CONTROLLABILITY RESULTS FOR THE MOORE–GIBSON–THOMPSON EQUATION ARISING IN NONLINEAR ACOUSTICS

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Abstract. We show that the Moore-Gibson-Thomson equation

$$\tau\partial_{ttt}y + \alpha\partial_{tt}y - c^2\Delta y - b\Delta\partial_t y = k\partial_{tt}(y^2) + \chi_{\omega(t)}u,$$

is controlled by a force that is supported on an moving subset $\omega(t)$ of the domain, satisfying a geometrical condition. Using the concept of approximately outer invertible map, a generalized implicit function theorem and assuming that $\gamma := \alpha - \frac{\tau c^2}{b} > 0$, the local null controllability in the nonlinear case is established. Moreover, the analysis of the critical value $\gamma = 0$ for the linear equation is included.

1. Introduction

Our concern in this paper is the study of controllability for the following equation of third order in time

$$\alpha y_{tt} - c^2\Delta y + \tau y_{ttt} - b\Delta y_t = F(t, y, y_t, y_{tt}), \tag{1.1}$$

where τ is a positive constant accounting for relaxation (the relaxation time), c is the speed of sound, $b = \delta + \tau c^2 \geq 0$, where δ is the diffusivity of sound. The case $b = 0$ and $F(t, y, y_t, y_{tt}) = \beta(y^2)_t$ is known as the Westervelt equation. This equation is employed for modeling the finite–amplitude nonlinear wave propagation in a soft tissue. In such case y represents the pressure of the acoustic field generated by a high–intensity focused ultrasound (HIFU). HIFU is a therapeutic method for a non-invasive ablation of benign and malignant tumors [45]. In such case, the first two terms in equation (1.1) describe the linear lossless wave propagating at a small-signal sound speed. The third term represents the loss due to thermal conduction and fluid viscosity. The nonlinear term F accounts for acoustic nonlinearity which may considerably affect thermal and mechanical changes within the tissue [22].

One of the main issues concerning this equation is the study of how a memory term creates damping mechanism and whether it causes energy decay. This issue is an ongoing research of I. Lasiecka, B. Kaltenbacher and co–workers [16, 23, 24, 25, 29]. Observe that thorough study of the linearized models is a good starting point for better understanding the nonlinear models. Actually, the work [36] has shown that, even in the linear case, rich dynamics appear. This model is known as the Moore-Gibson-Thomson equation [24].

However the study of controllability properties of the Moore-Gibson-Thomson equation (MGT equation) appears as an untreated topic in the literature that deserves to be investigated. The development of new knowledge on this area of research is an interesting and challenging open problem in all of its variants.

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The first goal of this paper is to study the interior controllability of the linear MGT equation, which is obtained from equation (1.1) with $F \equiv 0$, as follows

$$\left\{\begin{array}{rcll} \tau y_{ttt} + \alpha y_{tt} - c^2 \Delta y - b \Delta y_t & = & \chi_\omega u & , \text{ in } Q, \\ y & = & 0 & , \text{ on } \Gamma, \\ y(0) = y_0, \;\; y_t(0) = y_1, \;\; y_{tt}(0) & = & y_2 & , \text{ in } \Omega. \end{array}\right. \tag{1.2}$$

Here, $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$. We denote by $Q = \Omega \times (0, T)$ and $\Gamma = \partial\Omega \times (0, T)$, $T > 0$. Without loss of generality we assume that $\tau = 1$.

The control u is applied on an open subset ω of the domain Ω. This fact is modeled by the multiplicative factor χ_ω which stands for the characteristic function of the set ω that constitutes the support of the control.

Specifically, we consider the problem of null controllability of equation (1.2). In other words, given a final time $T > 0$ and initial data for the system (y_0, y_1, y_2) in a suitable functional setting (see Section 2), we analyze the existence of a control $u \in L^2((0, T) \times \omega)$ such that the corresponding solution of equation (1.2) satisfies the resting condition at the final time $t = T$:

$$y(x, T) = y_t(x, T) = y_{tt}(x, T) = 0 \quad \text{ in } \Omega. \tag{1.3}$$

Since the equation (1.2) has internal damping $-b\Delta y_t$ (also called structural damping), which produces a strong smoothing effect, resulting in the well posedness of equation (1.1) (see [25]), thus, it is expected to find poor control properties for equation (1.1). This is because the damping generates accumulation points in the spectrum. We refer to [36] for an extensive analysis and numerical validation of the spectrum of the MGT equation. Such a phenomenon was first noticed in [44] for the beam equation with internal damping, in [31] for the plate equation with internal damping, in [38] for the linearized Benjamin–Bona–Mahony equation, and in [37, 40] for the structurally damped wave equation.

These poor control properties can also be seen if we rewrite the equation (1.1) with $F = 0$ as follows

$$(y_t + \alpha y)_{tt} - b\Delta(\frac{c^2}{b} y + y_t) = 0. \tag{1.4}$$

The above expression motivates the introduction of the following change of variable

$$z = \frac{c^2}{b} y + y_t, \tag{1.5}$$

which implies that the equation (1.2) can be rewritten as a coupled system

$$\left\{\begin{array}{rcl} z_{tt} - b\Delta z + \gamma z_t - \gamma\beta z + \gamma\beta^2 y & = & 0, \\ y_t + \beta y & = & z, \end{array}\right. \tag{1.6}$$

where $\gamma := \alpha - \frac{c^2}{b} \geq 0$ denotes the critical coefficient and $\beta := \frac{c^2}{b}$. The critical coefficient γ is taken positive because in [12] the authors proved that the equation (1.2) exhibits chaotic behavior when γ is negative.

Then, the system under consideration can be seen as a wave equation with viscous damping (γz_t) coupled with a ODE, which implies that there exists vertical rays in the space–time variable (x, t) which do not propagate at all in the space variable x, thus also making the study of controllability–observability impossible in a cylindrical subset $\omega \times (0, T) \subset \Omega \times (0, T) = Q$. Regarding that, we note that the wave equation needs a geometric control condition (GCC) for the control of waves [3] and, because of the existence of vertical rays, we need to control the system with $\omega = \Omega$. However, from an applied point of view, one is interested in employing localized control $\omega \subset \Omega$. Hence, we cannot use controls supported in a cylindrical subset of Q. Such kind of phenomenon was observed in the study of the null controllability of viscoelastic equations with viscous Kelvin–Voigt and frictional dampings in [10], in which the system is not null controllable due to the existence of time–like characteristic hyperplanes, when the control domain is a nontrivial space–time cylinder.

