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Boundary controllability for the 1D Moore–Gibson–Thompson equation

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Abstract This article addresses the boundary controllability problem for a class of third order in time PDE, known as Moore–Gibson–Thompson equation, with a control supported on the boundary. It is shown that it is not spectrally controllable, which means that nontrivial finite linear combination of eigenvectors can be driven to zero in finite time. This implies that the Moore–Gibson–Thompson equation is not exact and null controllable. However, the approximate controllability will be proved.

Keywords Moore–Gibson–Thompson equation · Spectral controllability · Approximate controllability

1 Introduction

In this work we consider a third order in time equation with internal damping

$$\tau y_{ttt} + \alpha y_{tt} - c^2 y_{xx} - b y_{xxt} = 0, \tag{1}$$

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where τ, α, c and b are positive constants. This equation arises from nonlinear acoustics and is aimed at encompassing what [1] calls the *Moore–Gibson–Thompson* equation in the velocity potential. The physical meaning of the constants in (1) is the following: τ is a positive constant accounting for relaxation, c is the speed of the sound; $b = \delta + \tau c^2$, where δ is the diffusivity of the sound.

An operator-theoretical semigroup method to study the Moore–Gibson–Thompson equation (MGT-equation) was employed by Marchand et al. [2] and Kaltenbacher et al. [1]. These authors rewrite the MGT-equation as a first order abstract Cauchy problem $u' = Au + F$ in several state spaces. Then, well-posedness and exponential decay were obtained. It was proved (see [1, Theorem 1.1]) that for $b = 0$ the problem is ill-posed, in the sense that the matrix operator A does not generate a strongly continuous semigroup on the state space. On the other hand, a direct approach to well-posedness for the abstract MGT-equation without reduction of order was undertaken by Fernández et al. [3]. Later, in [3, Section 2] and [4], the MGT-equation has been studied from a different perspective, by means of the theory of linear viscoelasticity. As a consequence, well-posedness and qualitative properties can be studied from a representation of the solution of MGT-equation by means of certain variation of constant formula. In the context of a Banach space, this study seems to be possible using the theory of resolvent families deeply studied by Prüss [5]. It is worthwhile to mention that qualitative properties such as the existence of global attractors for the MGT-equation have also been

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studied, see for instance [6]. Results concerning the regularity of strong and mild solutions as well as compactness of trajectories have been provided in [3]. The study of the existence of asymptotically almost periodic and almost automorphic solutions can be found in [7, 8], respectively. Existence of mild solutions with non local initial conditions by methods of Hausdorff measure of non compactness, has been established in [8] and concerning ill-posedness, the presence of chaos for the linear MGT-equation has been discovered in [9]. A variation of the MGT-equation in order to include non-degenerate equations was introduced by Cai and Bu [10]. Other studies on decay and growth properties are due to Kalantarov and Yilmaz [11], inverse problems are studied by Liu and Triggiani [12, 13], addition of memory terms are studied by Lasiecka and co-authors [14–16], and the relation of the MGT-equation with thermoelasticity theory have been pointed out by Quintanilla [17]. Finally, some more recent references on MGT-equation can be consulted at [18–25].

However, despite the large number of qualitative properties studied, controllability has been little investigated, and we are aware only of the references [26] that show that the MGT-equation can be controlled by a force that is supported on a moving subset of the domain, and [27] that consider a quadratic control problem for the nonlinear MGT-equation.

Therefore, our main objective in this article is to present new advances in the understanding of the MGT-equation in this interesting and challenging topic.

Our first key observation, and starting point for this article, is the well-known fact that the damping term (also called structural damping) y_{xxt} produces a strong smoothing effect, which implies the generation of accumulations points for the spectrum of (1) with Dirichlet boundary conditions on the domain (0, 1), see [2]. Therefore, we can not expect good controllability properties for the MGT-equation. This condition was observed by Rusell [28] for the beam equation with internal damping, by Leugering, Schmidt and Meister [29] for the plate equation with internal damping, by Micu in [30] for the linearized Benjamin–Bona–Mahony equation, and by Martin, Rosier and Rouchon [31, 32] for the structurally damped wave equation.

