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Abstract Nonlinear Fractional Evolution Equations

Abstract: In this paper, we review some of the main results in the field of abstract nonlinear fractional evolution equations. We study solutions of the semilinear Cauchy problem in the subdiffusive and superdiffusive cases, mainly with the Caputo and Riemann-Liouville fractional order derivative, in the setting of the real semiaxis and real axis, and under various assumptions on the main data of the given equations. We consider in our analysis several kinds of perturbed systems e.g. delay, control and stochastic properties, even with nonlocal conditions and impulses. We provide a complete description of the representation of mild solutions in terms of associated solution families of operators.

Keywords: Semilinear fractional Cauchy problem; Caputo fractional derivative; Riemann-Liouville fractional derivative; mild solutions.

Classification: 35R11; 26A33; 37L05

For about two decades semilinear abstract Cauchy problems of the form

$$\begin{cases} {}^{C}D_t^{\alpha} u(t) &= & Au(t) + f(t, u(t)), \quad t \geq 0, \quad 0 < \alpha \leq 2; \\ u(0) &= & u_0; \quad \max\{0, \alpha - 1\}(u'(0) - u_1) = 0, \end{cases} \tag{1}$$

in a Banach space X have been studied in many different settings. Here A denotes the generator of a strongly continuous family of bounded and linear operators in X and f a nonlinear function.

This abstract and general approach, enables that the results can be applied to a broad variety of problems, including both ordinary fractional equations and partial fractional equations problems.

The basic idea for this type of approach is to consider the integral equation

$$u(t) = S_\alpha(t)x_0 + \int_0^t R_\alpha(t-s) f(s, u(s)) ds, \quad t \geq 0, \quad 0 < \alpha \leq 1, \tag{2}$$

where $\{R_\alpha(t)\}_{t>0}$ is an (α, α)-resolvent family generated by A and $S_\alpha(t) := (g_{1-\alpha} * R_\alpha)(t)$ is an α-resolvent family, also generated by the same operator A.

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DOI

The reason is that under appropriate conditions on the initial data x_0 and the forcing term f, a solution of the equation (2) corresponds to a strong (or classical) solution of the equation (1). See the Chapter "Abstract Linear Fractional Evolution Equations" for more information on this subject.

As a consequence of the definition of fractional derivative of order $\alpha > 0$, in the situation that $1 < \alpha \leq 2$ one should consider the integral equation

$$u(t) = S_\alpha(t)u_0 + (g_1 * S_\alpha)(t)u_1 + \int_0^t R_\alpha(t-s)f(s,u(s))ds \tag{3}$$

where $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent family generated by A and $R_\alpha(t) := (g_{\alpha-1} * S_\alpha)(t)$ is an (α,α)-resolvent family, generated by A. Note the inversion of roles between the families $S_\alpha(t)$ and $R_\alpha(t)$ in the cases $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$, respectively.

Solutions of the vector-valued integral equation (2) (resp. (3)) are called *mild* solutions of (1). It should be noted that the definition of mild solution for abstract fractional differential equations has been misunderstood for some researchers, contrasting with those known in the literature on the subject [11, Section 4]. This has been observed in some papers since some time ago [30, 47, 62].

If A is the generator of a C_0-semigroup $\{S_1(t)\}_{t\geq 0}$ and $0 < \alpha < 1$ then, using the subordination principle, we can show an explicit description of the families of bounded and linear operators in (2) as follows: The (α,α)-resolvent family generated by A is given by

$$R_\alpha(t) = t^{\alpha-1}P_\alpha(t) \text{ where } P_\alpha(t) := \alpha\int_0^\infty s\Phi_\alpha(s)S_1(st^\alpha)ds, \quad t > 0 \tag{4}$$

and the α-resolvent family generated by A is represented by

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s)S_1(st^\alpha)ds, \quad t > 0, \tag{5}$$

see subsection 2.2 in the Chapter "Abstract Linear Fractional Evolution Equations" for details. Here Φ_α are the functions of Wright type defined by

$$\Phi_\alpha(s) := \sum_{n=0}^\infty \frac{(-s)^n}{n!\alpha(-\alpha n + 1 - \alpha)} = \frac{1}{\pi\alpha}\sum_{n=1}^\infty (-s)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),$$

valid for $0 < \alpha < 1$ and $s \geq 0$. In the formulation (2) (resp. (3)) the unbounded operator A does only appear in terms of the family $R_\alpha(t)$ (resp. $S_\alpha(t)$), which makes e.g. the application of fixed point theorems for the solution of the integral equation possible.