A solution to this problem is to consider a moving distributed control with a long control time, such that the support of the control, which is moving, can visit all the domain. This technique was used to study the controllability properties by Castro and Zuazua in [8] and by Khapalov in [27] for parabolic equations, by Rosier and Zhang in [42] for the Benjamin–Bona–Mahony equation, by Martin, Rosier and Rouchon in [37] for the wave equation in the one–dimensional setting, by Chaves, Rosier and Zuazua in [10] for a system of viscoelasticity, by Lu, Zhang and Zuazua in [34] for the wave equation with memory, and by Chaves, Zhang and Zuazua in [11] for evolution equations with memory. In the one–dimensional case, the idea of considering a moving control domain, also called moving point control, was introduced by J. L. Lions in [33] for the wave equation.

For that reason, in this work we consider that the control is supported on an subset $\omega(t)$ of the domain. The support of the control u at time t may move in time and $\chi_{\omega(t)} = \chi_{\omega(t)}(x)$ stands for the characteristic function of the set $\omega(t)$. The control $u \in L^2(\omega)$ is an applied force localized in $\omega(t)$, where $\omega := \{(x,t) : \ x \in \omega(t), \ t \in (0,T)\}$.

Obviously, it is necessary to impose a certain geometric condition on ω so that the control domain can visit all the domain Ω. Inspired by [10, 34, 43] and by the coupled system (1.6) of a wave equation with a ODE, we consider the following geometrical condition on the moving control domain.

Definition 1.1. *We say that an open set $U \subset \overline{\Omega} \times (0,T)$ satisfies the* Moving Geometric Control Condition *(MGCC for short), if*

a) all rays of geometric optics of the wave equation enter into U before the time T.
b) for all $x_0 \in \Omega$, the vertical line $\{(x_0, s) : \ s \in \mathbb{R}\}$ enters into U before the time T and

$$\inf_{x\in\Omega} \sup_{(t_1,t_2)\times\{x\}\subset U} (t_2 - t_1) > 0. \tag{1.7}$$

A few remarks concerning the MGCC:

Remark 1.2. (1) *The condition a) is the basic assumption to be able to obtain the controllability of the wave equation, which follows the classical laws of Geometric Optics, see* [3, 47]*. This result was proved by means of microlocal analysis techniques.*
(2) *If we denote by T_0 the infimum of $T > 0$ such that U satisfies the MGCC, we obtain that the set $\mathcal{O} = \cup_{t\in(0,T)} U(t)$ is a control domain that satisfies the usual GCC for a time $T > T_0$.*
(3) *The condition b) needs that vertical rays, which do not propagate in space, also reach the control set and stay in it for some time. In practice this means that the cross section $U(t)$ of U has to move as time t evolves covering the whole domain. See* [10, 34, 43] *for more details concerning this condition.*

Under the previous condition on the control domain, we obtain the first main result of this paper. For sake of simplicity we denote by X the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem 1.3. *Let $T > 0$ be such that ω satisfy the MGCC and $\gamma \geq 0$. Then for all $(y_0, y_1, y_2) \in X \times X \times H_0^1(\Omega)$, there exists a function $u \in L^2(\omega)$ such that the solution of*

$$\left\{ \begin{array}{rcll} \tau y_{ttt} + \alpha y_{tt} - c^2 \Delta y - b \Delta y_t & = & \chi_{\omega(t)} u & , \text{ in } Q, \\ y & = & 0 & , \text{ on } \Gamma, \\ y(0) = y_0, \;\; y_t(0) = y_1, \;\; y_{tt}(0) & = & y_2 & , \text{ in } \Omega, \end{array} \right. \tag{1.8}$$

fulfills

$$y(T) = y_t(T) = y_{tt}(T) = 0, \qquad \text{in } \Omega. \tag{1.9}$$

The second goal of this paper is the analysis of the interior controllability for the nonlinear MGT equation which is obtained from (1.1) with $F(t, y, y_t, y_t) = k(y^2)_{tt}$, $k > 0$, as follows

$$\left\{ \begin{array}{rcll} \tau y_{ttt} + \alpha y_{tt} - c^2 \Delta y - b \Delta y_t & = & k(y^2)_{tt} + \chi_{\omega(t)} u & , \text{ in } Q, \\ y & = & 0 & , \text{ on } \Gamma, \\ y(0) = y_0, \;\; y_t(0) = y_1, \;\; y_{tt}(0) & = & y_2 & , \text{ in } \Omega. \end{array} \right. \tag{1.10}$$

We refer to [26, 28, 30] for the development of well posedness and exponential decay of energy to this nonlinear equation using tools of semigroup theory, and to the references [1, 7, 17, 18, 35] for studies on regularity, extensions, methods and qualitative properties of the linear equation.

As in the linear case, we consider the null controllability for (1.10) using again a moving control domain. Since (1.10) is a nonlinear type equation, we study a *local* controllability result. Namely, we say that equation (1.10) is locally null controlable at time $T > 0$ if there exists a neighborhood of the origin such that for any initial data (y_0, y_1, y_2) belonging to this neighborhood, there exists a control $u \in L^2(\omega)$ such that the solution y of (1.10) satisfies

$$y(T) = y_t(T) = y_{tt}(T) = 0, \qquad \text{in } \Omega.$$

The usual way to establish controllability of a nonlinear equation is to linearize the nonlinear problem into some coupled linear systems. Then, for the controllability result established for the linearized system, some fixed–point or implicit function results can be applied to establish controllability for the nonlinear system. The technique of fixed–point arguments (e.g. Kukatani's Theorem [19]) was used to study the null controllability in [4, 5] for a fluid–structure problem and in [21] for a chemotaxis system, and the implicit function Theorem was used, among others, in [13, 14] for two and three dimensional Navier–Stokes system, respectively. In all these articles, the main point of the proof was to establish a *Carleman estimates* of the linearized adjoint equation. This inequality together with appropriate regularity results provides the suitable spaces of functions for the definition of an operator where the implicit function Theorem, or fixed–point Theorem, can be applied.

Since in this article we prove the Theorem 1.3 without Carleman estimates, we cannot use the classic result of the implicit function Theorem as in [13, 14], where the operator has to be of class C^1 with surjective derivative. With the notion of outer invertible operator and Hadamard derivative, a generalized implicit function Theorem is employed in our case [15]. Thus, we only need to prove that the derivative of a suitable operator is compact.

Then, from the null controllability in the linear case Theorem 1.3 and a Generalized Implicit Function Theorem (see Section 4) we get the second main result of the paper, that is the local null controllability of the nonlinear MGT equation on a bounded domain.