A second observation is that (1) is an equation that models high-intensity focused ultrasound (HIFU), which is associated with biomedical problems [33], and therefore, a natural control function must be external.

Based on the above observations, it is natural to ask if it is possible to control the Eq. (1) from the boundary in any way. More precisely, given a time $T > 0$, some initial conditions (y_0, y_1, y_2) and final states $(y^{0,T}, y^{1,T}, y^{2,T})$, we analyze if it is possible to find a control $h = h(t)$ for the solutions of

$$\begin{cases} \tau y_{ttt} + \alpha y_{tt} - c^2 y_{xx} - b y_{xxt} = 0, & (0,1)\times(0,T), \\ y(0,t) = 0, \ y(1,t) = h(t), & t \in (0,T), \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \ y_{tt}(x,0) = y_2(x), & x \in (0,1), \end{cases} \tag{2}$$

in a way that we can obtain a desired final trajectory. Specifically, to find a control such that the solution of the equation goes from (y_0, y_1, y_2) at time $t = 0$ to $(y^{0,T}, y^{1,T}, y^{2,T})$ at time $t = T$. This definition corresponds to the exact controllability problem. When $(y^{0,T}, y^{1,T}, y^{2,T}) = (0,0,0)$, we say that the problem is null controllable. If we only can drive the solution to a neighborhood of $(y^{0,T}, y^{1,T}, y^{2,T})$, we say that the system is approximately controllable.

We will prove that the Eq. (2) is not spectrally controllable, which implies that is not exactly controllable in any space. However, we will see that (2) is approximately controllable. To our knowledge, this is the first investigation on boundary controllability for this system.

This article is organized as follows. In Sect. 2, we study the well-posedness of the Eq. (2) by using spectral methods. section 3 is devoted to prove the lack of exact controllability of (2). Finally, in Sect. 4, we establish the approximate controllability of the MGT-equation (2). The paper ends with a conclusion section.

2 Preliminaries

We first look at the well-posedness of the homogeneous MGT-equation, that is, we consider the problem

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2 y_{xx} - b y_{xxt} = 0, & (0,1)\times(0,T), \\ y(0,t) = 0, \ y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \ y_{tt}(x,0) = y_2(x), & x \in (0,1), \end{cases} \tag{3}$$

where τ, α, c and b are positive constants that satisfy $\gamma := \alpha - \beta \geq 0$, with $\beta := \frac{\tau c^2}{b}$. The initial conditions belongs to some function space to be specified below.

Let A be a self-adjoint positive operator on a Hilbert space H such that $D(A) \subset H$ is dense in H. Consider the operator matrix $\mathcal{P}$ with domain $D(\mathcal{P}) = D(A) \times D(A) \times H$ given by

$$\mathcal{P} := \begin{pmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ \frac{c^2}{\tau}A & \frac{b}{\tau}A & \frac{\alpha}{\tau}I \end{pmatrix}. \quad (4)$$

Let $\mathcal{H} := D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \times H$ be endowed with the graph norm. The following result is taken from [1, Theorem 1.2] where $A := -\partial_x^2$.

Theorem 1 *Let $\gamma \geq 0$. Then, the operator $-\mathcal{P}$ defined in (4) generates a strongly continuous group in $\mathcal{H}$. As a consequence, for every $Y_0 := (y_0, y_1, y_2) \in \mathcal{H}$, the system (3) has a unique strong solution Y given by $Y(t) = e^{-t\mathcal{P}}Y_0$, where $(e^{-t\mathcal{P}})_{t\geq 0}$ is the strongly continuous semigroup on $\mathcal{H}$ generated by $-\mathcal{P}$.*

