ASYMPTOTICALLY ALMOST AUTOMORPHIC SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In the present work, we introduce the concept of asymptotically almost automorphic functions on time scales and study their main properties. We study nonautonomous dynamic equations on time scales given by $x^{\Delta}(t) = A(t)x(t) + f(t)$ and $x^{\Delta}(t) = A(t)x(t) + f(t, x(t))$, $t \in \mathbb{T}$, where $\mathbb{T}$ is an invariant under translations time scale and $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$. We give new criteria ensuring the existence of an asymptotically almost automorphic solution for both equations.

1. Introduction

The theory of time scales is a recent subject of research, which was introduced by Stefan Hilger (see [19]). The study of time scales and their associated properties have proved to be a fruitful area of research over the past years. See, for instance, [1,4,5,12,16,20–22,24,26–28, 30,31] and the references therein. This is in part due to the interesting mathematical theory that has resulted from these investigations and also, due to the worthwhile applications that have arisen from them. It plays an important role to model realistic problems such as economics, population, physics (specially quantum physics), technology, among others. See, for instance, [5,12,24,31].

The qualitative properties of the solutions of dynamic equations on time scales have been attracting the attention of several researchers, specially concerning their periodicity. Periodic dynamic equations on time scales have been treated by many authors. See, for instance, [1,4,20,26,30]. Further, almost periodicity on time scales was formally introduced by Y. Li and C. Wang (2011) in [27] and after that, this theory has been a powerful source of research for other authors and interesting advances have been obtained using this concept. See, for instance, [16,21,22,28] and the references therein. On the other hand, the concept of almost automorphic functions on time scales was introduced formally recently in the literature by the authors in the paper [29]. Among others, in that paper, results concerning existence and uniqueness of almost automorphic solutions of nonautonomous dynamic equations on time scales were proved.

Meanwhile, the theory of asymptotically almost automorphic functions in continuous time is now a classical object of research that has been extensively investigated and important

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contributions have been obtained, for example, in the papers [6–11, 13–15, 25] and in the books [17, 18].

However, to the best of our knowledge, the concept of asymptotically almost automorphic functions on time scales as well as their main structural properties have not been introduced and investigated in the literature until now. Therefore, the purpose of this paper is to contribute to filling this important gap.

Since *dynamic equation on time scales* generalizes differential equation and difference equations, it is natural and desirable to introduce both from a theoretical point of view as from a practical, to investigate the properties of asymptotically almost automorphic functions on time scales such as the existence of asymptotically almost automorphic solutions of linear and nonlinear dynamic equations on time scales.

In this work, we will state and prove for the first time, structural theorems concerning the existence of asymptotically almost automorphic solutions of dynamic equations on time scales and prove some results concerning exponential dichotomies for such systems.

It is remarkable that in the discrete case, that is, which leads to difference equations, and which is included in our analysis, the concept of asymptotically almost automorphic functions on discrete time corresponds also to a new object of research in the literature. Therefore, we have included this case in several places as example for the development of this work.

The outline of this paper is as follows: the section 2 is devoted to present the preliminaries about the theory of time scales. In section 3, we present the definition of asymptotically almost automorphic functions on time scales (Definition 3.13) and we prove structural results concerning their main properties. Also, we investigate some remarkable properties of invariant under translations time scales. Then we present some structural results concerning the new theory of almost automorphic functions on time scales. They unify and complement the theory initiated by [29]. In section 4, we deal with the linear dynamic equation on time scales

$$x^{\Delta}(t) = A(t)x(t) + f(t), \quad t \in \mathbb{T},$$

where $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ and $f : \mathbb{T} \to \mathbb{R}^n$. We prove a general existence result of an asymptotically almost automorphic solution (Theorem 4.2). Finally, in section 5, we prove a result concerning existence and uniqueness of an asymptotically almost automorphic solution of

$$x^{\Delta}(t) = A(t)x(t) + f(t, x), \quad t \in \mathbb{T},$$

where $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ (Theorem 5.1).

2. Preliminaries

In this section, we present some basic concepts and results concerning time scales which will be essential to prove our main results. For more details, the reader may want to consult [2,3].

Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$.

Definition 2.1 (See [2, Definition 1.1])**.** For every $t \in \mathbb{T}$, we define the *forward* and *backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T} : \; s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : \; s < t\}.$$

In this definition, let us consider $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say that t is *right-scattered*. Otherwise, t is called *right-dense*. If $\rho(t) < t$, then t is called *left-scattered*, while if $\rho(t) = t$, then t is *left-dense*.

Definition 2.2. The *graininess function* $\mu : \mathbb{T} \to [0, +\infty)$ and the *backward graininess function* $\nu : \mathbb{T} \to [0, +\infty)$ are given, respectively, by

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = t - \rho(t) \ \text{ for every } \ t \in \mathbb{T}.$$

Definition 2.3 (See [2, Definition 1.57])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.4 (See [2, Definition 1.58])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$. We denote the class of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$.

If $f : \mathbb{T} \to \mathbb{R}$ is continuous at each right-dense point and each left-dense point, then f is said to be *continuous* on $\mathbb{T}$.

Given a pair of numbers $a, b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$, that is, $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T};\ a \leq t \leq b\}$. On the other hand, $[a, b]$ is the usual closed interval on the real line, that is, $[a, b] = \{t \in \mathbb{R};\ a \leq t \leq b\}$. We also denote $\mathbb{T}_+ = \mathbb{T} \cap [0, \infty)$.

We define the set $\mathbb{T}^\kappa$ which is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.5 (See [2, Definition 1.10])**.** Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$, we define the *delta-derivative* of f to be a number (if it exists) with the following property: given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(t) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s| \ \text{ for all } \ s \in U.$$

Definition 2.6. A *partition* of $[a, b]_{\mathbb{T}}$ is a finite sequence of points

$$\{t_0, t_1, \ldots, t_m\} \subset [a, b]_{\mathbb{T}} \ \text{ such that } \ a = t_0 < t_1 < \ldots < t_m = b.$$

Given such a partition, we put $\Delta t_i = t_i - t_{i-1}$. A tagged partition consists of a partition and a sequence of tags $\{\xi_1, \ldots, \xi_m\}$ such that $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$ for every $i \in \{1, \ldots, m\}$. The set of all tagged partitions of $[a, b]_{\mathbb{T}}$ will be denoted by the symbol $D(a, b)$.

If $\delta > 0$, then we denote by $D_\delta(a, b)$ the set of all tagged partitions of $[a, b]_{\mathbb{T}}$ such that for every $i \in \{1, \ldots, m\}$, either $\Delta t_i \leq \delta$, or $\Delta t_i > \delta$ and $\sigma(t_{i-1}) = t_i$. We point out that in the last case, the only way to choose a tag in $[t_{i-1}, t_i)_{\mathbb{T}}$ is to take $\xi_i = t_{i-1}$.

In the sequel, we present the definition of Riemann Δ-integrals. See [2, 3], for instance.

Definition 2.7. We say that $f : \mathbb{T} \to \mathbb{R}$ is *Riemann Δ-integrable* on $[a, b]_{\mathbb{T}}$, if there exists a number I with the following property: for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sum_i f(\xi_i)(t_i - t_{i-1}) - I \right| < \varepsilon,$$

for every $P \in D_\delta(a, b)$ independently of $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$ for $1 \leq i \leq n$. It is clear that such a number I is unique and is the Riemann Δ-integral of f from a to b.