A typical argument uses the contraction mapping principle. In such case we suppose that the nonlinearity f is continuous and globally Lipschitz. This argument has been refined in many directions, e.g. localizing the Lipschitz condition, or replacing it by compactness assumptions involving measures of non compactness and applying fixed point theorems for set contractions, or by allowing A to depend on t. This way also a qualitative theory for mild solutions of (1) can be developed.

There is a already a number of papers available where these ideas have been carried out in various settings, most of them in the autonomous one. In the non autonomous case, i.e. when A is time dependent, much work remains to be done.

In what follows, for transparency reasons, we separately discuss the cases

$$0 < \alpha \leq 1 \quad \text{and} \quad 1 < \alpha \leq 2.$$

1 The Semilinear Cauchy Problem: $0 < \alpha \leq 1$

1.1 Caputo fractional derivative

As standard model, we consider the semilinear Cauchy problem

$$\begin{cases} {}^C D_t^\alpha u(t) &= Au(t) + f(t, u(t)), \quad t \geq 0, \quad 0 < \alpha \leq 1, \\ u(0) &= u_0. \end{cases} \tag{1}$$

This is by far the most studied fractional abstract model in the existing literature. The analysis cover the existence and uniqueness of mild solutions for the fractional evolution equation (1) under different conditions on the operator A, the nonlinear term f and the initial condition u_0. The study of fractional controllability; fractional inclusions; and fractional stochastic evolution equations and inclusions are the most common subjects of research.

Recall that a mild solution of (1) is a solution of the integral equation

$$u(t) = S_\alpha(t)x_0 + \int_0^t R_\alpha(t-s)f(s, u(s))ds, \quad t \geq 0, \tag{2}$$

where $S_\alpha(t) = (g_{1-\alpha} * R_\alpha)(t)$, being $\{R_\alpha(t)\}_{t>0}$ an (α, α)-resolvent family generated by A. Existence and uniqueness of local and global mild solutions for (1) were investigated by Chen, Li, Chen and Feng [15] when A is the generator of an uniformly bounded and immediately norm continuous C_0-semigroup. As methods, they have mainly used Sadovskii's fixed point theorem and the technique of the measure of noncompactness. See also the paper [37] for a related result using the Kuratowski measure of noncompactness. However, until now, the most complete study of the semilinear problem (1) is contained in the monograph [21, Section 3] by Gal and Warma, which also includes several interesting applications, examples and historical remarks.

If A is the generator of a positive, compact and uniformly bounded C_0-semigroup, the existence of minimal and maximal mild solutions for the problem (1) with periodic boundary conditions has been studied by Mu and Li [55]. They used the method of upper and lower solutions coupled with a monotone iterative technique and the properties of positive C_0-semigroups.

Under the hypothesis that A is a sectorial operator on a Banach space X (i.e. A is the generator of an analytic semigroup), the authors Shu, Lai and Chen [62] proved the existence of unique mild solutions of (1) with impulses. Under the same hypothesis of sectoriality of A, Guswanto and Suzuki [27] studied the existence and uniqueness of a local mild solution for the problem (1) with nonlinear term in the form $f(u(t))$. They put some conditions on f and the initial data u_0 in terms of the fractional powers of A. By applying Banach's fixed point theorem, they obtain a unique local mild solution with smoothing effects, estimates, and a behavior at t close to 0. In the same line of ideas, and under some local Lipschitz conditions on f, De Andrade, Carvalho, Carvahlo-Neto and Marín-Rubio [18] proved an existence and uniqueness theorem for local mild solutions to (1), as well as continuation, non-continuation (due to blow-up) and global existence results. They also investigate critical cases by proving the existence of the so-called ϵ-regular mild solutions using a technique of fractional power spaces associated to the operator A. For some other results under the hypothesis of sectoriality on A, see the recent book of Zhou [75, Section 2.1.4].