Let us consider the following classical space $\mathcal{H}^s(0,1)$, defined for any $s \geq 0$

$$\mathcal{H}^s := \mathcal{H}^s(0,1) = \{y : (0,1) \to \mathbb{R};\ \sum_{n\geq 1} n^{2s}|\hat{y}_n|^2 < \infty\}, \quad (5)$$

where $\hat{y}_n$ is the n^{th} Fourier coefficient of any integrable function $y : (0,1) \to \mathbb{R}$ with respect to the orthonormal basis $\{\sqrt{2}\sin(n\pi x)\}_{n\geq 1}$ of $L^2(0,1)$. Endowed with the scalar product

$$(y,z)_s = \sum_{n\geq 1} n^{2s}\hat{y}_n\hat{z}_n,$$

$\mathcal{H}^s$ is a Hilbert space. Moreover, $\mathcal{H}^1 = H_0^1(0,1)$, $\mathcal{H}^2(0,1) = H^2(0,1) \cap H_0^1(0,1)$. In general, we have that for $s \leq 1/2$, $\mathcal{H}^s = H^s(0,1)$, $1/2 < s \leq 3/2$, $\mathcal{H}^s = H_0^s(0,1)$, and finally $3/2 < s \leq 2$, $\mathcal{H}^s = H^s(0,1) \cap H_0^1(0,1)$, where $H^s(0,1)$ and $H_0^s(0,1)$ are the classical Sobolev spaces. Besides, we denote by $\mathcal{H}^{-s}$ the dual space of $\mathcal{H}^s$ with respect to the pivot space $\mathcal{H}^0 = L^2(0,1)$.

Each pair (ν_n, φ_n) of eigenvalues and eigenfunctions of $-\partial_x^2$ with homogeneous Dirichlet conditions on $\mathcal{H}^2(0,1)$, generates a system of eigenvalues $\{\lambda_{n,j}\}_{n\in\mathbb{N}}$, $j = 1,2,3$, of $\mathcal{P}$ given as the roots of the following cubic equation:

$$\tau\lambda_{n,j}^3 + \alpha\lambda_{n,j}^2 + (\nu_n b)\lambda_{n,j} + \nu_n c^2 = 0. \quad (6)$$

From the work of Pellicer and Solà–Morales [34, Proposition 2], where the spectral properties of $\mathcal{P}$ was derived, we can obtain a series representation of the solution of (3). Indeed, if $\frac{1}{9} < \frac{\tau}{b} < 1$ any solution $y = y(x,t)$ of (3) can be written as

$$\begin{pmatrix} y(x,t) \\ y_t(x,t) \\ y_{tt}(x,t) \end{pmatrix} = \sum_{n\geq 1}\sum_{i=1}^{3} a_{n,i}e^{\lambda_{n,i}t}f_{n,i}, \quad (7)$$

where

$$f_{n,i} = \begin{pmatrix} \sqrt{2}\sin(n\pi x) \\ \lambda_{n,i}\sqrt{2}\sin(n\pi x) \\ \lambda_{n,i}^2\sqrt{2}\sin(n\pi x) \end{pmatrix}, \quad i = 1,2,3.$$

We observe that any set of initial conditions $(y_0, y_1, y_2) \in \mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^1$ can be written as

$$\begin{cases} y_0 = \sum_{n\geq 1} b_n\sqrt{2}\sin(n\pi x), \\ y_1 = \sum_{n\geq 1} c_n\sqrt{2}\sin(n\pi x), \\ y_2 = \sum_{n\geq 1} d_n\sqrt{2}\sin(n\pi x). \end{cases} \quad (8)$$

Then, we obtain from (7) that

$$b_n = \sum_{i=1}^{3} a_{n,i}, \quad c_n = \sum_{i=1}^{3} \lambda_{n,i}a_{n,i}, \quad d_n = \sum_{i=1}^{3} \lambda_{n,i}^2 a_{n,i}. \quad (9)$$