Definition 2.8 (See [2, Definition 2.25])**.** We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided
$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}$$
holds. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Now, let us define some operations in $\mathcal{R}(\mathbb{T}, \mathbb{R})$. Let $p, q \in \mathbb{T}$, then
$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \;\; \text{for all} \;\; t \in \mathbb{T}^{\kappa}$$
and
$$(\ominus p)(t) := \frac{-p(t)}{1 + \mu(t)p(t)} \;\; \text{for all} \;\; t \in \mathbb{T}^{\kappa}.$$
Clearly, $(\mathcal{R}, \oplus)$ is an Abelian group.

Definition 2.9 (See [2, Definition 2.30])**.** If $p \in \mathcal{R}$, then we define the *generalized exponential function* by
$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for} \;\; s, t \in \mathbb{T},$$
where the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is given by
$$\xi_h(z) = \frac{1}{h}\log(1 + zh),$$
where log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$. For the properties of the generalized exponential function, see [2, Theorem 2.36].

In what follows, we recall some definitions about matrix-valued functions on time scales. It can be found in [2].

Definition 2.10. Let A be an $n \times n$ matrix-valued function on $\mathbb{T}$. We say that A is *rd-continuous* on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$. We denote the class of all rd-continuous $n \times n$ matrix-valued function on $\mathbb{T}$ by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$.

We say that A is *delta-differentiable* at $\mathbb{T}$ if each entry of A is delta-differentiable on $\mathbb{T}$. In this case, we have
$$A^{\sigma}(t) = A(\sigma(t)) = A(t) + \mu(t)A^{\Delta}(t).$$

Definition 2.11. An $n \times n$ matrix-valued function A on a time scale $\mathbb{T}$ is called *regressive* (with respect to $\mathbb{T}$) provided
$$I + \mu(t)A(t) \;\; \text{is invertible} \;\; \text{for all} \;\; t \in \mathbb{T}^{\kappa},$$
and the class of all such regressive rd-continuous is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Let $A, B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$. Define $A \oplus B$ by
$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t) \;\; \text{for all} \;\; t \in \mathbb{T}^{\kappa}$$
and
$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) \;\; \text{for all} \;\; t \in \mathbb{T}^{\kappa}.$$
Clearly, $(\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n}), \oplus)$ is a group (see [2]).

Definition 2.12. (Fundamental Matrix) Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}$ is an $n \times n$ matrix valued function. The unique matrix-valued solution of the IVP

$$\begin{cases} Y^\Delta(t) = A(t)Y(t) \\ Y(t_0) = I, \end{cases}$$

where I denotes as usual the $n \times n$-identity matrix, is called the *fundamental matrix* at t_0 and it is denoted by $e_A(\cdot, t_0)$. For the properties of the fundamental matrix, see [2, Theorem 5.21].

Theorem 2.13 (Variation of Constants Formula [2, Theorem 5.21]). *Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$\begin{cases} y^\Delta(t) = A(t)y(t) + f(t) \\ y(t_0) = y_0 \end{cases} \tag{2.1}$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. Moreover, this solution is given by

$$y(t) = e_A(t, t_0)y_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Definition 2.14 (See [32, Definition 3.1]). Let $A(t)$ be $n \times n$ rd-continuous matrix-valued function on $\mathbb{T}$. We say that the linear system

$$x^\Delta(t) = A(t)x(t) \tag{2.2}$$

has an *exponential dichotomy* on $\mathbb{T}$ if there exist positive constants K and γ and a projection P such that

$$|\mathcal{X}(t)P\mathcal{X}^{-1}(s)| \leq Ke_{\ominus\gamma}(t, s), \quad s, t \in \mathbb{T}, \quad t \geq s,$$
$$|\mathcal{X}(t)(I - P)\mathcal{X}^{-1}(s)| \leq Ke_{\ominus\gamma}(s, t), \quad s, t \in \mathbb{T}, \quad t \leq s,$$

where $\mathcal{X}(t)$ is the fundamental solution matrix of (2.2) and I is the $n \times n$ identity matrix.

We will assume that the projection P commutes with $\mathcal{X}(t)$ for every $t \in \mathbb{T}$ in the previous definition.

Theorem 2.15 (See [2, Theorem 2.39]). *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c, \cdot)]^\Delta = -p[e_p(c, \cdot)]^\sigma$$

and

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Lemma 2.16 (See [28, Lemma 5.1]). *Let $\gamma > 0$, then for any fixed $s \in \mathbb{T}$ and $s = -\infty$, one has the following*

$$e_{\ominus\gamma}(t, s) \to 0 \quad as \quad t \to +\infty.$$

Lemma 2.17 (See [29, Theorem 2.14]). *If $\gamma > 0$, then $0 < e_{\ominus\gamma}(t, s) \leq 1$ for $t, s \in \mathbb{T}$ such that $t > s$.*

The next result describes the solution of the equation

$$x^\Delta(t) = A(t)x(t) + f(t). \tag{2.3}$$

Theorem 2.18 (See [27, Lemma 2.13]). *If the linear system* (2.2) *admits exponential dichotomy, then the system* (2.3) *has a bounded solution* $x(t)$ *as follows:*

$$x(t) = \int_{-\infty}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))f(s)\Delta s,$$

where $\mathcal{X}(t)$ *is the fundamental solution matrix of* (2.2).

3. Asymptotically almost automorphic functions on time scales

In this section, we introduce the concept of asymptotically almost automorphic functions on time scales and present their properties.

We start by recalling a definition of an invariant under translations time scale (see [29]). Throughout the paper, $\mathbb{T}$ will denote a time scale.

Definition 3.1. A time scale $\mathbb{T}$ is called *invariant under translations* if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T},\ \forall t \in \mathbb{T}\} \neq \{0\} \quad \text{and} \quad \Pi \neq \emptyset. \tag{3.1}$$

In the sequel, we present a result which ensures that an invariant under translations time scale preserves the properties of its elements when they are shifted by an element of Π.

Lemma 3.2. *Let* $\mathbb{T}$ *be invariant under translations. If* t *is right-dense, then for every* $h \in \Pi$, $t + h$ *is right-dense. Similarly, if* t *is right-scattered, then for every* $h \in \Pi$, $t + h$ *is right-scattered.*

Proof. In fact, if t is right-dense, then there exists a sequence (t_n) such that $t_n \in \mathbb{T}$ and

$$\lim_{n\to\infty} t_n = t \quad \text{and} \quad t_n > t \text{ for every } n \in \mathbb{N}.$$

Since $\mathbb{T}$ is invariant under translations, we obtain

$$t_n + h \in \mathbb{T} \quad \text{and} \quad t_n + h > t + h,$$

for every $h \in \Pi$. Then, we have

$$\lim_{n\to\infty} t_n + h = t + h$$

and thus, it follows that $\sigma(t + h) = t + h$, which implies that $t + h$ is right-dense. Now, suppose t is right-scattered and $t + h$ is right-dense for some $h \in \Pi$. Then, there exists $\gamma_n \in \mathbb{T}$ such that

$$\lim_{n\to\infty} \gamma_n = t + h \quad \text{and} \quad \gamma_n > t + h. \tag{3.2}$$

By the invariance of the time scale, it follows that $\gamma_n - h \in \mathbb{T}$ and by (3.2), we get

$$\gamma_n - h > t \quad \text{and} \quad \lim_{n\to\infty} \gamma_n - h = t.$$

It implies that $\sigma(t) = t$, contradicting the fact that t is right-scattered. Therefore, $t + h$ is right-scattered for every $h \in \Pi$. □

In what follows, we present a result which ensures the invariance of forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by the set Π, whenever the time scale is invariant under translations.