Theorem 1.4. *Let $T > 0$ be such that ω fulfills the MGCC and $\gamma > 0$. Then there exists $\rho > 0$ such that if*

$$\|(y_0, y_1, y_2)\|_{X \times X \times H_0^1(\Omega)} \leq \rho, \tag{1.11}$$

then there are $y \in C([0,T]; X \times X \times H_0^1(\Omega))$ *and* $u \in L^2(\omega)$ *such that the solution* y *of* (1.10) *fulfills*

$$y(T) = y_t(T) = y_{tt}(T) = 0, \qquad \text{in } \Omega. \tag{1.12}$$

Theorems 1.3 and 1.4 are, as far as we know, the first results concerning the control property of the linear and nonlinear MGT equation. In this sense, the main contribution of this work is not only the null controllability of the equation, but also to give new insights showing that, in order to obtain the controllability, it is necessary that the control moves in such a way that it can cross the whole domain. Concerning methods, we remark that to our knowledge, this is the first time that the results in [15] are used in control theory. This way, we propose a novel way to deal with controllability for nonlinear equations.

The remaining of this paper is organized as follows. In section 2 we present some basic results about the well posedness of the MGT equation which are needed for the controllability property. In section 3 we prove the first result of our work, namely Theorem 1.3, separating the cases $\gamma > 0$ and $\gamma = 0$, where γ denotes the critical coefficient. In section 4 we treat the nonlinear MGT equation and we prove Theorem 1.4. Finally, in section 5 we provide additional comments concerning the main conclusions of this paper and directions of future work.

2. Preliminaries

For the sake of completeness, we state the main results regarding the well posedness and regularity of solutions for the linear and nonlinear MGT equation.

2.1. Well posedness of the linear MGT equation.

In this section we present the well posedness result needed for studying the control system (1.8). We first review some results given by Marchand, McDevitt and Triggiani in [36] (also in [25]). Let us consider the equation

$$\begin{cases} y_{ttt} + \alpha y_{tt} - c^2 \Delta y - b\Delta y_t = f & , \text{ in } Q, \\ y = 0 & , \text{ on } \Gamma, \\ y(0) = y_0, \ \ y_t(0) = y_1, \ \ y_{tt}(0) = y_2 & , \text{ in } \Omega, \end{cases} \tag{2.1}$$

where (y_0, y_1, y_2) belongs to some function spaces to be specified below.

Using the change of variable (1.5), the problem (2.1) can be written as a coupled system

$$\begin{cases} z_{tt} - b\Delta z + \gamma z_t - \gamma\beta z + \gamma\beta^2 y = f & , \text{ in } Q, \\ y_t + \beta y = z & , \text{ in } Q, \\ y = z = 0 & , \text{ on } \Gamma, \\ z(0) = z_0, \ \ z_t(0) = z_1, \ \ y(0) = y_0 & , \text{ in } \Omega. \end{cases} \tag{2.2}$$

We know that the energy associated to the MGT equation decay exponentially when the critical coefficient γ is strictly positive and is conserved when γ is zero (see [25]). This can be seen from the coupled system (2.2). Indeed, if $\gamma > 0$ the first equation in (2.2) is a wave equation with viscous damping and it is well known that this equation has a exponential decay of the energy [46]. On the other hand, when $\gamma = 0$, the first equation is a pure wave equation which is conservative [47].

In the vector variable $Z = (z, z_t, y)^T$, the system (2.2) can be formally written as

$$Z_t = AZ + F, \tag{2.3}$$

where

$$A = \begin{bmatrix} 0 & I & 0 \\ b\Delta + \gamma\beta I & -\gamma I & -\gamma\beta^2 I \\ I & 0 & -\beta I \end{bmatrix}, \; F = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}, \tag{2.4}$$

where I is the identity operator and Δ the Laplace operator.

However, the writing (2.3)–(2.4) is purely formal. As it is well known, within the framework of Hilbert (or Banach) spaces of infinite dimension, a rigorous definition of the operator requires knowledge of not only the way in which the operator acts, but also his domain.

An adequate phase space, among others (see [36]), to solve the equation (2.3) is

$$H = X \times H_0^1(\Omega) \times X. \tag{2.5}$$

Remember that X denotes the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$. It is well know that if we consider the space $L^2(\Omega)$ and $\mathcal{A}u = -\Delta u$, then the operator $\mathcal{A}$ has domain $D(\mathcal{A}) = X$ when we consider homogeneous Dirichlet boundary conditions.

Then, the operator A defined by (2.4) has domain $D(A)$ given by (see [36])

$$D(A) = D(\mathcal{A}^{3/2}) \times D(\mathcal{A}) \times D(\mathcal{A}) \subset H. \tag{2.6}$$

With all these ingredients and assuming that $\gamma > 0$, we obtain that the operator A is the generator of a strongly continuous group e^{At} on H. Moreover, the operator

$$M = I \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -\beta \\ -\beta & 1 & \beta^2 \end{bmatrix} \tag{2.7}$$

is a homeomorphism between the spaces H and $X \times X \times H_0^1(\Omega)$.

Finally, if the initial data (y_0, y_1, y_2) of (2.1) belongs to $X \times X \times H_0^1(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$, then under the operator M we have that $(z_0, z_1, y_0) \in H$ and

$$(z, z_t, y) \in C([0, T]; H). \tag{2.8}$$

For a complete and extensive analysis, using methods of semigroup theory, of the well posedness of MGT equation, we refer to [36].

2.2. **Well posedness of the nonlinear MGT equation.** The nonlinear MGT equation read as follows

$$\left\{ \begin{array}{rcll} \tau y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t & = & \dfrac{d^2}{dt^2}\left(\dfrac{1}{c^2}\left[1 + \dfrac{D}{2E}\right] y^2\right) & , \text{ in } Q, \\ y & = & 0 & , \text{ on } \Gamma, \\ y(0) = y_0, \;\; y_t(0) = y_1, \;\; y_{tt}(0) & = & y_2 & , \text{ in } \Omega. \end{array} \right. \tag{2.9}$$

Here the positive parameters D, E represent the nonlinear interactions (see [39] for a physical interpretation).

Let us denote by $\mathcal{E}$ the following energy functional

$$\mathcal{E}(t) \equiv \|\Delta y(t)\|^2_{L^2(\Omega)} + \|\nabla y_t\|^2_{L^2(\Omega)} + \|y_{tt}\|^2_{L^2(\Omega)}. \tag{2.10}$$

For the nonlinear MGT equation (2.9) we have the following global results proved in [26], for small initial data.

