Bounded Solutions to Evolution Equations in Banach Spaces

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Contents

Chapter 1

Introduction

The main purpose of this thesis is the study of existence, uniqueness, regularity and qualitative properties of bounded solutions for some classes of abstract evolution equations in Banach spaces.

In the last years, the theory of (maximal) regularity for linear evolution equations in abstract spaces, has been applied to the study of solutions to nonlinear partial differential equations. See for instance, Amann [3, 4] and Denk-Hieber-Prüss [33], for more details. In the study of this theory, an useful tool to obtain characterizations of maximal regularity is based in some recent results in Fourier multipliers of operational type in abstract spaces. See Arendt-Bu [8, 9] (and the references therein). Using this results, we study in [66] and [69], characterizations of maximal regularity for two class of linear differential equations in periodic Lebesgue, Besov, Hölder and Triebel-Lizorkin vector-valued spaces. In the case of Lebesgue spaces, our results involve a geometrical condition on the underlying Banach space, whereas that in the Besov, Hölder and Triebel-Lizorkin spaces this geometric condition is not needed.

On the other hand, is well known the recent interest to obtain sufficient conditions to guarantee the existence and uniqueness of bounded solutions to some classes of differential equations, where the involved input data in the equations are, for instance, of type almost periodic or almost automorphic. See [64, 75, 77, 78, 80, 83], for further information. In [67], we study a class of integro-differential equation, and using a representation of the solution by means of a resolvent family, we obtain conditions to ensure the existence and uniqueness of solutions of the above mentioned type, among others. On the other hand, in [68] we consider a class of Volterra equation, where the input data of the equation are S^p-almost automorphic and we give sufficient conditions that guarantee the existence and uniqueness of mild solutions on the almost automorphic class. Finally, in [85] we consider a semilinear fractional differential equation, where the fractional derivative is understood in the Weyl's sense and we give sufficient conditions that ensure the existence and uniqueness of mild solutions on the almost automorphic and almost periodic class, among others.

In what follows, we will give a description of each chapter of this thesis.

In chapter 2, we summarize the preliminaries used in the thesis and we fix some notation.

In chapter 3, we are interested in the maximal regularity of solutions to the equation

$$\frac{d}{dt}(Mu(t)) = Au(t) + f(t), \qquad 0 \leq t \leq 2\pi, \tag{1.1}$$

where $(A, D(A))$ and $(M, D(M))$ are (unbounded) closed linear operators on a Banach space X, with $D(A) \subseteq D(M)$. The model (1.1), in case that $A = \Delta$ is the Laplacian and $M = m$ is the multiplication operator by a function $m(x)$, was first considered by Carroll and Showalter [23] and has been recently studied by Marinoschi [70]. This model describes, for example, the infiltration of water in unsaturated porous media, in which saturation might occur. The function m characterizes the porosity of the nonhomogeneous medium, while the fact that m is zero indicates the existence of impermeable intrusions in the soil. A study of solutions for this model, with $m(x) = 1$ and periodic conditions was made in [71] in case of a nonlinear convection, in connection with some results given in [52]. An interesting analysis of periodic solutions to a nonlinear model consisting in a degenerate diffusion equation of the form (1.1) with homogeneous Dirichlet boundary conditions, where A is a multivalued linear operator, has been given recently in the paper [42].

A detailed study of linear abstract degenerate differential equations, using both the semigroups generated by multivalued (linear) operators and extensions of the operational method from Da Prato and Grisvard has been described in the monograph [45].

Regularity of solutions in various vector-valued function spaces for the abstract equation

(1.1) with periodic conditions

$$Mu(0) = Mu(2\pi), \tag{1.2}$$

using the sum method have been studied in [10]. The obtained results gives sufficient conditions for periodicity, but leaves as an open problem to *characterize the maximal regularity* in terms of hypothesis of the modified resolvent operator $(\lambda M - A)^{-1}$ of the operators M and A.

On the other hand, Arendt and Bu [8], using operator-valued Fourier multiplier theorems, have derived spectral characterizations of maximal regularity in Lebesgue spaces for the equation (1.1) with $M = I$, the identity operator, and periodic conditions. Similar characterizations were then obtained for the scale of Besov spaces [9] and subsequently, the scale of Triebel-Lizorkin [17] spaces. See also [58] and references therein. This connection motivates the question whether it is possible to obtain a similar characterization for the problem (1.1)-(1.2). We note that, starting with the work [8], the problem of characterization of maximal regularity for evolution equations with periodic conditions have been studied intensively in the last years. See e.g. [14], [15], [16], [58], [63], [84] and references therein. For one side, the main novelty in this chapter relies in the presence of two non-commuting operators A and M, that are only related by the domain. There are only few papers dealing with this situation (see [65]), and the contents of this chapter can be considered as a progress on the treatment of such kind of problems. On the other side, our approach give immediate application to degenerate evolution equations, arising from applications. We notice that the results of this chapter has been recently published by the author in [66].

This chapter is organized as follows: In the first section of this chapter, we obtain a characterization for the existence and uniqueness of a strong L^p-solution for the problem (1.1)-(1.2) solely in terms of a property of boundedness for the sequence of operators $ikM(ikM - A)^{-1}$, under some kind of geometrical assumption on the Banach space X. We remark that no additional assumption on the operator A is required. In the next section, we prove a characterization is the context of Besov spaces. We notice that in this case an additional hypothesis on X is not longer required. In the particular case of Hölder spaces $C^s((0, 2\pi); X)$, $0 < s < 1$, we obtain that the following assertions are equivalent in general Banach spaces, provided $D(A) \subset D(M)$:

1. $ikM - A$ is bijective for all $k \in \mathbb{Z}$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - A)^{-1}|| < \infty$.

2. For every $f \in C^s((0, 2\pi); X)$ there exist a unique function $u \in C^s((0, 2\pi); D(A))$ such that $Mu \in C^{s+1}((0, 2\pi); X)$ and (1.1)-(1.2) holds for a.e. $t \in [0, 2\pi]$.

We remark that this result extends and improves [10, Theorem 2.1]. Finally, some concrete examples are examined in the last section.

In chapter 4, we study maximal regularity in Lebesgue, Besov and Triebel-Lizorkin vector-valued function spaces for the following class of differential equation with infinite delay

$$\frac{d}{dt}(Mu(t)) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \qquad 0 \leq t \leq 2\pi, \tag{1.3}$$

where $(A, D(A))$ and $(M, D(M))$ are (unbounded) closed linear operators defined on a Banach space X, with $D(A) \subseteq D(M)$, $a \in L^1(\mathbb{R}_+)$ an scalar-valued kernel and f an X-valued function defined on $[0, 2\pi]$.

The model (1.3) corresponds to problems related with viscoelastic materials; that is, materials whose stresses at any instant depend on the complete history of strains that the material has undergone (see [60]) or heat conduction with memory. For more details, see, for instance, [41], [45] and [87].

In case $M = I$ (the identity in X) and $a \equiv 0$, equation (1.3) with periodic conditions have been studied by Arendt-Bu, Bu-Kim and characterizations of the maximal regularity in Lebesgue, Besov and Triebel-Lizorkin vector-valued function spaces were obtained using the resolvent set of A. See [8], [9] and [17].

On the other hand, characterizations of maximal regularity for equation (1.3) in case $M = I$ and $a \in L^1(\mathbb{R})$ have been obtained by Keyantuo-Lizama [56] in Lebesgue and Besov vector-valued function spaces and by Bu-Fang [13] in Triebel-Lizorkin vector-valued spaces. We note that periodic solutions have been also studied by other authors, [86], and for example, in [22] using topological methods.

We apply the same method of chapter 3, to obtain characterizations of maximal regularity for the equation (1.3) in the Lebesgue, Besov and Triebel-Lizorkin vector-valued function spaces. The advantage of our approach is clear. We recover, as special cases, the results in [8], [9], [13], [17], [56] and [66].

The organization of the chapter is the following: In the first section, assuming that X is an UMD space, we characterize the existence and uniqueness of a strong L^p-solution for the problem (1.3)-(1.2) solely in terms of a property of R-boundedness for the sequence of operators $ikM(ikM - (1 + \tilde{a}(ik))A)^{-1}$. Here the tilde denotes Laplace transform of $a(t)$. In the second section, we obtain a characterization in the context of Besov spaces. We notice that, as particular case of this characterization, a simply condition to guarantee the existence and uniqueness of solution in Hölder spaces $C^s((0, 2\pi); X)$, $0 < s < 1$, in general Banach spaces X, is obtained.

In the next section of this chapter, we give the corresponding characterization in case of the scale of Triebel-Lizorkin vector-valued spaces. The difference with the scale of Besov vector valued spaces is only that we need more regularity of the sequence $\tilde{a}(ik)$. In the third section, the study a characterization in the context of Triebel-Lizorkin spaces. The fourth section conclude the chapter with two concrete examples. We observe that the results of this chapter can be found in the joint paper [69] and have been submitted for publication.

In the chapter 5, we consider the problem of existence, uniqueness and regularity of solutions for the following integro-differential equation

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.4}$$

where $\alpha, \beta \in \mathbb{R}$, $A : D(A) \subset X \to X$ is a closed linear operator defined on a Banach space X, and f belongs to a closed subspace of the space of continuous and bounded functions. Under appropriate additional assumptions on the scalars α, β, the operator A and on the forcing function f, we want to prove that the equation (1.4) has a unique solution u which behaves in the same way that f does. For example, we want to find conditions implying that u is almost periodic (resp. automorphic) if $f(\cdot, x)$ is almost periodic (resp. almost automorphic), that u is asymptotically periodic (resp. almost periodic) if $f(\cdot, x)$ is asymptotically periodic (resp. almost periodic), and that u is pseudo-almost periodic (resp. automorphic) if $f(\cdot, x)$ is pseudo-almost periodic (resp. automorphic).

This problem arises in several applied fields, like viscoelasticity or heat conduction with memory, and in such applications the operator A typically is the Laplacian in $X = L^2(\Omega)$, or the elasticity operator, the Stokes operator, or the biharmonic Δ^2, among others, and equipped with suitable boundary conditions. The exponential kernel $\alpha e^{-\beta t}$ is the typical choice when one consider Maxwell materials in viscoelasticity theory. In that context, $\alpha = \mu$ and $\beta = \mu/\nu$ where μ is the elastic modulus of the material and ν corresponds to their coefficient of viscosity. See for instance [72], [87, Section 9, Chapter II] and the references therein. Observe that the case $\alpha = 0$ leads with the semilinear problem

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.5}$$

which have been studied intensively by several authors; see e.g. the monograph [24] and references therein.

The problem of existence and uniqueness of almost periodic or almost automorphic solutions, as well as the study of their behavior at infinity, is not only a very natural one for the type of nonlinear evolution equations (1.4), but also there is a recent and increasing interest on this subject by many researchers; see [49, 74, 75, 76, 78, 79, 91] and references therein. In this chapter, we study in a unified way the existence and uniqueness of, among others, almost periodic, almost automorphic and compact almost automorphic solutions for (1.4). Even more, as immediate consequence of our method, necessary conditions for the asymptotic and pseudo-asymptotic behavior of the equation (1.4), under the hypothesis that A generates an *immediately norm continuous* C_0-semigroup on a Banach space X, are also established.

The chapter is organized as follows. In the first section, we study the linear case of equation (1.4), necessary for our method. Assuming that A generates an immediately norm continuous C_0-semigroup we are able to give a simply spectral condition on A in order to guarantee the existence of solutions in each class of function spaces introduced in section 5 of Chapter 2 (Theorem 5.2 and Corollary 5.3). It is remarkable that in the scalar case, that is $A = \rho I$, with $\rho \in \mathbb{R} \setminus \{0\}$, an explicit form of the solution for (1.4) is given by:

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s) f(s, u(s)) ds, \quad t \in \mathbb{R}, \tag{1.6}$$

where

$$S_\rho(t) = \frac{1}{2}\left(e^{t\frac{(\rho-\beta)+c}{2}} + e^{t\frac{(\rho-\beta)-c}{2}}\right) + \frac{(\beta+\rho)}{2c}\left(e^{t\frac{(\rho-\beta)+c}{2}} - e^{t\frac{(\rho-\beta)-c}{2}}\right),$$

and $c = \sqrt{(\beta+\rho)^2 + 4\alpha\rho}$. In particular, it shows that our results are a direct extension of the case $\alpha = 0$ studied in the literature but, notably, in our case the condition $\rho > 0$ even guarantee the existence of bounded solutions for the class of equations (1.4) in the linear case, in contrast with the case $\alpha = 0$ where $\rho < 0$ is necessary. Some examples and a picture of the situation completes this section. In section 2 of this chapter, we present our main results for the semilinear equation (1.4). There, using the previous results on the linear case and the Banach contraction principle, we present new results of existence of solutions that are directly based on the data of the problem. We finish the chapter with a concrete example, to show the feasibility of the abstract results. We note that the results of this chapter are included in the recently published paper by the author [67].

In chapter 6, we study almost automorphic solutions of an integral equation with infinite delay in a general Banach space X:

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s, u(s))] ds, \quad t \in \mathbb{R} \tag{1.7}$$

where the operator $A : D(A) \subset X \to X$ generates an integral resolvent and $a : \mathbb{R}_+ \to \mathbb{C}$ is an integrable function.

As in Chapter 5, a rich source of problems leading to the equation (1.7) is provided by the theory of viscoelastic material behavior. Some typical examples are provided by viscoelastic fluids and heat flow in materials of fading memory type: see for instance [29], [82] and [87]. The material kernel $a(t)$ reflects the properties of the medium under consideration. Note that, in the finite dimensional case, the system (1.7) contains as particular cases several systems with finite or infinite delay, already considered in the literature. See e.g. [30] and [51].

An equivalent form of equation (1.7) is given by

$$u(t) + \frac{d}{dt}(\alpha u(t) + \int_{-\infty}^{t} k(t-s) u(s) ds) = \int_0^{\infty} a(s) ds (Au(t) + f(t, u(t))), \quad t \in \mathbb{R} \tag{1.8}$$

for some $\alpha > 0$ and $k \in L^1(\mathbb{R}_+)$ nonnegative and nonincreasing, see [26, Section 2]. This integro-differential equation was studied in [27], were some results of [28] where used in order to obtain the existence and regularity of the solution u when A generates a contraction semigroup (not necessarily analytic) on X.

In the recent paper [39], the authors dealt with the existence of almost automorphic solutions to certain classes of fractional differential equations, which can be represented in the form ([31, Section 1]):

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[Au(s) + g(s, u(s))]ds; \quad u(0) = u_0, \quad 1 < \alpha < 2.$$

The aim of this chapter is to point out that similar results hold true for the class of the integral equations (1.7) (or equivalently (1.8)) containing the above equations as limiting special cases [87, Chapter II, section 11.5].

Specifically, we consider in this chapter the class of continuous data $f : \mathbb{R} \times X \to X$ of S^p- almost automorphic functions on t, and we look for solutions u belonging to the class of almost automorphic functions.

The concept of S^p-almost automorphy was introduced and applied to study the existence of solutions to some parabolic evolution equations by N'Guérékata and Pankov in [80]. We would like to point out that new and interesting results on S^p-almost automorphic mild solutions to evolution equations have recently appeared in [55], [36] and [61]. However, none of them include the existence of almost automorphic mild solutions for (1.7) or (1.8) with S^p-almost automorphic terms.

This chapter is organized as follows: In the first section we treat equation (1.7) when $f(t, u(t)) = g(t)$ is S^p-almost automorphic, that is, the linear case. We exploit in full strength the use of integral resolvents to obtain a representation of the solution, and then the use maximal regularity results from [39] (see Lemma 2.21 below). In particular, we improve in this section some results of [32]. The next section is devoted to our main results in the semilinear case. Using a crucial composition theorem from [39] (Theorem 2.22), we are able to prove a new and general existence and uniqueness theorem of almost automorphic solutions to the equation (1.7) (cf. Theorem 6.5). Finally, in this section, we point out that our results generalizes the existence results obtained in [32], as the space of S^p-almost automorphic functions contains the space $AA(X)$ of almost automorphic functions. In the last section of this chapter, several examples are examined. We note that the results of this chapter are contained in [68] and has been recently published.

In chapter 7, we consider the problem of existence, uniqueness and regularity of solutions for the following semilinear fractional differential equation

$$ {}_{-\infty}D^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.9} $$

where $\alpha > 0$, $A : D(A) \subset X \to X$ is the generator of an α-resolvent family defined on a Banach space X, f satisfy diverse Lipschitz type conditions and the fractional derivative is understood in the Weyl's sense. Under appropriate assumptions on A and f we want to prove that the equation (1.9) has a unique mild solution u which behaves in the same way that f. For example, we want to find conditions implying that u is almost periodic (resp. automorphic) if $f(\cdot, x)$ is almost periodic (resp. almost automorphic).

Fractional differential equations have been used by many researchers to adequately describe the evolution of a variety of physical and biological processes. Examples include the nonlinear oscillation of earthquake, electrochemistry, electromagnetism, viscoelasticity and rheology. See, for instance, [2, 53] and [59] for more details.

Sufficient conditions for the existence and uniqueness of mild solutions in the cases $\alpha = 1$, $\alpha = 2$, with f almost periodic or almost automorphic, among others, have been studied by several authors in [6, 7, 37, 46, 49, 78] for the case $\alpha = 1$, and in [6, 62, 81] for $\alpha = 2$. The fractional case, $\alpha > 0$,

$$ D^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.10} $$

where the fractional derivative is taken in the Riemann-Liouville's sense, have been studied in several papers. See [1, 5, 35, 75] and the references therein.

The problem of existence of almost periodic or almost automorphic solutions, among others, to (1.9) is very natural one. In this chapter, we study the existence and uniqueness of mild solutions for the equation (1.9) where the input data f belongs to some of above functions spaces. Concretely, we prove that if f is for example, almost periodic or almost automorphic and satisfies some Lipschitz type conditions, then the unique mild solution of the equation (1.9) belongs to the same function space that f and is given by

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s) f(s,u(s))ds, \quad t \in \mathbb{R}, \tag{1.11}$$

where $\{S_\alpha(t)\}_{t\geq 0}$ is the α-resolvent family generated by A.

Recently, Araya and Lizama in [5, Definition 3.2] defined for $1 \leq \alpha \leq 2$, the notion of *mild solution* for the equation (1.10) by (1.11). The authors showed that this definition of mild solution is the natural extension of the usual concept of mild solution in the boundary cases $\alpha = 1, \alpha = 2$. See [5, Remark 3.3]. But in the case $1 < \alpha < 2$, the authors do not checked that the expression (1.11) defines a mild solution for (1.10), due to the fact that in this case, there is no semigroup property ($\alpha = 1$) or existence of a cosine functional equation ($\alpha = 2$). In this chapter, we show that if we consider the Weyl's fractional derivative as in (1.9), instead of Riemann-Liouville fractional derivative, then effectively (1.11) defines a mild solution to (1.9). See Theorem 7.3 and Remark 7.5. Thus, finding the right concept of mild solution to the fractional problem (1.9).