Lemma 3.3. *Let* $\mathbb{T}$ *be invariant under translations and* $h \in \Pi$. *For every* $t \in \mathbb{T}$, *we have*

$$\sigma(t) + h = \sigma(t + h) \quad \textit{and} \quad \sigma(t) - h = \sigma(t - h). \tag{3.3}$$

Proof. If t is right-dense, then (3.3) follows immediately by Lemma 3.2. On the other hand, if t is right-scattered, then for $h \in \Pi$, $t+h$ is also right-scattered by Lemma 3.2, and $\sigma(t+h) > t+h$ by the definition. Then $\sigma(t+h) - h > t$ for every $t \in \mathbb{T}$. Since $\mathbb{T}$ is invariant under translations and $h \in \Pi$, it follows that $\sigma(t+h) - h \in \mathbb{T}$. By the definition of forward jump operator, it follows that $\sigma(t) \leq \sigma(t+h) - h$, which implies $\sigma(t+h) \geq \sigma(t) + h$. Reciprocally, since t is right-scattered, we have $\sigma(t) > t$. Then,

$$\sigma(t) + h > t + h. \tag{3.4}$$

Again, it is clear that $\sigma(t) + h \in \mathbb{T}$, then by (3.4) and from the definition of forward jump operator, it follows that $\sigma(t) + h \geq \sigma(t+h)$ for every $t \in \mathbb{T}$. Combining these two inequalities, we obtain the desired result. Similarly, one can prove the other equality $\sigma(t) - h = \sigma(t-h)$. □

Corollary 3.4. *Let $\mathbb{T}$ be invariant under translations and $h \in \Pi$. For every $t \in \mathbb{T}$, we have*

$$\mu(t+h) = \mu(t) = \mu(t-h)$$

Remark 3.5. Clearly, we have the analogues to the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ and the backward graininess function $\nu : \mathbb{T} \to [0, +\infty)$.

From now on, let us assume that X is a Banach space.

Definition 3.6 (See [29, Definition 3.15])**.** Let X be (real or complex) Banach space and $\mathbb{T}$ be an invariant under translation time scale. Then, an rd-continuous function $f : \mathbb{T} \to X$ is called *almost automorphic* on $\mathbb{T}$ if for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} f(t + \alpha_n) = \bar{f}(t)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{f}(t - \alpha_n) = f(t),$$

for every $t \in \mathbb{T}$.

Theorem 3.7. *Let $\mathbb{T}$ be invariant under translations and $f : \mathbb{T} \to X$ be almost automorphic function, then the composition $f \circ \sigma : \mathbb{T} \to X$ is also almost automorphic function.*

Proof. Since f is almost automorphic function, for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} f(t + \alpha_n) = \bar{f}(t)$$

is well-defined and exists for every $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{f}(t - \alpha_n) = f(t)$$

is well-defined and exists for every $t \in \mathbb{T}$. Thus, by Lemma 3.3, we get

$$\lim_{n\to\infty} f(\sigma(t + \alpha_n)) = \lim_{n\to\infty} f(\sigma(t) + \alpha_n) = \bar{f}(\sigma(t)),$$

since $\sigma(t) \in \mathbb{T}$ and f is almost automorphic. Reciprocally, we have

$$\lim_{n\to\infty} \bar{f}(\sigma(t - \alpha_n)) = \lim_{n\to\infty} \bar{f}(\sigma(t) - \alpha_n) = f(\sigma(t)),$$

for every $t \in \mathbb{T}$. Therefore, it follows that $f \circ \sigma$ is almost automorphic function. □

Theorem 3.8. *Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ be almost automorphic on time scales, that is, for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that*

$$\lim_{n\to\infty} A(t+\alpha_n) = \bar{A}(t)$$

exists and is well-defined for every $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{A}(t-\alpha_n) = A(t)$$

exists and is well-defined for every $t \in \mathbb{T}$. Let $\mathbb{T}$ be invariant under translations, then the fundamental matrix $e_A(t,s)$ of

$$\mathcal{X}^{\Delta}(t) = A(t)\mathcal{X}(t)$$

is also almost automorphic with relation to both variables. More precisely, for every sequence $(\alpha'_n) \in \Pi$, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} e_A(t+\alpha_n, s+\alpha_n) = e_{\bar{A}}(t,s)$$

exists and is well-defined for every $t, s \in \mathbb{T}$ and

$$\lim_{n\to\infty} e_{\bar{A}}(t-\alpha_n, s-\alpha_n) = e_A(t,s)$$

exists and is well-defined for every $t, s \in \mathbb{T}$.

Proof. The fundamental matrix of this system is given by $e_A(t,t_0)$. By the definition, we have

$$e_A(t+\alpha_n, s+\alpha_n) = \exp\left(\int_{s+\alpha_n}^{t+\alpha_n} \xi_{\mu(\tau)}(A(\tau))\Delta\tau\right).$$

Consider initially two cases: (i) $\mu(\tau) = 0$ for every $\tau \in [s+\alpha_n, t+\alpha_n]_{\mathbb{T}}$ and (ii) $\mu(\tau) > 0$ for every $\tau \in [s+\alpha_n, t+\alpha_n]_{\mathbb{T}}$.

If $\mu(\tau) = 0$ for every $\tau \in [s+\alpha_n, t+\alpha_n]_{\mathbb{T}}$, then

$$\begin{aligned}
\lim_{n\to\infty} e_A(t+\alpha_n, s+\alpha_n) &= \lim_{n\to\infty}\left[\exp\left(\int_{s+\alpha_n}^{t+\alpha_n} \xi_0(A(\tau))\Delta\tau\right)\right] \\
&= \lim_{n\to\infty}\left[\exp\left(\int_{s+\alpha_n}^{t+\alpha_n} A(\tau)\Delta\tau\right)\right] = \lim_{n\to\infty}\left[\exp\left(\int_{s}^{t} A(\tau+\alpha_n)\Delta\tau\right)\right] \\
&= \exp\left(\int_s^t \bar{A}(\tau)\Delta\tau\right) = \exp\left(\int_s^t \xi_0(\bar{A}(\tau))\Delta\tau\right) = e_{\bar{A}}(t,s),
\end{aligned}$$

for every $s, t \in \mathbb{T}$. Reciprocally, we obtain

$$\begin{aligned}
\lim_{n\to\infty} e_{\bar{A}}(t-\alpha_n, s-\alpha_n) &= \lim_{n\to\infty}\exp\left(\int_{s-\alpha_n}^{t-\alpha_n} \xi_0(\bar{A}(\tau))\Delta\tau\right) \\
&= \lim_{n\to\infty}\exp\left(\int_s^t \bar{A}(\tau-\alpha_n)\Delta\tau\right) = \exp\left(\int_s^t A(\tau)\Delta\tau\right) = e_A(t,s),
\end{aligned}$$

