^\infty |\langle \mathscr{F}(u(\cdot)x')(\rho), x^*\rangle| d\rho = \int_{\mathbb{R}\setminus[-1,1]} |\langle \mathscr{F}(u(\cdot)x')(\rho), x^*\rangle| d\rho \leq M_0\|x\|\|x^*\|. \tag{3.3}$$

On the other hand, again from the uniform boundedness of $H(i\cdot)$ in $\mathbb{R}$ we have that for each $t > 0$

$$S_0(t) := \int_{-\infty}^\infty \phi(\rho)H(i\rho)e^{i\rho t} d\rho = \int_{-2}^2 \phi(\rho)H(i\rho)e^{i\rho t} d\rho.$$

Hence, by the Riemann-Lebesgue lemma it follows that $S_0(t) \to 0$ in $B(\mathcal{H})$ as $t \to +\infty$.
Finally, for $x, x^* \in \mathcal{H}$ we have that

$$\begin{aligned}\langle S(t)x, x^*\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle H(i\rho)x, x^*\rangle e^{i\rho t}d\rho \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle \phi(\rho)H(i\rho)x, x^*\rangle e^{i\rho t}d\rho + \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle \psi(\rho)H(i\rho)x, x^*\rangle e^{i\rho t}d\rho.\end{aligned}$$

Integrating by parts in the second integral, we get

$$\begin{aligned}\langle S(t)x, x^*\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle \phi(\rho)H(i\rho)x, x^*\rangle e^{i\rho t}d\rho + \frac{1}{2\pi i t}\int_{-\infty}^{\infty}\langle (\psi(\rho)H(i\rho))' x, x^*\rangle e^{i\rho t}d\rho \\ &= \frac{1}{2\pi}\langle S_0(t)x, x^*\rangle + \frac{1}{2\pi i t}\int_{-\infty}^{\infty}\langle (\mathcal{F}(u(\cdot)))'(\rho)x, x^*\rangle e^{i\rho t}d\rho.\end{aligned}$$

Therefore $\|S(t)\| \le \frac{1}{2}\|S_0(t)\| + \frac{1}{2\pi t}M_0$, from which we obtain the result. □

Remark 3.7. Consider the following integral Volterra equation of scalar type:

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t), \tag{3.4}$$

where A is a closed and linear operator with domain $D(A)$ dense in X, $a \in L^1_{loc}(\mathbb{R}_+)$ is a scalar kernel and $f \in W^{1,1}(\mathbb{R}_+; X)$. It is well known that equation (3.4) is well-posed if and only if it admits a resolvent family, see for example [53]. In terms of the theory of (a,k)-regularized families, this correspond to an $(a,1)$-regularized family generated by A.
We recover the following result concerning stability of resolvent families which appeared in [42, Theorem 1].

Corollary 3.8. Suppose $a \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular; Let A be the generator of a resolvent family $\{S(t)\}_{t\ge 0}$ with finite growth bound in a Hilbert space $\mathcal{H}$, and the following conditions:

1. $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\Re\lambda \ge 0$ $\lambda \neq 0$.
2. $\lim_{\lambda\to 0} \lambda\hat{a}(\lambda) =: a(\infty) \neq 0$ and $0 \in \rho(A)$.
3. $(\lambda - \lambda\hat{a}(\lambda)A)^{-1}$ is uniformly bounded in $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$.

Then $\{S(t)\}_{t\ge 0}$ is uniformly stable.

Proof. We observe that the scalar kernel $k = \chi_{[0,\infty)}$ is 1-regular and satisfies the (**H**)-condition. It follows from (1) and (3) that (**H1**) and (**H3**) of Theorem 3.6 are satisfied, so that we only need to verify (**H2**). For this we note that by (2)

$$\lim_{\lambda\to 0} H(\lambda)x = \frac{1}{\lambda\hat{a}(\lambda)}\left(\frac{\lambda}{\lambda\hat{a}(\lambda)} - A\right)^{-1} = \frac{1}{a(\infty)}A^{-1}x,$$

and the conclusion follows. □

Recall that a strongly continuous family $S \equiv \{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ is called *uniformly integrable* if $t \mapsto \|S(t)\|$ is a measurable function and

$$\|S\|_{L^1} := \int_0^{\infty}\|S(t)\|dt < \infty.$$

Corollary 3.9. Suppose $a \in L^1_{loc}(\mathbb{R})$ and $k \in C(\mathbb{R}_+)$ are 1-regular and k satisfies the **(H)**-condition. Assume that A generates an (a,k)-resolvent family $S \equiv \{S(t)\}_{t\geq 0}$ with finite growth bound in a Hilbert space $\mathcal{H}$. If $\{S(t)\}_{t\geq 0}$ is uniformly integrable then $\{S(t)\}_{t\geq 0}$ is uniformly stable.

Proof. The fact that S is uniformly integrable implies that $H(\lambda)$ is well defined for all $\lambda \in \overline{\mathbb{C}}_+$. On the other hand, from the definition of an (a,k)-regularized family generated by A we can see that

$$\frac{1}{\hat{k}(\lambda)} H(\lambda)(I - \hat{a}(\lambda)A)x = x \quad \text{for all } \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\},\ x \in D(A),$$

then we obtain that $(I - \hat{a}(\lambda)A)$ is an injective operator. Now using that $H(\lambda)$ commutes with $(I - \hat{a}(\lambda)A)$ for all $x \in D(A)$ and $H(\lambda)D(A) \subset D(A)$, we conclude that the operator $(I - \hat{a}(\lambda)A)$ is surjective. Moreover $(I - \hat{a}(\lambda)A)^{-1}$ is bounded i.e. $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\Re\lambda \geq 0$, $\lambda \neq 0$. Then (**H1**) holds.
It follows from the above that $\hat{S}(\lambda) = H(\lambda)$ for all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$ and hence, by application of the dominated convergence theorem, we obtain

$$\lim_{\lambda \to 0} H(\lambda)x = \int_0^\infty S(t)x\,dt =: Bx.$$

Therefore (**H2**) is satisfied. Finally, let $\lambda \in \overline{\mathbb{C}}_+$ then we have

$$\|H(\lambda)x\| \leq \int_0^\infty \|S(t)x\|\,dt \leq \|S\|_{L^1}\|x\|$$

and we conclude that (**H3**) holds. Then by Theorem 3.6 $\{S(t)\}_{t\geq 0}$ is uniformly stable. □

Since the function $\chi_{[0,\infty)}$ is 1-regular and satisfies the (**H**)-condition, we recover the following result on stability of C_0-semigroups due to Gearhart, Greiner and Prüss (see [15, Theorem V. 1.11]), taking $k = a = \chi_{[0,\infty)}$ in Theorem 3.6.

Corollary 3.10. Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ with finite growth bound defined in a Hilbert space $\mathcal{H}$. The following conditions are equivalent.

1. $\{\lambda \in \mathbb{C} : \Re\lambda \geq 0\} \subset \rho(A)$ and $\sup_{\Re\lambda > 0} \|R(\lambda; A)\| < \infty$.
2. The semigroup $\{T(t)\}_{t\geq 0}$ is uniformly stable.

3.4. Examples and comments

Example 4.1. Suppose that A is the generator of an $(e_{-b}g_\alpha, e_{-b})$-regularized family $\{S(t)\}_{t\geq 0}$ for some $\alpha > 1$ and $b > 0$ satisfying the following conditions:

1. $(b + \lambda)^\alpha \in \rho(A)$ for all $\Re\lambda \geq 0$;
2. $\sup_{\Re\lambda > 0} \|(\lambda + b)^{\alpha-1}((\lambda + b)^\alpha - A)^{-1}\| < \infty$;

Then $\{S(t)\}_{t\geq 0}$ is uniformly stable.

Proof. It follows from Example 3.2 that $a = e_{-b}g_\alpha$ and $k = e_{-b}$ are 1-regular. On the other hand

$$\frac{1}{|\rho \hat{e}_{-b}(i\rho)|} = \frac{|b + i\rho|}{|\rho|} \leq M$$

for some $M > 0$ and for all $|\rho| \geq 1$, proving that the (**H**)-condition holds. By (1)

$$\lim_{\lambda\to 0} H(\lambda)x = b^{\alpha-1}(b^{\alpha} - A)^{-1}x$$

for all $x \in \mathcal{H}$, and therefore the above limit defines a bounded operator. Then (**H2**) holds. It also follows from (1) and (2) that (**H1**) and (**H3**) of Theorem 3.6 are satisfied. Then $\{S(t)\}_{t\geq 0}$ is uniformly stable. □

Let A be the generator of a strongly continuous cosine family $\{C(t)\}_{t\geq 0}$. It is well known that $\{C(t)\}_{t\geq 0}$ cannot be stable (because of the identity $I = 2C(t)^2 - C(2t)$ for $t \geq 0$). However, our above example in case $\alpha = 2$ shows that we can give a counterpart of Corollary 3.9 for strongly continuous cosine families of operators as follows.

Corollary 3.11. Let $b > 0$. Suppose that A is the generator of a strongly continuous cosine families of operators $\{C(t)\}_{t\geq 0}$ satisfying the following conditions:

1. $(b+\lambda)^2 \in \rho(A)$ for all $\Re\lambda \geq 0$.
2. $\sup_{\Re\lambda>0} \|(\lambda+b)((\lambda+b)^2 - A)^{-1}\| < \infty$.

Then $\{e^{-bt}C(t)\}_{t\geq 0}$ is uniformly stable.

Example 4.2. Note that Example 4.1 includes stability for the solution of the fractional differential equation

$$D^{\alpha}u(t) = Au(t);$$

with initial condition $u(0) = u_0$ or, equivalently, the solution of the integral equation of convolution type:

$$u(t) = u(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s)ds, \qquad t \geq 0.$$

Indeed, take $v(t) = e^{-bt}u(t)$ where $b > 0$ is given, then the above integral equation is equivalent to

$$v(t) = e^{-bt}v(0) + \int_0^t e^{-b(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Av(s)ds, \qquad t \geq 0. \tag{3.5}$$

Hence, we conclude that if the problem (3.5) is well-posed then, under the conditions (1) and (2), we have that $v(t) \to 0$ as $t \to \infty$.