Theorem 2.1. [26, Theorem 1.4] *Assume that the critical coefficient γ is strictly positive. Then for any $C > 0$ there exists $\rho_C = \rho_C(\gamma) > 0$, such that solutions of (2.9) corresponding to initial data $(y_0, y_1, y_2) \in X \times X \times H_0^1(\Omega)$ with*

$$\mathcal{E}(0) \le \rho_C \tag{2.11}$$

exists for all $t > 0$ *and satisfy*

$$(y, y_t, y_t) \in C^1((0,T); X \times X \times H_0^1(\Omega)) \cap C([0,T]; X \times H_0^1(\Omega) \times L^2(\Omega)). \tag{2.12}$$

and

$$\mathcal{E}(t) \leq C \quad \forall t > 0, \tag{2.13}$$

and depends continuously, with respect to the topology generated by $\mathcal{E}$*, on the initial data.*

Besides, there exist constants $\omega = \omega(\gamma) > 0$*,* $\overline{C} > 0$ *such that solutions of* (2.9) *satisfy*

$$\mathcal{E}(t) \leq \overline{C} e^{-\omega t} \quad \forall t > 0. \tag{2.14}$$

3. Controllability of the linear MGT equation

This section is devoted to prove Theorem 1.3. We separate the proof into two cases: $\gamma > 0$ and $\gamma = 0$. This is because the first one needs a much finer development, unlike the second that proves to be a little easier since this condition simplifies the system to study.

3.1. **Case** $\gamma > 0$**.**

In the present section, we reduce the null controllability problem of the equation (1.8) to the null controllability of a coupled system.

Consider the coupled system (2.2) with $f = \chi_{\omega(t)} u$. We observe that if these coupled system is null controllable then the equation (1.8) is null controllable. Indeed, if (2.2) is controllable then we obtain that for any initial data (z_0, z_1, y_0) there is a control $u \in L^2(\omega)$ such that the pair (z, y), solution of (2.2), satisfies

$$z(x,T) = z_t(x,T) = y(x,T) = 0 \quad \text{ in } \Omega.$$

This implies that

$$\begin{aligned} z(T) &= y_t(T) + \beta y(T), \\ z_t(T) &= y_{tt}(T) + \beta y_t(T), \end{aligned}$$

and we immediately obtain that

$$y_t(T) = 0, \quad y_{tt}(T) = 0.$$

Thus, we analyze the following controlled system

$$\left\{ \begin{array}{rcll} z_{tt} - b\Delta z + \gamma z_t - \gamma \beta z + \gamma \beta^2 y &=& \chi_{\omega(t)} u & , \text{ in } Q, \\ y_t + \beta y &=& z & , \text{ in } Q, \\ y \;=\; z &=& 0 & , \text{ on } \Gamma, \\ z(0) = z_0, \;\; z_t(0) = z_1, \;\; y(0) &=& y_0 & , \text{ in } \Omega. \end{array} \right. \tag{3.1}$$

Now, to study the controllability of the coupled system (3.1), we borrow some ideas from [34] and introduce the following subset of ω. For any $\epsilon > 0$ and $O \subset \mathbb{R}^{1+n}$, we denote by $\omega_\epsilon(O) := \{x \in \mathbb{R}^{1+n} : \operatorname{dist}(x, O) < \epsilon\}$. Let ω_ϵ be the following subset of ω

$$\omega_\epsilon := \omega \setminus \overline{\omega_\epsilon(\partial w \setminus \Gamma)}. \tag{3.2}$$

As ω satisfies the MGCC, there exists $\epsilon_0 > 0$ such that $w_{\frac{3}{2}\epsilon}$ and ω_{ϵ_0} still fulfills the MGCC, see [34]. Now, let $\xi \in C^\infty(\overline{Q})$ be given and satisfying the following set of conditions

$$\left\{ \begin{array}{l} 0 \leq \xi \leq 1, \\ \xi = 1 \text{ in } \omega_{\epsilon_0}, \\ \xi = 0 \text{ in } \omega \setminus \omega_{\frac{\epsilon}{2}}. \end{array} \right. \tag{3.3}$$

We observe that $\mathrm{supp}(\xi) \subset \overline{\omega}$. With this notation, we consider the following controlled system

$$\left\{\begin{array}{rcll} z_{tt} - b\Delta z + \gamma z_t - \gamma\beta z + \gamma\beta^2 y &=& \xi u & , \text{ in } Q, \\ y_t + \beta y &=& z & , \text{ in } Q, \\ y \;=\; z &=& 0 & , \text{ on } \Gamma, \\ z(0) = z_0, \quad z_t(0) = z_1, \quad y(0) &=& y_0 & , \text{ in } \Omega. \end{array}\right. \tag{3.4}$$

To study the null controllability of the coupled system (3.4), we introduce the adjoint system

$$\left\{\begin{array}{rcll} p_{tt} - b\Delta p - \gamma p_t - \gamma\beta p + \gamma\beta^2 q &=& 0 & , \text{ in } Q, \\ -q_t + \beta q &=& p & , \text{ in } Q, \\ p \;=\; q &=& 0 & , \text{ on } \Gamma, \\ p(T) = p_0, \quad p_t(T) = p_1, \quad q(T) &=& q_0 & , \text{ in } \Omega, \end{array}\right. \tag{3.5}$$

where $(p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$.

Let us first deduce a necessary and sufficient condition for the null controllability property of (3.4) to hold. By $\langle , \rangle_{U,U'}$ we denote the duality product between U and its dual U'.

Lemma 3.1. *The control $u \in L^2(\omega)$ drives the initial data $(z_0, z_1, y_0) \in X \times H_0^1(\Omega) \times X$ of system (3.4) to zero in time T if and only if*

$$\int_\omega \xi u(x,t)p(x,t)dxdt = \langle z_0, p_t(0)\rangle_{X,X'} + \gamma\beta^2\langle y_0, q(0)\rangle_{X,X'} - \langle z_1, p(0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - \gamma\langle z_0, p(0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}, \tag{3.6}$$

for all $(p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$, where (p,q) is the corresponding solution of (3.5).

Proof. By multiplying the first equation of (3.4) by p and then integrating by parts, we obtain

$$\begin{aligned} \int_\omega \xi u(x,t)p(x,t)dxdt &= \int_Q [z(p_{tt} - b\Delta p - \gamma p_t - \gamma\beta p) + \gamma\beta^2 py]dxdt \\ &+ \langle p_0, z_t(T)\rangle_{L^2(\Omega)} - \langle z_1, p(0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - \langle z(T), p_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} + \langle z_0, p_t(0)\rangle_{X,X'} \\ &+ \gamma\langle z(T), p_0\rangle_{L^2(\Omega)} - \gamma\langle z_0, p(0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}. \end{aligned} \tag{3.7}$$

Now, by multiplying the second equation of (3.4) by q and then integrating by parts, we have

$$\int_Q ypdxdt = \int_Q zqdxdt - \langle q_0, y(T)\rangle_{L^2(\Omega)} + \langle y_0, q(0)\rangle_{X,X'}. \tag{3.8}$$

Then, combining equations (3.7) and (3.8) and regarding the system (3.5), we deduce that

$$\begin{aligned} \int_\omega \xi u(x,t)p(x,t)dxdt &= \langle z_t(T), p_0\rangle_{L^2(\Omega)} - \langle z_1, p(0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \\ &- \langle z(T), p_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} + \langle z_0, p_t(0)\rangle_{X,X'} + \gamma\langle z(T), p_0\rangle_{L^2(\Omega)} - \gamma\langle z_0, p(0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \\ &- \gamma\beta^2\langle y(T), q_0\rangle_{L^2(\Omega)} + \gamma\beta^2\langle y_0, q(0)\rangle_{X,X'}. \end{aligned} \tag{3.9}$$

Now, from (3.9) it follows immediately that (3.6) holds if and only if (z_0, z_1, y_0) is controllable to zero and u is the corresponding control. □

Now, we introduce the classical concept of *observability* associated to the adjoint problem (3.5) for the study of the controllability. Since the operator A, defined in Section 2, is the generator of a strongly continuous group on H, borrowing the ideas for the observability inequality of the wave equation, we can define the following initial observability estimates.