for every $t, s \in \mathbb{T}$. Thus, in this case, $e_A(t,s)$ is almost automorphic with respect to both variables. Now, consider $\mu(\tau) > 0$ for every $\tau \in [s+\alpha_n, t+\alpha_n]_{\mathbb{T}}$, then

$$\begin{aligned}
\lim_{n\to\infty} e_A(t+\alpha_n, s+\alpha_n) &= \lim_{n\to\infty} \exp\left(\int_{s+\alpha_n}^{t+\alpha_n} \xi_{\mu(\tau)}(A(\tau))\Delta\tau\right)\\
&= \lim_{n\to\infty} \exp\left(\int_{s+\alpha_n}^{t+\alpha_n} \frac{1}{\mu(\tau)}\log(I+\mu(\tau)A(\tau))\Delta\tau\right)\\
&= \lim_{n\to\infty} \exp\left(\int_{s}^{t} \frac{1}{\mu(\tau+\alpha_n)}\log(I+\mu(\tau+\alpha_n)A(\tau+\alpha_n))\Delta\tau\right)\\
&= \exp\left(\int_{s}^{t} \frac{1}{\mu(\tau)}\log(I+\mu(\tau)\bar{A}(\tau))\Delta\tau\right) = e_{\bar{A}}(t,s)
\end{aligned}$$

for every $s, t \in \mathbb{T}$. Moreover,

$$\begin{aligned}
\lim_{n\to\infty} e_{\bar{A}}(t-\alpha_n, s-\alpha_n) &= \lim_{n\to\infty} \exp\left(\int_{s-\alpha_n}^{t-\alpha_n} \frac{1}{\mu(\tau)}\log(I+\mu(\tau)\bar{A}(\tau))\Delta\tau\right)\\
&= \lim_{n\to\infty} \exp\left(\int_{s}^{t} \frac{1}{\mu(\tau-\alpha_n)}\log(I+\mu(\tau-\alpha_n)\bar{A}(\tau-\alpha_n))\Delta\tau\right)\\
&= \exp\left(\int_{s}^{t} \frac{1}{\mu(\tau)}\log(I+\mu(\tau)A(\tau))\Delta\tau\right) = e_A(t,s),
\end{aligned}$$

for every $s, t \in \mathbb{T}$. Notice that in the case that there exist both $\tau_1, \tau_2 \in [s+\alpha_n, t+\alpha_n]_{\mathbb{T}}$ such that $\mu(\tau_1) = 0$ and $\mu(\tau_2) > 0$, it is only to combine the arguments presented previously for each case and the result follows as well. □

Now, we introduce the concept of asymptotically almost automorphic functions on time scales.

Definition 3.9. Let X be a (real or complex) Banach space, $\mathbb{T}$ be invariant under translations and $f : \mathbb{T} \to X$. We say that f is an *asymptotically almost automorphic function on time scales* if there is an almost automorphic function $f_1 : \mathbb{T} \to X$ and a continuous function $f_2 : \mathbb{T}_+ \to X$ such that $\lim_{t\to\infty} \|f_2(t)\| = 0$ and

$$f(t) = f_1(t) + f_2(t),$$

for every $t \in \mathbb{T}_+$. We say that f_1 and f_2 are called, respectively, the *principal* and *corrective* terms of the function f. We denote the set of all asymptotically almost automorphic functions $f : \mathbb{T}_+ \to X$ by $AAA_{\mathbb{T}}(X)$.

Remark 3.10. Clearly, every almost automorphic function on time scale restricted to $\mathbb{T}_+$ is asymptotically almost automorphic on time scales. It is enough to consider the corrective term equals zero.

The next result is a direct consequence of the definition. We omit its proof here, since it follows immediately.

Theorem 3.11. *Let $\mathbb{T}$ be invariant under translations and $f, f_1 : \mathbb{T}_+ \to X$ be asymptotically almost automorphic functions, then the sum of the functions $f + f_1$ and λf, $\lambda > 0$, is an arbitrary scalar are also asymptotically almost automorphic functions.*

In what follows, we present an important property of asymptotically almost automorphic functions. We omit its proof, since it is very similiar to the proof of [17, Theorem 2.5.4] with obvious adaptations.

Theorem 3.12. *Let $\mathbb{T}$ be invariant under translations, the decomposition of an asymptotically almost automorphic function $f : \mathbb{T}_+ \to X$ is unique.*

In the sequel, we present a result which describes some properties of asymptotically almost automorphic functions on time scales. We omit its proof, since it is analogous to the continuous case. See [17].

Theorem 3.13. *Let X be a (real or complex) Banach space, $\mathbb{T}$ be invariant under translations and let $f : \mathbb{T}_+ \to X$ and $u : \mathbb{T}_+ \to \mathbb{C}$ be asymptotically almost automorphic functions on time scales. Then the following statements hold.*

(i) $f_a : \mathbb{T}_+ \to X$ *defined by* $f_a(t) = f(t+a)$, *for a fixed* $a \in \Pi \cap [0, +\infty)$, *is asymptotically almost automorphic function on time scales;*
(ii) $uf : \mathbb{T}_+ \to X$ *defined as* $(uf)(t) = u(t)f(t)$ *is an asymptotically almost automorphic function on time scales.*

The next result follows immediately from the property of asymptotically almost automorphic function on time scales.

Theorem 3.14. *If $\mathbb{T}$ is invariant under translations and $f : \mathbb{T}_+ \to X$ is an asymptotically almost automorphic function on time scales, then*

$$\sup_{t \in \mathbb{T}_+} \|f(t)\| < \infty.$$

In the sequel, we present a result which describes a property of the composition of an asymptotically almost automorphic function and the forward jump operator when the time scale is invariant under translations.

Theorem 3.15. *Let $\mathbb{T}$ be invariant under translations and $f : \mathbb{T} \to X$ be an asymptotically almost automorphic function on time scales, then the function $f \circ \sigma : \mathbb{T} \to X$ is also asymptotically almost automorphic function on time scales.*

Proof. We start by recalling that if $\mathbb{T}$ is invariant under translations, then $\sup \mathbb{T} = +\infty$. The theorem follows immediately combining Theorem 3.7 and the fact that $\lim_{t \to \infty} \sigma(t) = \infty$. In this case, if f_1 and f_2 are, respectively, the principal and the corrective terms of f, it follows that $f_1 \circ \sigma$ and $f_2 \circ \sigma$ are, respectively, the principal and correctives terms of $f \circ \sigma$ and we have the desired result. □

The following result will be fundamental to prove our main results. Its proof is inspired in [17, Theorem 2.4.1].