Assuming that A is the generator of a compact semigroup, Chauhan and Dabas [14] proved existence of mild solutions for (1) with non local conditions

$$u(0) + g(u) = u_0, \tag{3}$$

and impulses. Moreover, in [14], the nonlinear term admits the general form $f(t, u(t), u(a_1(t)), ..., u(a_m(t)))$ where a_i are scalar functions defined on a finite interval. In the range $1/2 < \alpha < 1$, Ponce [59, Theorem 23] proved existence of mild solutions for (1) with the non local conditions (3) but assuming that A is the generator of an (α, α)-resolvent family $R_\alpha(t)$ and $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$.

We remark that the nonlocal condition (3) has a better effect on the solution of (1) and is more precise for physical measurements than the classical condition alone.

If the semigroup generated by A is non-compact, Gou and Li [24] investigated local and global existence of mild solution for (1) with impulses and an additional nonlinear term in the form

$$\int_0^t q(t-s)g(s,u(s))ds.$$

Existence results under general and weak assumptions on f by utilizing Schaefer and O'Regan fixed point theorems have been proved by Wang, Zhou and Feckan [71]. If A generates a bounded analytic semigroup, existence of mild solutions for fractional order equations with infinite delay and an integral nonlinearity in the form $\int_0^t a(t,s)f(s,u(s),u_s)ds$ has been analyzed by Aissani and Benchohra [7], by means of the application of Mönch's fixed point theorem combined with the Kuratowski measure of noncompactness.

An interesting way to deal with many kinds of nonlinearities at once, is the use of the notion of causal operator due to Tonelli [66]. This approach has been recently pursued by Agarwal, Asma, Lupulescu and O'Regan [2]. The main idea is consider the model

$$\begin{cases} {}^C D_t^\alpha u(t) &= Au(t) + (Qu)(t), \quad a.e.\ t \in [0,T], \\ u|_{[-\sigma,0]} &= \varphi \in C([-\sigma,T],X), \quad \sigma \geq 0, \end{cases} \tag{4}$$

where $Q : C([-\sigma,T],X) \to L^p([0,T],X)$ is a causal operator, i.e. for each $\tau \in [0,T)$ an for all $u,v \in L^p([-\sigma,T],X)$ with $u(t) = v(t)$ for every $t \in [0,\tau]$, we have $Qu(t) = Qv(t)$ for a.e. $t \in [0,\tau]$. Using this approach, the authors in [2] proved existence of mild solution of (4) assuming that A is the generator of an immediately norm continuous semigroup.

In [60], Sakthivel, Ren and Mahmudov established sufficient conditions for the approximate controllability of the problem

$${}^C D_t^\alpha u(t) = Au(t) + Bx(t) + f(t,u(t)), \quad t \in [0,T], \tag{5}$$

with initial condition $u(0) = u_0$. Here, the state variable $u(\cdot)$ takes values in the Hilbert space H; A is the generator of a C_0-semigroup; the control function $x(\cdot)$ is given in $L^2([0,T],U)$, U is a Hilbert space; and B is a bounded linear operator from U into H. The results are established under the assumption that the associated linear system is approximately controllable. Further, the authors extend their results to study the approximate controllability of fractional systems with nonlocal conditions as in (3). See also the papers [72, 33] for further results on this problem.

We point out that if the operator B is compact, or the C_0-semigroup generated by A is compact, then the controllability operator is also compact and hence the inverse of it does not exist if the state space is infinite dimensional. See [29] and [56]. Thus, the concept of exact controllability for fractional differential equations is too strong in infinite dimensional spaces and the notion of approximate controllability is more appropriate.

In [22] Ge, Zhou and Kou studied the approximate controllability of the semilinear fractional evolution equation (5) with nonlocal and impulsive conditions. The impulsive functions in that paper are supposed to be continuous and the nonlocal item is divided into two cases: Lipschitz continuous and only continuous, which generalizes previous contributions.