Hence,

$$\begin{pmatrix} a_{n,1} \\ a_{n,2} \\ a_{n,3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_{n,1} & \lambda_{n,2} & \lambda_{n,3} \\ \lambda_{n,1}^2 & \lambda_{n,2}^2 & \lambda_{n,3}^2 \end{pmatrix}^{-1} \begin{pmatrix} b_n \\ c_n \\ d_n \end{pmatrix} = N\begin{pmatrix} b_n \\ c_n \\ d_n \end{pmatrix}, \quad (10)$$

where

$$N = \begin{pmatrix} \frac{\lambda_{n,2}\lambda_{n,3}}{\xi} & \frac{-(\lambda_{n,2}+\lambda_{n,3})}{\xi} & \frac{1}{\xi} \\ \frac{-\lambda_{n,1}\lambda_{n,3}}{\xi} & \frac{\lambda_{n,1}+\lambda_{n,3}}{\xi} & \frac{-1}{\xi} \\ \frac{\lambda_{n,1}\lambda_{n,2}}{\xi} & \frac{-(\lambda_{n,1}+\lambda_{n,2})}{\xi} & \frac{1}{\xi} \end{pmatrix},$$

and $\xi = (\lambda_{n,1} - \lambda_{n,2})(\lambda_{n,1} - \lambda_{n,3})$.

Remark 1

1. For the sake of simplicity, throughout the remainder of the paper, we assume that $\frac{1}{9} < \frac{\tau}{b} < 1$.
2. The other two cases given in [34, Proposition 2], that is, $0 < \frac{\tau}{b} < \frac{1}{9}$ and $\frac{\tau}{b} = \frac{1}{9}$, can be naturally extended in the context of the representation of solutions.

With the above representation of any solution of (3), we deduce the following result. We recall that $\gamma = \alpha - \frac{\tau c^2}{b}$.

Proposition 1 *Let* $\gamma \geq 0$. *For every* $(y_0, y_1, y_2) \in \mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^1$, *the solution* y *of* (3) *belongs to* $C^2([0,T];\mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^1)$ *and* $\sum_{n\geq 1}\sum_{i=1}^{3} n(|a_{n,i}|) < +\infty$. *In particular,* $y_x(1,\cdot) \in C([0,T])$.

Proof The first assertion is clear. We observe that

$$y_x(1,t) = \sum_{n\geq 1}\sum_{i=1}^{3} a_{n,i} e^{\lambda_{n,i} t}\sqrt{2}(-1)^n n\pi.$$

As we know that for each $i = 1, 2, 3$ the eigenvalues satisfy $Re(\lambda_{n,i}) < 0$, and

$$\sum_{n\geq 1} n|a_{n,i}| \leq \left(\sum_{n\geq 1} n^{-2}\right)^{1/2}\left(\sum_{n\geq 1} n^4 |a_{n,i}|^2\right)^{1/2},$$

we obtain that $\sum_{n\geq 1}\sum_{i=1}^{3} n(|a_{n,i}|) < +\infty$ and $y_x(1,\cdot) \in C([0,T])$. □

Finally, concerning the boundary value control problem (2), from [35, Theorem 1.1] we have the following result about the existence and uniqueness of solutions for (2).

Proposition 2 [35, *Theorem* 1.1] *Let* $\gamma \geq 0$. *Let* $(y_0, y_1, y_2) \in \mathcal{H}^1 \times L^2 \times \mathcal{H}^{-1}$ *along with the compatibility condition* $y_2 - \partial_x^2 y_0 \in L^2(0,1)$ *and* $h \in L^2(0,T)$. *Then, the Eq.* (2) *has a unique solution* y *which belongs to* $C([0,T];L^2(0,1)) \cap C^1([0,T];\mathcal{H}^{-1}) \cap L^2([0,T];\mathcal{H}^{-2})$.