Theorem 3.16. *Let $\mathbb{T}$ be invariant under translations and $f : \mathbb{T} \to X$ be an almost automorphic function on time scales and its delta-derivative exists and is uniformly continuous on $\mathbb{T}$. Then $f^{\Delta}(t)$ is also almost automorphic on time scales.*

Proof. Let us consider two cases, that is, t is right-scattered and t is right-dense. Since f is almost automorphic function on time scales, for every sequence (α'_n) in Π, there exists a subsequence $(\alpha_n) \subset (\alpha'_n)$ such that

$$\lim_{n\to\infty} f(t+\alpha_n) = \bar{f}(t)$$

exists and is well-defined for every $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} \bar{f}(t-\alpha_n) = f(t)$$

exists and is well-defined for every $t \in \mathbb{T}$. If t is right-scattered, then

$$\begin{aligned}
\lim_{n\to\infty} f^{\Delta}(t+\alpha_n) &= \lim_{n\to\infty} \frac{f(\sigma(t+\alpha_n)) - f(t+\alpha_n)}{\mu(t+\alpha_n)} \\
&= \lim_{n\to\infty} \frac{f(\sigma(t+\alpha_n)) - f(t+\alpha_n)}{\mu(t)} \\
&= \frac{\bar{f}(\sigma(t)) - \bar{f}(t)}{\mu(t)} = \bar{f}^{\Delta}(t) := g(t),
\end{aligned}$$

for every $t \in \mathbb{T}$ and

$$\begin{aligned}
\lim_{n\to\infty} g(t-\alpha_n) &= \lim_{n\to\infty} \frac{\bar{f}(\sigma(t-\alpha_n)) - \bar{f}(t-\alpha_n)}{\mu(t-\alpha_n)} \\
&= \lim_{n\to\infty} \frac{\bar{f}(\sigma(t-\alpha_n)) - \bar{f}(t-\alpha_n)}{\mu(t)} \\
&= \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f^{\Delta}(t),
\end{aligned}$$

for every $t \in \mathbb{T}$, proving for this case. Now, consider that t is right-dense, then there exists a sequence $(t_n) \in \mathbb{T}$ such that

$$t_n \to t \quad \text{and} \quad t_n > t. \tag{3.5}$$

Let $\varepsilon > 0$, since $f^{\Delta}(t)$ is uniformly continuous, we can choose $\delta > 0$ such that for every pair $t_1, t_2 \in \mathbb{T}$ such that $|t_1 - t_2| < \delta$, we obtain $\|f^{\Delta}(t_1) - f^{\Delta}(t_2)\| < \varepsilon$. By (3.5), for this $\delta > 0$, we can choose $N \in \mathbb{N}$ such that $|t_n - t| < \delta$ for every $n > N$. It implies that

$$\frac{(f\,(t_n) - f(t))}{t_n - t} - f^{\Delta}(t) = \frac{1}{t_n - t}\int_t^{t_n} (f^{\Delta}(s) - f^{\Delta}(t))\Delta s.$$

Thus the sequence of almost automorphic functions $\dfrac{(f\,(t_n) - f(t))}{t_n - t}$ converges uniformly to $f^{\Delta}(t)$ on $\mathbb{T}$. By [29, Theorem 3.18], we get that $f^{\Delta}(t)$ is almost automorphic. □

Lemma 3.17. *Let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = +\infty$ and $f_n : \mathbb{T}_+ \to X$ be a sequence of continuous and bounded function satisfying $\lim_{t\to\infty} \|f_n(t)\| = 0$, for every $n \in \mathbb{N}_0$, which converges uniformly to a function $f : \mathbb{T}_+ \to X$. Then f is a continuous function and satisfies $\lim_{t\to\infty} \|f(t)\| = 0$.*

Proof. Let $\varepsilon > 0$. Since $\lim_{t\to\infty} \|f_n(t)\| = 0$ for every $n \in \mathbb{N}_0$, there exists a T sufficiently large such that for every $t > T$, we have

$$\|f_n(t)\| < \frac{\varepsilon}{2}$$

and by the uniform convergence of the sequence f_n, there exists a sufficiently large $N \in \mathbb{N}_0$ such that for $n > N$, we have

$$\|f_n(t) - f(t)\| < \frac{\varepsilon}{2}.$$

Thus, for $n > N$ and $t > T$, we have

$$\|f(t)\| \leq \|f(t) - f_n(t)\| + \|f_n(t)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

which implies that

$$\lim_{t\to\infty} \|f(t)\| = 0.$$

The continuity of the function f follows from Theorem 3.14 and by using the fact that the space of continuous and bounded functions $f : \mathbb{T}_+ \to X$ endowed with the supremum norm is a Banach space. $\square$

4. Asymptotically almost automorphic solutions of linear dynamic equations on time scales

In this section, our goal is to prove the existence of an asymptotically almost automorphic solution of first order linear dynamic equation on time scales given by

$$x^{\Delta}(t) = A(t)x(t) + f(t) \tag{4.1}$$

where $A : \mathbb{T} \to \mathbb{R}^{n\times n}$, $f : \mathbb{T} \to \mathbb{R}^n$ and its associated homogeneous equation

$$x^{\Delta}(t) = A(t)x(t). \tag{4.2}$$

Throughout this section, we assume that $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is almost automorphic.

The next theorem is one of the main results of this paper. It ensures the existence of an asymptotically almost automorphic solution of (4.1).

Theorem 4.1. *Let $\mathbb{T}$ be invariant under translations time scales and $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ be almost automorphic and nonsingular on $\mathbb{T}$ and $\{A^{-1}(t)\}_{t\in\mathbb{T}}$ and $\{(I + \mu(t)A(t))^{-1}\}_{t\in\mathbb{T}}$ are bounded. Also, suppose the equation* (4.2) *admits an exponential dichotomy with positive constants K and γ and $f \in \mathcal{C}_{rd}(\mathbb{T}_+, \mathbb{R}^n)$ is asymptotically almost automorphic function on time scales. Then the equation* (4.1) *has an asymptotically almost automorphic solution on $\mathbb{T}_+$.*

Proof. Since f is an asymptotically almost automorphic function on time scales, f can be written by

$$f(t) = f_1(t) + f_2(t),$$

for every $t \in \mathbb{T}$, where f_1 and f_2 are the principal and corrective terms of f, respectively. Thus the equation (4.1) can be rewritten as follows

$$x^{\Delta}(t) = A(t)x(t) + f_1(t) + f_2(t),$$

for every $t \in \mathbb{T}$. Then by Theorem 2.18, the equation (4.1) has a bounded solution given by

$$x(t) = \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))f(s)\Delta s$$

$$= \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))[f_1(s)+f_2(s)]\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))[f_1(s)+f_2(s)]\Delta s$$

where $t_0 \in \mathbb{T}_+$. Denoting by

$$x_1(t) := \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))f_1(s)\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))f_1(s)\Delta s$$

$$x_2(t) := \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))f_2(s)\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))f_2(s)\Delta s.$$

Since (4.2) admits an exponential dichotomy and f_1 is almost automorphic, it is not difficult to prove that $x_1(t)$ is an almost automorphic function on $\mathbb{T}$, by following the same steps as the proof of [29, Theorem 5.6]. It remains to show that x_2 satisfies $\lim_{t\to\infty} \|x_2(t)\| = 0$. Therefore, we get

$$\|x_2(t)\| \leq \int_{t_0}^{t} \|\mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))\|\|f_2(s)\|\Delta s + \int_{t}^{+\infty} \|\mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))\|\|f_2(s)\|\Delta s.$$