In the article [19] Debbouche and Torres studied the approximate controllability of (5), where the control function depends on multi-delay arguments and where the nonlocal condition is fractional. In [41], Liu and Fu studied controllability for (1) with a nonlinear term in the form $f(t, u(t)) + g(t)x(t)$ and a mixed nonconvex constraint on the control $x(\cdot)$. We also note the work of Mophou [53] where she studied the approximate controllability of a fractional semilinear differential equation (5) but involving the right fractional Caputo derivative.

Sufficient conditions for the approximate controllability of (1) with bounded delay, namely, the class

$$\left\{ \begin{array}{rcl} {}^C D_t^\alpha u(t) & = & Au(t) + Bx(t) + f(t, u(t-h)), \quad t \in (0, \tau], \quad \frac{1}{2} < \alpha \le 1, \\ u(t) & = & \varphi(t), \quad t \in [-h, 0], \end{array} \right.$$

have been considered by Kumar and Sakavanam [35].

Fractional semilinear differential inclusions in Banach spaces has been studied by Wang and Zhou [70]. By using the Bohnenblust–Karlin's fixed point theorem, an existence result of mild solutions for the multivalued version of (1) was obtained in [70] under the assumption that A generates a compact semigroup. In such paper, also controllability results are discussed. The paper of Liu and Liu [42] studied also the same topic but relaxing the conditions in the nonlinearity, admiting f non convex. See also [43, 69, 75] for additional research on this topic.

Under the general assumption that A is the generator of an (α, α)-resolvent family, relative controllability for a class of semilinear stochastic fractional differential equation with nonlocal conditions of the form

$$\left\{ \begin{array}{rcl} {}^C D_t^\alpha u(t) & = & Au(t) + Bx(t) + f(t, u(t)) + \sigma(t, u(t)) \frac{dw(t)}{dt}, \quad t \in [0, T], \\ u(0) + g(u) & = & u_0, \end{array} \right.$$

in Hilbert spaces, were studied by Guendouzi and Hamada [25]. See also the article of Saktivel, Revathi and Ren [61], where impulsive fractional stochastic differential equations with infinite delay under the same hypothesis has been studied. In [67], Toufik studied existence and controllability results for fractional stochastic semilinear differential inclusions of the above form. The results are obtained by using the Bohnenblust-Karlin fixed point theorem. More recently, Ahmed [6] proved the existence of mild solutions for a very general class of semilinear neutral fractional stochastic integro-differential equations with nonlocal conditions. Ahmed, derived sufficient conditions with the help of the Sadovskii fixed point theorem.

When $A = A(t)$ are bounded operators and $0 < \alpha < 1$, the problem of existence and uniqueness of solutions on an interval $[0, T]$ was studied by Balachandran and Park [9], as well as the existence and uniqueness of solutions with nonlocal condition of the form (3) where g is a given function satisfying certain Lipschitz type conditions. In the paper [10] the authors deal with fractional impulsive evolution equations. Under classical assumptions and by using the Banach contraction principle, the authors proved the existence and uniqueness of solutions.

1.2 Riemann-Liouville fractional derivative

We consider the model

$$^{RL}D_t^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \geq 0,\ 0 < \alpha \leq 1, \tag{6}$$

with initial condition $(g_{1-\alpha} * u)(0) = u_0$.

We observe that the definition of Riemann-Liouville fractional initial condition $(g_{1-\alpha} * u)(0) = u_0$ is difficult to interpret, but play an important role in some practical problems. Heymans and Podlubny [31] have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann Liouville fractional derivatives on the field of the viscoelasticity.