3 Lack of controllability

We recall that our problem is to study the controllability of

$$\begin{cases} \tau y_{ttt} + \alpha y_{tt} - c^2 y_{xx} - b y_{xxt} = 0, & (0,1)\times(0,T), \\ y(0,t) = 0,\ y(1,t) = h(t), & t \in (0,T), \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x),\ y_{tt}(x,0) = y_2(x), & x \in (0,1), \end{cases} \tag{11}$$

where $h \in H^1(0,T)$ is the boundary control and the initial conditions (y_0, y_1, y_2) belongs to $L^2(0,1)^3$.

We introduce the three classical notions of controllability and also the concept of spectral controllability needed in our main results. For a detailed analysis of controllability we refer the reader to the books [36, 37].

Definition 1 The system (11) is said to be:

1. *exactly controllable* if for any $(y_0, y_1, y_2) \in L^2(0,1)^3$ and for any vector $(y^{0,T}, y^{1,T}, y^{2,T}) \in L^2(0,1)^3$, there exists a control $h \in H^3(0,T)$ such that the solution of (11) satisfies $y(x,T) = y^{0,T}$, $y_t(x,T) = y^{1,T}$ and $y_{tt}(x,T) = y^{2,T}$, for all $x \in (0,1)$.
2. *null controllable* if for any $(y_0, y_1, y_2) \in L^2(0,1)^3$, there exists a control $h \in H^2(0,T)$ such that the solution of (11) satisfies the rest condition $y(x,T) = 0$, $y_t(x,T) = 0$ and $y_{tt}(x,T) = 0$, for all $x \in (0,1)$.
3. *approximate controllable* if for any $(y_0, y_1, y_2) \in L^2(0,1)^3$ and any functions $(y^{0,T}, y^{1,T}, y^{2,T}) \in L^2(0,1)^3$, there exists a control $h \in H^3(0,T)$ such that the solution of (11) satisfies $\|y(x,T) - y^{0,T}\|_2 + \|y_t(x,T) - y^{1,T}\|_2 + \|y_{tt}(x,T) - y^{2,T}\|_2 \leq \varepsilon$.
4. *spectrally controllable* if any finite linear combination of eigenvectors can be steered to zero by a control $h \in H^3(0,T)$.

For the study of the control properties of (11), we introduce the adjoint problem to (11) which is given by the solution z of

$$\begin{cases} -\tau z_{ttt} + \alpha z_{tt} - c^2 z_{xx} + b z_{xxt} = 0, & (0,1)\times(0,T), \\ z(0,t) = z(1,t) = 0, & t \in (0,T), \\ z(x,T) = z_0(x),\ z_t(x,T) = z_1(x),\ z_{tt}(x,T) = z_2(x), & x \in (0,1). \end{cases} \tag{12}$$

From Theorem 1, existence and uniqueness of the solution of (12) can be guaranteed.

Now, proceeding as what we did for Eq. (3), if the initial data $(z_0, z_1, z_2) \in \mathcal{H}^1 \times \mathcal{H}^1 \times L^2$ are written in Fourier series as follows

$$(z_0, z_1, z_2) = \left(\sum_{n\geq 1}\tilde{b}_n\sqrt{2}\sin(n\pi x), -\sum_{n\geq 1}\tilde{c}_n\sqrt{2}\sin(n\pi x), \sum_{n\geq 1}\tilde{d}_n\sqrt{2}\sin(n\pi x)\right), \tag{13}$$

and if $\tilde{a}_{n,i}$, $i = 1, 2, 3$, are given by (10) with b_n, c_n and d_n replaced by $\tilde{b}_n, \tilde{c}_n$ and $\tilde{d}_n$, then the solution of (12) is given by

$$z(x,t)=\sum_{n\geq 1}\sum_{i=1}^{3}(\tilde{a}_{n,i}e^{\lambda_{n,i}(T-t)})\sqrt{2}\sin(n\pi x). \tag{14}$$

Moreover, if $(z_0,z_1,z_2)\in\mathcal{H}^2\times\mathcal{H}^2\times\mathcal{H}^1$, we have by Proposition 1 that

$$z_x(1,t)=\sum_{n\geq 1}\sum_{i=1}^{3}(\tilde{a}_{n,i}e^{\lambda_{n,i}(T-t)})\sqrt{2}(-1)^n n\pi \tag{15}$$

is a continuous function for every $t\in[0,T]$.