The organization of this chapter is the following. In the first section, we consider the linear problem and we prove that if f belongs to the Schwartz class with values in $D(A)$ then, the unique strong solution of the equation (1.9) is given by (1.11) with $f(t,x) = f(t)$ for all $t \in \mathbb{R}, x \in X$. Using the Banach contraction principle, we give in the Section 2, sufficient conditions that guarantee the existence and uniqueness of almost periodic and almost automorphic mild solution, among others, to the semilinear fractional differential equation (1.9). We conclude the chapter with some applications of our results. We note that the results of this chapter can be found in the paper [85] and have been submitted for publication.

Chapter 2

Preliminaries

In this chapter we summarize the main concepts and results used in the thesis. Let X, Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ be the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote the domain by $D(A)$ and its resolvent set by $\rho(A)$. By $[D(A)]$ we denote the domain of A equipped with the graph norm. For $p \geq 1$, we denote by $L^p(\Omega; X)$ the Banach space of p-integrable functions defined from Ω to X.

2.1 UMD spaces

A Banach space X is said to have the *unconditional martingale difference (UMD)* property, or, briefly, X is a UMD space, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf(s) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t| \geq \varepsilon} \frac{f(s-t)}{t} dt.$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well known that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [12] and Burkholder [19]. Some examples of UMD-spaces include the Hilbert spaces, Sobolev spaces

$W_p^s(\Omega)$, $1 < p < \infty$, Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when X is a UMD-space. Moreover, a UMD-space is reflexive and therefore, $L^1(\Omega, \mu), L^\infty(\Omega, \mu)$ (if Ω is a infinite set) and $C^s([0, 2\pi]; X)$ are not UMD. More information on UMD spaces can be found in [12, 19] and [20].

2.2 R-bounded families of operators

The notion of R-boundedness has proved to be a significant tool in the study of abstract multipliers operators. See [33], [54] for more details. For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0, 1]$ i.e. $r_j(t) = sgn(\sin(2^j \pi t))$ and for $x \in X$, $r_j \otimes x$, denotes the vector-valued function $t \to r_j(t)x$.

Definition 2.1 *A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded, if there is a constant $C_p > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}, T_j \in \mathcal{T}, x_j \in X, j = 1, ..., N$ the inequality*

$$||\sum_{j=1}^{N} r_j \otimes T_j x_j||_{L^p((0,1);Y)} \leq C_p || \sum_{j=1}^{N} r_j \otimes x_j||_{L^p((0,1);X)} \tag{2.1}$$

is valid.

If (2.1) holds for some $p \in [1, \infty)$ then it holds for all $p \in [1, \infty)$. The smallest C_p in (2.1) is called *R-bound* of $\mathcal{T}$, we denote it by $R_p(\mathcal{T})$.

We remark that large classes of classical operators are R-bounded (cf. [47] and references therein). Hence, this assumption is not too restrictive for the applications that we consider in this thesis.

Remark 2.2

Several properties of R-bounded families can be founded in the monograph of Denk-Hieber-Prüss [33]. For the reader's convenience, we summarize here from [33, Section 3] some results.

(a) If $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded then it is uniformly bounded, with

$$\sup\{||T|| : T \in \mathcal{T}\} \leq R_p(\mathcal{T}).$$

(b) The definition of R-boundedness is independent of $p \in [1, \infty)$.

(c) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X, Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

(d) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X, Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(e) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \leq R_p(\mathcal{S})R_p(\mathcal{T})$.

(g) Let X, Y be Banach spaces and $\mathcal{T} \subset \mathcal{B}(X, Y)$ be R-bounded. If $\{\alpha_k\}_{k \in \mathbb{Z}}$ is a bounded sequence, then $\{\alpha_k T : k \in \mathbb{Z}, T \in \mathcal{T}\}$ is R-bounded.

2.3 Operator-valued multipliers

In this section, we recall some operator-valued Fourier multipliers theorems, that we shall use to characterize maximal regularity of problems with periodic boundary conditions in chapters 3 and 4.

We fix some notation. Given $1 \leq p < \infty$, we denote by $L^p_{2\pi}(\mathbb{R}, X)$ the space of all 2π-periodic Bochner measurable X-valued functions f, such that the restriction of f to $[0, 2\pi]$ is p-integrable.

For a function $f \in L^1_{2\pi}(\mathbb{R}, X)$ we denote by $\hat{f}(k), k \in \mathbb{Z}$ the k-th Fourier coefficient of f:

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t)dt,$$

for all $k \in \mathbb{Z}$.

Definition 2.3 *For $1 \leq p < \infty$, we say that a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an L^p-multiplier if, for each $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exists $u \in L^p_{2\pi}(\mathbb{R}, Y)$ such that*

$$\hat{u}(k) = M_k \hat{f}(k) \textit{ for all } k \in \mathbb{Z}.$$

It follows from the uniqueness theorem of Fourier series that u is uniquely determined by f. We recall the following results.

Proposition 2.4 ([8]) *Let X be a Banach space and $\{M_k\}_{k\in\mathbb{Z}}$ be an L^p-multiplier, where $1 \le p < \infty$. Then, the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Theorem 2.5 ([8]) *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.*

We shall need in the Chapters 3 and 4, the following Lemmas.

Lemma 2.6 ([8]) *Let $f, g \in L^p_{2\pi}(\mathbb{R}; X)$, where $1 \le p < \infty$ and A is a closed linear operator on a Banach space X. Then, the following assertions are equivalent.*

(i) $f(t) \in D(A)$ and $Af(t) = g(t)$, a.e.

(ii) $\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$, for all $k \in \mathbb{Z}$.

The following Lemma is analogue to [8, Lemma 2.1].

Lemma 2.7 *Let M be a closed linear operator, $u \in L^p_{2\pi}(\mathbb{R}; [D(M)])$ and $u' \in L^p_{2\pi}(\mathbb{R}; X)$ for $1 \le p < \infty$. Then, the following assertions are equivalent,*

(i) $\int_0^{2\pi} (Mu)'(t)dt = 0$ and there exist $x \in X$ such that $Mu(t) = x + \int_0^t (Mu)'(s)ds$ a.e. on $[0, 2\pi]$;

(ii) $\widehat{(Mu)'}(k) = ikM\hat{u}(k)$ for all $k \in \mathbb{Z}$.

2.4 n-Regular sequences

From [58] we recall the concept of n-regularity for $n = 1, 2, 3$. The general notion of n-regularity is the discrete analogue for the notion of n-regularity related to Volterra integral equations (see [87, Chapter I, Section 3.2]).

Definition 2.8 *A sequence* $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C} \setminus \{0\}$ *is said to be:*

(i) 1-regular, if the sequence $\left\{k\dfrac{(c_{k+1}-c_k)}{c_k}\right\}_{k\in\mathbb{Z}}$ *is bounded.*

(ii) 2-regular, if it is 1-regular and the sequence $\left\{k^2\dfrac{(c_{k+1}-2c_k+c_{k-1})}{c_k}\right\}_{k\in\mathbb{Z}}$ *is bounded.*

(iii) 3-regular, if it is 2-regular and the sequence $\left\{k^3\dfrac{(c_{k+2}-3c_{k+1}+3c_k-c_{k-1})}{c_k}\right\}_{k\in\mathbb{Z}}$ *is bounded.*

Note that if $\{c_k\}_{k\in\mathbb{Z}}$ is 1-regular, then $\lim_{|k|\to\infty} c_{k+1}/c_k = 1$. For more details on n-regularity of sequences, see [58].

We fix the notation for the Chapter 4. Let a be a complex valued function. We define the set

$$\rho_{M,a}(A) = \{\lambda \in \mathbb{C} : (\lambda M - (1+a(\lambda))A) : D(A)\cap D(M) \to X \\ \text{is invertible and } (\lambda M - (1+a(\lambda))A)^{-1} \in \mathcal{B}(X)\},$$

and denote by $\sigma_{M,a}(A)$ the complementary set $\mathbb{C}\setminus\rho_{M,a}(A)$. If $M = I$, is the identity operator on X and $a \equiv 0$, we denote simply the set $\rho_{M,a}(A)$ by $\rho(A)$ and as usual we call this set, the resolvent set of A. Denote by $\widetilde{a}(\lambda)$ the Laplace transform of a. In what follows, we always assume that $\widetilde{a}(ik)$ exists for all $k \in \mathbb{Z}$.

Henceforth, we use the following notation:

$$a_k := \widetilde{a}(ik)$$

and we suppose that $a_k \neq -1$ for all $k \in \mathbb{Z}$.

Remark 2.9 Note that by the Riemann-Lebesgue lemma, we have that the sequences $\{a_k\}_{k\in\mathbb{Z}}$ *and* $\{\frac{1}{1+a_k}\}_{k\in\mathbb{Z}}$ *are bounded.*

2.5 Vector-valued function spaces

In this section, we first recall the definition and basic properties of several function spaces of continuous and bounded functions, and then some recent results on uniform exponential stability of solutions for Volterra equations with special kernels.

We denote

$$BC(X) := \{f : \mathbb{R} \to X; f \text{ is continuous }, ||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)|| < \infty\},$$

where $(X, ||\cdot||)$ is a complex Banach space.

Let $P_\omega(X) := \{f \in BC(X) : f \text{ is continuous } : \exists\, \omega > 0, f(t+\omega) = f(t), \text{ for all } t \in \mathbb{R}\}$ be the space of all vector-valued periodic functions. We recall that a function $f \in BC(X)$ is said to be almost periodic (in the sense of Bohr) if for any $\varepsilon > 0$, there exists $\omega = \omega(\varepsilon) > 0$ such that every subinterval $\mathbb{R}$ of length ω contains at least one point τ such that $||f(t+\tau) - f(t)||_\infty < \varepsilon$. We denote by $AP(X)$ the set of all these functions. The space of compact almost automorphic functions will be denoted by $AA_c(X)$. Recall that function $f \in BC(X)$ belongs to $AA_c(X)$ if and only if for all sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ and $f(t) = \lim_{n\to\infty} g(t-s_n)$ uniformly over compact subsets of $\mathbb{R}$. Clearly the function g above is continuous on $\mathbb{R}$. Finally, a function $f \in BC(X)$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t-s_n), \qquad \text{for each } t \in \mathbb{R}.$$

We denote by $AA(X)$ the set of all almost automorphic functions. We recall the following properties.

Theorem 2.10 ([77]) *If $f, f_1, f_2 \in AA(X)$, then:*

(i) $f_1 + f_2 \in AA(X)$;

(ii) $\lambda f \in AA(X)$ *for any scalar* λ;

(iii) $f_\alpha \in AA(X)$ *where* $f_\alpha : \mathbb{R} \to X$ *is defined by* $f_\alpha(\cdot) = f(\cdot + \alpha)$;

(iv) the range $R_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* X*, and thus* f *is bounded in norm;*

(v) if $f_n \to f$ *uniformly on* $\mathbb{R}$ *where each* $f_n \in AA(X)$*, then* $f \in AA(X)$ *too.*

Almost automorphicity, as a generalization of the classical concept of an almost periodic function, was introduced in the literature by S. Bochner and recently studied by several authors, including [11, 18, 34, 49, 62, 78] among others. A complete description of their properties and further applications to evolution equations can be found in the monographs [79] and [77] by G. M. N'Guérékata.

We recall that $AA_c(X)$ and $AA(X)$ are Banach spaces under the norm $||\cdot||_\infty$ and

$$P_\omega(X) \subset AP(X) \subset AA_c(X) \subset AA(X) \subset BC(X).$$

Now we consider the set $C_0(X) := \{f \in BC(X) : \lim_{|t|\to\infty} ||f(t)|| = 0\}$, and define the space of asymptotically periodic functions as $AP_\omega(X) := P_\omega(X) \oplus C_0(X)$. Analogously, we define the space of asymptotically almost periodic functions,

$$AAP(X) := AP(X) \oplus C_0(X),$$

the space of asymptotically compact almost automorphic functions,

$$AAA_c(X) := AA_c(X) \oplus C_0(X),$$

and the space of asymptotically almost automorphic functions,

$$AAA(X) := AA(X) \oplus C_0(X).$$

We have the following natural proper inclusions

$$AP_\omega(X) \subset AAP(X) \subset AAA_c(X) \subset AAA(X) \subset BC(X).$$

Denote by $SAP_\omega(X) := \{f \in BC(X) : \exists \omega > 0, ||f(t+\omega) - f(t)|| \to 0 \text{ as } t \to \infty\}$. The class of functions in $SAP_\omega(X)$ is called S-asymptotically ω-periodic. Now, we consider the following set

$$P_0(X) := \{f \in BC(X) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} ||f(s)||ds = 0\},$$

and define the following classes of spaces: the space of pseudo-periodic functions

$$PP_\omega(X) := P_\omega(X) \oplus P_0(X),$$

the space of pseudo-almost periodic functions

$$PAP(X) := AP(X) \oplus P_0(X),$$

the space of pseudo-compact almost automorphic functions

$$PAA_c(X) := AA_c(X) \oplus P_0(X),$$

and the space of pseudo-almost automorphic functions

$$PAA(X) := AA(X) \oplus P_0(X).$$

As before, we also have the following relationship between them;

$$PP_\omega(X) \subset PAP(X) \subset PAA_c(X) \subset PAA(X) \subset BC(X).$$

Denote by $\mathcal{N}(\mathbb{R}, X)$ or simply $\mathcal{N}(X)$ the following function spaces

$$\begin{aligned} \mathcal{N}(X) : \;&=\; \{P_\omega(X), AP(X), AA_c(X), AA(X), AP_\omega(X), AAP(X), AAA_c(X), AAA(X), \\ & \quad PP_\omega(X), PAP(X), PAA_c(X), PAA(X), SAP_\omega(X), BC(X)\}. \end{aligned}$$

We recall that a strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is say to be uniformly integrable if

$$\int_0^\infty ||S(t)||dt < \infty.$$

The following Theorem is taken from [64].

Theorem 2.11 ([64]) *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a uniformly integrable and strongly continuous family. If f belongs to one of the spaces of $\mathcal{N}(X)$, then*

$$\int_{-\infty}^t S(t-s)f(s)ds,$$

belongs to the same space as f.

We define the set $\mathcal{N}(\mathbb{R}\times X;X)$ which consists of all continuous functions $f:\mathbb{R}\times X\to X$ such that $f(\cdot,x)\in\mathcal{N}(\mathbb{R},X)$ uniformly for each $x\in K$, where K is any bounded subset of X.

We recall from [64] that $\mathcal{M}(\mathbb{R},X)$, or simply $\mathcal{M}(X)$, denotes one of the spaces $P_\omega(X)$, $AP_\omega(X), PP_\omega(X),\ SAP_\omega(X), AP(X), AAP(X), PAP(X), AA(X), AAA(X), PAA(X)$. Define the set $\mathcal{M}(\mathbb{R}\times X,X)$ of all continuous functions $f:\mathbb{R}\times X\to X$ such that $f(\cdot,x)\in\mathcal{M}(\mathbb{R},X)$ uniformly for each $x\in K$, where K is any bounded subset of X. We have the following composition theorem.

Theorem 2.12 ([64]) *Let $f\in\mathcal{M}(\mathbb{R}\times X,X)$ be given and assume that there exists a constant L_f such that*

$$||f(t,u)-f(t,v)||\le L_f||u-v||,$$

for all $t\in\mathbb{R}$ and $u,v\in X$. If $\psi\in\mathcal{M}(X)$, then $f(\cdot,\psi(\cdot))\in\mathcal{M}(X)$.

Now, we present some recent results of uniform exponential stability of solutions to the homogeneous abstract Volterra equation

$$\begin{cases} u'(t)=Au(t)+\alpha\displaystyle\int_0^t e^{-\beta(t-s)}Au(s)ds,\ t\ge 0\\ u(0)=x, \end{cases} \tag{2.2}$$

where A is the infinitesimal generator of a C_0 semigroup on X, and $\alpha\neq 0,\ \beta>0$ with $\alpha+\beta>0$. We say that a solution of (2.2) is uniformly exponentially bounded if for some $\omega\in\mathbb{R}$, there exists a constant $M>0$ such that for each $x\in D(A)$, the corresponding solution $u(t)$ satisfies

$$||u(t)||\le Me^{-\omega t}||x||,\ t\ge 0. \tag{2.3}$$

In particular, we say that the solutions of (2.2) are *uniformly exponentially stable* if (2.3) holds for some $\omega>0$ and $M>0$.

Definition 2.13 *Let X be a Banach space. A function $T:\mathbb{R}_+\to\mathcal{B}(X)$ strongly continuous is said to be immediately norm continuous if $T:(0,\infty)\to\mathcal{B}(X)$ is continuous.*

Finally, we recall the following remarkable result from [25].

Theorem 2.14 ([25]) *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$ be given. Assume that*

(a) A generates an immediately norm continuous C_0-semigroup on a Banach space X.

(b) $\sup \{ Re(\lambda), \lambda \in \mathbb{C} : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A) \} < 0.$

Then, the solutions of the problem (2.2) are uniformly exponentially stable.

2.6 Stepanov bounded functions

In this section, we recall the class of almost automorphic functions in the Stepanov's sense.

Definition 2.15 ([83]) *The Bochner transform $f^b(t,s), t \in \mathbb{R}, s \in [0,1]$, of a function $f(t)$ on $\mathbb{R}$, with values in X, is defined by*

$$f^b(t,s) := f(t+s).$$

Definition 2.16 ([83]) *The space $BS^p(X)$ of all Stepanov bounded functions, with the exponent p, $1 \leq p < \infty$, consists of all measurable functions $f : \mathbb{R} \to X$ such that*

$$||f||_{S^p} := \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} ||f(\tau)||^p d\tau \Big)^{\frac{1}{p}} < \infty.$$

It is obvious that $L^p(\mathbb{R}; X) \subset BS^p(X) \subset L^p_{loc}(\mathbb{R}; X)$ and $BS^p(X) \subset BS^q(X)$ whenever $p \geq q \geq 1$.

Definition 2.17 ([80]) *The space $AS^p(X)$ of S^p-almost automorphic functions (S^p-a.a. for short) consists of all $f \in BS^p(X)$ such that $f^b \in AA(L^p([0,1]; X))$.*

In other words, a function $f \in L^p_{loc}(\mathbb{R}; X)$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p([0,1]; X)$ is almost automorphic, that is, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R}; X)$ such that

$$\lim_{n\to\infty} \Big(\int_0^1 ||f(t+s_n+s) - g(t+s)||^p ds \Big)^{\frac{1}{p}} = 0,$$

$$\lim_{n\to\infty} \Big(\int_0^1 ||g(t-s_n+s) - f(t+s)||^p ds \Big)^{\frac{1}{p}} = 0,$$

for each $t \in \mathbb{R}$.

Remark 2.18 *It is clear that if* $1 \leq p < q < \infty$ *and* $f \in L^q_{loc}(\mathbb{R}; X)$ *is* S^q*-almost automorphic, then* f *is* S^p*-almost automorphic. Also if* $f \in AA(X)$*, then* f *is* S^p*-almost automorphic for any* $1 \leq p < \infty$*.*

Denote as $l^\infty(X)$ the space of all bounded sequences with values in X. Recall that a sequence $x \in l^\infty(X)$ is said to be almost automorphic if for any sequence of integers (s'_n) there exists a subsequence (s_n) such that $\lim_{n\to\infty} \lim_{m\to\infty} x_{p-s_n-s_m} = x_p$.