In particular, if we consider the operator $Ax = \mu x$ for all $x \in \mathcal{H}$ where:

$$0 < \mu < b^{\alpha}, \tag{3.6}$$

then A generates an uniformly stable $(e_{-b}g_{\alpha}, e_{-b})$-regularized family $\{S(t)\}_{t\geq 0}$ given by

$$S(t) := e^{-bt}E_{\alpha}(\mu t^{\alpha}), \qquad t \geq 0,$$

where E_{α} denotes the Mittag-Leffler function. Indeed, we have that

$$\hat{S}(\lambda) = \left[\widehat{e^{-bt}E_{\alpha}(\mu t^{\alpha})}\right](\lambda) = \frac{(b+\lambda)^{\alpha-1}}{(b+\lambda)^{\alpha} - \mu} = \hat{k}(\lambda)(1 - \hat{a}(\lambda)\mu)^{-1}, \qquad \Re\lambda > 0,$$

where $a = e_{-b}g_{\alpha}$ and $k = e_{-b}$, which shows that $\{S(t)\}_{t\geq 0}$ is an $(e_{-b}g_{\alpha}, e_{-b})$-regularized family. We are going now to show properties (**H1**) and (**H2**) of Theorem 3.6. In fact, we note that $(b+\lambda)^{\alpha} \in \rho(A)$, for all $\Re\lambda \geq 0$ if and only if $(b+\lambda)^{\alpha} \neq \mu$, for all $\Re\lambda \geq 0$. If $(b+\lambda)^{\alpha} = \mu$ for some $\Re\lambda \geq 0$

then $\Re\lambda = \mu^{1/\alpha}\cos(2k\pi/\alpha) - b$ for $k \in \mathbb{Z}$. Therefore, condition (3.6) implies $\mu^{1/\alpha}\cos(2k\pi/\alpha) < b$, for all $k \in \mathbb{Z}$, and hence $(b+\lambda)^\alpha \in \rho(A)$ for all $\Re\lambda \geq 0$. We conclude that condition (**H1**) holds. On the other hand,

$$\hat{k}(\lambda)(1 - \hat{a}(\lambda)\mu)^{-1} = \frac{(b+\lambda)^{\alpha-1}}{(b+\lambda)^\alpha - \mu} \to 0$$

as $\lambda \to \infty$. In consequence, using the fact that $\hat{k}(\lambda)(1 - \hat{a}(\lambda)\mu)^{-1}$ is continuous in $\overline{\mathbb{C}_+}$, we conclude that $H(\lambda)$ is uniformly bounded in $\mathbb{C}_+$, namely, the condition (**H2**) holds. The claim follows.

It is illustrative to check the result in the following particular cases:

$\alpha = 1$: $0 < \mu < b$ implies $e^{-bt}E_1(\mu t) = e^{-(b-\mu)t} \to 0$ as $t \to \infty$;

$\alpha = \frac{1}{2}$: $0 < \mu < b^{1/2}$ implies $e^{-bt}E_{1/2}(\mu t^{1/2}) = e^{-(b-\mu^2)t}[1 - \mathrm{erf}(\mu t^{1/2})] \to 0$ as $t \to \infty$;

$\alpha = 2$: $0 < \mu < b^2$ implies $e^{-bt}E_2(\mu t^2) = e^{-bt}\cosh(\sqrt{\mu}t) = \frac{1}{2}[e^{-(b-\sqrt{\mu})t} + e^{-(b+\sqrt{\mu})t}] \to 0$ as $t \to \infty$;

$\alpha = 4$: $0 < \mu < b^4$ implies $e^{-bt}E_4(\mu t^4) = \frac{1}{2}e^{-bt}[\cos(\sqrt[4]{\mu}t) + \cosh(\sqrt[4]{\mu}t)] = \frac{1}{4}e^{-bt}[2\cos(\sqrt[4]{\mu}t) + e^{-(b-\sqrt[4]{\mu})t} + e^{-(b+\sqrt[4]{\mu})t}] \to 0$ as $t \to \infty$.

Example 4.3 Let $\mathcal{H}$ be a Hilbert space and for $0 < \alpha < 1$ consider the fractional relaxation equation.

$$u'(t) - AD^\alpha u(t) + u(t) = f(t), \qquad t > 0, \tag{3.7}$$

with initial condition $u(0) = u_0$ and f an appropriate $\mathcal{H}$-valued function. Equation (3.7) corresponds to the abstract version of the Basset problem (see [23]). We recall that the Basset equation arises in fluid dynamics concerning the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, see [43]. As stated in [23, Section 3], well-posedness of equation (3.7) is equivalent to the existence of an (a,k)-regularized family $\{S(t)\}_{t\geq 0}$ generated by A, with

$$a = g_{1-\alpha} - (g_{1-\alpha} * e_{-1}) \quad \text{and} \quad k = e_{-1}. \tag{3.8}$$

Moreover, is easy to check that the solution of the problem in terms of $\{S(t)\}_{t\geq 0}$ is given by

$$u(t) = u(0) - \int_0^t S(s)u(0)ds + \int_0^t S(t-s)f(s)ds. \tag{3.9}$$

Corollary 3.12. Let $0 < \alpha < 1$. Suppose that A generates an $(g_{1-\alpha} - (g_{1-\alpha} * e_{-1}), e_{-1})$-regularized family $\{S(t)\}_{t\geq 0}$ satisfying the following conditions.

1. $\frac{\lambda+1}{\lambda^\alpha} \in \rho(A)$ for all $\Re\lambda \geq 0$ and $\lambda \neq 0$.
2. $\sup_{\Re\lambda > 0} \|(1 + \lambda - \lambda^\alpha A)^{-1}\| < \infty$.

Then $\{S(t)\}_{t\geq 0}$ is uniformly stable.

Proof. It follows from the identities $\hat{a}(\lambda) = \frac{\lambda^\alpha}{1+\lambda}$ and $\hat{k}(\lambda) = \frac{1}{1+\lambda}$ that

$$\lim_{\lambda\to 0} H(\lambda)x = \lim_{\lambda\to 0}((1+\lambda) - \lambda^\alpha A)^{-1}x = x, \quad x \in \mathcal{H},$$

and the conclusion follows from Theorem 3.6. □

For instance, suppose that the operator A is scalar ($Ax = \mu x$, for some $\mu \in \mathbb{C}$ and for all $x \in \mathscr{H}$). We consider equation (3.7) with initial conditions, i.e.

$$u'(t) - \mu D^{\alpha} u(t) + u(t) = f(t), \quad u(0) = u_0.$$

We claim that if $\Re\mu < 0$ then A generates an uniformly stable (a,k)-regularized family $\{S(t)\}_{t\geq 0}$.

Firstly, we will show that there exist a bounded continuous function $S : [0,\infty) \to \mathbb{C}$ such that

$$\hat{S}(\lambda) = H(\lambda) = \frac{1}{1 + \lambda - \lambda^{\alpha}\mu} \quad \text{for all } \Re\lambda > 0,$$

and we will use [31, Proposition 3.1] to show that A generates an (a,k)-regularized family, where a and k are given in (3.8). For this, we note that $\frac{1+\lambda}{\lambda^{\alpha}} \neq \mu$ for all $\Re\lambda > 0$. Indeed, we have the identity

$$\Re\left(\frac{1+\lambda}{\lambda^{\alpha}}\right) = \frac{1}{|\lambda|^{\alpha}}[(1+\Re\lambda)\cos(\alpha\theta_{\lambda}) + (\Im\lambda)\sin(\alpha\theta_{\lambda})]$$

where $\theta_{\lambda} := \mathrm{Arg}_{[-\pi,\pi)}(\lambda)$. Then using that $0 < \alpha < 1$, we conclude that $\Re\left(\frac{1+\lambda}{\lambda^{\alpha}}\right) \geq 0$ for all $\Re\lambda > 0$. Since $\Re\mu < 0$ we obtain that $\frac{1+\lambda}{\lambda^{\alpha}} \neq \mu$ for all $\Re\lambda > 0$.

On the other hand, from the fact that $0 < \alpha < 1$ and the continuity of H and H' over $i\mathbb{R}$, we conclude that there exist $M > 0$ such that

$$|\lambda H(\lambda)| \leq M \quad and \quad |\lambda^2 H'(\lambda)| \leq M \quad \text{for all} \quad \Re\lambda > 0.$$

Hence by [3, Theorem 2.5.2] we get that there exists a bounded function $S \in C(\mathbb{R}_+)$ such that $\hat{S}(\lambda) = H(\lambda)$, for all $\Re\lambda > 0$.

Secondly, we will apply Corollary 3.12 to conclude that the solution is uniformly stable. The fact that $\Re\mu < 0$ and $\Re(\frac{1+\lambda}{\lambda^{\alpha}}) \geq 0$ for all $\Re\lambda \geq 0$, implies that the condition (1) of Corollary 3.12 is satisfied. Condition (2) follows from the fact that H is analytic over $\mathbb{C}_+$ and continuous in $i\mathbb{R}$. Therefore we have

$$\lim_{|\lambda|\to\infty} H(\lambda) = \lim_{|\lambda|\to\infty} \frac{1}{1 + \lambda - \lambda^{\alpha}\mu} = 0,$$

and hence the resolvent $\{S(t)\}_{t\geq 0}$ is uniformly stable. In particular, we conclude from (3.9) that if $\Re\mu < 0$ then the solution of the equation:

$$u'(t) - \mu D^{\alpha} u(t) + u(t) = 0, \quad u(0) = u_0,$$

satisfies $\mu D^{\alpha} u(t) - u(t) \to 0$ as $t \to \infty$.

CHAPTER 4

Maximal regularity of first order differential equation with finite delay

4.1. Introduction

Partial differential equations with delay are a subject which has been extensively studied in the last years. In an abstract way they can be written as.