Definition 3.2. *The system* (3.5) *is said to be (initially) observable on* ω *with weight* ξ *if the following observability inequality holds*

$$\|p(0)\|^2_{H^{-1}(\Omega)} + \|p_t(0)\|^2_{X'} + \|q(0)\|_{X'} \leq C\|\xi p\|^2_{L^2(\omega)}, \tag{3.10}$$

for any final data $(p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$.

By standard duality argument, we obtain the following result. See [19, 32, 47] and the references therein for a complete revision of this duality.

Proposition 3.3. *The system* (3.4) *is null controllable if and only if the system* (3.5) *is observable on* ω *with weight* ξ.

Proof. We start proving that the observability estimates implies the null controllability. Consider the functional $\mathcal{L} : L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega) \to \mathbb{R}$, defined by

$$\begin{aligned}\mathcal{L}(p_0, p_1, q_0) = \frac{1}{2}\int_\omega |\xi p|^2 dxdt + \langle z_1, p(0)\rangle_{H^1_0(\Omega), H^{-1}(\Omega)} + \gamma\langle z_0, p(0)\rangle_{H^1_0(\Omega), H^{-1}(\Omega)} \\ -\langle z_0, p_t(0)\rangle_{X,X'} - \gamma\beta^2\langle y_0, q(0)\rangle_{X,X'}.\end{aligned}$$

It is easy to prove that $\mathcal{L}$ is continuous, strictly convex and coercive. The coercivity of $\mathcal{L}$ follows from the observability estimate (3.10). Then, it is well known that $\mathcal{L}$ attains its minimum in $(\tilde{p}_0, \tilde{p}_1, \tilde{q}_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ (see [6]).

From the necessary condition of minimum of $\mathcal{L}$, we obtain that

$$\begin{aligned}\int_\omega \xi\tilde{p}p dxdt - \langle z_0, p_t(0)\rangle_{X,X'} - \gamma\beta^2\langle y_0, q(0)\rangle_{X,X'} + \langle z_1, p(0)\rangle_{H^1_0(\Omega), H^{-1}(\Omega)} \\ + \gamma\langle z_0, p(0)\rangle_{H^1_0(\Omega), H^{-1}(\Omega)} = 0,\end{aligned}$$

where the pair $(\tilde{p}, \tilde{q})$ is the solution of (3.5) with final data $(\tilde{p}_0, \tilde{p}_1, \tilde{q}_0)$. From Lemma 3.1, we deduce the null controllability of (3.4).

Next, we prove that the null controllability implies the observability inequality (3.10). Indeed, if it was not true, then there exists a sequence $\{p^k_0, p^k_1, q^k_0\}_{k\in\mathbb{N}}$, $(p^k_0, p^k_1, q^k_0) \neq (0, 0, 0)$ for all $k \in \mathbb{N}$, that belongs to $L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ such that the solution (p^k, q^k) of (3.5) corresponding to the final data (p^k_0, p^k_1, q^k_0) satisfy

$$0 \leq \int_\omega |\xi p^k|^2 dxdt < \frac{1}{k^2}(\|p^k(0)\|^2_{H^{-1}(\Omega)} + \|p^k_t(0)\|^2_{X'} + \|q^k(0)\|_{X'}) \tag{3.11}$$

$$:= \frac{1}{k^2}\|(p^k(0), p^k_t(0), q^k_0(0))\|^2_{H^{-1}(\Omega)\times X'\times X'}. \tag{3.12}$$

Let us write

$$(\tilde{p}^k_0, \tilde{p}^k_1, \tilde{q}^k_0) = \frac{\sqrt{k}}{\|(p^k(0), p^k_t(0), q^k(0))\|}(p^k_0, p^k_1, q^k_0)_{H^{-1}(\Omega)\times X'\times X'}.$$

We denote by $(\tilde{p}^k, \tilde{q}^k)$ the solution of (3.5) with final data $(\tilde{p}^k_0, \tilde{p}^k_1, \tilde{q}^k_0)$. We define the bounded linear operator $\mathcal{G} : L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega) \times X' \times X'$ as follows

$$\mathcal{G}(p_0, p_1, q_0) = (p(0), p_t(0), q(0)), \quad \forall (p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega).$$

Then, we obtain that

(3.13)

$$\begin{aligned}\|\mathcal{G}(\tilde{p}_0^k,\tilde{p}_1^k,\tilde{q}_0^k)\|^2_{H^{-1}(\Omega)\times X'\times X'} &= \frac{\sqrt{k}}{\|(p^k(0),p_t^k(0),q^k(0))\|_{H^{-1}(\Omega)\times X'\times X'}}\|\mathcal{G}(p_0^k,p_1^k,q_0^k)\|_{H^{-1}(\Omega)\times X'\times X'}\\ &= \sqrt{k}.\end{aligned}$$

Thus, we deduce

$$\int_\omega |\xi\tilde{p}^k|^2 dxdt < \frac{1}{k}. \tag{3.14}$$

On the other hand, since the system (3.4) is null controllable, from Lemma 3.1 we obtain that there is a control $u\in L^2(\omega)$ such that for any $(z_0,z_1,y_0)\in X\times H_0^1(\Omega)\times X$ we have

$$\begin{aligned}\int_\omega \xi u(x,t)\tilde{p}^k(x,t)dxdt &= \langle z_0,\tilde{p}_t^k(0)\rangle_{X,X'}+\gamma\beta^2\langle y_0,\tilde{q}^k(0)\rangle_{X,X'}\\ &\quad -\langle z_1,\tilde{p}^k(0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}-\gamma\langle z_0,\tilde{p}^k(0)\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}.\end{aligned}$$