By the exponential dichotomy, we obtain

$$\|x_2(t)\| \leq \int_{t_0}^{t} Ke_{\ominus\gamma}(t,\sigma(s))\|f_2(s)\|\Delta s + \int_{t}^{+\infty} Ke_{\ominus\gamma}(\sigma(s),t)\|f_2(s)\|\Delta s,$$

which implies that

$$\lim_{t\to+\infty} \|x_2(t)\| \leq \lim_{t\to+\infty} \int_{t_0}^{t} Ke_{\ominus\gamma}(t,\sigma(s))\|f_2(s)\|\Delta s. \tag{4.3}$$

Using the fact that $\lim_{t\to+\infty} \|f_2(t)\| = 0$, it follows that given $\varepsilon > 0$, there exists a T sufficiently large such that for every $t > T$, we get

$$\|f_2(t)\| < \varepsilon.$$

Then, we have

$$\begin{aligned}
\lim_{t\to+\infty} \int_{t_0}^{t} Ke_{\ominus\gamma}(t,\sigma(s))\|f_2(s)\|\Delta s &= \lim_{t\to+\infty} \Big[\int_{t_0}^{T} Ke_{\ominus\gamma}(t,\sigma(s))\|f_2(s)\|\Delta s + \\
&+ \int_{T}^{t} Ke_{\ominus\gamma}(t,\sigma(s))\|f_2(s)\|\Delta s \Big] \\
&\leq \lim_{t\to+\infty} \frac{K[e_{\ominus\gamma}(t,T) - e_{\ominus\gamma}(t,t_0)]M}{|\ominus\gamma|} + \\
&+ \lim_{t\to+\infty} \varepsilon \frac{K[e_{\ominus\gamma}(t,t) - e_{\ominus\gamma}(t,T)]}{|\ominus\gamma|} \\
&\leq \frac{\varepsilon K}{|\ominus\gamma|},
\end{aligned}$$

by Theorems 2.15 and 3.14 and by Lemmas 2.16 and 2.17, where $M = \sup_{t\in\mathbb{T}_+} \|f_2(t)\|$. Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{t\to+\infty} \int_{t_0}^{t} Ke_{\ominus\gamma}(t,\sigma(s))\|f_2(s)\|\Delta s = 0.$$

Therefore, by inequality (4.3), we get

$$\lim_{t\to+\infty} \|x_2(t)\| = 0$$

and the result follows as well. □

Remark 4.2. It is clear that the previous theorem remains valid for linear nabla dynamic equations on time scales. In other words, one can prove analogously that the nabla dynamic equation

$$x^{\nabla}(t) = A(t)x(t) + f(t), \tag{4.4}$$

where $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ and $f : \mathbb{T}_+ \to \mathbb{R}^n$, has an asymptotically almost automorphic solution on $\mathbb{T}_+$, under similar conditions to the ones presented in Theorem 4.1.

5. Asymptotically almost automorphic solutions of semilinear dynamic equations on time scales

In this section, consider the following semilinear dynamic equation

$$x^{\Delta}(t) = A(t)x(t) + f(t,x) \tag{5.1}$$

where $A : \mathbb{T} \to \mathbb{R}^{n\times n}$, $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ and its associated homogeneous equation

$$x^{\Delta}(t) = A(t)x(t). \tag{5.2}$$

In what follows, we present a result which ensures that if (5.1) has an asymptotically almost automorphic solution on $\mathbb{T}_+$, then (5.1) has also an almost automorphic solution on $\mathbb{T}$. Its proof is inspired in [17, Theorem 5.1.1].

Theorem 5.1. *Let $\mathbb{T}$ be invariant under translations and $f \in \mathcal{C}_{rd}(\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n)$ be almost automorphic on time scales with respect to the first variable. Assume $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ is an almost automorphic matrix function on time scales. Suppose also the following conditions are fullfiled:*

(i) *There exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le L\|x-y\| \quad \text{for every } \ x,y \in \mathbb{R}^n \text{ and } \ t \in \mathbb{T}.$$

(ii) *The equation (5.1) possesses an asymptotically almost automorphic solution $x(t)$ on $\mathbb{T}_+$, which the principal term of $x(t)$ is Δ-differentiable and the corrective term of $x(t)$ is Δ-differentiable and is uniformly continuous on $\mathbb{T}_+$.*

Then the principal term of $x(t)$ is also an almost automorphic solution of (5.1) on $\mathbb{T}_+$.

Proof. Let $x(t)$ be an asymptotically almost automorphic solution of (5.1). Then, we get

$$x(t) = x_1(t) + x_2(t), \quad t \in \mathbb{T}_+,$$

where x_1 and x_2 are its principal and corrective terms, respectively. Since $x(t)$ is a solution of (5.1), we obtain

$$x^\Delta(t) = A(t)[x_1(t) + x_2(t)] + f(t, x_1(t)) + (f(t, x(t)) - f(t, x_1(t))) \tag{5.3}$$

for each $t \in \mathbb{T}_+$. Notice that the function $F : \mathbb{T} \to X$ defined by $F(t) = f(t, x_1(t))$ is an almost automorphic function (see [29, Theorem 3.19]). Moreover, $A(t)x_1(t)$ is an almost automorphic function, since it is the product of two almost automorphic functions (see [29, Theorem 3.17]). Thus, $A(t)x_1(t) + f(t, x_1(t))$ is an almost automorphic function.

On the other hand, the function $H : \mathbb{T}_+ \to X$ defined by

$$H(t) = f(t, x(t)) - f(t, x_1(t))$$

satisfies

$$\|f(t, x(t)) - f(t, x_1(t))\| \leq L\|x(t) - x_1(t)\| = L\|x_2(t)\|.$$

Therefore,

$$\lim_{t\to\infty} \|f(t, x(t)) - f(t, x_1(t))\| \leq \lim_{t\to\infty} L\|x_2(t)\| = 0,$$

which implies that

$$\lim_{t\to\infty} \|f(t, x(t)) - f(t, x_1(t))\| = 0.$$

Also,

$$\lim_{t\to\infty} \|A(t)x_2(t)\| \leq \lim_{t\to\infty} \tilde{A}\|x_2(t)\| = 0,$$

where $\tilde{A} = \sup_{t\in\mathbb{T}} \|A(t)\|$. By (5.3), it is clear that $x^\Delta(t)$ is asymptotically almost automorphic function on time scales. On the other hand, we have

$$x^\Delta(t) = x_1^\Delta(t) + x_2^\Delta(t), \quad t \in \mathbb{T}_+. \tag{5.4}$$

Therefore, using the fact that x_1^Δ is almost automorphic (Theorem 3.16) and by equations (5.3), (5.4) and by the uniqueness of the decomposition of an asymptotically almost automorphic function, we obtain

$$x_1^\Delta(t) = A(t)x_1(t) + f(t, x_1(t))$$

for every $t \in \mathbb{T}_+$. Then, we get

$$x_2^\Delta(t) = A(t)x_2(t) + (f(t, x(t)) - f(t, x_1(t)))$$

and by the properties proved before, it follows that

$$\lim_{t\to\infty} \|x_2^\Delta(t)\| = 0.$$

Therefore, the theorem is proved. $\square$

Remark 5.2. It is clear that the previous theorem remains valid for linear nabla dynamic equations on time scales. In other words, one can prove analogously that if the nabla dynamic equation

$$x^\nabla(t) = A(t)x(t) + f(t, x(t)), \tag{5.5}$$

where $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ and $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, possesses an asymptotically almost automorphic solution $x(t)$ on $\mathbb{T}_+$, then if the principal term of $x(t)$ is ∇-differentiable and the corrective term of $x(t)$ is ∇-differentiable and uniformly continuous on $\mathbb{T}_+$, then the principal term of $x(t)$ is an almost automorphic solution of (5.5) on $\mathbb{T}$.