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{R}, \tag{4.1}$$

where $(A, D(A))$ is a (unbounded) linear operator on a Banach space X, $u_t(\cdot) = u(t+\cdot)$ on $[-r, 0]$, $r > 0$, and the delay operator is supposed to belong to $\mathscr{B}(L^p([-r, 0], X), X)$.
First studies on equation (4.1) goes back to J. Hale [19] and G. Webb [61]. A general and systematic study of linear delay equations with emphasis on the qualitative behavior and asymptotic properties can be found in the monograph by Bátkai an Piazzera [8]. See also [64]. The problem to find conditions for all solutions of (4.1) to be in the same space as f arises naturally from recent studies on maximal regularity and their application to nonlinear problems in the theory of evolution equations, see the monograph by Denk-Hieber-Prüss [13] and references therein. Maximal regularity of the problem (4.1) when f belongs to in a Hölder space is given by Lizama-Poblete [38].
In this work we are able to obtain necessary and sufficient conditions in order to guarantee well-posedness of the delay equation (4.1) in the $L^p(\mathbb{R}, D(A)) \cap W^{1,p}(\mathbb{R}, X)$ space $(1 < p < \infty)$ and under the condition that X is a UMD space. To obtain the results we use, for sufficient conditions the techniques of the multipliers of Fourier theory version in $L^p(\mathbb{R}, X)$ given by Weis ([62], [63], see also [11], [50]), and for necessary conditions we use an intermediate weight Lebesgue spaces $L^p_\alpha(\mathbb{R}, X)$ see ([45] and [55]). In addition also we will study the maximal regularity of equation (4.1) in a mild sense, that is: What are the sufficient and necessary conditions to ensure that for all $f \in L^p(\mathbb{R}, X)$ exists a unique solution $u \in f \in L^p(\mathbb{R}, X)$ of integral equation

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t Fu_s ds + \int_0^t f(s)ds,$$

for almost $t \in \mathbb{R}$.

4.2. Necessary conditions

In this section we will study necessary conditions under the hypothesis that the equation (4.1) has the property of maximal L^p-regularity. We start with the definition of maximal L^p-regularity of the problem (4.1) in a mild and an stronge sense.

Definition 4.1. For $f \in L^p(\mathbb{R}, X)$, we call $u \in L^p(\mathbb{R}, X)$ a *mild solution* of the equation (4.1) if $\int_0^t u(s)ds \in D(A)$ and there exists $x \in X$ such that,

$$u(t) = x + A\int_0^t u(s)ds + \int_0^t Fu_s ds + \int_0^t f(s)ds \tag{4.2}$$

for almost all $t \in \mathbb{R}$. Where A is a closed, linear operator on a Banach space X, $f \in L^p(\mathbb{R}, X)$ and $F : L^p([-r,0], X) \to X$ is a bounded operator.

We call u an *strong solution* of the equation (4.1), if $u \in W^{1,p}(\mathbb{R}, X) \cap L^p(\mathbb{R}, D(A))$ and u satisfies equation (4.1).

Definition 4.2. Let $1 < p < \infty$. The equation (4.1) has the property of *mild maximal L^p-regular*. If for all $f \in L^p(\mathbb{R}, X)$ there exists a unique *mild solution* of the equation (4.1).

Similarly we say that the equation (4.1) has the property of *maximal L^p-regular*. If for all $f \in L^p(\mathbb{R}, X)$ there exists a unique strong solution of the equation (4.1).

Given these definitions. We begin analyzing necessary conditions, when the property of *mild maximal L^p-regular* for equation (4.1) is satisfied. For this purpose we consider the solution operator for the equation (4.2) in $L^p(\mathbb{R}, X)$ (in a mild sence) which is defined by

$$\begin{aligned} D(M_p) &:= \{f \in L^p(\mathbb{R}, X) : \textit{ exists a unique } u_f \in L^p(\mathbb{R}, X) \textit{ such that} \\ &\quad\; u_f \textit{ is a mild solution of equation } (4.1)\} \\ M_p f &:= u_f \end{aligned}$$

Remark 4.3. We note that M_p is a closed operator, moreover if $D(M_p) = L^p(\mathbb{R}, X)$ then M_p is bounded, by the closed graph theorem.

For $\alpha > 0$ we define the following weighted L^p_α-space

$$L^p_\alpha(\mathbb{R}, X) := \{f : \mathbb{R} \to X : \textit{measurable}, \|f\|_{p,\alpha} < \infty\},$$

where the norm is given by $\|f\|_{\alpha,p} := \left(\int_{\mathbb{R}} \|e^{-\alpha|t|} f(t)\|^p dt\right)^{\frac{1}{p}}$.
Further we define, the following mapping

$$\begin{aligned} \tilde{} : L^p_\alpha(\mathbb{R}, X) &\to L^p(\mathbb{R}, X) \\ u &\to \tilde{u}, \text{ where } \tilde{u}(t) := e^{-\alpha|t|}u(t). \end{aligned}$$

Remark 4.4. It is easy to see that $\tilde{}$ is an isomorphism between $L^p_\alpha(\mathbb{R}, X)$ and $L^p(\mathbb{R}, X)$. On the other hand we will need apply $\tilde{}$ to the object Fu_t this means that

$$\widetilde{Fu_t} = e^{-\alpha|t|}Fu(t+\cdot) = Fe^{-\alpha|t|}u(t+\cdot) = F(\widetilde{u_t}).$$

Its important to see that in general $F(\widetilde{u_t}) \neq F\tilde{u}_t$. For example if $X = \mathbb{R}$ and $F : L^p([-r,0], X) \to X$ defined by $F(f(\cdot)) = \int_{-r}^0 f(s)ds$. We obtain:

$$F(\widetilde{u_t}) = Fe^{-\alpha|t|}u(t+\cdot) = \int_{-r}^0 e^{-\alpha|t|}u(t+s)ds$$

and

$$F\tilde{u}_t = Fe^{-\alpha|t+\cdot|}u(t+\cdot) = \int_{-r}^0 e^{-\alpha|(t+s)|}u(t+s)ds.$$

If $u \equiv 1$, $r \neq 0$ and t fixed, then

$$F(\widetilde{u_t}) \neq F\tilde{u}_t.$$

Lemma 4.5. Let $p \in (1,\infty)$, $\alpha > 0$ and $f \in L^p_\alpha(\mathbb{R}, X)$. Then $u \in L^p_\alpha(\mathbb{R}, X)$ is a *mild solution* of the equation (4.1), if and only if, $\widetilde{u} \in L^p(\mathbb{R}, X)$ is a mild solution of the equation

$$\widetilde{u}'(t) = A\widetilde{u} + F(\widetilde{u_t}) + \widetilde{f}(t) - \alpha sig(t)\widetilde{u}(t). \tag{4.3}$$

Proof. If $u \in L^p_\alpha(\mathbb{R}, X)$ then follows,

$$\begin{aligned}
\|Fu_t\|^p_{p,\alpha} &= \int_{\mathbb{R}} \|e^{-\alpha|t|}Fu_t\|^p dt \\
&\leq \int_{\mathbb{R}} e^{-\alpha|t|p}\|F\|^p\|u_t\|^p_{L^p([-r,0],X)} dt = \|F\|^p \int_{\mathbb{R}} e^{-\alpha|t|p}\int_{-r}^0 \|u(t+\theta)\|^p d\theta dt \\
&= \|F\|^p \int_{-r}^0 \int_{\mathbb{R}} e^{-\alpha|t|p}\|u(t+\theta)\|^p dt d\theta = \|F\|^p \int_{-r}^0 \int_{\mathbb{R}} e^{-\alpha|s-\theta|p}\|u(s)\|^p ds d\theta \\
&= \|F\|^p (\int_{-r}^0 \int_{(-\infty,\theta)} e^{-\alpha|s-\theta|p}\|u(s)\|^p ds d\theta + \int_{-r}^0 \int_{[-\theta,0]} e^{-\alpha|s-\theta|p}\|u(s)\|^p ds d\theta \\
&\quad + \int_{-r}^0 \int_{(0,\infty)} e^{-\alpha|s-\theta|p}\|u(s)\|^p ds d\theta\ ) \\
&\leq \|F\|^p (\int_{-r}^0 \int_{(-\infty,\theta)} e^{r\alpha p} e^{-\alpha|s|p}\|u(s)\|^p ds d\theta + \int_{-r}^0 \int_{[-\theta,0]} e^{-\alpha|s-\theta|p}\|u(s)\|^p ds d\theta \\
&\quad + \int_{-r}^0 \int_{(0,\infty)} e^{-\alpha|s|p}\|u(s)\|^p ds d\theta\ ) \\
&\leq \|F\|^p (re^{r\alpha p}\|u\|^p_{\alpha,p} + r\|u\|^p_{\alpha,p} + \int_{-r}^0 \int_{[-\theta,0]} \frac{e^{-\alpha|s-\theta|p}}{e^{-\alpha|s|p}} e^{-\alpha|s|p}\|u(s)\|^p ds d\theta) \\
&\leq \|F\|^p (re^{r\alpha p}\|u\|^p_{\alpha,p} + r\|u\|^p_{\alpha,p} + \int_{-r}^0 \int_{-r}^0 C_{r,p} e^{-\alpha|s|p}\|u(s)\|^p ds d\theta) \\
&\leq \|F\|^p (re^{r\alpha p}\|u\|^p_{\alpha,p} + r\|u\|^p_{\alpha,p} + rC_{r,p}\|u\|^p_{\alpha,p}) < \infty
\end{aligned}$$

where $C_{r,p} := \sup\limits_{(t,\theta)\in[-r,0]\times[-r,0]} \frac{e^{-\alpha|s-\theta|p}}{e^{-\alpha|s|p}}$. Then we conclude that $Fu. \in L^p_\alpha(\mathbb{R}, X)$.
Let $u \in L^p_\alpha(\mathbb{R}, X)$ be a *mild solution* of the equation (4.1) , i.e. there exist $x \in X$ such that the integrated equation (4.2) holds for almost all $t \in \mathbb{R}$. With integration by parts, it follows

$$\begin{aligned}
& A\int_0^t \widetilde{u}(s)ds + \int_0^t F(\widetilde{u_s})ds + \int_0^t \widetilde{f}(s)ds \\
= \ & A(e^{-\alpha|t|}\int_0^t u(s)ds + \alpha\int_0^t sig(s)e^{-\alpha|s|}\int_0^s u(r)drds) \\
+ \ & (e^{-\alpha|t|}\int_0^t Fu_s ds + \alpha\int_0^t sig(s)e^{-\alpha|s|}\int_0^s Fu_r drds) \\
+ \ & (e^{-\alpha|t|}\int_0^t f(s)ds + \alpha\int_0^t sig(s)e^{-\alpha|s|}\int_0^s f(r)drds) \\
= \ & e^{-\alpha|t|}\left(A\int_0^t u(s)ds + \int_0^t Fu_s ds + \alpha + \int_0^t f(s)ds\right) \\
+ \ & \alpha sig(t)\int_0^t \left(e^{-\alpha|s|}A\int_0^s u(r)dr + \int_0^s Fu_r dr + \alpha + \int_0^s f(r)dr\right)ds \\
= \ & e^{-\alpha|t|}(u(t) - x) + \alpha sig(t)\int_0^t e^{-\alpha|s|}(u(s) - x)ds \\
= \ & \widetilde{u}(t) + \alpha sig(t)\int_0^t \widetilde{u}(s)ds - x
\end{aligned}$$

for almost all $t \in \mathbb{R}$. Thus, $\widetilde{u}$ is a *mild solution* of the equation (4.3). The reverse can be proved similar. □