Now, let us define the duality product between $X\times H_0^1(\Omega)\times X$ and $H^{-1}(\Omega)\times X'(\Omega)\times X'$ by

$$\langle(\phi_0,\phi_1,\phi_2),(\varphi_0,\varphi_1,\varphi_2)\rangle = \langle\phi_1,\varphi_0\rangle_{X,X'}+\gamma\beta^2\langle\phi_2,\varphi_2\rangle_{X,X'}-\langle\phi_0,\gamma\varphi_0+\varphi_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}.$$

Therefore, we obtain

$$\int_\omega \xi u(x,t)\tilde{p}^k(x,t)dxdt = \langle(z_0,z_1,y_0),\mathcal{L}(\tilde{p}_0^k,\tilde{p}_1^k,\tilde{q}_0^k)\rangle.$$

From the previous computation and (3.14), we deduce that $\mathcal{L}(p_0^k,p_1^k,q_0^k)$ tends to zero weakly in $X'\times H^{-1}(\Omega)\times X'$. By the Uniform Boundedness Principle the sequence

$$\{\mathcal{L}(p_0^k,p_1^k,q_0^k)\}_{k\in\mathbb{N}}$$

is uniformly bounded in $X'\times H^{-1}(\Omega)\times X'$. This fact contradicts (3.13) and the proof is finished. □

Thus, our main result, namely Theorem 1.3, is equivalent to prove the observability estimates (3.10). This can be seen in the following Theorem.

Theorem 3.4. *Suppose that ω fulfills the MGCC. Then the system* (3.5) *is observable on ω with weight ξ.*

The main idea for the proof of this Theorem is to introduce an alternative functional setting for the controllability and observability problems (3.4) and (3.5), respectively. Then, in this new setting we can use a finer observability inequality given by Lü, Zhang and Zuazua [34] and conclude the assertion.

We observe that the inequality (3.10) contains terms involving norms in negative Sobolev spaces, which makes the analysis even more difficult. For this reason, we will consider the controllability and observability problems for (3.4) and (3.5), respectively, in the following functional setting.

Definition 3.5. *(i) We say that the system* (3.4) *is null controllable if for any initial data $(z_0,z_1,y_0)\in L^2(\Omega)\times H^{-1}(\Omega)\times L^2(\Omega)$, there is a control function $u\in L^2(0,T;X')$ such that the corresponding solution (z,y) of* (3.4) *satisfies*

$$z(T)=z_t(T)=y(T)=0 \quad \text{in } \Omega. \tag{3.15}$$

(ii) The system (3.5) *is said to be initially observable on* ω *with weight* ξ *if for any final data* $(p_0, p_1, q_0) \in X \times H_0^1(\Omega) \times X$*, there exists a constant* $C > 0$ *such that*

$$\|p(0)\|^2_{H_0^1(\Omega)} + \|p_t(0)\|^2_{L^2(\Omega)} + \|q(0)\|_{L^2(\Omega)} \leq C\|\xi p\|^2_{H^2(\omega)}. \tag{3.16}$$

In this new setting, let us start with a classical characterization of the null controllability for the control system (3.4). The proof is identical to that given in Lemma 3.1, so we omit it.

Lemma 3.6. *The control* $u \in L^2(0, T; X')$ *drives the initial data* $(z_0, z_1, y_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ *of system* (3.4) *to zero in time* T *if and only if*

$$\begin{aligned}\langle \xi p, u\rangle_{L^2(0,T;X),L^2(0,T;X')} = \langle p_t(0), z_0\rangle_{L^2(\Omega)} + \gamma\beta^2\langle q(0), y_0\rangle_{L^2(\Omega)} \\ - \langle p(0), z_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - \gamma\langle p(0), z_0\rangle_{L^2(\Omega)},\end{aligned} \tag{3.17}$$

for all $(p_0, p_1, q_0) \in X \times H_0^1(\Omega) \times X$ *where* (p, q) *is the corresponding solution of* (3.5)*.*

As usual (see [47]), the relation (3.17) can be seen as an optimality condition for the critical points for a certain functional.

Proposition 3.7. *Assume that the coupled system* (3.5) *with final data* $(p_0, p_1, q_0) \in X \times H_0^1(\Omega) \times X$ *is initially observable in* ω *with weight* ξ*. Then, the problem* (3.4) *is null controllable for any initial data* $(z_0, z_1, y_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$*.*

Proof. Assume that (3.5) is initially observable with final data $(p_0, p_1, q_0) \in X \times H_0^1(\Omega) \times X$. Since we have changed the functional setting of our controllability and observability problem, and in order to define a certain functional (as the Proposition 3.3), we consider the Hilbert space $\mathcal{V}$ which is the completion of

$$\left\{(p_0, p_1, q_0) \in X \times H_0^1(\Omega) \times X : \int_\omega |(\partial_{tt} + \Delta)(\xi p)|^2 dxdt < \infty\right\}, \tag{3.18}$$

with respect to the norm (from the observability hypothesis)

$$\|(p_0, p_1, q_0)\|^2_{\mathcal{V}} := \int_\omega |(\partial_{tt} + \Delta)(\xi p)|^2 dxdt, \tag{3.19}$$

where (p, q) is the solution of (3.5) with final data (p_0, p_1, q_0).

First, we observe that $\mathcal{V} \subset H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. Indeed, if (p, q) is a solution to (3.5), then necessarily it is also solution of the following problem

$$\left\{\begin{array}{rcll} \phi_{tt} - b\Delta\phi - \gamma\phi_t - \gamma\beta\phi + \gamma\beta^2\varphi & = & 0 & , \text{ in } Q, \\ -\varphi_t + \beta\varphi & = & \phi & , \text{ in } Q, \\ \phi = \varphi & = & 0 & , \text{ on } \Gamma, \\ \phi(0) = p(0), \quad \phi_t(0) = p_t(0), \quad \varphi(0) & = & q(0) & , \text{ in } \Omega. \end{array}\right. \tag{3.20}$$

From the observability inequality (3.16), we know that if $(p_0, p_1, q_0) \in \mathcal{V}$, then

$$(p(0), p_t(0), q(0)) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega). \tag{3.21}$$

We deduce that $(p(0), p_t(0), q(0)) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$. From the well–posedness of (3.20) we obtain that

$$(\phi, \varphi) \in [C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))] \times C^1([0, T]; L^2(\Omega)). \tag{3.22}$$

Insomuch as $(p(0),p_t(0))\in H_0^1(\Omega)\times L^2(\Omega)$ and $\varphi\in C^1([0,T];L^2(\Omega))$, from the wave equation of (3.20) we have that

$$\phi\in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega)). \tag{3.23}$$

Thus we obtain that $(p_0,p_1,q_0)=(\phi(T),\phi_t(T),\varphi(T))\in H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$.