Using the above preliminaries, we can obtain the first main result in this paper.

Theorem 2 *Let $\gamma\geq 0$. The control problem (11) is not spectrally controllable in $L^2(0,1)$.*

Proof We prove that no nontrivial finite linear combination of eigenvectors can be driven to zero in finite time. Consider three sequences $\{b_n\}_{n\in\mathbb{N}}$, $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ with $b_n=c_n=d_n=0$ for all $n>N$.

Now, suppose that (11) is spectrally controllable. It means that there exists a boundary control $h\in H^1(0,T)$ such that the solution y of (11) with initial data

$$y_0=\sum_{n\geq 1}b_n\sqrt{2}\sin(n\pi x),\ y_1=\sum_{n\geq 1}c_n\sqrt{2}\sin(n\pi x),$$
$$y_2=\sum_{n\geq 1}d_n\sqrt{2}\sin(n\pi x),$$

satisfy $y(x,T)=y_t(x,T)=y_{tt}(x,T)=0$.

We observe that if we consider $y=y(x,t)$ and $z=z(x,t)$ the solutions of (11) and (12), respectively, scaling in (11) by z and integrating over $(0,1)\times(0,T)$, we obtain

$$0=\int_0^T\int_0^1\Big(\tau y_{ttt}+\alpha y_{tt}-c^2y_{xx}-by_{xxt}\Big)zdxdt.$$

Then, integrating by parts both in time and space, we get

$$0=\int_0^T\int_0^1\Big(-\tau z_{ttt}+\alpha z_{tt}-c^2z_{xx}+bz_{xxt}\Big)ydxdt-\int_0^T[c^2h(t)+bh'(t)]z_x(1,t)dt$$
$$-\tau\int_0^1[zy_{tt}-z_ty_t+z_{tt}y]\Big|_0^T dx-\alpha\int_0^1[zy_t-z_ty]\Big|_0^T dx+b\int_0^1 z_{xx}y\Big|_0^T dx$$

Using the fact that z is the solution of the backward problem (12), we deduce

$$-\int_0^T[c^2h(t)+bh'(t)]z_x(1,t)dt=\tau\int_0^1[zy_{tt}-z_ty_t+z_{tt}y]\Big|_0^T dx+\alpha\int_0^1[zy_t-z_ty]\Big|_0^T dx-b\int_0^1 z_{xx}y\Big|_0^T dx. \tag{16}$$

From (15) and (16), we obtain that

$$-\int_0^T[c^2h(t)+bh'(t)]\sqrt{2}(n\pi)(-1)^n e^{-\lambda_{n,1}t}dt$$
$$=\tau(d_n-\lambda_{n,1}c_n-\lambda_{n,1}^2d_n)-\alpha(c_n+\lambda_{n,1}b_n)-bn^2\pi^2b_n, \tag{17}$$

$$-\int_0^T[c^2h(t)+bh'(t)]\sqrt{2}(n\pi)(-1)^n e^{-\lambda_{n,2}t}dt$$
$$=\tau(d_n-\lambda_{n,2}c_n-\lambda_{n,2}^2d_n)-\alpha(c_n+\lambda_{n,2}b_n)-bn^2\pi^2b_n, \tag{18}$$

$$-\int_0^T[c^2h(t)+bh'(t)]\sqrt{2}(n\pi)(-1)^n e^{-\lambda_{n,3}t}dt$$
$$=\tau(d_n-\lambda_{n,3}c_n-\lambda_{n,3}^2d_n)-\alpha(c_n+\lambda_{n,3}b_n)-bn^2\pi^2b_n, \tag{19}$$

for each $n\geq 1$.