Example 2.19 ([80])

Let (a_n) be an almost automorphic sequence and $\epsilon_0 \in (0, 1/2)$. Let $f(t) = a_n$ if $t \in (n - \epsilon_0, n + \epsilon_0)$ and $f(t) = 0$ otherwise. Then, $f \in AS^q(X)$ for all $q \in [1, \infty)$ but f is not in $AA(X)$.

Definition 2.20 ([36]) *A function* $f : \mathbb{R} \times X \to X$, $(t, u) \mapsto f(t, u)$ *with* $f(\cdot, u) \in L^p_{loc}(\mathbb{R}, X)$ *for each* $u \in X$ *is said to be* S^p*-almost automorphic in* $t \in \mathbb{R}$ *uniformly for* $u \in X$*, if for every sequence of real numbers* $(s'_n)_{n\in\mathbb{N}}$*, there exist a subsequence* $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ *and a function* $g : \mathbb{R} \times X \to X$ *with* $g(\cdot, u) \in L^p_{loc}(\mathbb{R}, X)$ *such that*

$$\lim_{n\to\infty} \Big(\int_0^1 ||f(t + s_n + s, u) - g(t + s, u)||^p ds \Big)^{\frac{1}{p}} = 0,$$

$$\lim_{n\to\infty} \Big(\int_0^1 ||g(t - s_n + s, u) - f(t + s, u)||^p ds \Big)^{\frac{1}{p}} = 0,$$

for each $t \in \mathbb{R}$ *and for each* $u \in X$*. We denote by* $AS^p(\mathbb{R} \times X, X)$ *the set of all such functions.*

Lemma 2.21 ([39]) *Let* $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ *be a strongly continuous family of bounded linear operators such that*

$$||S(t)|| \leq \phi(t), \quad \text{for all } t \in \mathbb{R}_+,$$

where $\phi \in L^1(\mathbb{R}_+)$ *is nonincreasing. Then, for each* $f \in AS^1(X)$*,*

$$\int_{-\infty}^t S(t - s) f(s) ds \in AA(X).$$

Theorem 2.22 ([39, 38]) *Assume that*

(i) $f \in AS^p(\mathbb{R} \times X, X)$ *with* $p > 1$*;*

(ii) there exists a nonnegative function $L \in AS^r(\mathbb{R})$ *with* $r \geq max\{p, p/(p-1)\}$ *such that for all* $u, v \in X$ *and* $t \in \mathbb{R}$*,*

$$||f(t,u) - f(t,v)|| \leq L(t)||u - v||;$$

(iii) $x \in AS^p(X)$ *and* $K = \overline{\{x(t) : t \in R\}}$ *is compact in* X*.*

Then, there exists $q \in [1, p)$ *such that* $f(\cdot, x(\cdot)) \in AS^q(X)$*.*

The following definition is taken from [87, Definition 1.6, p.46].

Definition 2.23 *Let* X *be a complex Banach space,* A *a closed linear operator in* X*, with non-empty resolvent set, and* $a \in L^1_{loc}(\mathbb{R}_+)$ *a scalar kernel* $\not\equiv 0$*. A family* $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ *is called an integral resolvent with generator* A *if the following conditions are satisfied.*

(i) $S(\cdot)x \in L^1_{loc}(\mathbb{R}_+; X)$ *for each* $x \in X$ *and* $||S(t)|| \leq \psi(t)$ *a.e. on* $\mathbb{R}_+$*, for some* $\psi \in L^1_{loc}(\mathbb{R}_+)$*;*

(ii) $S(t)$ *commutes with* A *for each* $t \geq 0$*;*

(iii) the following integral resolvent equation holds

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)xds. \tag{2.4}$$

for all $x \in D(A)$ *and a.a.* $t \geq 0$*.*

We will see that the concept of integral resolvent is directly and naturally related with the solution of the equation (1.7) by means of a kind of variation of parameters formula (cf. Definition 6.4). On the other hand, our definition of a solution (mild) for equation (1.7) that we will give is motivated by the linear case (see (6.3) in Proposition 6.1).

2.7 Weyl fractional calculus

In this section, we recall the definition and some basic properties of the Weyl's fractional calculus.

We denote by $\mathcal{S} \equiv \mathcal{S}(\mathbb{R}; X)$, the *Schwartz class* on $\mathbb{R}$, which consists of all functions $f : \mathbb{R} \to X$ which are infinitely differentiable and satisfy

$$\sup_{t \in \mathbb{R}} \left\| t^m \frac{d^n}{dt^n} f(t) \right\| < \infty,$$

for any $m, n \in \mathbb{N} \cup \{0\}$.

Given $\alpha > 0$ and $f \in \mathcal{S}$, the *Weyl fractional integral*, ${}_{-\infty}D^{-\alpha} f$ of order $\alpha > 0$ is defined by

$${}_{-\infty}D^{-\alpha} f(t) := \int_{-\infty}^{t} g_\alpha(t-s) f(s) ds, \; t \in \mathbb{R},$$

where $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. The operator ${}_{-\infty}D^{-\alpha} : \mathcal{S} \to \mathcal{S}$ is one to one, and its inverse, the *Weyl fractional derivative* of order $\alpha > 0$, ${}_{-\infty}D^{\alpha}$ is given by

$${}_{-\infty}D^{\alpha} f(t) := \frac{d^n}{dt^n} \int_{-\infty}^{t} g_{n-\alpha}(t-s) f(s) ds, \; t \in \mathbb{R},$$

where $n = [\alpha] + 1$. We remark that ${}_{-\infty}D^{-\alpha} {}_{-\infty}D^{\alpha} f(t) = f(t)$, for all $t \in \mathbb{R}$ and ${}_{-\infty}D^{n} = \frac{d^n}{dt^n}$ holds if $n \in \mathbb{N} \cup \{0\}$. More details of fractional calculus can be found, for example, in [73].

Chapter 3

Periodic solutions for a class of degenerate differential equations

In this chapter, using operator-valued Fourier multipliers theorems, we obtain necessary and sufficient conditions to guarantee existence and uniqueness of periodic solutions to the abstract equation $\frac{d}{dt}(Mu(t)) = Au(t) + f(t)$, where A and M are closed linear operators defined on a complex Banach space X, with the periodic conditions $Mu(0) = Mu(2\pi)$, in terms of either boundedness or R-boundedness of the modified resolvent operator determined by the equation. The results are obtained in the scales of periodic Besov and periodic Lebesgue vector-valued spaces.

3.1 A characterization on vector-valued Lebesgue spaces

We consider the problem

$$\begin{cases} \dfrac{d}{dt}(Mu(t)) = Au(t) + f(t), & 0 \le t \le 2\pi, \\ \\ Mu(0) = Mu(2\pi), \end{cases} \tag{3.1}$$

where $A : D(A) \subseteq X \to X$ and $M : D(M) \subseteq X \to X$ are closed linear operators, $D(A) \subseteq D(M)$ and $f \in L^p_{2\pi}(\mathbb{R}, X), p \geq 1$. For a given closed operator M, and $1 \leq p < \infty$, we define the set

$$\begin{aligned} H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) &= \{u \in L^p_{2\pi}(\mathbb{R}; [D(M)]) : \exists v \in L^p_{2\pi}(\mathbb{R}; X), \\ &\quad \hat{v}(k) = ikM\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}. \end{aligned}$$

If $M = I$, we denote $H^{1,p}_{per}(\mathbb{R}; X)$; see [8]. Now, we introduce the following definition.

Definition 3.1 *We say that a function $u \in H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$ is a strong L^p-solution of problem (3.1) if $u(t) \in D(A)$ and equation (3.1) holds for a.e. $t \in [0, 2\pi]$.*

Denote the *M-resolvent set of A* by

$$\rho_M(A) = \{\lambda \in \mathbb{C} : (\lambda M - A) : D(A) \to X \text{ is bijective and } (\lambda M - A)^{-1} \in \mathcal{B}(X)\}.$$

We begin with the following result.

Proposition 3.2 *Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators defined on a UMD space X. Suppose that $D(A) \subseteq D(M)$. Then, the following assertions are equivalent*

(i) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$;

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. Define $M_k = ikM(ikM - A)^{-1}$. Since A is closed, by the identity $M_k = A(ikM - A)^{-1} + I$ and the Closed Graph Theorem we conclude that M_k is a bounded operator for each $k \in \mathbb{Z}$. By Proposition 2.4 it follows that (i) implies (ii). Conversely, by Theorem 2.5 is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. In fact, we note the following

$$\begin{aligned} k[M_{k+1} - M_k] &= k\Big[i(k+1)M[i(k+1)M - A]^{-1} - ikM[ikM - A]^{-1}\Big] \\ &= kM\Big[i(k+1)[i(k+1)M - A]^{-1} - ik[ikM - A]^{-1}\Big] \\ &= kM(i(k+1)M - A)^{-1}\Big[i(k+1)(ikM - A) - ik(i(k+1)M - A)\Big] \cdot \\ &\quad \cdot (ikM - A)^{-1} \end{aligned}$$

$$= \quad kM(i(k+1)M - A)^{-1}[-iA](ikM - A)^{-1}.$$

Using the identity $A(ikM - A)^{-1} = ikM(ikM - A)^{-1} - I$, we obtain

$$k[M_{k+1} - M_k] = -ikM(i(k+1)M - A)^{-1}\Big[ikM(ikM - A)^{-1} - I\Big]. \tag{3.2}$$

Therefore, since the products and sums of R-bounded sequences is R-bounded, by (d) and (g) in Remark 2.2, the proof is finished. ∎

The following is one of the main results in this chapter. It corresponds to an extension of [8, Theorem 2.3] in case $M = I$.

Theorem 3.3 *Let X be a UMD space and $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ linear closed operators satisfying $D(A) \subseteq D(M)$. Then, the following assertions are equivalent*

(i) For every $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exist a unique strong L^p-solution of (3.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$;

(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. $(i) \Rightarrow (ii)$ Follows the same lines of [8, Theorem 2.3]. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt}y$. By hypothesis, there exists $u \in H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + f(t)$. Taking Fourier transform on both sides, we have $\hat{u}(k) \in D(A)$ and,

$$\begin{aligned} ikM\hat{u}(k) &= A\hat{u}(k) + \hat{f}(k) \\ &= A\hat{u}(k) + y. \end{aligned}$$

Thus, $(ikM - A)\hat{u}(k) = y$ for all $k \in \mathbb{Z}$ and therefore $(ikM - A)$ is surjective. Let $x \in D(A)$. If $(ikM - A)x = 0$, then $u(t) = e^{ikt}x$ defines a periodic solution of (3.1). In fact, since $u(t) = e^{ikt}x$ we obtain $(Mu)'(t) - Au(t) = ike^{ikt}Mx - e^{ikt}Ax = e^{ikt}(ikM - A)x = 0$. Hence $u \equiv 0$ by the assumption of uniqueness, and thus $x = 0$. Therefore, $(ikM - A)$ is bijective. Now, we must prove that $(ikM - A)^{-1}$ is a bounded operator for all $k \in \mathbb{Z}$. Suppose that $(ikM - A)$ has no bounded inverse. Then, for each $k \in \mathbb{Z}$ there exists a sequence $(y_{n,k})_{n\in\mathbb{Z}} \subset X$ such that $||y_{n,k}|| \leq 1$ and

$$||(ikM - A)^{-1}y_{n,k}|| \geq n^2, \text{ for all } n \in \mathbb{Z}.$$

Thus, we obtain that the sequence $x_k := y_{k,k}$, satisfies

$$||(ikM - A)^{-1}x_k|| \geq k^2, \text{ for all } k \in \mathbb{Z}.$$

Let $f(t) := \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{ikt}\frac{x_k}{k^2}$. Observe that $f \in L^p_{2\pi}(\mathbb{R}, X)$ and so, by hypothesis, there exists a unique strong solution $u \in L^p_{2\pi}(\mathbb{R}, X)$ of (3.1). One can check that $u(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}}(ikM - A)^{-1}e^{ikt}\frac{x_k}{k^2}$. Since

$$\left\|(ikM - A)^{-1}e^{ikt}\frac{x_k}{k^2}\right\| \geq 1, \text{ for all } k \in \mathbb{Z}\setminus\{0\},$$

we obtain $u \notin L^p_{2\pi}(\mathbb{R}, X)$. A contradiction. Thus, we conclude that $(ikM - A)^{-1}$ is a bounded operator for all $k \in \mathbb{Z}$, and therefore $ik \in \rho_M(A)$ for all $k \in \mathbb{Z}$.

We will see that $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Using the Closed Graph Theorem, we have that there exist a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}; X)$ such that

$$||(Mu)'||_{L^p} + ||Au||_{L^p} \leq C||f||_{L^p}.$$

Note that for $f(t) = e^{itk}y, y \in X$, the solution u of (3.1) is given by $u(t) = (ikM - A)^{-1}e^{ikt}y$. Hence,

$$||ikM(ikM - A)^{-1}y|| \leq C||y||.$$

So, we have that $ikM(ikM - A)^{-1}$ is a bounded operator for all $k \in \mathbb{Z}$. Let $f \in L^p_{2\pi}(\mathbb{R}, X)$, by hypothesis, there exist $u \in H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + f(t)$. Taking Fourier transform on both sides, and using that $(ikM - A)$ is bijective, we have $\hat{u}(k) \in D(A)$ and $\hat{u}(k) = (ikM - A)^{-1}\hat{f}(k)$. Now, since $u \in H^{1,p}_{per,M}(\mathbb{R}, [D(M)])$ and by definition of $H^{1,p}_{per,M}(\mathbb{R}, [D(M)])$, there exist $v \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{v}(k) = ikM\hat{u}(k)$ for all $k \in \mathbb{Z}$. Therefore, we have $\hat{v}(k) = ikM\hat{u}(k) = ikM(ikM - A)^{-1}\hat{f}(k)$.

(ii) $\Rightarrow$ *(i)* Define $M_k = ikM(ikM - A)^{-1}$. Suppose that $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Let $f \in L^p_{2\pi}(\mathbb{R}, X)$. Then, there exist $u \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{u}(k) = ikM(ikM-A)^{-1}\hat{f}(k)$, for all $k \in \mathbb{Z}$. Now by the identity $I = ikM(ikM-A)^{-1} - A(ikM-A)^{-1}$ it follows that

$$\begin{aligned}
\hat{u}(k) &= ikM(ikM - A)^{-1}\hat{f}(k) \\
&= (I + A(ikM - A)^{-1})\hat{f}(k).
\end{aligned}$$

So, we obtain $\widehat{(u-f)}(k) = A(ikM-A)^{-1}\hat{f}(k)$. Putting $v := u - f$, we have $v \in L^p_{2\pi}(\mathbb{R}, X)$, and $\hat{v}(k) = A(ikM-A)^{-1}\hat{f}(k)$. Observe that A^{-1} is an isomorphism of X onto $D(A)$ (seen as a Banach space with the graph norm). Therefore, $A^{-1}\hat{v}(k) = (ikM-A)^{-1}\hat{f}(k)$. Let $w := A^{-1}v$. Since A^{-1} is a bounded operator, we obtain that $w \in L^p_{2\pi}(\mathbb{R}, [D(A)])$, $\hat{w}(k) \in D(A)$ and $\hat{w}(k) = (ikM-A)^{-1}\hat{f}(k)$. So,

$$\begin{aligned} ikM\hat{w}(k) - A\hat{w}(k) &= ikM(ikM-A)^{-1}\hat{f}(k) - A(ikM-A)^{-1}\hat{f}(k) \\ &= (ikM-A)(ikM-A)^{-1}\hat{f}(k) \\ &= \hat{f}(k). \end{aligned}$$

Now, observe that we have

$$\hat{u}(k) = ikM(ikM-A)^{-1}\hat{f}(k) = ikM\hat{w}(k),$$

for all $k \in \mathbb{Z}$. Therefore, $w \in H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$. Moreover $Mw(0) = Mw(2\pi)$, since $w(0) = w(2\pi)$ and $w(t) \in D(A)$. Since A and M are closed operators and $\widehat{(Mw)}'(k) = ikM\hat{w}(k) = A\hat{w}(k) + \hat{f}(k)$, for all $k \in \mathbb{Z}$, one has $(Mw)'(t) = Aw(t) + f(t)\, a.e.$ by Lemmas 2.6 and 2.7. So w is a strong L^p-solution of (3.1).

Now, to see the uniqueness, let $u \in H^{1,p}_{per,M}(\mathbb{R}, [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$ such that $(Mu)'(t) = Au(t)$. Then $\hat{u}(k) \in D(A)$, and $(ikM-A)\hat{u}(k) = 0$, for all $k \in \mathbb{Z}$. Since $(ikM-A)$ is bijective for all $k \in \mathbb{Z}$, we obtain $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$, and thus $u \equiv 0$.

(ii) $\Leftrightarrow$ *(iii)* Proposition 3.2. ■

Corollary 3.4 *Let H be a Hilbert space, $A : D(A) \subset H \to H$, and $M : D(M) \subset H \to H$ closed linear operators satisfying $D(A) \subseteq D(M)$. Then, for $1 < p < \infty$, the following assertions are equivalent*

(i) For every $f \in L^p_{2\pi}(\mathbb{R}, H)$, there exists a unique strong L^p-solution of (3.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\sup_k ||ikM(ikM-A)^{-1}|| < \infty$.

Proof. Follows from Theorem 3.3, and the fact that in Hilbert spaces the concepts of R-boundedness and boundedness are equivalent [33]. ■

The solution $u(\cdot)$ given in Theorem 3.3 actually satisfies the following maximal regularity property.

Corollary 3.5 *In the context of Theorem 3.3, if condition (iii) is fulfilled, we have* $(Mu)', Au \in L^p_{2\pi}(\mathbb{R}, X)$. *Moreover, there exists a constant* $C > 0$ *independent of* $f \in L^p_{2\pi}(\mathbb{R}; X)$ *such that*

$$||(Mu)'||_{L^p} + ||Au||_{L^p} \leq C||f||_{L^p}. \tag{3.3}$$

Remark 3.6 We remark that from the inequality (3.3) we deduce that the operator L defined by:

$$(Lu)(t) = (Mu)'(t) - Au(t) \quad \text{with domain} \quad D(L) = H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)]),$$

is an isomorphism onto. Indeed, since A *and* M *are closed, the space* $H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}, [D(A)])$ *becomes a Banach space under the norm*

$$|||u||| := ||u||_p + ||(Mu)'||_p + ||Au||_p.$$

We note that such isomorphisms are crucial for the handling of nonlinear evolution equations (see [3]).

3.2 Maximal regularity on the scale of vector-valued Besov Spaces

In this section, we study the existence and uniqueness of solutions to (3.1) in $B^s_{p,q}((0, 2\pi); X)$, the vector-valued periodic Besov spaces for $1 \leq p \leq \infty, s > 0$, where X is a Banach space. For the definition and main properties of these spaces we refer to [9] or [57]. For the scalar case, see [21], [88]. Contrary to the L^p case, the multiplier theorems established for vector-valued Besov spaces are valid for arbitrary Banach spaces X; see [4], [9] and [48]. Special cases here allow one to treat Hölder-Zygmund spaces. Specifically, we have $B^s_{\infty,\infty} = \mathcal{C}^s$ for

$s > 0$. Moreover, if $0 < s < 1$ then $B^s_{\infty,\infty}$ is just the usual Hölder space C^s. We summarize some useful properties of $B^s_{p,q}((0, 2\pi); X)$. See [9, Section 2] for a proof.