Lemma 4.6. Let $p \in (1,\infty)$, $\alpha > 0$ and $f \in L^p_\alpha(\mathbb{R}, X)$ be given. If $\widetilde{u} \in L^p(\mathbb{R}, X)$ is a *mild solution* of the equation (4.1) associated to the function $\widetilde{f}$. Then

$$M_p\widetilde{f} = \widetilde{u} + \alpha M_p\left(sig(\cdot)\widetilde{u}(\cdot) - \frac{1}{\alpha}F\widetilde{(u.)} + \frac{1}{\alpha}F\widetilde{u}.\right)$$

Proof. By hypothesis, there exists $x_1 \in X$ such that

$$\widetilde{u}(t) = x_1 + A\int_0^t \widetilde{u}(s)ds + \int_0^t F\widetilde{(u_s)}ds + \int_0^t \widetilde{f}(s)ds - \alpha sig(t)\int_0^t \widetilde{u}(s)ds.$$

Let $v := M_p\left(sig(\cdot)\widetilde{u}(\cdot) - \frac{1}{\alpha}F\widetilde{u}. + \frac{1}{\alpha}F(\widetilde{u}).\right)$ i.e. there exists $x_2 \in X$ such that

$$\begin{aligned} v(t) &= x_2 + A\int_0^t v(s)ds + \int_0^t Fv_s ds + \int_0^t \left(sig(s)\widetilde{u}(s) - \frac{1}{\alpha}F\widetilde{(u_s)} + \frac{1}{\alpha}F\widetilde{u}_s\right)ds \\ &= x_2 + A\int_0^t v(s)ds + \int_0^t Fv_s ds + sig(t)\int_0^t \widetilde{u}(s)ds - \int_0^t \frac{1}{\alpha}F\widetilde{(u_s)}ds + \int_0^t \frac{1}{\alpha}F\widetilde{u}_s ds. \end{aligned}$$

It follows that

$$\begin{aligned} &= \widetilde{u}(t) + \alpha v(t) \\ &= x_1 + \alpha x_2 + A\int_0^t (\widetilde{u}(s) + \alpha v(s))ds + \int_0^t F(\widetilde{u} + \alpha v)_s ds + \int_0^t \widetilde{f}(s)ds, \end{aligned}$$

hence, $\widetilde{u} + \alpha v$ is a *mild solution* of (4.3). i.e. $M_p\widetilde{f} = \widetilde{u} + \alpha M_p\left(sig(\cdot)\widetilde{u}(\cdot) - \frac{1}{\alpha}F\widetilde{(u.)} + \frac{1}{\alpha}F\widetilde{u}.\right)$. □

For the next theorem we define for $\lambda \in \mathbb{C}$ the operator

$$\begin{aligned} Fg(\lambda) : X &\to X \\ x &\to Fg(\lambda)x = F(g(\lambda,\theta)x), \end{aligned}$$

where $g(\lambda,\theta) := e^{\lambda\theta}$ for $\lambda \in \mathbb{C}$, $\theta \in [-r, 0]$. We note note that

$$\sup_{s\in\mathbb{R}} \|Fg(is)\| \le \|F\|$$

Theorem 4.7. *Let* $1 < p < \infty$. *Suppose that the equation (4.1) has the property of mild maximal* L^p*-regular. Then for all* $\xi \in \mathbb{R}$ *the operator* $(i\xi - (Fg)(i\xi) - A)$ *is invertible and there exists a constant* $c > 0$ *such that*

$$\|(i\xi - (Fg)(i\xi) - A)^{-1}\| < c.$$

Proof. We define the linear operator $R : L_p(\mathbb{R}, X) \to L_p(\mathbb{R}, X)$ where

$$R(u)(t) := \alpha M_p(sig(t)u(t) + \frac{1}{\alpha}F\widetilde{u}_t - \frac{1}{\alpha}F\widetilde{(u_t)}). \tag{4.4}$$

We note that if $u \in L^p(\mathbb{R}, X)$ then,

$$
\begin{aligned}
\|F\widetilde{u}_t - F(\widetilde{u_t})\|^p &= \int_{\mathbb{R}} \|F(e^{-\alpha|t+\cdot|}u(t+\cdot)) - F(e^{-\alpha|t|}u(t+\cdot))\|^p dt \\
&\le \|F\|^p \int_{\mathbb{R}} \|e^{-\alpha|t+\cdot|}u(t+\cdot) - e^{-\alpha|t|}u(t+\cdot)\|^p_{L^p([-r,0],X)} dt \\
&= \|F\|^p \int_{\mathbb{R}} \int_{-r}^{0} \|e^{-\alpha|t+\theta|}u(t+\theta) - e^{-\alpha|t|}u(t+\theta)\|^p d\theta dt \\
&= \|F\|^p \int_{-r}^{0} \int_{\mathbb{R}} \|e^{-\alpha|t+\theta|}u(t+\theta) - e^{-\alpha|t|}u(t+\theta)\|^p dt d\theta \\
&= \|F\|^p \int_{-r}^{0} \int_{\mathbb{R}} \|u(s)\|^p |e^{-\alpha|s|} - e^{-\alpha|s-\theta|}|^p dt d\theta \\
&\le \|F\|^p \int_{-r}^{0} \int_{\mathbb{R}} \|u(s)\|^p |e^{\alpha r} - 1|^p dt d\theta \\
&\le |e^{\alpha r} - 1|^p |r| \|F\|^p \|u\|^p_{L^p(\mathbb{R},X)},
\end{aligned}
$$

then we obtain that $sig(\cdot)u(\cdot) - \frac{1}{\alpha}F(\widetilde{u.}) + \frac{1}{\alpha}F\widetilde{u}. \in L^p(\mathbb{R}, X)$ for all $u \in L^p(\mathbb{R}, X)$ and the operator is well defined. Since M_p is a closed operator we conclude that R so is. By hypothesis, $D(M_p) = L_p(\mathbb{R}, X)$, then from the closed graph theorem R is a bounded operator. Moreover

$$
\|R\| \le \alpha \|M_p\| + \frac{1}{\alpha}|e^{\alpha r} - 1| \sqrt[p]{|r|} \|F\|, \tag{4.5}
$$

if α is small enough then $\|R\| < 1$ and we obtain that the mapping $\widetilde{u} \to \widetilde{u} + R\widetilde{u}$ is invertible. Hence the mapping

$$
\begin{aligned}
M_{p,\alpha} \quad &: \quad L^p_\alpha(\mathbb{R}, X) \to L^p_\alpha(\mathbb{R}, X) \\
&\quad\;\; M_{p,\alpha} f := \widetilde{()}^{-1} (I + R)^{-1} M_p \widetilde{f}
\end{aligned}
$$

maps each $f \in L^p_\alpha(\mathbb{R}, X)$ to the unique *mild solution* $M_{p,\alpha} f = u \in L^p_\alpha(\mathbb{R}, X)$ of equation (4.1) associated to the function f. Again with the closed graph theorem, we have that $M_{p,\alpha}$ is bounded. Now let $y \in X$ and $\xi \in \mathbb{R}$ and define

$$
f^s(t) := e^{i\xi(s+t)} y = e^{i(\xi s)} f^0(t)
$$

for all $s, t \in \mathbb{R}$ where $f^0(t) = e^{i\xi t} y$. Remark that $f^s \notin L^p(\mathbb{R}, X)$ but $f^s \in L^p_\alpha(\mathbb{R}, X)$ for all $\alpha > 0$ with

$$
\|f^s\|_{p,\alpha} = \left(\frac{2}{p\alpha}\right)^{\frac{1}{p}} \|y\|.
$$

By the considerations above, there exists for each f^s a unique *mild solution* $u^s \in L^p_\alpha(\mathbb{R}, X)$ of equation (4.1) associated to the function f^s. Now we show that for all $s \in \mathbb{R}$ it holds that $u^s(t) = u^0(s+t) = e^{i\xi s} u^0(t)$ for almost $t \in \mathbb{R}$. In fact, since u^s solves (4.2) we obtain that

$$
\begin{aligned}
e^{-i\xi s} u^s(t) - e^{-i\xi s} u^s(0) &= A \int_0^t e^{-i\xi s} u^s(r) dr + \int_0^t e^{-i\xi s} F(u^s)_r dr + \int_0^t e^{-i\xi s} f^0(r) dr \\
&= A \int_0^t e^{-i\xi s} u^s(r) dr + \int_0^t F(e^{-i\xi s} u^s(\cdot))_r dr + \int_0^t e^{-i\xi s} f^0(r) dr
\end{aligned}
$$

for all $t \in \mathbb{R}$. Hence $e^{-i\xi s} u^s$ is a *mild solution* of the equation (4.1) associated to the f^0. From the uniqueness of solutions, it follows $u^s = e^{i\xi s} u^0$. For the second equality, let $\overline{u}(t) := u^0(t+s)$.