Now, we introduce a definition of solution of (5.1) in a strict sense. Here, we will restrict ourselves for this concept of solution for (5.1).

Definition 5.3. We say that $x : \mathbb{T} \to \mathbb{R}^n$ is a *solution* of (5.1) if x satisfies the following equation

$$x(t) = \int_{-\infty}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))f(s,x(s))\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))f(s,x(s))\Delta s, \tag{5.6}$$

where $\mathcal{X}$ is the fundamental matrix of (5.2).

It is clear from [29, Remark 6.2] that the previous definition makes sense.

In what follows, we prove our main result of this section.

Theorem 5.4. *Let $\mathbb{T}$ be invariant under translations and $f \in \mathcal{C}_{rd}(\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n)$ be almost automorphic with respect to the first variable. Assume that $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ is almost automorphic and nonsingular matrix function, the sets $\{A^{-1}(t)\}_{t\in\mathbb{T}}$ and $\{(I+\mu(t)A(t))^{-1}\}_{t\in\mathbb{T}}$ are bounded. Suppose also the equation (5.2) admits an exponential dichotomy on $\mathbb{T}$ with positive constants K and γ and there exists a constant $0 < L < \dfrac{\gamma}{2K(2+\tilde{\mu}\gamma)}$ such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \quad \text{for every } \ x, y \in \mathbb{R}^n \ \text{ and } \ t \in \mathbb{T},$$

where $\tilde{\mu} = \sup_{t\in\mathbb{T}} \|\mu(t)\|$. Then the system (5.1) has a unique solution which is asymptotically almost automorphic on $\mathbb{T}_+$.

Proof. Let $t_0 \in \mathbb{T}$. Define an operator $T : AAA(\mathbb{T}, X) \to AAA(\mathbb{T}, X)$ as follows

$$(Tx)(t) = \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))f(s,x(s))\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))f(s,x(s))\Delta s.$$

Let us prove that T is well-defined. Take $x \in AAA(\mathbb{T}, \mathbb{R}^n)$, we can rewrite $x(t) = x_1(t)+x_2(t)$, where $x_1 \in AA(\mathbb{T}, X)$ and x_2 is a continuous function which satisfies

$$\lim_{t\to\infty} \|x_2(t)\| = 0. \tag{5.7}$$

Therefore, we get

$$(Tx)(t) = \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))[f(s,x(s)) - f(s,x_1(s))]\Delta s$$

$$+ \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))f(s,x_1(s))\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))f(s,x_1(s))\Delta s$$

$$- \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))[f(s,x(s)) - f(s,x_1(s))]\Delta s.$$

Since x_1 is almost automorphic, it is not difficult to show that

$$\int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))f(s,x_1(s))\Delta s - \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))f(s,x_1(s))\Delta s$$

is almost automorphic. It is only to use the similar arguments than the ones found in the proof of [29, Theorem 6.3]. It remains to prove that

$$\lim_{t\to+\infty} \Big\| \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))[f(s,x(s))-f(s,x_1(s))]\Delta s$$

$$-\int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))[f(s,x(s))-f(s,x_1(s))]\Delta s\Big\| = 0.$$

Define the following function

$$U(t) := \int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))[f(s,x(s))-f(s,x_1(s))]\Delta s$$

$$-\int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))[f(s,x(s))-f(s,x_1(s))]\Delta s.$$

By the exponential dichotomy and the Lipschitz condition, we have

$$\|U(t)\| \le \int_{t_0}^{t} Ke_{\ominus\gamma}(t,\sigma(s))L\|x_2(s)\|\Delta s + \int_{t}^{+\infty} Ke_{\ominus\gamma}(t,\sigma(s))L\|x_2(s)\|\Delta s.$$

Since $x_2(t)$ is continuous and satisfies (5.7), we obtain that given $\varepsilon > 0$, there exists $\tilde{T}$ sufficiently large such that for $t > \tilde{T}$, we get

$$\|x_2(t)\| < \varepsilon,$$

and also, by Theorem 3.14, $x_2(t)$ is a bounded function. Thus, we have for $t > \tilde{T}$

$$\begin{aligned}
\|U(t)\| &\le \int_{t_0}^{\tilde{T}} Ke_{\ominus\gamma}(t,\sigma(s))L\|x_2(s)\|\Delta s + \int_{\tilde{T}}^{t} Ke_{\ominus\gamma}(t,\sigma(s))L\|x_2(s)\|\Delta s\\
&\le \frac{KL[e_{\ominus\gamma}(t,\tilde{T})-e_{\ominus\gamma}(t,t_0)]M}{|\ominus\gamma|} + \varepsilon\frac{KL[e_{\ominus\gamma}(t,t)-e_{\ominus\gamma}(t,\tilde{T})]}{|\ominus\gamma|},
\end{aligned}$$

where $M = \sup_{t\in\mathbb{T}}\|x_2(t)\|$. Therefore, we obtain

$$\begin{aligned}
\lim_{t\to+\infty}\|U(t)\| &\le \lim_{t\to\infty}\frac{KL[e_{\ominus\gamma}(t,\tilde{T})-e_{\ominus\gamma}(t,t_0)]M}{|\ominus\gamma|} + \varepsilon\lim_{t\to\infty}\frac{KL[e_{\ominus\gamma}(t,t)-e_{\ominus\gamma}(t,\tilde{T})]}{|\ominus\gamma|}\\
&= \varepsilon\frac{KL}{|\ominus\gamma|}
\end{aligned}$$

by Lemmas 2.16 and 2.17. Since $\varepsilon > 0$ is arbitrary, we obtain that

$$\lim_{t\to+\infty}\|U(t)\| = 0.$$

Thus, T is well-defined. It remains to prove that T is a contraction. Let $z, y \in AAA(\mathbb{T}, \mathbb{R}^n)$. Then, we get

$$\begin{aligned}
\|Tz - Ty\| &= \|\int_{t_0}^{t} \mathcal{X}(t)P\mathcal{X}^{-1}(\sigma(s))[f(s,z) - f(s,y)]\Delta s \\
&- \int_{t}^{+\infty} \mathcal{X}(t)(I-P)\mathcal{X}^{-1}(\sigma(s))[f(s,z) - f(s,y)]\Delta s\| \\
&\leq \int_{t_0}^{t} Ke_{\ominus\gamma}(t,\sigma(s))L\|z-y\|\Delta s + \int_{t}^{+\infty} Ke_{\ominus\gamma}(\sigma(s),t)L\|z-y\|\Delta s \\
&\leq \frac{1}{|\ominus\gamma|}[Ke_{\ominus\gamma}(t,t) - Ke_{\ominus\gamma}(t,t_0)]L\|z-y\|_\infty + \int_{t}^{+\infty} Ke_{\gamma}(t,\sigma(s))L\|z-y\|\Delta s \\
&\leq \frac{1}{|\ominus\gamma|}[K - Ke_{\ominus\gamma}(t,t_0)]L\|z-y\|_\infty + \frac{1}{\gamma}[K - Ke_{\gamma}(t,+\infty)]L\|z-y\|_\infty,
\end{aligned}$$