(i) If $(X, ||\cdot||)$ is a Banach space and $s > 0$, then $B^s_{p,q}((0, 2\pi); X)$ is a Banach space;

(ii) If $s > 0$, then $B^s_{p,q}((0, 2\pi); X) \hookrightarrow L^p((0, 2\pi); X)$, and the natural injection from $B^s_{p,q}((0, 2\pi); X)$ into $L^p((0, 2\pi); X)$ is a continuos linear operator;

(iii) Let $s > 0$. Then $f \in B^{s+1}_{p,q}((0, 2\pi); X)$ if and only if f is differentiable a.e. and $f' \in B^s_{p,q}((0, 2\pi); X)$.

We begin with the definition of operator-valued Fourier multipliers in the context of periodic Besov spaces.

Definition 3.7 *Let $1 \le p \le \infty$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is a $B^s_{p,q}$-multiplier if for each $f \in B^s_{p,q}((0, 2\pi); X)$ there exists a function $g \in B^s_{p,q}((0, 2\pi); Y)$ such that*

$$M_k \hat{f}(k) = \hat{g}(k), \quad k \in \mathbb{Z}.$$

The following concept was studied in [58].

Definition 3.8 *We say that $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ satisfies the Marcinkiewicz condition of order 2 if*

$$\sup_{k\in\mathbb{Z}} ||M_k|| < \infty, \ \sup_{k\in\mathbb{Z}} ||k(M_{k+1} - M_k)|| < \infty, \tag{3.4}$$

$$\sup_{k\in\mathbb{Z}} ||k^2(M_{k+1} - 2M_k + M_{k-1})|| < \infty. \tag{3.5}$$

We recall the following operator-valued Fourier multiplier theorem on Besov spaces.

Theorem 3.9 ([9]) *Let X, Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X, Y)$ a sequence that satisfies the Marcinkiewicz condition of order 2. Then, for $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier.*

We next prove the following result, which is the analogue to Proposition 3.2.

Proposition 3.10 *Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators. Suppose that $D(A) \subseteq D(M)$. Then, the following assertions are equivalent*

(i) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM-A)^{-1}\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier for $1 \le p, q \le \infty$;

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM-A)^{-1}|| < \infty$.

Proof. $(i) \Rightarrow (ii)$. Follows the same lines as the proof in [56, Proposition 3.4]. $(ii) \Rightarrow (i)$ For $k \in \mathbb{Z}$, define $M_k = ikM(ikM-A)^{-1}$. From the identity (3.2) we obtain:

$$\sup_{k\in\mathbb{Z}} ||k(M_{k+1} - M_k)|| < \infty, \tag{3.6}$$

proving (3.4). To verify (3.5), we notice:

$$\begin{aligned}
& k^2[M_{k+1} - 2M_k + M_{k-1}] = \\
= \;& k^2\Big[i(k+1)M[i(k+1)M-A]^{-1} - 2ikM[ikM-A]^{-1} \\
+ \;& i(k-1)M[i(k-1)M-A]^{-1}\Big] \\
= \;& k^2M[i(k+1)M-A]^{-1}\Big[i(k+1)[ikM-A] - 2ik[i(k+1)M-A] \\
+ \;& i(k-1)[i(k+1)M-A][i(k-1)M-A]^{-1}[ikM-A]\Big][ikM-A]^{-1} \\
= \;& k^2M[i(k+1)M-A]^{-1}\Big[i(k+1)[ikM-A] - 2ik[ikM-A] - 2ikiM \\
+ \;& i(k-1)[i(k-1)M-A][i(k-1)M-A]^{-1}[ikM-A] \\
+ \;& 2i\cdot i(k-1)M[i(k-1)M-A]^{-1}[ikM-A]\Big][ikM-A]^{-1} \\
= \;& k^2M[i(k+1)M-A]^{-1}\Big[\big(i(k+1) - 2ik + i(k-1) + 2iM_{k-1}\big)[ikM-A] - 2ikiM\Big]\cdot \\
& \cdot[ikM-A]^{-1} \\
= \;& k^2M[i(k+1)M-A]^{-1}\Big[2iM_{k-1}[ikM-A] - 2ikiM\Big][ikM-A]^{-1} \\
= \;& kM[i(k+1)M-A]^{-1}\Big[2ikM_{k-1}[ikM-A] - 2ikikM\Big][ikM-A]^{-1} \\
= \;& kM[i(k+1)M-A]^{-1}\Big[2ikM_{k-1}\cdot I - 2ikikM[ikM-A]^{-1}\Big] \\
= \;& kM[i(k+1)M-A]^{-1}\Big[2ikM_{k-1} - 2ikM_k\Big] \\
= \;& kM[i(k+1)M-A]^{-1}\Big[-2ik(M_k - M_{k-1})\Big] \\
= \;& kM[i(k+1)M-A]^{-1}\Big[-2i(k-1)(M_k - M_{k-1}) - 2i(M_k - M_{k-1})\Big].
\end{aligned}$$

Since, $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is bounded, and $\{M_k\}_{k\in\mathbb{Z}}$ is bounded by hypothesis, we conclude from the above identity that,

$$\sup_{k\in\mathbb{Z}} ||k^2(M_{k+1} - 2M_k + M_{k-1})|| < \infty. \tag{3.7}$$

So, $\{M_k\}_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order 2 and therefore, by Theorem 3.9, $\{M_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier.

■

Definition 3.11 *Let $1 \leq p, q \leq \infty$ and $s > 0$. A function $u \in B^s_{p,q}((0, 2\pi); [D(A)])$ is said to be a strong $B^s_{p,q}$-solution of problem (3.1) if $Mu \in B^{s+1}_{p,q}((0, 2\pi); X)$ and equation (3.1) holds for a.e. $t \in (0, 2\pi)$.*

The next Theorem is the main result of this section. It extends [9, Theorem 5.1] with $M = I$.

Theorem 3.12 *Let $1 \leq p, q \leq \infty$ and $s > 0$. Let X be a Banach space and let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ linear closed operators satisfying $D(A) \subseteq D(M)$. Then, the following assertions are equivalent*

(i) For every $f \in B^s_{p,q}((0, 2\pi); X)$ there exist a unique strong $B^s_{p,q}$-solution of (3.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\{ikM(ikM - A)^{-1}\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier;

(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - A)^{-1}|| < \infty$.

Proof. $(ii) \Leftrightarrow (iii)$. Follows from Proposition 3.10.

$(i) \Rightarrow (iii)$. Suppose that for every $f \in B^s_{p,q}((0, 2\pi); X)$ there exist a unique strong $B^s_{p,q}$-solution of (3.1). Fix $x \in X$ and $k \in \mathbb{Z}$. Define $f(t) = e^{itk}x$. Then $f \in B^s_{p,q}((0, 2\pi); X)$. By hypothesis there exist $u \in B^s_{p,q}((0, 2\pi); [D(A)])$ with $Mu \in B^{s+1}_{p,q}((0, 2\pi); X)$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + f(t)$ a.e. $t \in (0, 2\pi)$. By Lemma 2.7 we have $ikM\hat{u}(k) = A\hat{u}(k) + x$. Following the same reasoning that in the proof of Theorem 3.3, we obtain that $ik \in \rho_M(A)$ for all $k \in \mathbb{Z}$. Let $M_k := ikM(ikM - A)^{-1}$. We will see that $\{M_k\}_{k\in\mathbb{Z}}$ is bounded.

Using the Closed Graph Theorem, we have that there exist a constant C independent of f such that

$$||Mu||_{B^{s+1}_{p,q}((0,2\pi);X)} + ||Au||_{B^s_{p,q}((0,2\pi);[D(A)])} \leq C||f||_{B^s_{p,q}((0,2\pi);X)}.$$

Note that for $f(t) = e^{itk}x$, the solution u of (3.1) is given by $u(t) = (ikM - A)^{-1}e^{ikt}x$. Hence,

$$\sup_{k\in\mathbb{Z}} ||ikM(ikM - A)^{-1}x|| \leq C||x||.$$

$(iii) \Rightarrow (i)$. Suppose that $\{ik\}_{k\in\mathbb{Z}} \subset \rho_M(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - A)^{-1}|| < \infty$. Define $M_k := ikM(ikM - A)^{-1}$ and $N_k := (ikM - A)^{-1}$ for $k \in \mathbb{Z}$. Since $\sup_{k\in\mathbb{Z}} ||M_k|| < \infty$, we have by Proposition 3.10 that $\{M_k\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Now, we will see that $\{N_k\}_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order 2. First note that, since $0 \in \rho_M(A)$, A^{-1} is an bounded operator, and hence the identity $ikM(ikM - A)^{-1} = A(ikM - A)^{-1} + I$ imply $N_k = A^{-1}(M_k - I)$. So, $\sup_{k\in\mathbb{Z}} ||N_k|| < \infty$. Now, observe that

$$\begin{aligned} k\big[N_{k+1} - N_k\big] &= k\big[(i(k+1)M - A)^{-1} - (ikM - A)^{-1}\big] \\ &= A^{-1}k\big[M_{k+1} - M_k\big]. \end{aligned}$$

Hence, by (3.6) we get $\sup_{k\in\mathbb{Z}} ||k(N_{k+1} - N_k)|| < \infty$. In the same way, we have

$$\begin{aligned} k^2\big[N_{k+1} - 2N_k + N_{k-1}\big] &= k^2\big[A^{-1}M_{k+1} - A^{-1} - 2[A^{-1}(M_k - I)] + A^{-1}M_{k-1} - A^{-1}\big] \\ &= A^{-1}k^2\big[M_{k+1} - 2M_k + M_{k-1}\big]. \end{aligned}$$

Therefore, using (3.7), we obtain

$$\sup_{k\in\mathbb{Z}} ||k^2(N_{k+1} - 2N_k + N_{k-1})|| < \infty.$$

So, $\{N_k\}_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order 2 and, by Theorem 3.9, $\{N_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier. We conclude that $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers. Let $f \in B^s_{p,q}((0,2\pi);X)$. There exists $u, v \in B^s_{p,q}((0,2\pi);X)$ such that $\hat{u}(k) = ikM(ikM - A)^{-1}\hat{f}(k)$ and $\hat{v}(k) = (ikM - A)^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$. So, we have $ikM\hat{v}(k) = \hat{u}(k)$ for all $k \in \mathbb{Z}$. By Lemma 2.6 we obtain $(Mv)' = u$ a.e. Since $u \in B^s_{p,q}((0,2\pi);X)$ we have

$(Mv)' \in B^s_{p,q}((0,2\pi);X)$ and so, $Mv \in B^{s+1}_{p,q}((0,2\pi);X)$. Also, since $(ikM - A)$ is bijective for all $k \in \mathbb{Z}$ and $\hat{v}(k) = (ikM - A)^{-1}\hat{f}(k)$, we have $v(t) \in D(A)$ and $ikM\hat{v}(k) - \hat{f}(k) = A\hat{v}(k)$, for all $k \in \mathbb{Z}$. So, one has $(Mv)'(t) = Av(t) + f(t)$ a.e. $t \in (0,2\pi)$ by Lemma 2.6. Uniqueness follows the same way as in the proof of Theorem 3.3.

■

Remark 3.13 Note that the Besov spaces $B^s_{\infty,\infty}((0,2\pi);X)$ *corresponds to the familiar Hölder spaces* C^s, *if* $0 < s < 1$. *Hence, Theorem 3.12 extends and improves Theorem 2.1 in [10] where* X *was considered a reflexive Banach space only.*

3.3 Examples

We conclude the chapter with some applications of the above results.

Example 3.14

Let us consider the periodic boundary value problem

$$\frac{\partial(m(x)u)}{\partial t} - \Delta u = f(t,x), \text{ in } [0,2\pi] \times \Omega \tag{3.8}$$

$$u = 0, \text{ in } [0,2\pi] \times \partial\Omega \tag{3.9}$$

$$m(x)u(0,x) = m(x)u(2\pi,x) \text{ in } \Omega, \tag{3.10}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x) \geq 0$ is a given measurable bounded function on Ω and f is a function on $[0,2\pi] \times \Omega$. The initial value problem $m(x)u(0,x) = v_0$ relative to (3.8)- (3.9) has been studied in [43], [44] both in the spaces $H^{-1}(\Omega)$, $L^2(\Omega)$ and in $L^p(\Omega), p > 1$. The periodic problem (3.8)-(3.10) has been studied in [10] in the spaces $H^{-1}(\Omega)$ and $L^2(\Omega)$.

Let M be the multiplication operator by m. If we take $X = H^{-1}(\Omega)$ then by [10, p.38] (see also references therein), we have that there exists a constant $c > 0$ such that

$$||M(zM - \Delta)^{-1}|| \leq \frac{c}{1+|z|},$$

whenever $Re(z) \geq -c(1+|Im(z)|)$. In particular, in the imaginary axis we have $||M(ikM - \Delta)^{-1}|| \leq \frac{c}{1+|k|}$, for all $k \in \mathbb{Z}$. Therefore, Theorem 3.12 applies immediately, obtaining existence and uniqueness of solutions of (3.8)-(3.10) in periodic Besov spaces, complementing the results in [10]. On the other hand, and because $H^{-1}(\Omega)$ is a Hilbert space, Corollary 3.4 also applies, obtaining that for all $f \in L^p_{2\pi}(\mathbb{R}, H^{-1}(\Omega))$ the periodic problem (3.8)-(3.10) has precisely one strong solution u with maximal regularity.

Example 3.15

Consider, for $t \in [0, 2\pi]$ and $x \in [0, \pi]$, the problem

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2}{\partial x^2}+1\Big)u(t,x) = -a\frac{\partial^2}{\partial x^2}u(t,x) - ku(t,x) + f(t,x) \tag{3.11}$$

$$u(t,0) = u(t,\pi) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,\pi) = 0 \tag{3.12}$$

$$\Big(\frac{\partial^2}{\partial x^2}+1\Big)u(0,x) = \Big(\frac{\partial^2}{\partial x^2}+1\Big)u(2\pi,x), \tag{3.13}$$

where a is positive constant and $-2a < k < 4a$. If we take $X = C_0([0,\pi]) = \{u \in C([0,\pi]) : u(0) = u(\pi)\}$ and K the realization of $\frac{\partial^2}{\partial x^2}$ with domain

$$D(K) = \{u \in C^2([0,\pi]) : u(0) = u(\pi) = \frac{\partial^2}{\partial x^2}u(0) = \frac{\partial^2}{\partial x^2}u(\pi) = 0\},$$

then we take $M = K + I$, $A = aM + (k-a)I$. By [10, p.39, Ex.1.2] we have, as in the Example 3.14:

$$||M(ikM - A)^{-1}|| \leq \frac{c}{1+|k|}$$

for all $k \in \mathbb{Z}$. Therefore, Theorem 3.12 applies, and hence for all $f \in B^s_{p,q}((0,2\pi), C_0([0,\pi]))$, $s > 0$, $1 \leq p, q \leq \infty$ the problem (3.11)-(3.13) has a unique strong solution u with regularity $\frac{\partial^2 u}{\partial x^2} \in B^s_{p,q}((0,2\pi), C_0([0,\pi]))$. In particular, because the class of Besov spaces contains the class of Hölder spaces, our result recover and extends Example 1.2 in [10].

Remark 3.16 Following a similar method of proof, and using the operator-valued Fourier multiplier theorem stated in [17, Theorem 3.2], an analogous result like Theorem 3.12 for the scale of Triebel-Lizorkin spaces can be proved.

Chapter 4

Periodic solutions for a class of degenerate integro-differential equations with infinite delay

Let A and M be closed linear operators defined on a complex Banach space X and $a \in L^1(\mathbb{R}_+)$ a scalar kernel. As in the Chapter 3, we use operator-valued Fourier multipliers techniques to obtain necessary and sufficient conditions to guarantee the existence and uniqueness of periodic solutions to the equation

$$\frac{d}{dt}(Mu(t)) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad 0 \leq t \leq 2\pi,$$

with initial condition $Mu(0) = Mu(2\pi)$, solely in terms of spectral properties of the data. Our results are obtained in the scales of periodic Besov, Triebel-Lizorkin and Lebesgue vector-valued function spaces and there is no special properties in the existence of bounded inverse of A or M or in the commutativity of A with M.

4.1 Maximal regularity on vector-valued Lebesgue spaces

To characterize the maximal regularity, we begin with the study of the relation between multipliers and R-boundedness of sequences of operators. Consider the problem

$$\begin{cases} \dfrac{d}{dt}(Mu(t)) = Au(t) + \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)ds + f(t),\ 0 \le t \le 2\pi, \\ \\ Mu(0) = Mu(2\pi), \end{cases} \tag{4.1}$$

where $(A, D(A))$ and $(M, D(M))$ are closed linear operators on X, $D(A) \subseteq D(M)$, $a \in L^1(\mathbb{R}_+)$ is a scalar-valued kernel and $f \in L^p_{2\pi}(\mathbb{R}, X), p \ge 1$.

Lemma 4.1 *Let X be a UMD-space. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. By Remarks 2.9 and 2.2 (g), $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is R-bounded. Moreover,

$$k\left(\frac{1}{1+a_{k+1}} - \frac{1}{1+a_k}\right) = -k\left(\frac{a_{k+1}-a_k}{a_k}\right) \cdot a_k \cdot \frac{1}{1+a_{k+1}} \cdot \frac{1}{1+a_k}.$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular, we conclude the proof of Lemma by Remark 2.9 and Theorem 2.5. ∎

The following Proposition is an extension of [66, Proposition 3.2].

Proposition 4.2 *Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Let $A : D(A) \subseteq X \to X$ and $M : D(M) \subseteq X \to X$ be linear closed operators defined on a UMD space X. Suppose that $D(A) \subseteq D(M)$. Then, the following assertions are equivalent*

(i) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\widetilde{a}}(A)$ and $\{ikM(ikM-(1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$;

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\widetilde{a}}(A)$ and $\{ikM(ikM-(1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. Define $N_k := (ikM - (1+a_k)A)^{-1}$ and $M_k := ikM(ikM - (1+a_k)A)^{-1}$, $k \in \mathbb{Z}$. By the Closed Graph Theorem we can show that if $ik \in \rho_{M,\widetilde{a}}(A)$, then M_k are bounded operators for each $k \in \mathbb{Z}$. By Proposition 2.4 it follows that (i) implies (ii). Conversely, by Theorem 2.5 is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. In fact, we note the following

$$\begin{aligned} k[M_{k+1} - M_k] &= k\big[i(k+1)MN_{k+1} - ikMN_k\big] \\ &= ikMN_{k+1}\big[(k+1)[ikM - (1+a_k)A] - \\ &\quad k[i(k+1)M - (1+a_{k+1})A]\big]N_k \qquad (4.2) \\ &= ikMN_{k+1}\big[k(a_{k+1} - a_k)A - (1+a_k)A\big]N_k \\ &= ikMN_{k+1}\left[k\tfrac{(a_{k+1}-a_k)}{1+a_k}\right](1+a_k)AN_k - M_k(1+a_k)AN_k. \end{aligned}$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular, the sequence $\{k(\frac{a_{k+1}-a_k}{1+a_k})\}_{k\in\mathbb{Z}}$ is bounded. The identity $(1+a_k)AN_k = M_k - I$, imply that $\{(1+a_k)AN_k\}_{k\in\mathbb{Z}}$ is R-bounded. We conclude the proof using the Remark 2.2. ∎

From Chapter 3, we recall that for a given closed operator M, and $1 \le p < \infty$, the set $H^{1,p}_{per,M}(\mathbb{R};[D(M)])$ is defined by

$$\begin{aligned} H^{1,p}_{per,M}(\mathbb{R};[D(M)]) &= \{u \in L^p_{2\pi}(\mathbb{R};[D(M)]) : \exists v \in L^p_{2\pi}(\mathbb{R};X), \\ &\qquad \hat{v}(k) = ikM\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}. \end{aligned}$$

Now, we introduce the following definition of solution for (4.1).