Assuming that A is the generator of an (α, α)-resolvent family $\{R_\alpha(t)\}_{t>0}$, the appropriate and more general notion of mild solution for (6) is a locally integrable function u such that $g_{1-\alpha} * u$ is absolutely continuous and satisfy the integral equation

$$u(t) = R_\alpha(t)u_0 + \int_0^t R_\alpha(t-s)f(s, u(s))ds, \quad t > 0, \tag{7}$$

see [57, Lemma 4] and the Chapter "Abstract Linear Fractional Evolution Equations". In the paper [59, Theorem 26], Ponce proved existence of mild solutions for the problem (6) with nonlocal initial conditions. He assumes that A is the generator of an immediately norm continuous (α, α)-resolvent family and that the resolvent operator $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$. In the article [57], Pan, Li and Zhao investigated the solvability and optimal controllability for the following semilinear control system

$$^{RL}D_t^{\alpha}u(t) = Au(t) + f(t, u(t)) + Bx(t), \quad t \in J := (0, T], \tag{8}$$

with initial condition $(g_{1-\alpha} * u)(0) = u_0$. Here, the control function $x(\cdot)$ is given in a suitable admissible control set, and B is a linear operator from a separable reflexive Banach space Y into X. If A is the generator of a compact C_0-semigroup, then sufficient conditions for the existence of mild solutions of (8) are proved. Further, optimal control results corresponding to the admissible control sets are shown. More recently, approximate controllability was studied in [32] when A is the generator of an (α, α)-resolvent family. See also the paper of Liu and Li [44] where existence of mild solutions of (8) in the space $C_{1-\alpha}(J; X) := \{u : t^{1-\alpha}u(t) \in C(J; X)\}$ as well as approximate controllability is studied. The article [39] consider also impulses.

If A is almost sectorial, i.e. $A \in \Theta_{\omega}^{p}(X)$ where $-1 < p < 0 < \omega < \pi/2$, and the associated C_0-semigroup generated by A is compact, then Zhou [75, Section 2.1.3] proved that, under suitable conditions on f, the problem (6) has at least one mild solution in $B_r^{(\alpha)}(J)$, for every $x_0 \in D(A^{\beta})$ with $\beta > 1 + p$. Here $B_r^{(\alpha)}(J)$ is the ball of radius r of the Banach space $X^{(\alpha)}(J) := \{u \in C(J; X) : \lim_{t \to 0^+} t^{1+\alpha p}u(t)$ exists and is finite $\}$ provided with the norm $\|u\|_{\alpha} := \sup_{t \in J} t^{1+\alpha p}\|u(t)\|$.

Fractional evolution inclusions for (6) with nonconvex right hand side has been only recently studied by Liu, Bin and Liu [38]. Assuming that A is the generator of an (α, α)-resolvent family, they proved existence of the extreme solution and the relationship of the solution sets between the original problem and the convexified problem.

It should be noted that in recent years a practical and useful way to treat the Caputo and Riemann-Liouville fractional abstract Cauchy problem, simultaneously, has been investigated by means of the Hilfer fractional derivative. The notion of mild solution in such case is a solution of

$$u(t) = (g_{\gamma(1-\alpha)} * R_{\alpha})(t)u_0 + \int_0^t R_{\alpha}(t-s)f(s, u(s))ds, \quad t > 0, \tag{9}$$

where $0 \leq \gamma \leq 1$ and $0 < \alpha \leq 1$. Here $\{R_{\alpha}(t)\}_{t>0}$ is an (α, α)-resolvent family generated by A. Note that $(g_{\gamma(1-\alpha)} * R_{\alpha})(t)$ is an $(\alpha, \alpha + \gamma(1-\alpha))$-resolvent family with the same generator. When $\gamma = 0$, the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative and (9) is the same that (7) ($g_0 \equiv$ Dirac delta). When $\gamma = 1$, the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative and (9) reduces to (2) in Section 1.1. For more details on this type of approach for the study of existence of mild solutions for nonlinear fractional nonautonomous evolution equations of Sobolev type with delay, see the recent paper of Gou and Li [24].