Since u^0 is a *mild solution* of the equation (4.1), we get

$$\begin{aligned}
\overline{u}(t)-\overline{u}(0) &= u^0(s+t)-u^0(s)\\
&= A\int_0^{t+s} u^0(r)dr+\int_0^{t+s} F(u^0)_r dr+\int_0^{t+s} f^0(r)dr\\
&\quad -A\int_0^{s} u^0(r)dr+\int_0^{s} F(u^0)_r dr+\int_0^{s} f^0(r)dr\\
&= A\int_s^{t+s} u^0(r)dr+\int_s^{t+s} F(u^0)_r dr+\int_s^{t+s} f^0(r)dr\\
&= A\int_0^{t} \overline{u}(r)dr+\int_0^{t} F(\overline{u})_r dr+\int_0^{t} f^0(r)dr.
\end{aligned}$$

Again from the uniqueness of solutions it follows that $\overline{u}=u^s$ i.e $u^0(t+s)=u_s(t)$ for all $s,t\in\mathbb{R}$. Now define $z:=u^0(0)$. Then $u^0(t)=e^{i\xi t}z$ is a *mild solution* of equation (4.1) associated to the function f^0. Since $\int_0^t u^0(r)dr\in D(A)$, it follows that $z\in D(A)$. Now u^0 is differentiable and $(e^{i\xi(\cdot)}z)\in L^p([-r,0],X)$ then we have

$$\begin{aligned}
i\xi e^{i\xi t}z=u'(t) &= Au^0(t)+F(e^{i\xi(\cdot)}z)_t+f^0(t) \qquad (4.6)\\
&= Au^0(t)+Fg(i\xi)e^{i\xi t}z+f^0(t)
\end{aligned}$$

in particular, evaluating $t=0$ in the previous equation we obtain

$$i\xi z=u'(t)=Az+Fg(i\xi)z+y$$

Hence $(i\xi z-Fg(i\xi)z-Az)=y$ and $(i\xi z-Fg(i\xi)z-Az)$ is surjective since y was chosen arbitrary. To prove injectivity, suppose that exists $z\neq 0\in D(A)$ such that $Fg(i\xi)z+Az=i\xi z$, then we define $u(t)=e^{i\xi t}z\in L^p_\alpha(\mathbb{R},X)$. Then it is easy to see that u is a solution of the homogeneous equation (4.1) thus by uniqueness of solutions $z=0$. Further, since $M_{p,\alpha}$ is bounded, we obtain

$$\left(\frac{2}{p\alpha}\right)^{\frac{1}{p}}\|z\|=\|u^0\|_{p,\alpha}\leq\|M_{p,\alpha}\|\|f^0\|_{p,\alpha}=\left(\frac{2}{p\alpha}\right)^{\frac{1}{p}}\|y\|,$$

then if $c:=\|M_{p,\alpha}\|$ we conclude that for all $\xi\in\mathbb{R}$

$$\|(i\xi z-Fg(i\xi)-A)^{-1}y\|\leq c$$

□

We finish this section, studying necessary conditions when the equation (4.1) has the property of *maximal L^p-regular*. In the next we consider the solution operator for the equation (4.1) defined by

$$\begin{aligned}
D(M_{w,p}) &:= \{f\in L^p(\mathbb{R},X):\ \textit{exists a unique}\ u_f\ \textit{strong solution of the problem}\ (4.1)\}\\
M_{w,p}f &:= u_f
\end{aligned}$$

We note that, $M_{w,p}$ is a closed operator, moreover if $D(M_{w,p})=L^p(\mathbb{R},X)$ then $M_{w,p}$ is bounded, by the closed graph theorem.

For $\alpha>0$ we define the following weighted $W^{1,p}_\alpha(\mathbb{R},X)$-space

$$W^{1,p}_\alpha(\mathbb{R},X)=\{f:\mathbb{R}\to X\,f,f'\in L^p_\alpha(\mathbb{R},X)\}$$

$$\|f\|_{W^{1,p}_\alpha(\mathbb{R},X)}=\|f\|_{p,\alpha}+\|f'\|_{p,\alpha}$$

Further we define, the following mapping

$$\begin{aligned}\tilde{}: W^{1,p}_\alpha(\mathbb{R},X) &\to W^{1,p}(\mathbb{R},X) \\ u &\to \tilde{u}, \text{ where } \tilde{u}(t) := e^{-\alpha|t|}u(t)\end{aligned} \tag{4.7}$$

It is easy to see that $\tilde{}$ is an isomorphism between $W^{1,p}_\alpha(\mathbb{R},X)$ and $W^{1,p}(\mathbb{R},X)$. Particularly if $f \in D(M_{w,p})$ we obtain that $M_{w,p}f$ is a *mild solution* of equation (4.1).

The following theorem gives some necessary conditions when we have that the problem (4.1) is *maximal L^p-regular.*

Theorem 4.8. *Suppose the problem (4.1) is* maximal L^p-regular. *Then for all $\xi \in \mathbb{R}$ the operator $i\xi - (Fg)(i\xi) - A$ is invertible and there exists a constant $c > 0$ such that*

$$\|(i\xi - (Fg)(i\xi) - A)^{-1}\| < \frac{c}{1+|\xi|}.$$

Proof. The proof follows the steps of Theorem 4.7 proof. By hypothesis $D(M_{w,p}) = L^p(\mathbb{R},X)$. Given $f \in L^p_\alpha(\mathbb{R},X)$ we consider $u := M_{w,p}\tilde{f}$ which is solution of (4.1), particularly is a *mild solution* of (4.1) associated to the function f. Then it follows from lemma 4.5 and lemma 4.6,

$$\tilde{u}'_f(t) = A\tilde{u} + F(\widetilde{u_t}) + \tilde{f}(t) - \alpha sig(t)\tilde{u}(t)$$

$$M_{w,p}\tilde{f} = \tilde{u} + \alpha M_{w,p}\left(sig(\cdot)\tilde{u}(\cdot) - \frac{1}{\alpha}F(\widetilde{u.}) + \frac{1}{\alpha}F\tilde{u}.\right).$$

In view of equation (4.5), if α is small enough then $\|R\| < 1$, where $R : L^p(\mathbb{R},X) \to L^p(\mathbb{R},X)$ and

$$Ru := \alpha M_{w,p}\left(sig(\cdot)\tilde{u}(\cdot) - \frac{1}{\alpha}F(\widetilde{u.}) + \frac{1}{\alpha}F\tilde{u}.\right),$$

then we obtain that the mapping $\tilde{u} \to \tilde{u} + R\tilde{u}$ is invertible.
Now we consider the operator

$$\begin{aligned} M_{w,p,\alpha} \quad : \quad & L^p_\alpha(\mathbb{R},X) \to W^{1,p}_\alpha(\mathbb{R},X) \\ & M_{w,p,\alpha}f := \tilde{()}^{-1}(I+R)^{-1}M_p\tilde{f},\end{aligned}$$

which maps each $f \in L^p_\alpha(\mathbb{R},X)$ to the unique solution $M_{w,p,\alpha}f = u \in W^{1,p}_\alpha(\mathbb{R},X) \cap L^p_\alpha(\mathbb{R},D(A))$ of equation. (4.1). Again with the closed graph theorem, we have that $M_{w,p,\alpha}$ is bounded i.e.

$$\|u_f\|_{W^{1,p}_\alpha(\mathbb{R},X)} = \|u_f\|_{L^p_\alpha(\mathbb{R},X)} + \|u'_f\|_{L^p_\alpha(\mathbb{R},X)}x \le \|M_{w,p,\alpha}\|\|f\|_{p,\alpha}$$

Now let $y \in X$ and $\xi \in \mathbb{R}$ and define

$$f^s(t) := e^{i\xi(s+t)}y = e^{i(\xi s)}f^0(t)$$

for all $s,t \in \mathbb{R}$ where $f^0(t) = e^{i\xi t}y$. Remark that $f^s \notin L^p(\mathbb{R},X)$ but $f^s \in L^p_\alpha(\mathbb{R},X)$ for all $\alpha > 0$ with

$$\|f^s\|_{p,\alpha} = \left(\frac{2}{p\alpha}\right)^{\frac{1}{p}}\|y\|.$$

By the considerations above, there exists for each f^s a unique solution $u^s \in W^{1,p}_\alpha(R,X) \cap L^p_\alpha(\mathbb{R},D(A))$ of equation (4.1) associated to the function f. Similar as in the proof of Theorem 4.7 one can show that that for all $s \in \mathbb{R}$ it holds that $u^s(t) = u^0(s+t) = e^{i\xi s}u^0(t)$ for almost $t \in \mathbb{R}$. Now define $z := u^0(0)$. Then $u^0(t) = e^{i\xi t}z$ is a solution of equation (4.1), then we have

$$\begin{aligned} i\xi e^{i\xi t} z = u'(t) &= Au^0(t) + F(e^{i\xi(\cdot)}z)_t + f^0(t) \\ &= Au^0(t) + Fg(i\xi)e^{i\xi t}z + f^0(t) \end{aligned} \tag{4.8}$$

in particular, evaluating $t = 0$ in the previous equation we obtain

$$i\xi z = u'(t) = Az + Fg(i\xi)z + y$$

Hence $(i\xi z - Fg(i\xi)z - Az) = y$ and $(i\xi z - Fg(i\xi)z - Az)$ is surjective since y was chosen arbitrary.
To prove injectivity, suppose that exists $z \neq 0 \in D(A)$ such that $Fg(i\xi)z + Az = i\xi z$, then we define $u(t) = e^{i\xi t}z \in L^p_\alpha(\mathbb{R}, X)$. Then it is easy to see that u is a solution of the homogeneous equation (4.1) thus by uniqueness of solutions $z = 0$. Finally, since $M_{w,p,\alpha}$ is bounded, we obtain

$$\begin{aligned} \left(\frac{2}{p\alpha}\right)^{\frac{1}{p}} (1+|\xi|)\|(i\xi - Fg(i\xi) - A)^{-1}y\| &= \left(\frac{2}{p\alpha}\right)^{\frac{1}{p}} (|\xi|\|z\| + \|z\|) \\ &= \|u_f\|_{W^{1,p}_\alpha(\mathbb{R},X)} \leq \|M_{w,p,\alpha}\|\|f\|_{p,\alpha} = \left(\frac{2}{p\alpha}\right)^{\frac{1}{p}} \|M_{w,p,\alpha}\|\|y\|, \end{aligned}$$

thus choosing $c := \|M_{w,p,\alpha}\|$ it follows that

$$\|(i\xi - Fg(i\xi) - A)^{-1}y\| \leq \frac{c}{1+|\xi|}.$$

□

Remark 4.9. We note that if the operator $F \equiv 0$ we recover de Mielk theorem [45, Satz 2.2].