Definition 4.3 *We say that a function $u \in H^{1,p}_{per,M}(\mathbb{R};[D(M)]) \cap L^p_{2\pi}(\mathbb{R};[D(A)])$ is a strong L^p-solution of (4.1) if $u(t) \in D(A)$ and equation (4.1) holds for a.e. $t \in [0,2\pi]$.*

The following is the main result of this section. For $a \in L^1(\mathbb{R}_+)$, A a closed operator and $u \in L^p_{2\pi}(\mathbb{R};[D(A)])$ denote by $F(t) := (a\dot{*}Au)(t) = \int_{-\infty}^t a(t-s)Au(s)ds$. An easy computation show that $\hat{F}(k) = A\widetilde{a}(ik)\hat{u}(k), k \in \mathbb{Z}$, where the hat $\hat{}$ denotes Fourier transform (see [56] for a proof of this assertion).

Theorem 4.4 *Let X be a UMD space. Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, the following assertions are equivalent*

(i) For every $f \in L^p_{2\pi}(\mathbb{R}, X)$, there exist a unique strong L^p-solution of (4.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$;

(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.

Proof. $(i) \Rightarrow (ii)$. We follow the same lines of [8, Theorem 2.3] and [56]. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt}y$. By hypothesis, there exists $u \in H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + (a\dot{*}Au)(t) + f(t)$. Taking Fourier transform on both sides, we have $\hat{u}(k) \in D(A)$ and,

$$\begin{aligned} ikM\hat{u}(k) &= (1 + a_k)A\hat{u}(k) + \hat{f}(k) \\ &= (1 + a_k)A\hat{u}(k) + y. \end{aligned}$$

Thus, $(ikM - (1 + a_k)A)\hat{u}(k) = y$ for all $k \in \mathbb{Z}$ and we conclude that $(ikM - (1 + a_k)A)$ is surjective. Let $x \in D(A)$. If $(ikM - (1 + a_k)A)x = 0$, then $u(t) = e^{ikt}x$ defines a periodic solution of (4.1). Hence $u \equiv 0$ by the assumption of uniqueness, and thus $x = 0$. Therefore, $(ikM - (1 + a_k)A)$ is bijective. Following the same reasoning that in the proof of Theorem 3.3, we obtain that $ik \in \rho_{M,\tilde{a}}(A)$ for all $k \in \mathbb{Z}$. We will see that$\{ikM(ikM - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.

Using the Closed Graph Theorem, we have that there exist a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}; X)$ such that

$$||(Mu)'||_{L^p} + ||a\dot{*}Au||_{L^p} + ||Au||_{L^p} \leq C||f||_{L^p}.$$

Note that for $f(t) = e^{itk}y, y \in X$, the solution u of (4.1) is given by $u(t) = (ikM - (1 + a_k)A)^{-1}e^{ikt}y$. Hence,

$$||ikM(ikM - (1 + a_k)A)^{-1}y|| \leq C||y||.$$

We obtain that for $k \in \mathbb{Z}$, $ikM(ikM-(1+a_k)A)^{-1}$ is a bounded operator. Let $f \in L^p_{2\pi}(\mathbb{R}, X)$, by hypothesis, there exist $u \in H^{1,p}_{per,M}(\mathbb{R}; [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + (a*Au)(t) + f(t)$. Taking Fourier transform on both sides, and using that $(ikM - (1 + a_k)A)$ is invertible, we have $\hat{u}(k) \in D(A)$ and $\hat{u}(k) = (ikM - (1 + a_k)A)^{-1}\hat{f}(k)$. Now, since $u \in H^{1,p}_{per,M}(\mathbb{R}; [D(M)])$ and by definition of $H^{1,p}_{per,M}(\mathbb{R}; [D(M)])$, there exist $v \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{v}(k) = ikM\hat{u}(k)$ for all $k \in \mathbb{Z}$. Therefore, we have $\hat{v}(k) = ikM\hat{u}(k) = ikM(ikM - (1 + a_k)A)^{-1}\hat{f}(k)$.

$(ii) \Rightarrow (i)$. Define $M_k = ikM(ikM - (1 + a_k)A)^{-1}$ and $N_k = (ikM - (1 + a_k)A)^{-1}$. Suppose that $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. For $f \in L^p_{2\pi}(\mathbb{R}, X)$ there exist $u \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{u}(k) = ikM(ikM - (1 + a_k)A)^{-1}\hat{f}(k)$, for all $k \in \mathbb{Z}$. The identity $I = M_k - (1 + a_k)AN_k$ imply that

$$\begin{aligned}\hat{u}(k) &= ikM(ikM - (1 + a_k)A)^{-1}\hat{f}(k) \\ &= (I + (1 + a_k)AN_k)\hat{f}(k).\end{aligned}$$

So, we obtain $\widehat{(u - f)}(k) = (1 + a_k)AN_k\hat{f}(k)$. By Lemma 4.1, the sequence $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Thus, for $u - f \in L^p_{2\pi}(\mathbb{R}, X)$ there exists $v \in L^p_{2\pi}(\mathbb{R}, X)$ such that $\hat{v}(k) = \frac{1}{1+a_k}\widehat{(u - f)}(k) = AN_k\hat{f}(k)$. Since that $0 \in \rho_{M,\tilde{a}}(A)$ we obtain that $A^{-1} \in \mathcal{B}(X)$, and therefore $w := A^{-1}v \in L^p_{2\pi}(\mathbb{R}, X)$ and $\hat{w}(k) = N_k\hat{f}(k)$. Hence $ikM\hat{w}(k) - (1 + a_kA)\hat{w}(k) = \hat{f}(k)$. Now, observe that for all $k \in \mathbb{Z}$, we have

$$\hat{u}(k) = ikM(ikM - (1 + a_k)A)^{-1}\hat{f}(k) = ikM\hat{w}(k).$$

Thus $w \in H^{1,p}_{per,M}(\mathbb{R}, [D(M)]) \cap L^p_{2\pi}(\mathbb{R}; [D(A)])$. Moreover $Mw(0) = Mw(2\pi)$, since $w(0) = w(2\pi)$ and $w(t) \in D(A)$. Since A and M are closed operators and $\widehat{(Mw)'}(k) = ikM\hat{w}(k) = (1 + a_k)A\hat{w}(k) + \hat{f}(k)$, for all $k \in \mathbb{Z}$, one has $(Mw)'(t) = Aw(t) + (a*Au)(t) + f(t)\,a.e.$ by Lemmas 2.6 and 2.7. Thus, we conclude that w is a strong L^p-solution of (4.1). Finally, the uniqueness follows the same way as in the proof of Theorem 3.3.

$(ii) \Leftrightarrow (iii)$. Proposition 4.2. ■

In the Hilbert case, we obtain a simple condition to existence and uniqueness of solutions of (4.1). Since in Hilbert spaces the concepts of R-boundedness and boundedness are equivalent [33], the proof of the next Corollary follows from Theorem 4.4.

Corollary 4.5 *Let H be a Hilbert space, $A : D(A) \subset H \to H$, and $M : D(M) \subset H \to H$ closed linear operators satisfying $D(A) \subseteq D(M)$. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular. Then, for $1 < p < \infty$, the following assertions are equivalent*

(i) For every $f \in L^p_{2\pi}(\mathbb{R}, H)$, there exists a unique strong L^p-solution of (4.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1 + a_k)A)^{-1}|| < \infty$.

Note that the solution $u(\cdot)$ given in Theorem 4.4 satisfies the following maximal regularity property.

Corollary 4.6 *In the context of Theorem 4.4, if condition (iii) is fulfilled, we have $(Mu)'$, $Au, a\dot{*}Au \in L^p_{2\pi}(\mathbb{R}, X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p_{2\pi}(\mathbb{R}; X)$ such that*

$$||(Mu)'||_{L^p} + ||Au||_{L^p} + ||a\dot{*}Au||_{L^p} \leq C||f||_{L^p}. \tag{4.3}$$

Remark 4.7 The Fejer's Theorem (see [8, Proposition 1.1]) can be used to construct the solution u given in the Theorem 4.4. More precisely, if $M_k = ikM(ikM - (1 + a_k)A)^{-1}$ satisfies the condition (ii) or (iii) in the Theorem 4.4, then for $f \in L^p_{2\pi}(\mathbb{R}, X)$, the solution $u \in L^p_{2\pi}(\mathbb{R}, X)$ of (4.1) is given by

$$u(\cdot) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes M_k \hat{f}(k),$$

where $e_k(t) := e^{ikt}$, $t \in \mathbb{R}$ and the convergence holds in $L^p_{2\pi}(\mathbb{R}, X)$.

4.2 Maximal regularity on vector-valued Hölder and Besov spaces

In this section, we formulate analogous theorems to the Section 1, in the context of Hölder and Besov Spaces.

We study the existence and uniqueness of solutions to (4.1) in $B^s_{p,q}((0,2\pi);X)$, the vector-valued periodic Besov spaces for $1 \le p \le \infty, s > 0$, where X is a Banach space. As in the Chapter 3, is remarkable that in this case, there are no geometrical conditions on the Banach space X. Moreover, we recall that in the Chapter 3, some useful properties of $B^s_{p,q}((0,2\pi);X)$ are summarized and the definition and properties of operator-valued Fourier multipliers in the context of periodic Besov spaces are studied.

The following Proposition is the analogous version of the Proposition 4.2.

Proposition 4.8 *Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, the following assertions are equivalent*

(i) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM-(1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier for $1 \le p, q \le \infty$;

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM-(1+a_k)A)^{-1}|| < \infty$.

Proof. $(i) \Rightarrow (ii)$. Follows the same lines as the proof in [56, Proposition 3.4]. $(ii) \Rightarrow (i)$ For $k \in \mathbb{Z}$, define $M_k = ikM(ikM-(1+a_k)A)^{-1}$ and $N_k = (ikM-(1+a_k)A)^{-1}$. From the identity (4.2) we obtain:

$$\sup_{k\in\mathbb{Z}} ||k(M_{k+1}-M_k)|| < \infty, \tag{4.4}$$

proving (3.4). To verify (3.5), note that:

$$\begin{aligned}
&k^2[M_{k+1}-2M_k+M_{k-1}] = \\
&= ik^2M\Big[(k+1)N_{k+1}-2kN_k+(k-1)N_{k-1}\Big] \\
&= ik^2MN_{k+1}\Big[(k+1)N_k^{-1}-2kN_{k+1}^{-1}+(k-1)N_{k+1}^{-1}N_{k-1}N_k^{-1}\Big]N_k \\
&= ik^2MN_{k+1}\Big[(k+1)N_k^{-1}-2k\big[(ikM-(1+a_k)A)-(a_{k+1}-a_k)A+iM\big]+ \\
&\quad (k-1)\big[(i(k-1)M-(1+a_{k-1})A)+2iM-(a_{k+1}-a_{k-1})A\big]N_{k-1}N_k^{-1}\Big]N_k \\
&= ik^2MN_{k+1}\Big[(k+1)N_k^{-1}-2kN_k^{-1}+2k(a_{k+1}-a_k)A-2ikM+ \\
&\quad \big[(k-1)N_{k-1}^{-1}+2i(k-1)M-(k-1)(a_{k+1}-a_{k-1})A\big]N_{k-1}N_k^{-1}\Big]N_k \\
&= ik^2MN_{k+1}\Big[(k+1)I-2kI+2k(a_{k+1}-a_k)AN_k-2ikMN_k+ \\
&\quad \big[(k-1)I+2i(k-1)MN_{k-1}-(k-1)(a_{k+1}-a_{k-1})AN_{k-1}\big]\Big]
\end{aligned}$$

$$
\begin{aligned}
&= ik^2 MN_{k+1}\Big[2k(a_{k+1}-a_k)AN_k - 2(M_k - M_{k-1}) - (k-1)(a_{k+1}-a_{k-1})AN_{k-1}\Big] \\
&= ikMN_{k+1}\Big[2k^2(a_{k+1}-a_k)AN_k - 2k(M_k - M_{k-1}) - \\
&\qquad k(k-1)(a_{k+1}-a_{k-1})AN_{k-1}\Big] \\
&= ikMN_{k+1}\Big[2k^2\frac{(a_{k+1}-a_k)}{1+a_k}(M_k - I) - 2k(M_k - M_{k-1}) - \\
&\qquad k(k-1)\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}(M_{k-1}-I)\Big].
\end{aligned}
$$

Using the identities,

$$2k^2(a_{k+1}-a_k) = k^2(a_{k+1}-2a_k+a_{k-1}) + k^2(a_{k+1}-a_{k-1}),$$

and

$$k(k-1)(a_{k+1}-a_{k-1}) = k^2(a_{k+1}-a_{k-1}) - k(a_{k+1}-a_{k-1}),$$

we obtain

$$
\begin{aligned}
&2k^2\frac{(a_{k+1}-a_k)}{1+a_k}[M_k - I] - k(k-1)\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I] = \\
&= k^2\frac{(a_{k+1}-2a_k+a_{k-1})}{1+a_k}[M_k - I] + k^2\frac{(a_{k+1}-a_{k-1})}{1+a_k}\cdot \\
&\quad \cdot\Big[\frac{(M_k - M_{k-1}) + (a_k - a_{k-1})I + a_{k-1}M_k - a_kM_{k-1}}{1+a_{k-1}}\Big] + k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I] \\
&= k^2\frac{(a_{k+1}-2a_k+a_{k-1})}{1+a_k}[M_k - I] + k\frac{(a_{k+1}-a_{k-1})}{1+a_k}\Big[\frac{1}{1+a_{k-1}}k(M_k - M_{k-1}) \\
&\quad + k\frac{(a_k - a_{k-1})}{1+a_{k-1}}I + \frac{k}{1+a_{k-1}}[a_{k-1}M_k - a_kM_{k-1}]\Big] + k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}}[M_{k-1}-I].
\end{aligned}
$$

Since the identities

$$\frac{k}{1+a_{k-1}}\Big[a_{k-1}M_k - a_kM_{k-1}\Big] = \frac{a_{k-1}}{1+a_{k-1}}k[M_k - M_{k-1}] + k\frac{(a_{k-1}-a_k)}{1+a_{k-1}}M_{k-1},$$

$$k\frac{(a_{k+1}-a_{k-1})}{1+a_k} = k\frac{(a_{k+1}-a_k)}{1+a_k} + \frac{k}{k-1}(k-1)\frac{(a_k - a_{k-1})}{a_{k-1}}a_{k-1}\frac{1}{1+a_k},$$

$$k\frac{(a_{k+1}-a_{k-1})}{1+a_{k-1}} = k\frac{(a_{k+1}-a_k)}{1+a_{k-1}} + k\frac{(a_k - a_{k-1})}{1+a_{k-1}},$$

and

$$k[M_k - M_{k-1}] = \frac{k}{k-1}(k-1)[M_k - M_{k-1}],$$

are valid, and from the fact that $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is bounded and $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular, we conclude from the above identities and Remark 2.9 that,

$$\sup_{k\in\mathbb{Z}} ||k^2(M_{k+1} - 2M_k + M_{k-1})|| < \infty. \tag{4.5}$$

Thus, $\{M_k\}_{k\in\mathbb{Z}}$, satisfies the Marcinkiewicz condition of order 2 and therefore, by Theorem 3.9, $\{M_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier.

■

Lemma 4.9 *Let X be a Banach space. Suppose that the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier for $1 \le p, q \le \infty$.*

Proof. Define $m_k := \frac{1}{1+a_k}$, $k \in \mathbb{Z}$. By Remark 2.9, the sequence $\{m_k\}_{k\in\mathbb{Z}}$ is bounded. Moreover, $\{m_k\}_{k\in\mathbb{Z}}$ satisfies the identities,

$$k[m_{k+1} - m_k] = -k\frac{a_{k+1} - a_k}{1 + a_k}\frac{1}{1 + a_{k+1}},$$

and

$$\begin{aligned} k^2[m_{k+1} - 2m_k + m_{k-1}] &= -\frac{1}{(1 + a_{k+1})(1 + a_k)(1 + a_{k-1})}k^2(a_{k+1} - 2a_k + a_{k-1}) + \\ &\quad \frac{2}{(1 + a_{k+1})(1 + a_k)(1 + a_{k-1})}k(a_k - a_{k-1})k(a_{k+1} - a_k) - \\ &\quad \frac{a_k}{(1 + a_{k+1})(1 + a_k)(1 + a_{k-1})}k^2(a_{k+1} - 2a_k + a_{k-1}). \end{aligned}$$

Since $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular, we conclude that $\{m_k\}_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order 2 and therefore $\{m_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier. ■

As in the Chapter 3, we have the following definition.

Definition 4.10 *Let $1 \le p, q \le \infty$ and $s > 0$. A function $u \in B^s_{p,q}((0, 2\pi); [D(A)])$ is said to be a strong $B^s_{p,q}$-solution of (4.1) if $Mu \in B^{s+1}_{p,q}((0, 2\pi); X)$ and equation (4.1) holds for a.e. $t \in (0, 2\pi)$.*

The next Theorem, is the main result of this section and is the analogous version of Theorem 4.4 in the context of Besov spaces. We remark that there are no special conditions in the space X.

Theorem 4.11 *Let $1 \leq p, q \leq \infty$ and $s > 0$. Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular. Then, the following assertions are equivalent*

(i) For every $f \in B^s_{p,q}((0, 2\pi); X)$ there exist a unique strong $B^s_{p,q}$-solution of (4.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\widetilde{a}}(A)$ and $\{ikM(ikM - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier;

(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\widetilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1 + a_k)A)^{-1}|| < \infty$.