2 The Semilinear Cauchy Problem: $1 < \alpha \leq 2$

2.1 Caputo fractional derivative

In this section, we deal with the problem

$$^{C}D_t^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \geq 0,\ 1 < \alpha \leq 2, \tag{1}$$

with initial conditions $u(0) = u_0$ and $u'(0) = u_1$. Recall that a mild solution of (1) is understood as a solution of the integral equation

$$u(t) = S_\alpha(t)u_0 + (g_1 * S_\alpha)(t)u_1 + \int_0^t R_\alpha(t-s)f(s, u(s))ds \tag{2}$$

where $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent family generated by A and $R_\alpha(t) := (g_{\alpha-1} * S_\alpha)(t)$ is an (α, α)-resolvent family with the same generator.

If A is the generator of a strongly continuous cosine family $\{S_2(t)\}_{t\geq 0}$ and $1 < \alpha < 2$ then, using the subordination principle, we have the following explicit representation:

$$S_\alpha(t) = \int_0^\infty \Phi_{\alpha/2}(s)S_2(st^{\alpha/2})ds.$$

Existence and uniqueness of mild solutions of (1) with non-local initial conditions have been proved by Ponce [59, Theorems 20 and 21] under the hypothesis that A is the generator of an α-resolvent family and $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$.

Assuming that A is the generator of an α-resolvent family, Li [36] proved the existence of mild solutions of (1) with a nonlinear term in the form

$$\int_0^t h(t, s, u(s))ds + g(t).$$

The paper by Mophou and N'Guérékata [54] is concerned with the semilinear differential system of fractional order with infinite delay:

$$ {}^C D_t^\alpha u(t) = Au(t) + Bx(t) + f(t, u_t), \quad t \in (0, T], \tag{3} $$

$u(t) = \phi(t)$, $t \in (-\infty, 0]$. The authors proved that the system is controllable when A generates an α-resolvent family $(S_\alpha(t))_{t\geq 0}$ on a complex Banach space X and the control $x \in L^2([0, T]; X)$. This problem has been also recently studied by Shukla, Sukanavam and Pandey [64].

Assuming that A is the generator of a compact (α, α)-resolvent family, Shukla, Sukavanam and Pandey proved in [63] that the problem

$$ {}^C D_t^\alpha u(t) = Au(t) + Bx(t) + f(t, u(t)), \quad t \in [0, T], $$

with initial conditions $u(0) = u_0$ and $u'(0) = u_1$ is approximately controlable. They used Schauder's fixed point theorem in order to achieve their results. We remark that necessary and sufficient conditions for the compactness of $\{R_\alpha(t)\}_{t\geq 0}$ have been recently studied by Lizama, Pereira and Ponce [48, 59]. Under essentially the same hypothesis on A, Guendouzi and Farahi considered in [26] the approximate controllability for a class of fractional semilinear stochastic dynamic systems with nonlocal conditions in Hilbert spaces of the form

$$ \begin{aligned} {}^C D_t^\alpha u(t) &= Au(t) + Bx(t) + f(t, u(t), u(b_1(t), ..., u(b_m(t))) \\ &\quad + \sigma(t, u(t), u(a_1(t), ..., a_n(t)) \frac{dw(t)}{dt}, \quad t \in [0, T]. \end{aligned} $$

Existence of solutions of (1) in the non autonomous case, i.e. when $A = A(t)$ are bounded linear operators, and considering impulses and anti-periodic boundary value conditions in the equation, have been proved by Agarwal and Ahmad [3]. The contraction mapping principle and Krasnoselskii's fixed point theorem are applied to prove the main results. An extension of such result to the case of mixed boundary values has been studied by Zhang, Wang and Song [74].

Dos Santos et al. [20] studied the existence of mild solutions for abstract fractional neutral equations of the type (1), but that includes an additional integral term and a state-dependent delay, by using the Leray–Schauder alternative fixed point theorem. If the integral term is not present, the resulting mild solutions coincides with those given by (2). Very recently, Tamilalagan and Balasubramaniam [65] consider a class of fractional stochastic differential inclusions, that includes problem (1), driven by fractional Brownian motion in Hilbert space with Hurst parameter $H \in (1/2, 1)$. Sufficient conditions for the existence and asymptotic stability of mild solutions are derived in mean square moment by employing (α, α)-resolvent families and Bohnenblust–Karlin's fixed point theorem. For other contributions in this direction, see the references in [65].