Now, we are in position to prove that the functional J has a unique minimizer. From the Direct Method of Calculus of Variations, we know that it is sufficient to prove that J is continuous, strictly convex and coercive. The first two assertions are immediate. The coercivity of J follows from the observability inequality (3.16) and from the positivity of all the constants involved. Indeed,

$$\begin{aligned} J(p_0,p_1,q_0)\geq{}& \frac{1}{2}\int_\omega|(\partial_{tt}+\Delta)(\xi p)|^2dxdt-\|p(0)\|_{H_0^1(\Omega)}\|z_1\|_{H^{-1}(\Omega)}-\gamma\|p(0)\|_2\|z_0\|_2\\ &-\|p_t(0)\|_2\|z_0\|_2-\gamma\beta^2\|q(0)\|_2\|y_0\|_2\\ \geq{}& C_1\|\xi p\|^2_{H^2(w)}-(\|p(0)\|_{H_0^1(\Omega)}\|z_1\|_{H^{-1}(\Omega)}+\|p_t(0)\|_2\|z_0\|_2\\ &+\gamma\beta^2\|q(0)\|_2\|y_0\|_2+\gamma\|p(0)\|_2\|z_0\|_2)\\ \geq{}& C_1\|\xi p\|^2_{H^2(w)}-C_2\|\xi p\|_{H^2(w)}(\|z_1\|_{H^{-1}(\Omega)}+\|z_0\|_2+\gamma\beta^2\|y_0\|_2\\ &+\gamma\|z_0\|_2). \end{aligned}$$

By the previous computation we obtain that J is coercive. This implies that J has a unique minimizer $(\hat p_0,\hat p_1,\hat q_0)\in\mathcal{V}$.

Now, since J achieve its minimum at $(\hat p_0,\hat p_1,\hat q_0)$, then for any $(p_0,p_1,q_0)\in X\times H_0^1(\Omega)\times X$ and $h\in\mathbb{R}$ we have necessarily that

$$\lim_{h\to0}\frac{J(\hat p_0+hp_0,\hat p_1+hp_1,\hat q_0+hq_0)-J(\hat p_0,\hat p_1,\hat q_0)}{h}=0. \tag{3.24}$$

Let us develop the numerator of the previous limit.

$$\begin{aligned} &J(\hat p_0+hp_0,\hat p_1+hp_1,\hat q_0+hq_0)-J(\hat p_0,\hat p_1,\hat q_0)\\ &=\frac{1}{2}\int_\omega|(\partial_{tt}+\Delta)(\xi(\hat p+hp))|^2dxdt+\langle\hat p(0)+hp(0),z_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}\\ &\quad+\gamma\langle\hat p(0)+hp(0),z_0\rangle_{L^2(\Omega)}-\langle\hat p_t(0)+hp_t(0),z_0\rangle_{L^2(\Omega)}-\gamma\beta^2\langle\hat q(0)+hq(0),y_0\rangle_{L^2(\Omega)}\\ &\quad-\frac{1}{2}\int_\omega|(\partial_{tt}+\Delta)(\xi\hat p)|^2dxdt-\langle\hat p(0),z_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}-\gamma\langle\hat p(0),z_0\rangle_{L^2(\Omega)}\\ &\quad+\langle\hat p_t(0),z_0\rangle_{L^2(\Omega)}+\gamma\beta^2\langle\hat q(0),y_0\rangle_{L^2(\Omega)}\\ &=h\int_\omega(\partial_{tt}+\Delta)(\xi\hat p)(\partial_{tt}+\Delta)(\xi p)dxdt+\frac{h^2}{2}\int_\omega|(\partial_{tt}+\Delta)(\xi p)|^2dxdt\\ &\quad+h\langle p(0),z_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}+h\gamma\langle p(0),z_0\rangle_{L^2(\Omega)}\\ &\quad-h\langle p_t(0),z_0\rangle_{L^2(\Omega)}-h\gamma\beta^2\langle q(0),y_0\rangle_{L^2(\Omega)}. \end{aligned}$$

Thus, replacing in (3.24) we obtain

$$\begin{aligned} 0=&\int_\omega(\partial_{tt}+\Delta)(\xi\hat p)(\partial_{tt}+\Delta)(\xi p)dxdt+\langle p(0),z_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}\\ &+\gamma\langle p(0),z_0\rangle_{L^2(\Omega)}-\langle p_t(0),z_0\rangle_{L^2(\Omega)}-\gamma\beta^2\langle q(0),y_0\rangle_{L^2(\Omega)}. \end{aligned} \tag{3.25}$$

From (3.25) we can observe that if $u=(\partial_{tt}+\Delta)^2(\xi\hat p)$ belongs to $L^2(0,T;X')$, then necessarily u is a control which leads the initial data (z_0,z_1,y_0) to zero in time T. Indeed, if

$u = (\partial_{tt} + \Delta)^2(\xi\hat{p}) \in L^2(0,T;X')$, then

$$\langle \xi p, u\rangle_{L^2(0,T;X),L^2(0,T;X')} = \int_\omega (\partial_{tt} + \Delta)(\xi\hat{p})(\partial_{tt} + \Delta)(\xi p)dxdt,$$

and from (3.25) we deduce that

$$\begin{aligned}\langle \xi p, u\rangle_{L^2(0,T;X),L^2(0,T;X')} = -\langle p(0), z_1\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - \gamma\langle p(0), z_0\rangle_{L^2(\Omega)} \\ + \langle p_t(0), z_0\rangle_{L^2(\Omega)} + \gamma\beta^2\langle q(0), y_0\rangle_{L^2(\Omega)}.\end{aligned}$$

Lemma 3.6 implies that u is a control of the system (3.4).

Therefore, to finish the proof, we need to prove that $u = (\partial_{tt} + \Delta)^2(\xi\hat{p})$ belongs to $L^2(0,T;X')$. Indeed, from the definition of $\mathcal{V}$ we have

$$(\partial_{tt} + \Delta)(\xi\hat{p}) \in L^2(\omega). \tag{3.26}$$

Because $(\hat{p}_0, \hat{p}_1, \hat{q}_0)$ belongs to $\mathcal{V} \subset H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, we obtain that the solution $(\hat{p}, \hat{q})$ of (3.5) corresponding to the final data $(\hat{p}_0, \hat{p}_1, \hat{q}_0)$ satisfies

$$\hat{p} \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \quad \hat{q} \in C^1([0,T]; L^2(\Omega)). \tag{3.27}$$

Now, we develop the term $(\partial_{tt} + \Delta)(\xi\hat{p})$:

$$\begin{aligned}\partial_{tt}(\xi\hat{p}) &= 2\xi_t\hat{p}_t + \xi_{tt}\hat{p} + \xi\hat{p}_{tt}\\ \Delta(\xi\hat{p}) &= \hat{p}\Delta\xi + 2\nabla\xi\cdot\nabla\hat{p} + \xi\Delta\hat{p}.\end{aligned}$$