Proof. $(i) \Rightarrow (iii)$. Suppose that for every $f \in B^s_{p,q}((0, 2\pi); X)$ there exist a unique strong $B^s_{p,q}$-solution of (4.1). Fix $x \in X$ and $k \in \mathbb{Z}$. Define $f(t) = e^{itk}x$. Then $f \in B^s_{p,q}((0, 2\pi); X)$. By hypothesis there exist $u \in B^s_{p,q}((0, 2\pi); [D(A)])$ with $Mu \in B^{s+1}_{p,q}((0, 2\pi); X)$ such that $u(t) \in D(A)$ and $(Mu)'(t) = Au(t) + (a\dot{*}Au)(t) + f(t)$ a.e. $t \in (0, 2\pi)$. By Lemma 2.7 we have $ikM\hat{u}(k) = A\hat{u}(k) + a_kA\hat{u}(k) + x$. Following the same reasoning that in the proof of Theorem 4.4, we obtain that $ik \in \rho_{M,\widetilde{a}}(A)$ for all $k \in \mathbb{Z}$. Let $M_k := ikM(ikM-(1+a_k)A)^{-1}$. We will see that $\{M_k\}_{k\in\mathbb{Z}}$ is bounded. Using the Closed Graph Theorem, we have that there exist a constant C independent of f such that

$$||Mu||_{B^{s+1}_{p,q}((0,2\pi);X)} + ||Au||_{B^s_{p,q}((0,2\pi);[D(A)])} + ||a\dot{*}Au||_{B^s_{p,q}((0,2\pi);[D(A)])} \leq C||f||_{B^s_{p,q}((0,2\pi);X)}.$$

Note that for $f(t) = e^{itk}x$, the solution u of (4.1) is given by $u(t) = (ikM-(1+a_k)A)^{-1}e^{ikt}x$. Hence,

$$\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1 + a_k)A)^{-1}x|| \leq C||x||.$$

$(ii) \Rightarrow (i)$. Define $M_k = ikM(ikM - (1 + a_k)A)^{-1}$ and $N_k = (ikM - (1 + a_k)A)^{-1}$. Suppose that $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\widetilde{a}}(A)$ and $\{M_k\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier. For $f \in B^s_{p,q}((0, 2\pi); X)$ there exist $u \in B^s_{p,q}((0, 2\pi); X)$ such that $\hat{u}(k) = ikM(ikM-(1+a_k)A)^{-1}\hat{f}(k)$, for all $k \in \mathbb{Z}$. The identity $I = M_k - (1 + a_k)AN_k$ imply that

$$\begin{aligned} \hat{u}(k) &= ikM(ikM - (1 + a_k)A)^{-1}\hat{f}(k) \\ &= (I + (1 + a_k)AN_k)\hat{f}(k). \end{aligned}$$

So, we obtain $\widehat{(u-f)}(k) = (1+a_k)AN_k\hat{f}(k)$. By Lemma 4.9, the sequence $\{\frac{1}{1+a_k}I\}_{k\in\mathbb{Z}}$ is an $B^s_{p,q}$-multiplier. Thus, for $u-f \in B^s_{p,q}((0,2\pi);X)$ there exists $v \in B^s_{p,q}((0,2\pi);X)$ such that $\hat{v}(k) = \frac{1}{1+a_k}\widehat{(u-f)}(k) = AN_k\hat{f}(k)$. Since that $0 \in \rho_{M,\tilde{a}}(A)$ we obtain that $A^{-1} \in \mathcal{B}(X)$, and therefore $w := A^{-1}v \in B^s_{p,q}((0,2\pi);X)$ and $\hat{w}(k) = N_k\hat{f}(k)$. Hence $ikM\hat{w}(k) - (1+a_kA)\hat{w}(k) = \hat{f}(k)$. Observe that for all $k \in \mathbb{Z}$, we have

$$\hat{u}(k) = ikM(ikM - (1+a_k)A)^{-1}\hat{f}(k) = ikM\hat{w}(k).$$

Thus, by uniqueness of Fourier coefficients, $u(t) = (Mw)'(t)$. Since $u \in B^s_{p,q}((0,2\pi);X)$, then $(Mw)' \in B^s_{p,q}((0,2\pi);X)$ and therefore, $Mw \in B^{s+1}_{p,q}((0,2\pi);X)$. Moreover $Mw(0) = Mw(2\pi)$, since $w(0) = w(2\pi)$ and $w(t) \in D(A)$.

Since A and M are closed operators and $\widehat{(Mw)}'(k) = ikM\hat{w}(k) = (1+a_k)A\hat{w}(k) + \hat{f}(k)$, for all $k \in \mathbb{Z}$, one has $(Mw)'(t) = Aw(t) + (a \dot{*} Au)(t) + f(t)\, a.e.$ by Lemmas 2.6 and 2.7. We conclude that $w \in B^s_{p,q}((0,2\pi);X)$ is a strong $B^s_{p,q}$-solution to (4.1). Finally, the uniqueness follows the same way as in the proof of Theorem 3.3.

$(iii) \Leftrightarrow (ii)$. Follows from Proposition 4.8. ■

4.3 Maximal regularity on vector-valued Triebel-Lizorkin spaces

In this section, we study the existence and uniqueness of solutions to (4.1) in the context of Triebel-Lizorkin spaces; $F^s_{p,q}((0,2\pi);X)$, where X is a Banach space, $1 \le p,q \le \infty$ and $s \in \mathbb{R}$. More details of theses spaces can be found in [17] and the references therein.

The next definition and theorem are the analogous versions mentioned in the Sections 1 and 2.

Definition 4.12 *Let $1 \le p,q \le \infty, s \in \mathbb{R}$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is a $F^s_{p,q}$-multiplier if for each $f \in F^s_{p,q}((0,2\pi);X)$ there exists a function $g \in F^s_{p,q}((0,2\pi);Y)$ such that*

$$\hat{g}(k) = M_k\hat{f}(k), \quad k \in \mathbb{Z}.$$

We recall the following result due to Bu-Kim[17].

Theorem 4.13 ([17]) *Let X, Y be Banach spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. Assume that*

$$\sup_{k\in\mathbb{Z}} ||M_k|| < \infty, \ \sup_{k\in\mathbb{Z}} ||k(M_{k+1} - M_k)|| < \infty, \tag{4.6}$$

$$\sup_{k\in\mathbb{Z}} ||k^2(M_{k+1} - 2M_k + M_{k-1})|| < \infty, \tag{4.7}$$

$$\sup_{k\in\mathbb{Z}} ||k^3(M_{k+2} - 3M_{k+1} + 3M_k - M_{k-1})|| < \infty, \tag{4.8}$$

where $1 \leq p < \infty$ and $1 \leq q \leq \infty$, $s \in \mathbb{R}$. Then $\{M_k\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier.

Remark 4.14 We remark that, if X, Y are UMD spaces in the above theorem, then the conditions (4.6) and (4.7) are sufficient for $\{M_k\}_{k\in\mathbb{Z}}$ to be an $F^s_{p,q}$-multiplier.

The definition of solution of the equation (4.1) in the Triebel-Lizorkin spaces is the same that in the Besov case. The proof of following theorem is similar to Theorem 4.11. We omit the details.

Theorem 4.15 *Let $1 \leq p, q \leq \infty$ and $s > 0$. Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ be linear closed operators on a Banach space X. Suppose that $D(A) \subseteq D(M)$ and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is 3-regular. Then, the following assertions are equivalent*

(i) For every $f \in F^s_{p,q}((0, 2\pi); X)$ there exist a unique strong $F^s_{p,q}$-solution of (4.1);

(ii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\{ikM(ikM - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier;

(iii) $\{ik\}_{k\in\mathbb{Z}} \subset \rho_{M,\tilde{a}}(A)$ and $\sup_{k\in\mathbb{Z}} ||ikM(ikM - (1 + a_k)A)^{-1}|| < \infty$.

4.4 Applications

We conclude the chapter, with some applications of our results.

Example 4.16

Let us consider the boundary value problem

$$\begin{aligned} \frac{\partial(m(x)u(t,x))}{\partial t} - \Delta u &= \int_{-\infty}^{t} a(t-s)\Delta u(s,x)ds + f(t,x), \text{ in } [0,2\pi]\times\Omega \qquad (4.9)\\ u &= 0, \text{ in } [0,2\pi]\times\partial\Omega, \qquad (4.10)\\ m(x)u(0,x) &= m(x)u(2\pi,x) \text{ in } \Omega, \qquad (4.11)\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x)\geq 0$ is a given measurable bounded function on Ω and f is a function on $[0,2\pi]\times\Omega$.

Let M be the multiplication operator by m. If we take $X = H^{-1}(\Omega)$ then by [10, p.38] (see also references therein), we have that there exists a constant $c>0$ such that

$$||M(zM-\Delta)^{-1}|| \leq \frac{c}{1+|z|},$$

whenever $Re(z)\geq -c(1+|Im(z)|)$. Thus, the inequality

$$||ikM(ikM-((1+a_k)\Delta)^{-1}|| = \frac{|k|}{|1+a_k|}\left\|M\left(\frac{ik}{1+a_k}M-\Delta\right)^{-1}\right\| \leq c,$$

holds, if $Re\big(\frac{ik}{1+a_k}\big)\geq -c(1+|Im\big(\frac{ik}{1+a_k}\big)|)$, for all $k\in\mathbb{Z}$, that is, if

$$k\beta_k \geq -c((1+\alpha_k)^2+\beta_k^2+|k(1+\alpha_k)|), \qquad (4.12)$$

is valid for all $k\in\mathbb{Z}$, where α_k and β_k denotes the real and imaginary part of a_k, respectively. In particular, if $a(t) := \frac{t^{b-1}}{\Gamma(b)}$, with b an even integer, then one can check that $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular and $\beta_k = 0$ for all $k\in\mathbb{Z}$, thus the inequality (4.12) holds. Therefore, by Theorem 4.11 (or Corollary 4.5), we conclude that that for all $f\in L^p_{2\pi}(\mathbb{R},H^{-1}(\Omega))$ there exists a unique solution for (4.9)-(4.10).

Example 4.17

Consider, for $t\in[0,2\pi]$ and $x\in[0,\pi]$, the problem

$$\begin{aligned} \frac{\partial}{\partial t}\Big(\frac{\partial^2}{\partial x^2}+1\Big)u(t,x) &= -b\frac{\partial^2}{\partial x^2}u(t,x) - cu(t,x) + \qquad (4.13)\\ &\quad \int_{-\infty}^{t} a(t-s)\big(b\frac{\partial^2}{\partial x^2}+c\big)u(s,x)ds + f(t,x) \qquad (4.14)\\ u(t,0)=u(t,\pi) &= \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,\pi) = 0 \qquad (4.15)\\ \Big(\frac{\partial^2}{\partial x^2}+1\Big)u(0,x) &= \Big(\frac{\partial^2}{\partial x^2}+1\Big)u(2\pi,x), \qquad (4.16)\end{aligned}$$

where b is positive constant and $-2b < c < 4b$. If we take $X = C_0([0,\pi]) = \{u \in C([0,\pi]) : u(0) = u(\pi)\}$ and K the realization of $\frac{\partial^2}{\partial x^2}$ with domain

$$D(K) = \{u \in C^2([0,\pi]) : u(0) = u(\pi) = \frac{\partial^2}{\partial x^2}u(0) = \frac{\partial^2}{\partial x^2}u(\pi) = 0\},$$

then we take $M = K + I,\ A = bM + (c-b)I$. By [10, p.39, Ex.1.2] we have that:

$$||M(zM - A)^{-1}|| \leq \frac{d}{1+|z|}$$

for all $Re(z) \geq -d(1 + |Im(z)|)$, and d being a suitable positive constant. Therefore, as in the Example 4.17, if for all $k \in \mathbb{Z}$, the inequality

$$k\beta_k \geq -d((\alpha_k - 1)^2 + \beta_k^2 + |k(\alpha_k - 1)|), \tag{4.17}$$

is valid, then for all $f \in B^s_{p,q}((0,2\pi), C_0([0,\pi])), s > 0, 1 \leq p,q \leq \infty$, by Theorem 4.11, we conclude that the problem (4.13)-(4.16) has a unique strong solution u with regularity $\frac{\partial^2 u}{\partial x^2} \in B^s_{p,q}((0,2\pi), C_0([0,\pi]))$. In particular, if $a(t) := e^{\gamma t}$, where $\gamma \in \mathbb{R}$, we can check that $\{a_k\}_{k\in\mathbb{Z}}$ is 2-regular and the inequality (4.17) holds with $d = 1$.

Chapter 5

Bounded solutions for a class of semilinear integro-differential equations

In this chapter, we study the existence and uniqueness of bounded solutions for the semilinear integro-differential equation with infinite delay

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s)ds + f(t, u(t)) \quad t \in \mathbb{R}; \quad \alpha, \beta \in \mathbb{R},$$

where A be the generator of an immediately norm continuous C_0-semigroup defined on a Banach space X and $f : \mathbb{R} \times X \to X$ satisfy a Lipschitz type condition. Sufficient conditions are established for the existence and uniqueness of an almost periodic, almost automorphic and asymptotically almost periodic solution, among other types of distinguished solutions. These results have significance in viscoelasticity theory. Finally, an example is presented to illustrate the feasibility and effectiveness of the results.

5.1 The linear case

Let $\alpha, \beta \in \mathbb{R}$ be given. In this section we study bounded solutions for the linear integro-differential equation

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s)ds + f(t), \quad t \in \mathbb{R}, \tag{5.1}$$

where A generates an immediately norm continuous C_0-semigroup on a Banach space X. To begin our study, we note in the next proposition that under the given hypothesis on A, it is possible to construct for (5.1) a strongly continuous family of bounded and linear operators, that commutes with A and satisfy certain "resolvent equation". This class of strongly continuous families has been studied extensively in the literature of abstract Volterra equations; see e.g. Prüss [87] and references therein.

Proposition 5.1 *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that*
(a) A generates an immediately norm continuous C_0-semigroup on a Banach space X;
(b) $\sup \{ Re(\lambda), \lambda \in \mathbb{C} : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A) \} < 0$.

Then, there exists a uniformly exponentially stable and strongly continuous family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ such that $S(t)$ commutes with A, that is, $S(t)D(A) \subset D(A)$, $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \geq 0$ and

$$S(t)x = x + \int_{0}^{t} b(t-s)AS(s)xds, \text{ for all } x \in X, \ t \geq 0, \tag{5.2}$$

where $b(t) := 1 + \frac{\alpha}{\beta}[1 - e^{-\beta t}], \ t \geq 0$.

Proof. For $t \geq 0$ and $x \in X$ define $S(t)x := u(t;x)$ where $u(t;x)$ is the unique solution of equation (2.2). See [40, Corollary 7.22, p.449] for the existence of such solution and their strong continuity. We will see that $S(\cdot)x$ satisfies the resolvent equation (5.2). Since $S(t)x$ is the solution of (2.2), we have that $S(t)x$ is differentiable and satisfies

$$S'(t)x = AS(t)x + \alpha \int_{0}^{t} e^{-\beta(t-s)} AS(s)xds. \tag{5.3}$$

Integrating (5.3), we conclude from Fubini's theorem that,

$$\begin{aligned}
S(t)x - x &= \int_0^t AS(s)xds + \alpha \int_0^t \int_0^s e^{-\beta(s-\tau)} AS(\tau)xd\tau ds \\
&= \int_0^t AS(s)xds + \alpha \int_0^t \int_\tau^t e^{-\beta(s-\tau)} AS(\tau)xdsd\tau \\
&= \int_0^t AS(s)xds + \alpha \int_0^t \int_0^{t-\tau} e^{-\beta v} AS(\tau)xdvd\tau \\
&= \int_0^t AS(s)xds + \frac{\alpha}{\beta} \int_0^t (1 - e^{-\beta(t-\tau)}) AS(\tau)xd\tau \\
&= \int_0^t 1 + \frac{\alpha}{\beta}[1 - e^{-\beta(t-\tau)}] AS(\tau)xd\tau \\
&= \int_0^t b(t-\tau)AS(\tau)xd\tau.
\end{aligned}$$

The commutativity of $S(t)$ with A follows in the same way as that of [87, p. 31,32]. The uniform exponential stability follows from Theorem 2.14. ∎

We recall that a function $u \in C^1(\mathbb{R}; X)$ is called a strong solution of (5.1) on $\mathbb{R}$ if $u \in C(\mathbb{R}; D(A))$ and (5.1) holds for all $t \in \mathbb{R}$. On the other hand, if $f \in BC(X)$, the expression

$$u(t) = \int_{-\infty}^t S(t-s)f(s)ds,$$

for all $t \in \mathbb{R}$, where $\{S(t)\}_{t\geq 0}$ is given in the Proposition 5.1, is called a *mild solution* of (5.1).

Theorem 5.2 *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and*

$$\sup\left\{Re(\lambda) : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)\right\} < 0.$$

If $f \in \mathcal{N}(X)$, then the unique mild solution of the problem (5.1) belongs to the same space as that of f.

Proof. By Proposition 5.1, the family $\{S(t)\}_{t\geq 0}$ is uniformly exponentially stable and therefore u is well defined. Since S satisfies the resolvent equation

$$S(t)x = \int_0^t b(t-s)AS(s)xds + x, \; x \in X,$$

where $b(t) = 1 + \frac{\alpha}{\beta}[1 - e^{-\beta t}]$, we have that b is differentiable and the above equation shows that for each $x \in X$, $S'(t)x$ exists and

$$S'(t)x = AS(t)x + \alpha \int_0^t e^{-\beta(t-s)} AS(s)xds.$$

It remains to prove that u is a mild solution of (5.1). Since A is a closed operator, using Fubini's theorem, we have

$$\begin{aligned}
u'(t) &= S(0)f(t) + \int_{-\infty}^t S'(t-s)f(s)ds \\
&= f(t) + \int_{-\infty}^t \Big[AS(t-s)f(s) + \alpha \int_0^{t-s} e^{-\beta(t-s-\tau)} AS(\tau)f(s)d\tau\Big] ds \\
&= f(t) + \int_{-\infty}^t AS(t-s)f(s)ds + \alpha \int_{-\infty}^t \int_0^{t-s} e^{-\beta(t-s-\tau)} AS(\tau)f(s)d\tau ds \\
&= f(t) + Au(t) + \alpha \int_{-\infty}^t \int_s^t e^{-\beta(t-v)} AS(v-s)f(s)dvds \\
&= f(t) + Au(t) + \alpha \int_{-\infty}^t \int_{-\infty}^v e^{-\beta(t-v)} AS(v-s)f(s)dsdv \\
&= f(t) + Au(t) + \alpha \int_{-\infty}^t e^{-\beta(t-v)} \int_{-\infty}^v AS(v-s)f(s)dsdv \\
&= f(t) + Au(t) + \alpha \int_{-\infty}^t e^{-\beta(t-v)} Au(v)dv.
\end{aligned}$$

■

Note that, if $u(t) \in D(A)$ for all $t \in \mathbb{R}$, then a mild solution is a strong solution.

In the case of Hilbert spaces, we can use a result of You [90] which characterizes norm continuity of C_0-semigroups, obtaining the following result.

Corollary 5.3 *Let A be the generator of a C_0-semigroup on a Hilbert space H. Let $s(A) := \sup\{Re(\lambda) : \lambda \in \sigma(A)\}$ denote the spectral bound of A. Let $\beta > 0, \alpha \neq 0, \alpha + \beta > 0$ be given. Assume that*

(a) $\lim_{\mu \in \mathbb{R}, |\mu| \to \infty} ||(\mu_0 + i\mu - A)^{-1}|| = 0$ *for some* $\mu_0 > s(A)$;

(b) $\sup \big\{Re(\lambda) : \lambda(\lambda + \beta)(\lambda + \alpha + \beta)^{-1} \in \sigma(A)\big\} < 0$.

If $f \in \mathcal{N}(X)$, then the unique mild solution of the problem (5.1) belongs to the same space as that of f.

Remark 5.4 In the case $A = \rho I$, $\rho \in \mathbb{C}$ we obtain from (5.2), using Laplace transform, that for each $x \in X$:

$$S_\rho(t)x = e^{\frac{(\rho-\beta)t}{2}} \left(\cosh\left(\frac{t\sqrt{(\beta+\rho)^2+4\rho\alpha}}{2} \right) + \frac{\sinh\left(\frac{t}{2}\sqrt{(\beta+\rho)^2+4\rho\alpha} \right)(\beta+\rho)}{\sqrt{(\beta+\rho)^2+4\rho\alpha}} \right) x. \tag{5.4}$$

The following result shows the remarkable fact that in this case the conditions of the above abstract result can be considerably relaxed.