Remark 4.10. We denote

$$\rho(A,F) = \{\lambda \in \mathbb{C} : (\lambda - Fg(\lambda) - A) \text{ is an invertible and bounded operator}\},$$

and for $\lambda \in \rho(A,F)$ we denote $R(\lambda, A, F) := (\lambda - Fg(\lambda) - A)^{-1}$. We note that if $F \equiv 0$ then $\rho(A,F) = \rho(A)$, where $\rho(A)$ is the resolvent of the operator A and $R(\lambda, A, F) = R(\lambda, A) = (\lambda - A)^{-1}$. Theorem 4.8 can be read in the followings sense. If the problem (4.1) is *maximal L^p-regular*, then $i\mathbb{R} \subseteq \rho(A,F)$ and $\|R(i\xi, A, F)\| \leq \frac{c}{1+|\xi|}$ for all $\xi \in \mathbb{R}$.

We want to look more carefully $\rho(A,F)$.

Proposition 4.11. *Suppose that $i\mathbb{R} \subseteq \rho(A,F)$ and there exists $c > 0$ such that $\|R(\lambda, A, F)\| \leq \frac{c}{1+|\xi|}$. Then, there exists $b > 0$ such that,*

$$V_b := \{z \in \mathbb{C} : |\Re z| \leq \frac{1}{b}(1 + |\Im z|)\} \subseteq \rho(\lambda, A, F)$$

and $\|(\lambda - Fg(\lambda) - A)^{-1}\| \leq \frac{b}{1+|z|}$ for all $\lambda \in V_b$.

Proof. Let $R(\lambda) := R(\lambda, A, F)$, then we have

$$\begin{aligned} R(\lambda) &= R(\lambda)R^{-1}(i\xi)R(i\xi) = [(\lambda - Fg(\lambda) - A)R(i\xi)]^{-1}R(i\xi) \\ &= [(\lambda - i\xi + Fg(i\xi) - A - Fg(\lambda) + i\xi - Fg(i\xi))R(i\xi)]^{-1}R(i\xi) \\ &= [I + ((\lambda - i\xi + Fg(i\xi) - Fg(\lambda))R(i\xi)]^{-1}R(i\xi). \end{aligned}$$

Now let $\lambda = \nu + i\xi \in \mathbb{C}$ where $\nu = \Re\lambda$ and $\xi = \Im\lambda$. Then by hypothesis we have

$$\|((\lambda - i\xi + Fg(i\xi) - Fg(\lambda))M(i\xi)\| \quad \le \quad \frac{c}{1+|\xi|}[|\nu| + \|F\|\|1 - e^{\nu\theta}\|_{L^p([-r,0],X)}], \tag{4.9}$$

on the other hand using $|e^x - 1| \le |x|e^x + |x|$, for all $x \in \mathbb{R}$ we obtain,

$$\|1 - e^{\nu\theta}\|_{L^p([-r,0],X)} \le \|e^{(\cdot)\nu}|(\cdot)\nu| + |(\cdot)\nu|\|_{L^p([-r,0],X)} \le |\nu|\sqrt[p]{|r|} + |\nu|. \tag{4.10}$$

Then its follows from equations (4.9), (4.10), that $R(\lambda)$ is invertible if

$$\frac{c}{1+|\xi|}[|\nu| + \|F|\nu|(\sqrt[p]{|r|} + 1) < 1$$

Hence if $b > c(1+\|F\|(\sqrt[p]{|r|}+1))$ then for all $\lambda \in V_b$, $R(z)$ is invertible. Now if $\widetilde{c} := 1+\|F\|(\sqrt[p]{|r|}+1)$ we obtain the following estimate:

$$\begin{aligned}
\|R(\lambda)\| &\le \|[I + ((\lambda - i\xi + Fg(i\xi) - Fg(\lambda))R(i\xi)]^{-1}\|\|R(i\xi)\| \\
&\le \frac{1}{1 - \frac{c\widetilde{c}|\nu|}{1+|\xi|}}\frac{c}{1+|\xi|} \le \frac{1}{1 - \frac{c\widetilde{c}}{b}}\frac{c}{1+|\xi|} \le \frac{1}{1 - \frac{c\widetilde{c}}{b}}\frac{c}{1+|\xi|}\frac{1+|\nu|+|\xi|}{1+|\lambda|} \\
&= \frac{c}{1 - \frac{c\widetilde{c}}{b}}\frac{1+|\nu|+|\xi|}{1+|\xi|}\frac{1}{1+|\lambda|} \le \frac{c}{1 - \frac{c\widetilde{c}}{b}}(\frac{1}{b} + 1)\frac{1}{1+|\lambda|}.
\end{aligned}$$

Then we choose $b = \frac{c(1+\widetilde{c})+\sqrt{(c(1+\widetilde{c})^2)+4c^2}}{2}c$ which is a positive solution of equation $b = \frac{c}{1-\frac{c\widetilde{c}}{b}}(\frac{1}{b}+1)$ also $b > c\widetilde{c} = c(1+\|F\|(\sqrt[p]{|r|}+1))$. Then under the above choice of b we obtain that for all $\lambda \in V_b$, $(\lambda - Fg(\lambda) - A)$ is invertible and

$$\|(\lambda - Fg(\lambda) - A)^{-1}\| \le \frac{b}{1+|\lambda|}$$

□

Remark 4.12. Suppose that $F \equiv 0$. If the problem (4.1) is *maximal L^p-regular*, then there exists b such that $V_b \subset \rho(A)$ and $\|R(\lambda, A)\| \le \frac{b}{1+|b|}$ for all $\lambda \in V_b$. This means that the operator A is a bisectorial invertible operator see [55, pag. 17].

4.3. Maximal Regularity

In the next, we study sufficient conditions to ensure that the equation (4.1) has the property of *maximal L^p-regular*. Let $\mathscr{F}$ be the Fourier transform on $L^1(\mathbb{R}, X)$ defined by

$$\mathscr{F}u(s) = \int_{\mathbb{R}} e^{-st}u(t)dt, \quad s \in \mathbb{R}, \tag{4.11}$$

for all $u \in L^1(\mathbb{R}, X)$.

We denote by $\mathscr{D}(\mathbb{R}, X)$ the space of X-valued C-functions with compact support and let $\mathscr{D}'(\mathbb{R}, X) = \mathscr{L}(\mathscr{D}(\mathbb{R}, X), X)$ be the space of X-valued distributions. Similarly, let $\mathscr{S}(\mathbb{R}, X)$ the Schwartz space of smooth rapidly decreasing X-valued functions on $\mathbb{R}$ and $\mathscr{S}'(\mathscr{S}, X) = \mathscr{L}(\mathscr{D}(\mathbb{R}, X), X)$.

Then, given a function $M \in L^1_{loc}(\mathbb{R}, \mathscr{L}(X))$, we may define the pseudo-diferential operator $M(D) : \mathscr{F}^{-1}\mathscr{D}(\mathbb{R}, X) \to \mathscr{D}'(\mathbb{R}, X)$ by

$$M(D)\phi = \mathscr{F}^{-1} M \mathscr{F} \phi, \tag{4.12}$$

for all $\phi \in \mathscr{D}(\mathbb{R}, X)$. Since $\mathscr{F}^{-1}\mathscr{D}(\mathbb{R}, X)$ is dense in $L^p(\mathbb{R}, X)$, $M(D)$ is defined on a dense subset of $L^p(\mathbb{R}, X)$.

Now, one can ask what conditions have to be imposed on M so that $M(D)$ extends to a bounded linear operator on $L^p(\mathbb{R}, X)$. In the scalar case, the famous Mikhlin multiplier theorem gives a satisfactory answer ([46], see also [21]). This theorem was then extended by Bourgain [9], McConnell [44] and Zimmermann [65] to a vector-valued version on UMD spaces. Pisier proved in [50] that the operator-valued case of Mikhlin's theorem is only valid in Hilbert spaces. But recently a new operator-valued version was found by Weis ([62] and [63], see also [12] and [50]) in which boundedness of the multiplier function is replaced by $\mathscr{R}$-boundedness. This version is also useful to study the first-order equation on the line, but first, we will give the definition of $\mathscr{R}$-boundedness (see [50]).

A family $\mathscr{M} \subset \mathscr{L}(X)$ of bounded, linear operators on a Banach space X is called $\mathscr{R}$-bounded if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, all elements $x_j \in X$, selections $T_j \in \mathscr{M}$ $(1 \leq j \leq n)$ and n independent symmetric $\{1;1\}$-valued random variables ϵ_j on a probability space (Ω, Σ, μ) the following inequality holds:

$$\| \sum_{j=1}^{n} \epsilon_j T_j x j \|_{L^1(\Omega, X)} \leq C \| \sum_{j=1}^{n} \epsilon_j x j \|_{L^1(\Omega, X)}. \tag{4.13}$$

The $\mathscr{R}$-bound of the family $\mathscr{M}$ is given by

$$\mathscr{R}(\mathscr{M}) := \inf\{C \geq 0 : (4.13) \text{ holds } \}. \tag{4.14}$$

We turn now to the operator-valued version of Mikhlin's theorem due to Weis ([124],[125], see also [39]).

Theorem 4.13. *Let X be a UMD space and $1 < p < \infty$. Suppose $M(D) \in C^1(\mathbb{R}\backslash\{0\}, \mathscr{L}(X))$ such that the following conditions are satisfied:*

1. $\mathscr{R}(\{M(s) : s \in \mathbb{R}\backslash\{0\}\}) < \infty$
2. $\mathscr{R}(\{sM'(s) : s \in \mathbb{R}\backslash\{0\}\}) < \infty$

Then, the pseudo-differential operator $M(D)$ defined by (4.12) can be extended to a bounded, linear operator on $L^p(\mathbb{R}; X)$.

If we apply the above Theorem 4.13 to equation (4.1), we obtain the following criterion for maximal regularity on UMD spaces.

Theorem 4.14. *Let X be a UMD space. Suppose that $i\mathbb{R} \subseteq \rho(A, F)$ and $\|R(i\xi, A, F)\| \leq \frac{c}{1+|\xi|}$ for all $\xi \in \mathbb{R}$. If the set $\{isR(is, A, F) : \mathbb{R}\backslash\{0\}\}$ is $\mathscr{R}$-bounded. Then the problem (4.1) is* maximal L^p-regular.