by Theorem 2.15 and by the properties of the generalized exponential function. Therefore, we obtain

$$\begin{aligned}
\|Tz - Ty\| &\leq \frac{1}{\frac{\gamma}{1+\tilde{\mu}\gamma}}[K - Ke_{\ominus\gamma}(t,t_0)]L\|z-y\|_\infty + \frac{1}{\gamma}[K - Ke_{\gamma}(t,+\infty)]L\|z-y\|_\infty \\
&\leq \frac{1+\tilde{\mu}\gamma}{\gamma}\left[|K| + |Ke_{\ominus\gamma}(t,t_0)|\right]L\|z-y\|_\infty + \frac{1}{\gamma}KL\|z-y\|_\infty \\
&\leq L\|z-y\|_\infty\left(\frac{2K(1+\tilde{\mu}\gamma)}{\gamma} + \frac{K}{\gamma}\right) \\
&\leq L\left(\frac{2K(2+\tilde{\mu}\gamma)}{\gamma}\right)\|z-y\|_\infty
\end{aligned}$$

by Lemmas 2.16 and 2.17. By hypothesis, we obtain that $\beta := L\left(\frac{2K(2+\tilde{\mu}\gamma)}{\gamma}\right) < 1$, which implies that T is a contraction. Thus, by Banach Fixed-Point Theorem, T has a unique fixed point. By the definition of T and Definition 5.3, we obtain that the equation (5.1) has a unique solution which belongs to $AAA(\mathbb{T}, \mathbb{R}^n)$. □

References

[1] L. Bi, M. Bohner, M. Fan, *Periodic solutions of functional dynamic equations with infinite delay*, Nonlinear Anal., 68 (2008), 170-174.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[3] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[4] M. Bohner, M. Fan and J. Zhang, *Periodicity of scalar dynamic equations and applications to population models*, J. Math. Anal. Appl. 300, (2007), 1-9.

[5] D. Brigo and F. Mercurio, *Discrete time vs continuous time stock-price dynamics and implications for option pricing*, Finance and Stochastics 4 (2000), 147-159.

[6] D. Bugajewski and G. M. N'Guérékata, *On the topological structure of almost automorphic and asymptotically almost automorphic solutions of differential and integral equations in abstract spaces*, Nonlinear Anal. 59 (2004), 1333-1345.

[7] S. Calzadillas and C. Lizama, *Bounded mild solutions of perturbed Volterra equations with infinite delay*, Nonlinear Analysis., 72 (2010), 3976-3983.

[8] T. Caraballo and D. Cheban, *Levitan/Bohr almost periodic and almost automorphic solutions of second order monotone differential equations*, J. Diff. Equations, 251 (2011), 708-727.

[9] T. Caraballo and D. Cheban, *Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. I*, J. Diff. Equations, 246(1) (2009), 108-128.

[10] T. Caraballo and D. Cheban, *Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. II*, J. Diff. Equations, 246(3) (2009), 1164-1186.

[11] Z. Chen and W. Lin, *Square-mean pseudo almost automorphic process and its application to stochastic evolution equations*, J. Func. Anal. 261 (2011), 69-89.

[12] F. B. Christiansen and T. M. Fenchel, *Theories of populations in biological communities*, vol. 20 of Lectures Notes in Ecological Studies, Springer-Verlag, Berlin, 1977.

[13] H-S Ding, T-J Xiao and J. Liang, *Asymptotically almost automorphic solutions for some integrodifferential equations with nonlocal initial conditions*, J. Math. Anal. Appl. 338 (2008), 141-151.

[14] M. Fu and Z. Liu, *Square-mean almost automorphic solutions for some stochastic differential equations*, Proc. Amer. Math. Soc. 138 (2010), 3689-3701.

[15] J. A. Goldstein and G. M. N. Guérékata, *Almost automorphic solutions of semilinear evolution equations*, Proc. Amer. Math. Soc., 133 (8) (2005), 2401-2408.

[16] Y. Guan and K. Wang, *Translation properties of time scales and almost periodic functions*, Math. Comp. Model., 57 (5-6) (2013), 1165–1174.

[17] G. M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/Plenium Publishers, New York, 2001.

[18] G. M. N'Guérékata, *Topics in Almost Automorphy*, Springer-Verlag, New York-Boston-London-Moscow, 2005.

[19] S. Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, PhD thesis, Universität Würzburg, 1988.

[20] S. Hong and J. Liu, *Phase spaces and periodic solutions of set functional dynamic equations with infinite delay*, Nonlinear Anal. 74 (9) (2011), 2966-2984.

[21] M. Hu and L. Wang, *Unique Existence Theorem of Solution of Almost Periodic Differential Equations on Time Scales*, Discrete Dyn. in Nature and Society, (2012), Article ID 240735, 8 pages, doi:10.1155/2012/240735.

[22] M. Hu and L. Wang, *Existence and exponential stability of almost periodic solution for recurrent neural networks on time scales*, Int. J. Comput. Math. Sci., 6 (2012), 98-102.

[23] S. Keller, *Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung in Rahmen der Analysis auf Zeitskalin*, PhD thesis, Universität Augsburg, 1999.

[24] I. Klapper and H. Qian, *Remarks on discrete and continuous large-scale models of DNA dynamics*, Biophysical Journal, 74 (1998), 2504-2514.

[25] J. Liang, J. Zhang, T-J Xiao, *Composition of pseudo almost automorphic and asymptotically almost automorphic functions*, J. Math. Anal. Appl. 340 (2008), 1493-1499.

[26] Xi-Lan Liu and Wan-Tong Li, *Periodic solutions for dynamic equations on time scales*, Nonlinear Analysis. 67 (2007), 1457-1463.

[27] Y. Li and C. Wang, *Almost Periodic Functions on Time Scales and Applications*, Discrete Dyn. in Nature and Society, (2011), 20 pages.

[28] Y. Li and C. Wang, *Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales*, Adv. Difference Equ., 2012 (2012), 24 pages.

[29] C. Lizama and J. G. Mesquita, *Almost automorphic solutions of dynamic equations on time scales*, J. Funct. Anal., 265 (2013), 2267-2311.

[30] H. R. Sun and W. T. Li, *Existence theory for positive solutions to one-dimensional p-Laplacian boundary value problems on time scales*, J. Diff. Equations, 240 (2007), 217-248.

[31] C. C. Tisdell and A. Zaidi, *Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling*, Nonlinear Anal., 68 (2008), 3504-3524.

[32] J. Zhang, M. Fan and Huaiping Zhu, *Existence and roughness of exponential dichotomies of linear dynamic equations on time scales*, Comput. Math. Appl. 59 (2010), 2658 – 2675.

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