Now, to conclude the proof we define the complex function F as follows

$$F(z):=\int_0^T[c^2h(t)+bh'(t)]e^{izt}dt. \tag{20}$$

According to Paley–Wiener Theorem (see e.g. [38, Theorem 7.22]), we have that F is an entire function which satisfies for each $i=1,2,3$, $F(i\lambda_{n,i})=0$, for all $n>N$. We know that the real eigenvalue, $\lambda_{n,1}$, satisfy the asymptotic behavior

$$\lambda_{n,1}\longrightarrow-\frac{\tau c^2}{b},\quad \text{as } n\to\infty.$$

Then, we deduce that F is zero on a set with finite accumulation points, which implies that $F\equiv 0$. We observe that from (17)–(19) and using the fact that

$\lambda_{n,1}$ is real and $\lambda_{n,2}$ and $\lambda_{n,3}$ are complex conjugates, we obtain that $b_n = c_n = d_n = 0$, for $n \geq 1$. □

4 Approximate controllability

In this Section, due the lack of spectral controllability, we aim to prove that the system (11) is approximately controllable with a control supported at the boundary. Let us start pointing out that, from the linearity of the system (11), to study the approximate controllability condition (see Definition 1) is sufficient to consider zero initial datum $(y_0, y_1, y_2) = (0, 0, 0)$.

Our second main result read as follows.

Theorem 3 *Let $\gamma \geq 0$. The problem (11) is approximately controllable in $L^2(0,1)$ for any time $T \geq 0$.*

Proof Since the approximate controllability means a density result, we will use one consequence of Hahn–Banach Theorem [39, Corollary 1.8]. To do that, let us define the set of reachable states

$$R((0,0,0),T) := \{(y(\cdot,T), y_t(\cdot,T), y_{tt}(\cdot,T)) : y \text{ solution of (11) with } h \in H^1(0,T)\}.$$

Then, from [39, Corollary 1.8], to get the desired approximate controllability result, is enough to prove that every continuous linear function on $L^2(0,1)^3$ that vanishes on $R((0, 0, 0), T)$, must vanish everywhere on $L^2(0,1)^3$.

Thus, let $(w_0, w_1, w_2) \in L^2(0,1)^3$ and assume that

$$0 = \int_0^1 y(\cdot,T) w_2 dx + \int_0^1 y_t(\cdot,T) w_1 dx + \int_0^1 y_{tt}(\cdot,T) w_0 dx. \tag{21}$$

We define $(z_0, z_1, z_2) \in \mathcal{H}^2 \times \mathcal{H}^2 \times \mathcal{H}^1$ such that

$$\tau z_2 - \alpha z_1 - b\partial_x^2 z_0 = w_2, \quad -\tau z_1 + \alpha z_0 = w_1, \quad \tau z_0 = w_0.$$

Let z be the solution of (12) with initial conditions (z_0, z_1, z_2). Thus, from (16) we obtain

$$\int_0^T [c^2 h(t) + b h'(t)] z_x(1,t) dt = 0, \tag{22}$$

for any $h \in H^1(0,T)$. Now, we need to prove that $\tilde{b}_n = \tilde{c}_n = \tilde{d}_n = 0$, for each n.

Indeed, using the expression (15) for $z_x(1,t)$, we obtain from (22) that for each $i = 1, 2, 3$

$$\begin{aligned} 0 &= \int_0^T [c^2 h(t) + b h'(t)] z_x(1,t) dt \\ &= \sqrt{2}\pi \int_0^T [c^2 h(t) + b h'(t)] \sum_{n \geq 1} \tilde{a}_{n,i} e^{\lambda_{n,i}(T-t)} (-1)^n n dt. \end{aligned} \tag{23}$$