2.2 Riemann-Liouville fractional derivative

We consider the model problem

$$ {}^{RL}D_t^{\alpha} u(t) = Au(t) + f(t, u(t)), \quad t \geq 0, \ 1 < \alpha \leq 2, \tag{4} $$

with initial conditions $(g_{2-\alpha} * u)(0) = u_0$ and $(g_{2-\alpha} * u)'(0) = u_1$. We note that in this case, controllability, stability analysis and other qualitative and quantitative properties have not received much attention from researchers, and hence many problems are still open.

For the problem (4), a mild solution should be understood as a solution of the following integral equation

$$ u(t) = L_\alpha(t)u_0 + R_\alpha(t)u_1 + \int_0^t R_\alpha(t-s)f(s, u(s))ds, \quad t \geq 0, $$

where $\{L_\alpha(t)\}_{t\geq 0}$ is an $(\alpha, \alpha-1)$-resolvent family generated by A and $R_\alpha(t) := (g_1 * L_\alpha)(t)$ is an (α, α)-resolvent family, with the same generator A. However, except for the information provided by the general theory of (a, k)-regularized families, there exists little material in the literature concerning the family of operators $\{L_\alpha(t)\}_{t\geq 0}$. See the Chapter "Abstract Linear Fractional Evolution Equations" for details.

When A is the generator of an $(\alpha, \alpha - 1)$-resolvent family and $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$, Ponce [59, Theorems 24 and 25] proved the existence of at least one mild solution for (4) with nonlocal initial conditions. He assumed that f satisfies a Carathéodory type condition and then uses Krasnoselskii theorem.

When $u_0 = 0$ a unified approach to (4) with Riemann-Liouville and Caputo fractional order derivative of order $1 < \alpha \leq 2$ has been developed by Mei, Peng and Gao [52]. They used the Hilfer fractional derivative and obtain a representation of the homogeneous problem (4) with $u_0 = 0$ by means of $(\alpha, \alpha + \gamma(2-\alpha))$-resolvent families, where $0 \leq \gamma \leq 1$. This method can be used to develop a more complete theory for the linear and nonlinear problem, at least in this case. In a very general form, the linear problem (4) is included in the paper [34] by Kostic, where strong and mild solutions are considered.

3 The Semilinear Cauchy Problem on the line: $0 < \alpha \le 1$

We consider the model problem

$$_{-\infty}D_t^{\alpha} u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},\ 0 < \alpha \le 1, \tag{1}$$

where $_{-\infty}D_t^{\alpha}$ is the Liouville-Weyl fractional derivative. Suppose that A is the generator of an (α, α)-resolvent family $\{R_\alpha(t)\}_{t \ge 0}$. In such general case, a mild solution of (1) is a solution of the equation

$$u(t) = \int_{-\infty}^{t} R_\alpha(t-s) f(s, u(s)) ds, \quad t \in \mathbb{R}, \tag{2}$$

whenever the above integral exists. The existence can be proved, for example, when the nonlinear term is bounded and A is the generator of an exponentially stable C_0- semigroup $\{S_1(t)\}_{t \ge 0}$ since in such case one can appeal to the representation

$$R_\alpha(t) = t^{\alpha-1} P_\alpha(t) \text{ where } P_\alpha(t) := \alpha \int_0^{\infty} s \Phi_\alpha(s) S_1(st^\alpha) ds, \quad t > 0,$$

and use [75, Property 1.10 (ii) and (iii)] and [75, Property 1.11 (v)]. Using this representation, and under the hypothesis that A is the generator of an exponentially stable C_0-semigroup, which is in addition positive or compact, Zhou [75, Section 2.2.3] established some sufficient conditions for the existence and uniqueness of periodic solutions, S-asymptotically periodic solutions, and other types of bounded solutions when $f : \mathbb{R} \times X \to X$ satisfies some ordering hypothesis on X or Lipschitz conditions in f. The main methods are the monotone iterative technique and Banach contraction principle.