Theorem 5.5 *Let $A := \rho I$ where $\rho \in \mathbb{R}$ be given. Suppose that $\rho < \beta$ and $(\alpha+\beta)\rho < 0$. Let $f \in \mathcal{N}(X)$. Consider the equation*

$$u'(t) = \rho u(t) + \rho\alpha \int_{-\infty}^{t} e^{-\beta(t-s)} u(s)ds + f(t), \quad t \in \mathbb{R}. \tag{5.5}$$

Then the equation (5.5) has a unique solution u which belongs to the same space as that of f and is given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s)f(s)ds, \quad t \in \mathbb{R}, \tag{5.6}$$

where $\{S_\rho(t)\}_{t\geq 0}$ is defined by (5.4).

Proof. Let $f \in \Omega$ where Ω is one of the spaces in $\mathcal{N}(X)$. Since $A = \rho I$ generates an immediately norm continuous C_0-semigroup and $\sigma(A) = \{\rho\}$, we have that $\lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)$ if and only if $\lambda^2 + \lambda(\beta-\rho) - \rho(\alpha+\beta) = 0$. We claim that $S_\rho(t)$ is integrable. In fact, we can rewrite $S_\rho(t)$ in (5.4) as follows:

$$S_\rho(t) = \frac{1}{2}\left(e^{t\frac{(\rho-\beta)+c}{2}} + e^{t\frac{(\rho-\beta)-c}{2}} \right) + \frac{(\beta+\rho)}{2c}\left(e^{t\frac{(\rho-\beta)+c}{2}} - e^{t\frac{(\rho-\beta)-c}{2}} \right),$$

where $c = \sqrt{(\beta+\rho)^2+4\alpha\rho}$. Therefore,

$$|S_\rho(t)| \;\leq\; \frac{1}{2}\left(e^{tRe\left(\frac{(\rho-\beta)+c}{2}\right)} + e^{tRe\left(\frac{(\rho-\beta)-c}{2}\right)} \right) + \frac{|\beta+\rho|}{2c}\left(e^{tRe\left(\frac{(\rho-\beta)+c}{2}\right)} + e^{tRe\left(\frac{(\rho-\beta)-c}{2}\right)} \right),$$

where $\beta-\rho > c$ because $(\beta+\alpha)\rho < 0$ and $\beta > \rho$. Hence $\rho-\beta-c < \rho-\beta+c < 0$. We conclude that $S_\rho(t)$ is integrable, proving the claim. Therefore, by Theorem 5.2, there exists a unique solution of equation (5.5) which belongs to the same space as that of f and is explicitly given by (5.6). ■

Remark 5.6 Observe that the case $\alpha = 0$ implies $\rho < 0$.

Example 5.7

Let $\rho = -1, \alpha = 1, \beta = 1$. Hence, by Theorem 5.5, for any $f \in \mathcal{N}(X)$ there exists a unique solution $u \in \mathcal{N}(X)$ of the equation

$$u'(t) = -u(t) - \int_{-\infty}^{t} e^{s-t} u(s)ds + f(t), \quad t \in \mathbb{R}, \tag{5.7}$$

given by

$$u(t) = \int_{-\infty}^{t} e^{-(t-s)} \cos(t-s) f(s)ds, \quad t \in \mathbb{R},$$

since, $S_{-1}(t) = e^{-t}\cos(t)$.

Example 5.8

This example is taken from [25, Remark 3.9]. Let $\rho = 1$, $\alpha = -3$ and $\beta = 2$. From Theorem 5.5, if $f \in \mathcal{N}(X)$, then there exists a unique solution $u \in \mathcal{N}(X)$ of the equation

$$u'(t) = u(t) - 3\int_{-\infty}^{t} e^{-2(t-s)} u(s)ds + f(t), \quad t \in \mathbb{R}, \tag{5.8}$$

given by

$$u(t) = \int_{-\infty}^{t} S_1(t-s) f(s)ds, \quad t \in \mathbb{R},$$

where, $S_1(t) = e^{-\frac{t}{2}}\left(\cos\left(t\frac{\sqrt{3}}{2}\right) + \sqrt{3}\sin\left(t\frac{\sqrt{3}}{2}\right)\right)$. It is remarkable that even when in this case the associated C_0-semigroup $T(t)x = e^t x$ is not exponentially stable, the resolvent family $S_1(t)$ does have this property.

A complete description of the area in the plane where we can choose α and β in order to have exponential stability for $S_\rho(t)$ for $\rho \in \mathbb{R} \setminus \{0\}$, is shown in the following figure. Note that, depending on the sign of ρ, there are two distinguished cases.

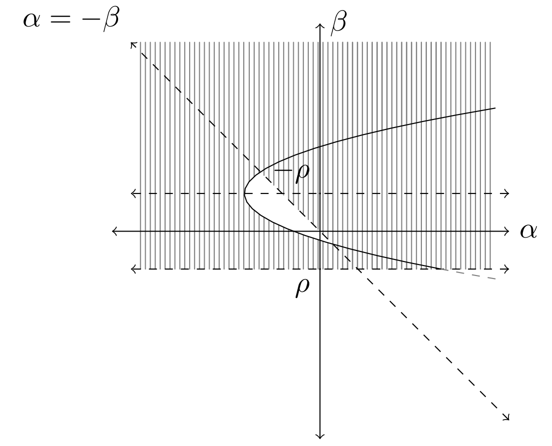

Figure 1: $\rho < 0$

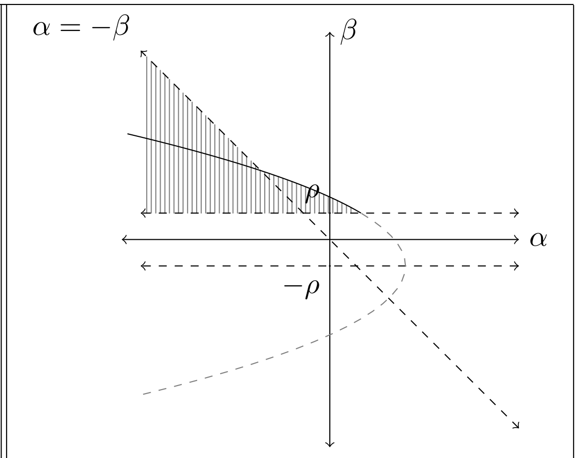

Figure 2: $\rho > 0$

Consider $A = \rho I$ for $\rho < 0$ and observe that the area shown hatched in Figure 1 includes the sector $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Hence, the area for exponential stability of $S_\rho(t)$ is considerably bigger than those guaranteed in Theorem 5.2. Note the exception of a sector located between the parabola $\beta^2 + 2\rho\beta + \rho^2 = -4\alpha\rho$ and the line $\alpha = -\beta$. Figure 2 considers the case $\rho > 0$. It shows the area where the stability of the C_0-semigroup is not necessary, in general, for the exponential stability of $S_\rho(t)$. In particular, note that the point $(-3, -2)$ belongs to the hatched area when $\rho = 1$ (cf. Example 5.8).

5.2 The semilinear problem

In this section we study the existence and uniqueness of solutions in $\mathcal{M}(X)$ for the semilinear integro-differential equation

$$u'(t) = Au(t) + \alpha \int_{-\infty}^{t} e^{-\beta(t-s)} Au(s)ds + f(t, u(t)), \quad t \in \mathbb{R}. \tag{5.9}$$

Definition 5.9 *A function $u : \mathbb{R} \to X$ is said to be a mild solution to equation (5.9) if*

$$u(t) = \int_{-\infty}^{t} S(t-s) f(s, u(s))ds,$$

for all $t \in \mathbb{R}$, where $\{S(t)\}_{t \geq 0}$ is given in Proposition 5.1.

Recall that $\mathcal{M}(X)$ denotes one of the spaces $P_\omega(X)$, $AP_\omega(X)$, $PP_\omega(X)$, $AP(X)$, $AAP(X)$, $PAP(X)$, $AA(X)$, $AAA(X)$ or $PAA(X)$ defined in Section 2.

If the hypothesis of Proposition 5.1 are valid, then there a strongly continuous family $\{S(t)\}_{t\geq 0}$ such that $||S(t)|| < Me^{-\omega t}$, where $M, \omega > 0$. Thus, the next theorem holds.

Theorem 5.10 *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and*

$$\sup\left\{Re(\lambda) : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)\right\} < 0. \tag{5.10}$$

If $f \in \mathcal{M}(\mathbb{R} \times X, X)$ satisfies

$$||f(t,u) - f(t,v)|| \leq L||u - v||, \tag{5.11}$$

for all $t \in \mathbb{R}$ and $u, v \in X$, with $L < \frac{\omega}{M}$. Then the equation (5.9) has a unique mild solution $u \in \mathcal{M}(X)$.

Proof. Define the operator $F : \mathcal{M}(X) \mapsto \mathcal{M}(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^{t} S(t-s)f(s,\varphi(s))\,ds, \quad t \in \mathbb{R}. \tag{5.12}$$

By Theorems 2.11 and 2.12 we have that F is well defined. For $\varphi_1, \varphi_2 \in \mathcal{M}(X)$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned}
||(F\varphi_1)(t) - (F\varphi_2)(t)|| &\leq \int_{-\infty}^{t} ||S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]||ds \\
&\leq LM\int_{-\infty}^{t} e^{-\omega(t-s)}\| \cdot \|\varphi_1(s) - \varphi_2(s)\|ds \\
&= LM\|\varphi_1 - \varphi_2\|_\infty \int_0^\infty e^{-\omega\tau}d\tau \\
&= L\frac{M}{\omega}||\varphi_1 - \varphi_2||_\infty.
\end{aligned}$$

Hence, by the contraction principle F has a unique fixed point $u \in \mathcal{M}(X)$. ∎

An immediate consequence are the following corollaries.

Corollary 5.11 *Let $\beta > 0, \alpha \neq 0$ and $\alpha + \beta > 0$. Assume that A generates an immediately norm continuous C_0-semigroup on a Banach space X and the spectral condition (5.10). If $f \in AP(\mathbb{R} \times X, X)$ (resp. $AA(\mathbb{R} \times X, X)$) satisfies the Lipschitz condition (5.11) with $L < \frac{\omega}{M}$, then the equation (5.9) has a unique mild almost periodic solution (resp. almost automorphic solution).*

In Hilbert spaces, we have the following result.

Corollary 5.12 *Let A be the generator of a C_0-semigroup on a Hilbert space H. Let $s(A) := \sup\{Re(\lambda) : \lambda \in \sigma(A)\}$ denote the spectral bound of A. Let $\beta > 0, \alpha \neq 0, \alpha + \beta > 0$ be given. Assume that*

(a) $\lim_{\mu \in \mathbb{R}, |\mu| \to \infty} ||(\mu_0 + i\mu - A)^{-1}|| = 0$ *for some* $\mu_0 > s(A)$;

(b) $\sup\{Re(\lambda) : \lambda(\lambda + \beta)(\lambda + \alpha + \beta)^{-1} \in \sigma(A)\} < 0$.

If $f \in \mathcal{M}(\mathbb{R} \times H, H)$ satisfies

$$||f(t,u) - f(t,v)|| \leq L||u - v||,$$

for all $t \in \mathbb{R}$ and $u, v \in X$, where $L < \frac{\omega}{M}$. Then the equation (5.9) has a unique mild solution $u \in \mathcal{M}(H)$.

In the special case $A = \rho I$ we obtain the following consequence of Theorem 5.5.

Theorem 5.13 *Let $A := \rho I$ where $\rho \in \mathbb{R}$ be given. Suppose that $\rho < \beta$ and $(\alpha + \beta)\rho < 0$. Let $f \in \mathcal{M}(\mathbb{R} \times X, X)$. Consider the equation*

$$u'(t) = \rho u(t) + \rho\alpha \int_{-\infty}^{t} e^{-\beta(t-s)} u(s)ds + f(t, u(t)) \quad t \in \mathbb{R}. \tag{5.13}$$

Then the equation (5.13) has a unique solution $u \in \mathcal{M}(X)$ given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t - s) f(s, u(s))ds, \quad t \in \mathbb{R}, \tag{5.14}$$

where $\{S_\rho(t)\}_{t \geq 0}$ is defined in (5.4).

Remark 5.14 Observe that in case $\alpha = 0$ we must have $\rho < 0$ and hence we recover results on existence of solutions for the equation

$$u'(t) = \rho u(t) + f(t, u(t)), \quad t \in \mathbb{R}$$

in the spaces previously defined. See for instance [64, 49] and [78].

5.3 An application

We finish this chapter with the following application.

Example 5.15

Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \displaystyle\int_{-\infty}^{t} e^{-(t-s)} \dfrac{\partial^2 u}{\partial x^2}(s,x) ds + f(t, u(t)) \\ u(0,t) = u(\pi, t) = 0, \end{cases} \tag{5.15}$$

with $x \in [0,\pi], t \in \mathbb{R}$. Let $X = L^2[0,\pi]$ and define $A := \frac{\partial^2}{\partial x^2}$, with domain $D(A) = \{g \in H^2[0,\pi] : g(0) = g(\pi) = 0\}$. Then (5.15) can be converted into the abstract form (5.9) with $\alpha = \beta = 1$. It is well known that A generates an immediately norm continuous C_0- semigroup $T(t)$ on X and $\sigma(A) = \{-n^2 : n \in \mathbb{N}\}$.

Since we must have $\lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)$ we need to solve the equations $\frac{\lambda(\lambda+1)}{\lambda+2} = -n^2$, obtaining (see [25])

$$\lambda_1 = -1 \pm i, \; \lambda_2 = \frac{-5 \pm 7i}{2},$$

and

$$\lambda_n = \frac{-(n^2+1) \pm \sqrt{(n^2-3)^2 - 8}}{2} \leq -2,$$

for all $n \geq 3$. We conclude that

$$\sup\left\{Re(\lambda) : \lambda(\lambda+\beta)(\lambda+\alpha+\beta)^{-1} \in \sigma(A)\right\} = -1.$$

Hence, by Proposition 5.1, there exists $M, \omega > 0$ such that $||T(t)|| \leq Me^{-\omega t}$ for all $t \geq 0$ and from Theorem 5.10 we obtain that if $f \in \mathcal{M}(\mathbb{R} \times X, X)$ satisfies

$$||f(t,u) - f(t,v)|| \leq L||u - v||,$$

for all $t \in \mathbb{R}$ and $u, v \in X$, where $L < \frac{\omega}{M}$, then equation (5.15) has a unique mild solution $u \in \mathcal{M}(X)$. In particular, if $f(t,\varphi)(s) = b(t)\sin(\varphi(s))$, for all $\varphi \in X$, $t \in \mathbb{R}$ with $b \in \mathcal{M}(X)$, then we observe that $t \to f(t,\varphi)$ belongs to $\mathcal{M}(X)$, for each $\varphi \in X$, and we have

$$||f(t,\varphi_1) - f(t,\varphi_2)||_2^2 \leq \int_0^\pi |b(t)|^2 |\sin(\varphi_1(s)) - \sin(\varphi_2(s))| ds \leq |b(t)|^2 ||\varphi_1 - \varphi_2||_2^2.$$

In consequence, problem (5.15) has a unique mild solution in $\mathcal{M}(X)$ if $||b||_\infty < 1$, (by Theorem 5.10).

Chapter 6

Almost automorphic solutions for a class of Volterra equations

Given $a \in L^1(\mathbb{R})$ and A a closed linear operator defined on a Banach space X, we prove in this chapter, the existence of an almost automorphic mild solution to the semilinear integral equation $u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s, u(s))]ds$ for each $f : \mathbb{R} \times X \to X$ S^p-almost automorphic in t, uniformly in $x \in X$, and satisfying a Lipschitz type condition. For the scalar linear case, we prove that $a \in L^1(\mathbb{R})$ completely monotonic is already sufficient.

6.1 Almost automorphic solutions for the linear equation

In this section we consider the existence and uniqueness of almost automorphic solutions to the evolution equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + g(s)]ds, \quad t \in \mathbb{R}, \tag{6.1}$$

where A is the generator of an integral resolvent family and $a \in L^1(\mathbb{R})$.

Proposition 6.1 *Let $a \in L^1(\mathbb{R})$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ on X, which satisfies*

$$||S(t)|| \leq \phi(t), \quad \textit{for all } t \in \mathbb{R}_+,$$

where $\phi \in L^1(\mathbb{R}_+)$ is nonincreasing. If $f \in AS^1(X)$ and takes values on $D(A)$ then the unique bounded solution of the problem

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{6.2}$$

is almost automorphic and is given by

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s)ds, \quad t \in \mathbb{R}. \tag{6.3}$$

Proof. Since $f(t) \in D(A)$ for all $t \in \mathbb{R}$, we obtain $u(t) \in D(A)$ for all $t \in \mathbb{R}$ (see [87, Proposition 1.2]). Then applying (2.4) and Fubini's theorem we obtain

$$\begin{aligned}
\int_{-\infty}^{t} a(t-s)Au(s)ds &= \int_{-\infty}^{t} a(t-s)A\int_{-\infty}^{s} S(s-\tau)f(\tau)d\tau ds \\
&= \int_{-\infty}^{t}\int_{-\infty}^{s} a(t-s)AS(s-\tau)f(\tau)d\tau ds \\
&= \int_{-\infty}^{t}\int_{\tau}^{t} a(t-s)AS(s-\tau)f(\tau)dsd\tau \\
&= \int_{-\infty}^{t}\int_{0}^{t-\tau} a(t-\tau-p)AS(p)dpf(\tau)d\tau \\
&= \int_{-\infty}^{t} (S(t-\tau)f(\tau) - a(t-\tau)f(\tau))d\tau \\
&= u(t) - \int_{-\infty}^{t} a(t-\tau)f(\tau)d\tau.
\end{aligned}$$

The statement follows by Lemma 2.21. ∎

Recall that a C^∞-function $f : (0,\infty) \to \mathbb{R}$ is called completely monotonic if

$$(-1)^n f^{(n)}(\lambda) \geq 0, \text{ for all } \lambda > 0, n \in \mathbb{N}_0.$$

We remark that such functions naturally occur in areas such as probability theory, numerical analysis, and elasticity. Our main result for the case $X = \mathbb{R}$ is the following theorem. It is remarkable that the hypotheses given, are completely based on the data of the problem.

Theorem 6.2 *Let $a \in L^1(\mathbb{R}_+)$ be a scalar, completely monotonic function on $\mathbb{R}_+$. Let $\rho > 0$ be given. If $g \in AS^1(\mathbb{R})$ then:*

a) There is $S_\rho \in L^1(\mathbb{R}_+)$ completely monotonic such that equation (2.4) is satisfied with $A = -\rho$.

b) The equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[-\rho u(s) + g(s)]ds, \quad t \in \mathbb{R}, \tag{6.4}$$

has an unique almost automorphic solution given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s)g(s)ds, \quad t \in \mathbb{R}.$$

Proof. By the hypothesis on the scalar kernel $a(t)$ and [51, Theorem 2.8, p.147] we have that $\log(a)$ is convex on $\mathbb{R}_+$. Moreover, since $a(t)$ is positive and nonincreasing, it follows by [87, Lemma 4.1, p.98] that there exists $S_\rho \in L^1(\mathbb{R}_+)$ completely monotone, such that equation (2.4) is satisfied with $A = -\rho$, that is

$$S_\rho(t) = a(t) - \rho \int_0^t a(t-s)S_\rho(s)ds. \tag{6.5}$$

Hence (a) follows. Part (b) is an immediate consequence of Lemma 2.21, since S_ρ is nonincreasing. ■

In the case where $g \in AA(\mathbb{R})$ we have the following result that improves [32, Corollary 3.7]. We recall that $\widetilde{a}(\lambda)$ denotes the Laplace transform of $a(t)$.