Then, it follows that for any $f(t) \in \mathrm{span}(e^t)^{\perp}$ there exists $h \in H^1(0,T)$ such that $(f(\cdot), z_x(1,\cdot))_{L^2(0,T)} = ([c^2 h(\cdot) + b h'(\cdot)], z_x(1,\cdot))_{L^2(0,T)} = 0$, which implies that $z_x(1,\cdot) \in \mathrm{span}(e^t)^{\perp\perp} = \mathrm{span}(e^t)$. Thus, there exists $\alpha \in \mathbb{R}$ such that

$$z_x(1,t) = \sum_{n \geq 1} \sum_{i=1}^{3} (\tilde{a}_{n,i} e^{\lambda_{n,i}(T-t)}) \sqrt{2} (-1)^n n\pi = \alpha e^t, \tag{24}$$

which is equivalent to

$$\sum_{n \geq 1} \sum_{i=1}^{3} (\overline{a}_{n,i} e^{\lambda_{n,i}\tau}) + \overline{a}_0 e^{-\tau} = 0, \quad \tau \in [0,T], \tag{25}$$

where $\overline{a}_{n,i} = \tilde{a}_{n,i} \sqrt{2} (-1)^n n\pi$ and $\overline{a}_0 = -\alpha e^T$.

In consequence, we need to prove that the coefficients $\overline{a}_{n,i}$, for $i = 1, 2, 3$, are all zero. For the case $i = 1$, we know that the real eigenvalue $\lambda_{n,1}$ is negative. Then, from the Lemma 1 in [32], we obtain that $\overline{a}_{n,1} = 0$, for all $n \geq 1$.

For $i = 2, 3$ we have that the complex eigenvalues $\lambda_{n,i}$ are complex conjugates and all with negative real parts. On the other hand, consider a sequence $\{\alpha_n\}_{n \geq 1}$ of negative real numbers and the complex function

$$F(z) = \sum_{n \geq 1} \beta_n e^{\alpha_n z}, \tag{26}$$

where $\{\beta_n\}_{n \geq 1}$ is a sequence of complex numbers. Then, F is an analytic function on the halfplane $\{z \in \mathcal{C} : Re(z) > 0\}$. From the analytic continuation property we obtain that $F(1 + it) = 0$ for any $t \in \mathbb{R}$. This implies that for each $i = 2, 3$

$$\sum_{n \geq 1} \overline{a}_{n,i} e^{Re(\lambda_{n,i}) + Im(\lambda_{n,i}) it} = 0, \quad \forall t \in \mathbb{R}. \tag{27}$$

Borrowing the ideas for the proof of Lemma 1 in [32], we deduce that $\overline{a}_{n,i} = 0$, for any $n \geq 1$ and $i = 2, 3$. This complete the proof of Theorem 3. □

5 Conclusions and further comments

In this article, we have presented an analysis of controllability for the 1D MGT-equation. We have proved that the boundary control problem is not spectrally controllable in the Lebesgue space $L^2(0,1)$ (and consequently cannot be exact and null controllable) but that it is approximately controllable in such space, with control supported in the boundary, for any time $T \geq 0$. In order to prove the last statement, we equivalently verify the unique continuation property. Let us observe that the lack and approximate controllability result proved in this article is valid only in the one-dimensional framework. Indeed, the proof of both main results are based strongly in the identity (16). At this point we replace the explicit representation of the normal derivative for the solution of the adjoint problem at $x = 1$, which implies that it does not depend on the space variable. However, in the multidimensional case, the normal derivative must be evaluated at the boundary of the domain. Therefore, the normal derivative depends on spatial variable, which makes the technique presented in this paper unusable.

With respect to the non-linear MGT equation, the local approximately controllability around the origin could be proven using the approximately controllability for the linearized equation (Theorem 3) and a perturbation argument.

Finally, since the equation is neither controllable to zero nor exactly, and it is only possible to prove approximate controllability, an interesting future work in this direction would be to study the controllability of MGT-equation, posed on a periodic domain, using an interior control of the moving type. That is, to consider a moving distributed control with a long control time, such that the support of the control, which is moving, can visit all the domain. See for instance [31, 40–42] for some recent works in this direction.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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