Proof. Let $f \in \mathscr{F}^{-1}\mathscr{D}(\mathbb{R}, X)$. If we take Fourier transform of Equation (4.1), we obtain

$$is\mathscr{F}u(s) = A\mathscr{F}u(s) + Fe^{\lambda(\cdot)}\mathscr{F}u(s) + \mathscr{F}f(s) \; or \; \mathscr{F}u(s) = (\lambda - Fg(\lambda) - A)^{-1}\mathscr{F}f(s)$$

Let, $M(s) := R(is, A, F)$, then we have

$$M(s) - M(s') = M(s)(is' - is - (Fg(is') - Fg(is))M(s)$$

then we obtain that

$$M'(s) = iM(s)(I - F\left(\frac{d}{ds}g(is)\right))M(s),$$

and we conclude that $M \in C^1(\mathbb{R},\mathscr{L}(X))$. It follows from the hypothesis $\{isR(is,A,F) : \mathbb{R}\backslash\{0\}\}$ is $\mathscr{R}$-bounded and the fact that $\sup_{s\in\mathbb{R}} \|M(s)\| < \infty$, $\sup_{s\in\mathbb{R}} \|F\left(\frac{d}{ds}g(is)\right)\| < \infty$ that the conditions (1), (2) of Theorem 4.13 are satisfies and consequently we obtain that $Mf \in L^p(\mathbb{R},X)$. On the other hand, if we consider $\overline{M} = AM$ we have,

$$AM(s) = isM(s) - Fg(is)M(s) - I,$$

furthermore

$$\overline{M}'(s) = iM(s) + isM'(s) - iF\left(\frac{d}{ds}g(is)\right)M(s) - Fg(is)M'(s),$$

the above equality implies that $\overline{M} \in C^1(\mathbb{R},X)$ and $\overline{M}$ satisfies the conditions (1), (2) of Theorem 4.13. Then we conclude that $AMf \in L^p(\mathbb{R},X)$ or $Mf \in D(A)$. Let $f \in L^p(\mathbb{R},X)$, it follows from Theorem 4.13 that the operator M can be extended to $L^p(\mathbb{R},X)$. We define $u = Mf$, by closeness of operator A we obtain $Mf \in L^p(\mathbb{R},D(A)) \cap W^{1,p}(\mathbb{R},X)$ and is an strong solution of equation (4.1).

In order to prove uniqueness, suppose that

$$u'(t) = Au(t) + Fu_t, \quad t \in \mathbb{R}, \tag{4.15}$$

where $u \in W^{1,p}(\mathbb{R},X) \cap L^p(\mathbb{R},D(X))$.

We claim that $\hat{u}_{\cdot} \in L^p([-r,0],X)$. In fact, if $\Re\lambda > 0$

$$\begin{aligned}
\left\|\int_0^\infty e^{-\lambda t}u_t dt\right\|^p_{L^p([-r,0],X)} &= \int_{-r}^0 \left\|\int_0^\infty e^{-\lambda t}u(t+\theta)dt\right\|^p d\theta = \int_{-r}^0 \left\|\int_\theta^\infty e^{-\lambda(s-\theta)}u(s)ds\right\|^p d\theta \\
&\le \int_{-r}^0 \left(\int_\theta^\infty e^{-\Re\lambda(s-\theta)}\|u(s)\|ds\right)^p d\theta \\
&\le \int_{-r}^0 \left(\left(\int_\theta^\infty e^{-\Re\lambda(s-\theta)q}ds\right)^{\frac{1}{q}} \left(\int_\theta^\infty \|u(s)\|^p ds\right)^{\frac{1}{p}}\right)^p d\theta \\
&\le \|u\|_p^p \int_{-r}^0 \left(\int_\theta^\infty e^{-\Re\lambda(s-\theta)q}ds\right)^{\frac{1}{q}} d\theta \\
&\le \|u\|_p^p \int_{-r}^0 (\int_{-r}^\infty e^{-\Re\lambda(s-\theta)q}ds)^{\frac{1}{q}} d\theta < \infty,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then we obtain $\hat{u}_{\cdot} \in L^p([-r,0],X)$ for $Re\lambda > 0$. Analogously we obtain the claim for $\Re\lambda < 0$. For $\Re\lambda \neq 0$, with an easy computation, we obtain the Carleman transforms

$$\hat{u}_{\cdot}(\lambda) = g\hat{u}(\lambda) + gh, \tag{4.16}$$

where $g(\theta) = e^{\lambda\theta}$ and $h(\theta) = \int_\theta^0 e^{-\lambda t}u(t)dt$ where $\theta \in [-r,0]$. Note that $gh \in L^p(\mathbb{R},X)$. Since F is bounded, we obtain from (4.16) that

$$\widehat{Fu.}(\lambda) = F\hat{u}.(\lambda) = Fg\hat{u}(\lambda) + Fgh. \tag{4.17}$$

Since, $\hat{u}'(\lambda) = \lambda\hat{u}(\lambda) - u(0)$ for $\Re\lambda \neq 0$. Using the fact that A is closed, from (4.17) and (4.15) it follows

$$(\lambda - Fg(\lambda) - A)\hat{u}(\lambda) = Fgh + u(0),$$

for all $\lambda \in \mathbb{C}\backslash i\mathbb{R}$, it follows that the Carleman spectrum $sp_C(u)$ of u is empty. Hence $u \equiv 0$ by [4, Theorem 4.8.2] □

Corollary 4.15. Let X be a UMD space. Suppose that $i\mathbb{R} \subseteq \rho(A,F)$ and $\|R(i\xi,A,F)\| \le \frac{c}{1+|\xi|}$ for all $\xi \in \mathbb{R}$. If the set $\{R(is,A,F) : \mathbb{R}\backslash\{0\}\}$ is $\mathscr{R}$-bounded. Then the problem (4.1) is *mild maximal L^p-regular.*

Since bounded sets correspond to $\mathscr{R}$-bounded sets if, and only if, X is a Hilbert space, we get as a simple consequence from the previous theorem that the necessary condition of Theorem 4.8 is also sufficient if the underlying Banach space is a Hilbert space

Corollary 4.16. Let X be a Hilbert space. Then the problem (4.1) is *maximal L^p-regular* if and only if $i\mathbb{R} \subseteq \rho(A,F)$ and there exists $c > 0$ such that $\|R(i\xi,A,F)\| < \frac{c}{1+|\xi|}$ for all $\xi \in \mathbb{R}$.

Now we give the main result of this work.

Theorem 4.17. *Assume that X is a UMD-space. Let $1 < p < \infty$. The following assertion are equivalent.*

1. *Problem (4.1) is* maximal L^p-regular.
2. *$i\mathbb{R} \subseteq \rho(A,F)$ and $\|R(i\xi,F,A)\| \le \frac{c}{1+|\xi|}$ and the set $\{isR(i\xi,A,F) : \xi \in \mathbb{R}\setminus\{0\}\}$ is $\mathscr{R}$-bounded.*

Proof. (2) ⇒ (1) is given by Theorem 4.14. (1) ⇒ (2) Assume that the problem (4.1) is *maximal L^p-regular.* Then by Theorem 4.8, one has $i\mathbb{R} \subseteq \rho(A,F)$. Then by the Closed Graph Theorem the operator solution $M_{w,p} : L^p(\mathbb{R},X) \to W^{1,p}(\mathbb{R},X) \cap L^p(\mathbb{R},D(A))$ is a bounded operator. We claim that if $f \in \mathscr{F}^{-1}\mathscr{D}(\mathbb{R},X)$ then $M_{w,p}f \in \mathscr{S}(\mathbb{R},D(A))$. In fact, let $u := M_{w,p}f \in W^{1,p}(\mathbb{R},X) \cap L^p(\mathbb{R},D(A))$ and $\phi \in \mathscr{S}(\mathbb{R})$. Then

$$\begin{aligned}
\int_{\mathbb{R}} \phi(s)R(is,A,F)\hat{f}(s)ds &= \int_{\mathbb{R}} \phi(s)R(is,A,F)\int_{\mathbb{R}} f(t)e^{-ist}dtds \\
&= \int_{\mathbb{R}} \phi(s)R(is,A,F)\int_{\mathbb{R}} (u'(t) - Fu_t - Au(t))e^{-ist}dtds \\
&= \int_{\mathbb{R}} \phi(s)R(is,A,F)(is - Fg(is) - A)\hat{u}(s)ds \\
&= \int_{\mathbb{R}} \phi(s)\hat{u}(s)ds = \int_{\mathbb{R}} \phi(s)\int_{\mathbb{R}} e^{-ist}u(t)dtds = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(s)e^{-ist}u(t)dsdt \\
&= \int_{\mathbb{R}} \hat{\phi}(t)u(t)dt
\end{aligned}$$

We recall that we may identify $L^p(\mathbb{R},D(A))$ with a subspace of $\mathscr{S}'(R,D(A))$ by letting

$$< v,\phi >= \int_{\mathbb{R}} v(t)\phi(t)dt$$

for $v \in L^p(\mathbb{R},D(A))$, $\phi \in S(\mathbb{R})$. Thus, the identity above, says that $\mathscr{F}u = R(i\cdot,A,F)\hat{f}(\cdot) \in \mathscr{D}(\mathbb{R},D(A))$. Hence $u \in \mathscr{S}(\mathbb{R},D(A))$. Thus the function M with values in $\mathscr{L}(\mathbb{R},D(A))$ given by $M(s) = R(is,A,F)$ is an $L^p(\mathbb{R},X) \to L^p(\mathbb{R},D(A))$ multiplier. It follows from Clément-Prüss [11] se also [29], that the set $\{M(s), s \in \mathbb{R}\} \subseteq \mathscr{L}(X,D(A))$ is $\mathscr{R}$-Bounded. Since $F - A : D(A) \to X$ is an isomorphism, the set $\{(F-A)M(s), s \in \mathbb{R}\}$ is $\mathscr{R}$-bounded. This implies that

$$(F-A)M(s) = FM(s) - AM(s) = FM(s) - isM(s) + Fg(is)M(s) + I,$$

then

$$isM(s) = FM(s) - (F-A)M(s) + Fg(is)M(s) + I,$$

using the fact that the operators F and $Fg(i\cdot)$ are bounded we obtain that the set $\{isM(s), s \in \mathbb{R}\}$ is $\mathscr{R}$-bounded. This completes the proof. □

4.4. Examples and comments

In this section we compare the equation (4.1) with the the differential equation

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}. \tag{4.18}$$

We says A is $\mathscr{R}$-bisectorial if A is bisectorial and the set

$$\{sR(is, A) : s \in R\backslash\{0\}\}$$

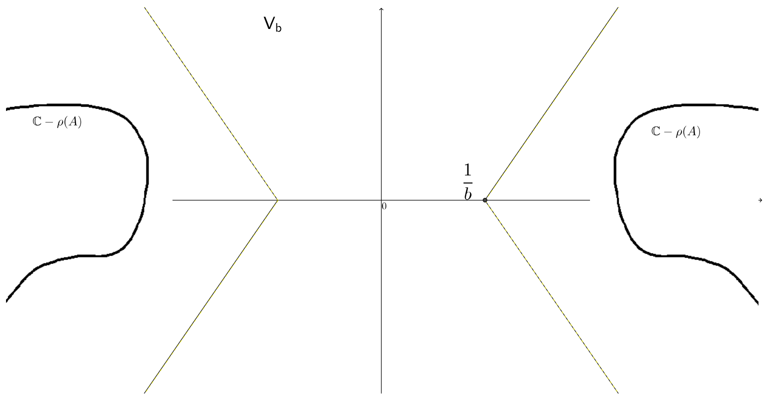
is $\mathscr{R}$-bounded. For more information about the bisectorial operator see [55, page 17–19]. The following lemma is a characterization of the invertible bisectorial operator.