Theorem 6.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be an almost automorphic function and let $\rho > 0$ be a real number. Suppose $a \in L^1(\mathbb{R}_+)$, and $\widetilde{a}(\lambda) \neq -\frac{1}{\rho}$ for all $Re(\lambda) \geq 0$. Then*

a) There is $S_\rho \in L^1(\mathbb{R}_+)$ such that equation (2.4) is satisfied with $A = -\rho$;

b) The equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[-\rho u(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{6.6}$$

has a unique almost automorphic solution given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

Proof. The proof is a direct consequence of the half-plane Paley-Wiener theorem [51, Theorem 4.1 p.45] and [32, Lemma 3.1] (see also the references therein). ∎

6.2 Almost automorphic mild solutions for nonlinear equations

In this section we consider the existence and uniqueness of almost automorphic mild solutions to the nonlinear evolution equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s, u(s))]ds, \quad t \in \mathbb{R}, \tag{6.7}$$

where A is the generator of an integral resolvent family and $a \in L^1(\mathbb{R})$. The following definition is motivated by the linear case.

Definition 6.4 *Let A be the generator of an integral resolvent family $\{S(t)\}_{t\geq 0}$. A function $u : \mathbb{R} \to X$ is said to a mild solution to the equation (6.7) if the function $s \to S(t-s)f(s, u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and*

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s, u(s))ds, \text{ for all } t \in \mathbb{R}. \tag{6.8}$$

The following result gives conditions under which we have the existence of a unique almost automorphic mild solution with S^p- almost automorphic terms. Note that we assume that the function f is bounded by a Lipschitz function $L(t)$ with respect to the first argument uniformly in the second argument. Moreover, we have to assume a control on the S^1-norm of the Lipschitz function.

Theorem 6.5 *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ such that*

$$\|S(t)\| \leq \phi(t), \text{ for all } t \geq 0,$$

where $\phi \in L^1(\mathbb{R}_+)$ is nonincreasing with $0 < \phi_0 := \sum_{k=0}^{\infty} \phi(k) < \infty$. Suppose that

(i) $f \in AS^p(\mathbb{R} \times X, X)$ with $p > 1$;

(ii) there exists a nonnegative function $L \in AS^r(\mathbb{R})$ with $r \geq max\{p, p/(p-1)\}$ such that for all $u, v \in X$ and $t \in \mathbb{R}$,

$$||f(t,u) - f(t,v)|| \leq L(t)||u - v||.$$

If $||L||_{S^1} < \phi_0^{-1}$, then the equation (6.7) has a unique almost automorphic mild solution.

Proof. We define the operator $F : AA(X) \mapsto AA(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^{t} S(t-s)f(s,\varphi(s))\,ds, \quad t \in \mathbb{R}. \tag{6.9}$$

Since $\varphi \in AA(X)$, we have that $\overline{\{\varphi(t) : t \in \mathbb{R}\}}$ is compact in X. By Theorem 2.22, there exists $q \in [1, p)$ such that $f(\cdot, \varphi(\cdot)) \in AS^q(X) \subset AS^1(X)$. Then, by Lemma 2.21, we conclude that F is well defined. Then for $\varphi_1, \varphi_2 \in AA(X)$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned}
\|F\varphi_1(t) - F\varphi_2(t)\| &\leq \int_{-\infty}^{t} ||S(t-s)|| \cdot ||f(s,\varphi_1(s)) - f(s,\varphi_2(s))||ds \\
&\leq \int_{0}^{\infty} L(t-\tau)\|S(\tau)\| \cdot \|\varphi_1(t-\tau) - \varphi_2(t-\tau)\|d\tau \\
&\leq ||\varphi_1 - \varphi_2||_\infty \int_{0}^{\infty} L(t-\tau)\phi(\tau)d\tau \\
&= ||\varphi_1 - \varphi_2||_\infty \sum_{k=0}^{\infty} \int_{k}^{k+1} L(t-\tau)\phi(\tau)d\tau \\
&\leq ||\varphi_1 - \varphi_2||_\infty \sum_{k=0}^{\infty} \phi(k) \int_{k}^{k+1} L(t-\tau)d\tau \\
&= ||\varphi_1 - \varphi_2||_\infty \phi_0 \int_{t-k}^{t-k-1} L(s)ds \\
&\leq ||\varphi_1 - \varphi_2||_\infty \phi_0 ||L||_{S^1}.
\end{aligned}$$

This proves that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in AA(X)$, such that $Fu = u$, that is $u(t) = \int_{-\infty}^{t} S(t-s)f(s,u(s))ds.$

∎

We remark that in the case of $L(t) \equiv L$, by following the proof of previous theorem, one can get the same conclusion.

6.3 Applications

We finish the chapter with some applications of the above results.

Example 6.6

Let $a(t) = e^{-bt}, b > 0$ and $\rho > 0$. Then $a(t)$ is completely monotonic and $\widetilde{a}(\lambda) = \frac{1}{\lambda+b} \neq -\frac{1}{\rho}$ for all $Re(\lambda) \geq 0$. Moreover a direct calculation using the Laplace transform gives $S_\rho(t) = e^{-(b+\rho)t}$. Hence for any $g \in AS^1(\mathbb{R})$ (resp. $g \in AA(\mathbb{R})$) there exists a unique almost automorphic solution of the equation

$$u(t) = \int_{-\infty}^{t} e^{-b(t-s)}[-\rho u(s) + g(s)]ds, \quad t \in \mathbb{R},$$

given by

$$u(t) = \int_{-\infty}^{t} e^{(t-s)(b+\rho)} g(s)ds, \quad t \in \mathbb{R}.$$

The remarkable fact is that we only need $g \in AS^1(\mathbb{R})$ instead of $g \in AA(\mathbb{R})$ to have the existence of almost automorphic solutions.

Example 6.7

Let $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-bt}, b > 0, \alpha > 0$ and $\rho > 0$. We note that $a(t)$ is not completely monotonic but, under the condition $\cos(\pi/\alpha) \leq \frac{b}{\rho^{1/\alpha}}$ and since $(-\rho)^{\frac{1}{\alpha}} = [\cos(\frac{\pi}{\alpha}) + i\sin(\frac{\pi}{\alpha})]\rho^{\frac{1}{\alpha}}$, we have $\lambda \neq (-\rho)^{\frac{1}{\alpha}} - b$ for all $Re(\lambda) \geq 0$, that is $\widetilde{a}(\lambda) = \frac{1}{(\lambda+b)^\alpha} \neq -\frac{1}{\rho}$, for all $Re(\lambda) \geq 0$. A calculation using the Laplace transform shows $S_\rho(t) = t^{\alpha-1}e^{-bt}E_{\alpha,\alpha}(\rho t^\alpha)$, where $E_{\alpha,\alpha}$ denotes the generalized Mittag-Leffler function (see e.g. [50]).

Hence for any $g \in AA(\mathbb{R})$ there exists a unique almost automorphic solution of the equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} e^{-b(t-s)}[-\rho u(s) + g(s)]ds, \quad t \in \mathbb{R},$$

given by

$$u(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} e^{-b(t-s)} E_{\alpha,\alpha}(\rho(t-s)^\alpha) g(s)ds, \quad t \in \mathbb{R}.$$

Note that this example improves [32, Example 3.6] where only $1 < \alpha < 2$ was considered. In fact, $1 < \alpha < 2$ implies immediately that the more general condition $\cos(\pi/\alpha) \leq \frac{b}{\rho^{1/\alpha}}$ holds.

Example 6.8

Let $b > 0$ and consider the perturbed problem

$$u(t) = \int_{-\infty}^{t} e^{-b(t-s)}[Bu(s) + bu(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{6.10}$$

where B is the generator of an exponentially stable C_0-semigroup $T(t)$. Taking $a(t) = e^{-bt}$ and $A := B + bI$ we obtain (6.10) in the form of equation (6.2). By (2.4) we have

$$\widetilde{S}(\lambda) = (\frac{1}{\widetilde{a}(\lambda)} - A)^{-1} = (\lambda + b - B - b)^{-1} = (\lambda - B)^{-1} = \widetilde{T}(\lambda).$$

Hence, by uniqueness of the Laplace transform, we obtain in this case that the integral resolvent $S(t)$ is identical to the C_0-semigroup $T(t)$. We conclude by Proposition 6.1 that given $f \in AS^1(X)$ taking values on $D(B)$, the unique bounded solution of the problem (6.10) is almost automorphic and given by

$$u(t) = \int_{-\infty}^{t} T(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

Chapter 7

Mild solutions to abstract differential equations involving the Weyl fractional derivative

In this chapter, we study the existence and uniqueness of bounded mild solutions to the fractional differential equation ${}_{-\infty}D^{\alpha}u(t) = Au(t) + f(t, u(t))$, $t \in \mathbb{R}$, where $\alpha > 0$, A is the generator of an α-resolvent family and the fractional derivative is taken in the sense of Weyl. Sufficient conditions for the existence of mild solutions in wide classes of functions spaces are provided.

Definition 7.1 ([5]) *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X and $\alpha > 0$. We say that A is the generator of an α-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \;\; Re(\lambda) > \omega, x \in X.$$

In this case, $\{S_\alpha(t)\}_{t\geq 0}$ is called the α-resolvent family generated by A.

We remark that, by the uniqueness of Laplace transform, a 1-resolvent family is a C_0-semigroup, whereas that a 2-resolvent family corresponds to a sine family. As in [5, Proposition 2.4] we have the following Proposition.

Proposition 7.2 *Let $\alpha > 0$ and $\{S_\alpha(t)\}_{t\geq 0}$ be an α-resolvent family on X with generator A. Then, the following holds:*

(a) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$;

(b) Let $x \in D(A)$ and $t \geq 0$. Then,

$$S_\alpha(t)x = g_\alpha(t)x + \int_0^t g_\alpha(t-s)AS_\alpha(s)xds.$$

In particular, $\frac{d}{dt}S_\alpha(t)x$ exists.

(c) Let $x \in X$ and $t \geq 0$. Then $\int_0^t g_\alpha(t-s)S_\alpha(s)xds \in D(A)$ and

$$S_\alpha(t)x = g_\alpha(t)x + A\int_0^t g_\alpha(t-s)S_\alpha(s)xds.$$

In particular $S_\alpha(0) = g_\alpha(0)$.

7.1 The linear case

We study in this section the existence of solutions for the linear fractional differential equation

$$_{-\infty}D^\alpha u(t) = Au(t) + f(t), \quad t \in \mathbb{R},\ \alpha > 0, \tag{7.1}$$

where A generates an α-resolvent family.

We recall that a function $u \in \mathcal{S}$ is said to be a *strong solution* to equation (7.1) on $\mathbb{R}$, if $u(t) \in D(A)$ and (7.1) holds for all $t \in \mathbb{R}$.

The following is the main result in this section.

Theorem 7.3 *Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, for some $\alpha > 0$ satisfying*

$$||S_\alpha(t)|| \leq \phi_\alpha(t),\ t \geq 0,$$

where $\phi_\alpha \in \mathcal{S}(\mathbb{R}_+;\mathbb{R})$. If $f \in \mathcal{S}$ and takes values on $D(A)$, then the unique strong solution of the equation (7.1) belongs to $\mathcal{S}$ and is given by

$$u(t) = \int_{-\infty}^{t} S_\alpha(t-s)f(s)ds,\ t \in \mathbb{R}.$$

Proof. Let $u(t) := \int_{-\infty}^{t} S_\alpha(t-s)f(s)ds$, $t \in \mathbb{R}$. Since $f(t) \in D(A)$ for all $t \in \mathbb{R}$, we obtain $u(t) \in D(A)$ for all $t \in \mathbb{R}$ (see [87, Proposition 1.2]). Let $k, m \in \mathbb{N} \cup \{0\}$. Since $\{S_\alpha(t)\}_{t\geq 0}$ is integrable, we have by the dominated convergence theorem that

$$||t^m u^{(k)}(t)|| \leq \int_0^\infty \phi_\alpha(s)||t^m f^{(k)}(t-s)||ds.$$

As in [89, p. 142] one can prove that $f \in \mathcal{S}$ imply that u belongs to $\mathcal{S}$. Now, we need to verify that u is a strong solution of (7.1). Let $n = [\alpha]+1$. We obtain by the Proposition 7.2 and from Fubini's theorem that

$$\begin{aligned}
{}_{-\infty}D^\alpha u(t) &= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)u(s)ds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} S_\alpha(s-r)f(r)drds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\Big[g_\alpha(s-r)f(r) + \\
&\qquad \int_0^{s-r} g_\alpha(s-r-v)AS_\alpha(v)f(r)dv\Big]drds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\,{}_{-\infty}D^{-\alpha}f(s)ds + \\
&\qquad \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_0^{s-r} g_\alpha(s-r-v)AS_\alpha(v)f(r)dvdrds.
\end{aligned}$$

Since ${}_{-\infty}D^{-\alpha}\,{}_{-\infty}D^\alpha f(t) = f(t)$, for all $t \in \mathbb{R}$, we obtain

$$\begin{aligned}
{}_{-\infty}D^\alpha u(t) &= f(t) + \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_0^{s-r} g_\alpha(s-r-v)AS_\alpha(v)f(r)dvdrds \\
&= f(t) + \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_r^{s} g_\alpha(s-w)AS_\alpha(w-r)f(r)dwdrds \\
&= f(t) + \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_{-\infty}^{w} g_\alpha(s-w)AS_\alpha(w-r)f(r)drdwds \\
&= f(t) + \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} g_\alpha(s-w)\int_{-\infty}^{w} AS_\alpha(w-r)f(r)drdwds.
\end{aligned}$$

Since A is a closed operator and $u \in \mathcal{S}$ we conclude

$$\begin{aligned} {}_{-\infty}D^{\alpha}u(t) &= f(t) + \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} g_{\alpha}(s-w)Au(w)dwds \\ &= f(t) + \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\,{}_{-\infty}D^{-\alpha}Au(s)ds \\ &= f(t) + Au(t). \end{aligned}$$

■

7.2 Bounded mild solutions to semilinear case

In this section we study the existence and uniqueness of solutions in $\mathcal{M}(X)$ for the semilinear differential equation

$$ {}_{-\infty}D^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},\ \alpha > 0. \tag{7.2}$$

The following definition is motivated by the linear case.

Definition 7.4 *Let A be the generator of an α-resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$. A function $u : \mathbb{R} \to X$ is said to be a mild solution to equation (7.2) if the function $s \to S_{\alpha}(t-s)f(s, u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and*

$$u(t) = \int_{-\infty}^{t} S_{\alpha}(t-s)f(s, u(s))ds,$$

for each $t \in \mathbb{R}$.

Remark 7.5 We note that, as in [5, Remark 3.3], one can check that the above definition of mild solution to (7.2) is the same that the usual concept of mild solution in the cases $\alpha = 1$ and $\alpha = 2$. We emphasize that by Theorem 7.3, the above definition of mild solution for (7.2) is the natural extension for $\alpha > 0$. It happens in view of the fractional derivative in the sense of Weyl that we are considering in this chapter.

Recall that $\mathcal{M}(X)$ denote one of the spaces $P_{\omega}(X)$, $AP_{\omega}(X)$, $PP_{\omega}(X)$, $SAP_{\omega}(X)$, $AP(X)$, $AAP(X), PAP(X), AA(X), AAA(X)$ or $PAA(X)$ defined in section 5 of chapter 2. The following Theorem is an extension of [78, Theorem 3.2] in the case $\alpha = 1$.

Theorem 7.6 *Let $\alpha > 0$. Assume that A generates an α-resolvent family satisfying*

$$||S_\alpha(t)|| \leq \phi_\alpha(t),\ t \geq 0,$$

where $\phi_\alpha \in L^1(\mathbb{R}_+)$. If $f \in \mathcal{M}(\mathbb{R} \times X, X)$ satisfies

$$||f(t,u) - f(t,v)|| \leq L||u - v||, \tag{7.3}$$

for all $t \in \mathbb{R}$ and $u, v \in X$, where $L < ||\phi_\alpha||_1^{-1}$. Then the equation (7.2) has a unique mild solution $u \in \mathcal{M}(X)$.

Proof. Define the operator $F : \mathcal{M}(X) \mapsto \mathcal{M}(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^{t} S_\alpha(t-s) f(s, \varphi(s))\, ds, \quad t \in \mathbb{R}. \tag{7.4}$$

By Theorems 2.11 and 2.12 we have that F is well defined. For $\varphi_1, \varphi_2 \in \mathcal{M}(X)$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned} ||(F\varphi_1)(t) - (F\varphi_2)(t)|| &\leq \int_{-\infty}^{t} ||S_\alpha(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]||ds \\ &\leq \int_{-\infty}^{t} L\|S_\alpha(t-s)\| \cdot \|\varphi_1(s) - \varphi_2(s)\|ds \\ &\leq L\|\varphi_1 - \varphi_2\|_\infty \int_{-\infty}^{t} \phi_\alpha(t-s)ds \\ &= L\|\varphi_1 - \varphi_2\|_\infty ||\phi_\alpha||_1. \end{aligned}$$

This prove that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in \mathcal{M}(X)$ such that $Fu = u$. ■

7.3 Applications

To illustrate the above results, we conclude the chapter with the following applications.

Example 7.7

Let $\rho > 0$ be a real number and $1 < \alpha < 2$. As in [5, page 3701], we denote

$$l(\alpha, \rho) := \left(\frac{2}{\alpha \rho^{\alpha}} - \frac{1}{\rho^{\alpha}} - \frac{2}{\alpha \rho^{\alpha} \cos(\pi/\alpha)} \right)^{-1}.$$

If f belongs to some of space of $\mathcal{M}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and satisfies

$$||f(t,x) - f(t,y)|| \leq L||x - y||, \tag{7.5}$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}$, with $L < l(\alpha, \rho)$, then the equation

$${}_{-\infty}D^{\alpha}u(t) = -\rho^{\alpha}u(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

has a unique mild solution which belongs to the same space as that of f.

Example 7.8

Let $\alpha > 0$. Suppose that A generates an α-resolvent family satisfying

$$||S_{\alpha}(t)|| \leq \phi_{\alpha}(t),\ t \geq 0,$$

where $\phi_{\alpha} \in L^1(\mathbb{R}_+)$. Suppose that $f(t, u) \equiv g(t)$ for all $t \in \mathbb{R}$ and $u \in X$. If g belongs to some of space of $\mathcal{M}(X)$, we conclude from the Theorem 7.6 that the equation

$${}_{-\infty}D^{\alpha}u(t) = Au(t) + g(t), \quad t \in \mathbb{R}, \tag{7.6}$$

has a unique mild solution which belongs to the same space as that of g.

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