Bounded mild solutions to (1) when in the nonlinear term we add a perturbation in the form

$$\int_{-\infty}^{t} a(t-s) A u(s) ds$$

have been studied by Ponce [58]. However, until now the study of the model (1) is still undeveloped and much work remains to be done.

4 The Semilinear Cauchy Problem on the line: $1 < \alpha \le 2$

We consider the problem

$$ {}_{-\infty}D_t^{\alpha} u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},\ 1 < \alpha \le 2. \tag{1} $$

A mild solution of (1) is a fixed point of the equation

$$ u(t) = \int_{-\infty}^{t} R_\alpha(t-s) f(s, u(s)) ds, \quad t \in \mathbb{R}, $$

whenever the above integral exists. Here $\{R_\alpha(t)\}_{t\ge 0}$ denotes an (α,α)-resolvent family generated by A. This extension of the notion of mild solution from the border cases $\alpha = 1$ and $\alpha = 2$ to the intermediary case $1 < \alpha < 2$ was first noted by Araya and Lizama [8]. Assuming that A is the generator of an integrable (α,α)-resolvent family $\{R_\alpha(t)\}_{t\ge 0}$ i.e. such that

$$ \|R_\alpha(t)\| \le \varphi_\alpha(t), \quad t > 0, \quad \varphi_\alpha \in L^1(\mathbb{R}_+), $$

and that f satisfies a global Lipschitz condition, it is proved in [8] the existence and uniqueness of an almost automorphic mild solution of the semilinear equation (1). See also the recent paper of Liu, Cheng and Zhang [40] that establish the existence of anti-periodic mild solutions. After the paper [8], Cuevas and Lizama [17] studied almost automorphic mild solutions of the equation (1) with forcing term $f(t, u(t)) := D_t^{\alpha-1} g(t, u(t))$. In such case, a mild solution of (1) is a fixed point of the equation

$$ u(t) = \int_{-\infty}^{t} S_\alpha(t-s) g(s, u(s)) ds, \quad t \in \mathbb{R}, $$

where $\{S_\alpha(t)\}_{t\ge 0}$ is an α-resolvent family generated by A. It should be observed that according to Cuesta [16], a such family $\{S_\alpha(t)\}_{t\ge 0}$ exists and is integrable whenever A is a sectorial operator of negative type.

Following this approach, in [5] Agarwal, Cuevas and Soto proved sufficient conditions for the existence and uniqueness of a pseudo-almost periodic solutions of the equation (1) with forcing term $f(t, u(t)) := D_t^{\alpha-1} g(t, u(t))$. With the same forcing term, Cao, Yang and Huang [12], proved existence of anti-periodic mild solutions, Chang, Zhang and N'Guérékata [13] proved the existence of weighted pseudo almost automorphic mild solutions and He, Cao and Yang

[28] established sufficient criteria for the existence and uniqueness of a weighted Stepanov-like pseudo-almost automorphic mild solution. For a review of regularity results in several classes of vector-valued subspaces of the space of continuous and bounded functions, see the paper [49] by Lizama and Poblete. Wang and Xia [68] proved the existence and uniqueness of (μ, ν)-pseudo almost automorphic mild solution. See also the paper [73] by Xia, Fan and Agarwal where the same property is investigated but with nonlinearity in the form $D_t^{\alpha-1} f(t, Bu(t))$, being B a bounded linear operator. Also, in the reference [53], Mophou studied the existence and uniqueness of weighted pseudo almost automorphic mild solution to the semilinear fractional equation (1) with $f(t, u(t)) := g(t, u(t), Bu(t))$. This extends a previous paper of Agarwal, de Andrade and Cuevas [4] where the case $B = 0$ was considered. The results obtained are utilized to study the existence and uniqueness of a weighted pseudo almost automorphic solution to fractional diffusion wave equation with Dirichlet conditions.

Although existence and uniqueness of solutions of this equation has been studied in several subspaces of the vector-valued space of bounded functions, still some development in other lines of research could be interesting to pursue, as for instance discrete settings. In this line, recently some papers have appeared [1, 45, 46, 50, 51].

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