Lemma 4.18. [55, lemma 1.1.2] For a closed linear operators A, the following are equivalent:

(i) The operator A is bisectorial with $0 \in \rho(A)$.

(ii) There exists a constant b > 0 such that $V_b := \{z \in \mathbb{C} : |\Re(z)| \leq \frac{1}{b}(1 + |\Im(z))|\} \subset \rho(A)$ and $\|R(z)\| \leq \frac{b}{1+|z|}$ for all $z \in V_b$.

This setting is described in the following picture



We have the following result.

Theorem 4.19. *[5, Theorem 2.4] Assume that X is a UMD-space. Let* $1 < p < \infty$. *The following assertions are equivalent*

1. *Problem (4.18) is* maximal L^p-regular.

2. *A is $\mathscr{R}$-bisectorial and invertible.*

Remark 4.20. If A is bisectorial and invertible then there exists $c > 0$ such that

$$\|R(i\xi, A)\| \leq \frac{c}{1 + |\xi|}, \quad \textit{for all } \xi \in \mathbb{R}.$$

Lemma 4.21. Suppose that the problem (4.18) is *maximal L^p-regular*. If $\|F\| \leq \frac{1}{c}$ then the problem (4.1) is *maximal L^p-regular.*

Proof. By Theorem 4.19 we have that $i\mathbb{R} \subseteq \rho(A)$ and there exists $c > 0$ such that $\|R(i\xi, A)\| \leq \frac{1}{1+|\xi|}$, for all $\xi \in \mathbb{R}$.

$$\begin{aligned} R(i\xi, A, F) &= (i\xi - Fg(i\xi) - A)^{-1} \\ &= (I - (i\xi - A)^{-1} Fg(i\xi))^{-1}(i\xi - A)^{-1}, \end{aligned}$$

then we have that $i\mathbb{R} \subseteq \rho(A, F)$ and using $0 < 1 - c\|F\| < 1$ we obtain

$$\|R(i\xi, A, F)\| \leq \frac{c}{1 - c\|F\| + |\xi|} \leq \frac{c}{(1 - c\|F\|) + (1 - c\|F\|)|\xi|} = \frac{C'}{1 + |\xi|},$$

where $C' = \frac{c}{1-c\|F\|} > 0$. On the other hand the set

$$\{isR(is, A, F) : s \in \mathbb{R} \setminus \{0\}\} = \{(I - (i\xi - A)^{-1} Fg(i\xi))^{-1} isR(is, A) s \in \mathbb{R} \setminus \{0\}\},$$

then it follows from $(I - (i\xi - A)^{-1} Fg(i\xi))^{-1} : X \to X$ is an isomorphic we obtain that $\{isR(is, A, F) : s \in \mathbb{R} \setminus \{0\}\}$ is $\mathscr{R}-bounded$. Then it follows from Theorem 4.17 that the problem (4.1) is *maximal* L^p*-regular.* □

Example 4.22. Consider a bisectorial closed and invertible operator A and a Hilbert space X. For $a > 0$ we define $F : L^p([-r, 0], X) \to X$ by

$$F(v(\cdot)) = a \int_{-r}^{0} v(s) ds$$

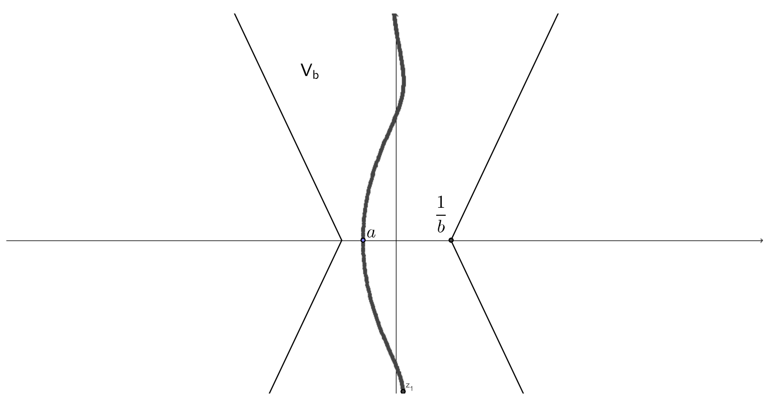
and we consider the equation

$$u'(t) = Au(t) + Fu_t + f(t) \qquad t \in \mathbb{R}, \tag{4.19}$$

where $f \in L^p(\mathbb{R}, X)$.
We will establish a connection between $\rho(A)$ and the operator F to ensure that the above equation has the property of maximal L^p-regular. In view of corollary (4.16) we need that $i\mathbb{R} \subset \rho(A, F)$ and ensure that there exist $c > 0$ such that $R(is, A, F) < \frac{c}{1+|s|}$ for all $s \in \mathbb{R}$. We observe that

$$(is - Fg(is) - A)^{-1} = (is + a\frac{e^{-isr} - 1}{is} - A)^{-1}$$

In view of the lemma 4.18, if we plot the set $\{s \in \mathbb{R} \setminus \{0\} : is + a\frac{e^{-isr}-1}{is}\}$ in the complex plane we obtain the following picture.



We observe that if $\frac{1}{b} > a$ we obtain that $i\mathbb{R} \subset \rho(A,F)$. On the other hand

$$\begin{aligned} \|R(is,A,F)\| &= \|R(is+a\frac{e^{-isr}-1}{is},A)\| \\ &\leq \frac{c}{1+|is+a\frac{e^{-isr}-1}{is}|} \\ &= \frac{c}{1+\frac{\sqrt{s^4-2s^2a\cos(sr)+2s^2a+2a^2-2a^2\cos(sr)}}{|s|}} \\ &\leq \frac{c}{1+|s|}. \end{aligned}$$

In resume the unique condition that we need to ensure the property of maximal L^p-regular of the problem (4.19) under initial conditions is that

$$\frac{1}{b} > a.$$

However if we use lemma 4.21 to ensure the maximal L^p-regular for the problem (4.19) probably we will estimate the norm of operator F. In fact $\|F\| < a\sqrt[q]{r}$ (where $\frac{1}{q}+\frac{1}{p}=1$) then if

$$a\sqrt[q]{r} < \frac{1}{c}$$

we obtain that the equation (4.19) has the property of maximal L^p-regular. The first method is more exactly because its does not depend of the delay r.

What happens if the space X is not a Hilbert space? In this case we have to ensure that the set $\{R(is,A,F), s\in\mathbb{R}\setminus\{0\}\}$ is $\mathscr{R}$-bounded using the condition that the set $\{R(is,A) : s\in\mathbb{R}\setminus\{0\}\}$ so is, for example. The question about. What conditions we need to be imposed to ensure that the set $\{R(is,A,F), s\in\mathbb{R}\setminus\{0\}\}$ is $\mathscr{R}$-bounded ? is opened.

Definition 4.23. The operator A is called sectorial if $(-\infty,0)\subset\rho(A)$ and $\sup_{\lambda>0}\|\lambda(\lambda+A)^{-1}\|<\infty$. If the set $\{\lambda(\lambda+A)^{-1} : \lambda>0\}$ is even $\mathscr{R}$-bounded, the we call $\mathscr{R}$-sectorial.

Consider the second order equation with finite delay,

$$u''(t) = Au(t) + Fu_t + f(t) \quad t\in\mathbb{R}, \tag{4.20}$$

where A an operator $\mathscr{R}$-sectorial.
Assume that A is densely defined sectorial and invertible. Then $-A^{1/2}$ generates a bounded holomorphic C_0-semigroup. In particular, $A^{1/2}$ is sectorial as well and invertible. We consider $V := D(A^{1/2})$ as a Banach space with the graph norm. Then $A^{1/2} : D(A^{1/2}) \to X$ is an isomorphism. Consider the Banach space $\mathscr{X} = V\times X$. Then the above problem can be viewed by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \mathscr{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \mathscr{F}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \quad t\in\mathbb{R},$$

where $\mathscr{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ and $\mathscr{F} = \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}$.
Arendt and Duelli has proved in [5, Proposition 3.3] that if A is R-sectorial, then $\mathscr{A}$ is $\mathscr{R}$-bisectorial. Now we use the lema 4.21 for the next example.

Example 4.24. (elliptic equations with a finite delay). Let $1 < p, q < \infty$ and let $A = (-\triangle_q + I)$ on $L^q(\mathbb{R}^N)$ with domain $D(A) = W^{2,q}(\mathbb{R}^N)$. Consider the equation,

$$u''(t) = -\triangle_q u(t) + u(t) + Fu_t + f(t), t \in \mathbb{R}. \tag{4.21}$$

where F is considered in the example 4.22. The above equation is equivalent to the equation,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \mathscr{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \mathscr{F} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \quad t \in \mathbb{R},$$

where $\mathscr{X} = V \times X$, $\mathscr{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ and $\mathscr{F} = \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}$. The operator A is $\mathscr{R}$-sectorial and by [5, Proposition 3.3] we obtain that the operator $\mathscr{A}$ is $\mathscr{R}$-bisectorial. Then if we use the lemma 4.21 we obtain that if the constant a is small enough ($ar^{\frac{2p}{p-1}} < \frac{1}{c}$) then the equation 4.21 has the property of maximal L^p-regular.

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