From (3.27), we obtain that

$$2\xi_t\hat{p}_t + \xi_{tt}\hat{p} \in C([0,T]; L^2(\Omega)), \quad \hat{p}\Delta\xi + 2\nabla\xi\cdot\nabla\hat{p} \in C([0,T]; L^2(\Omega)). \tag{3.28}$$

Since $(\hat{p}, \hat{q})$ is the solution of (3.5), we deduce the following

$$(\xi\hat{p})_{tt} - b\Delta(\xi\hat{p}) - \gamma(\xi\hat{p})_t - \gamma\beta(\xi\hat{p}) + \gamma\beta^2(\xi\hat{q}) = 2\xi_t\hat{p}_t + \xi_{tt}\hat{p} - b\hat{p}\Delta\xi - 2b\nabla\xi\cdot\nabla\hat{p} - \gamma\xi_t\hat{p}$$

and

$$-(\xi\hat{q})_t + \beta^2(\xi\hat{q}) = \xi\hat{p} - \xi_t\hat{q}.$$

From the definition of ξ, we obtain that the pair $(\xi\hat{p}, \xi\hat{q})$ satisfies the following system

$$\left\{\begin{array}{rcll}(\xi\hat{p})_{tt} - b\Delta(\xi\hat{p}) - \gamma(\xi\hat{p})_t - \gamma\beta(\xi\hat{p}) + \gamma\beta^2(\xi\hat{q}) &=& 2\xi_t\hat{p}_t + \xi_{tt}\hat{p} - b\hat{p}\Delta\xi & \\ && -2b\nabla\xi\cdot\nabla\hat{p} - \gamma\xi_t\hat{p} & , \text{ in } Q,\\ -(\xi\hat{q})_t + \beta(\xi\hat{q}) &=& (\xi\hat{p}) - \xi_t\hat{q} & , \text{ in } Q,\\ \xi\hat{p} \ = \ \xi\hat{q} &=& 0 & , \text{ on } \Gamma,\\ (\xi\hat{p})(T) = 0, \ \ (\xi\hat{p})_t(T) = 0, \ \ (\xi\hat{q})(T) &=& 0 & , \text{ in } \Omega.\end{array}\right. \tag{3.29}$$

For what comes next we need the following computations. Using the first equation of (3.5) we obtain

$$\begin{aligned}(\partial_{tt} + \Delta)(\xi_t\hat{p}_t) = \xi_{ttt}\hat{p}_t + 2\xi_{tt}[b\Delta\hat{p} + \gamma\hat{p}_t + \gamma\beta\hat{p} - \gamma\beta^2\hat{q}] + \xi_t[b\Delta\hat{p}_t + \gamma\beta\hat{p}_t - \gamma\beta^2\hat{q}_t]\\ + \xi_t[b\Delta\hat{p} + \gamma\hat{p}_t + \gamma\beta\hat{p} - \gamma\beta^2\hat{q}] + \hat{p}_t\Delta\xi_t + 2\nabla\xi_t\cdot\nabla\hat{p}_t + \xi_t\Delta\hat{p}_t.\end{aligned} \tag{3.30}$$

Then, we have that $(\partial_{tt} + \Delta)(\xi_t\hat{p}_t) \in C([0,T] : X')$. Similarly, we deduce that

$$(\partial_{tt} + \Delta)(2\xi_t\hat{p}_t + \xi_{tt}\hat{p} - b\hat{p}\Delta\xi - 2b\nabla\xi\cdot\nabla\hat{p} - \gamma\xi_t\hat{p}) \in C([0,T]; X'). \tag{3.31}$$

Also, we obtain

$$\begin{aligned}(\partial_{tt} + \Delta)(\xi\hat{p} - \xi_t\hat{q}) = 2\xi_t\hat{p}_t + \xi_{tt}\hat{p} + \xi(b\Delta\hat{p} + \gamma\hat{p}_t + \gamma\beta\hat{p} - \gamma\beta^2\hat{q}) + \hat{p}\Delta\xi\\ + 2\nabla\xi\cdot\nabla\hat{p} + \xi\Delta\hat{p} - \xi_{ttt}\hat{q} - 2\xi_{tt}\hat{q}_t - \xi_t(\hat{p}_t - \beta\hat{q}_t) - \hat{q}\Delta\xi_t - 2\nabla\xi_t\cdot\nabla\hat{q} - \xi_t\Delta\hat{q} \in C([0,T]; X').\end{aligned} \tag{3.32}$$

Now, write $\psi = (\partial_{tt}+\Delta)(\xi\hat{p})$ and $\eta = (\partial_{tt}+\Delta)(\xi\hat{q})$. Then, using (3.29) and (3.31)–(3.32), we obtain that (ψ,η) solves

$$\left\{\begin{array}{rcll} \psi_{tt} - b\Delta\psi - \gamma\psi_t - \gamma\beta\psi + \gamma\beta^2\eta &=& (\partial_{tt}+\Delta)(2\xi_t\hat{p}_t + \xi_{tt}\hat{p} - b\hat{p}\Delta\xi & \\ && -2b\nabla\xi\cdot\nabla\hat{p} - \gamma\xi_t\hat{p}) & , \text{ in } Q, \\ -\eta_t + \beta\eta &=& (\partial_{tt}+\Delta)(\psi - \xi_t\hat{q}) & , \text{ in } Q, \\ \psi \;=\; \eta &=& 0 & , \text{ on } \Gamma, \\ \psi(T)=0, \quad \psi_t(T)=0, \quad \eta(T) &=& 0 & , \text{ in } \Omega. \end{array}\right. \tag{3.33}$$

Then we conclude from (3.32) and the second equation of (3.33) that

$$\eta \in C^1([0,T];X'). \tag{3.34}$$

Besides, from (3.26) we obtain that $\Delta(\partial_{tt}+\Delta)(\xi\hat{p}) \in L^2(0,T;X')$. Using that $\psi = (\partial_{tt}+\Delta)(\xi\hat{p})$, it follows from (3.30), (3.31), (3.32), (3.33) and (3.34) that $\partial_{tt}(\partial_{tt}+\Delta)(\xi\hat{p}) \in L^2(0,T;X')$. Therefore, we can conclude that

$$u = (\partial_{tt}+\Delta)^2(\xi\hat{p}) \in L^2(0,T;X').$$

This completes the proof. □

Now, from [34] we deduce the following finer observability estimate.

Proposition 3.8 (see [34])**.** *If the system* (3.4) *with initial data* $(z_0,z_1,y_0) \in L^2(\Omega)\times H^{-1}(\Omega)\times L^2(\Omega)$ *is null controllable, then the solutions* (p,q) *of* (3.5) *satisfies*

$$\|p(0)\|^2_{H^1_0(\Omega)} + \|p_t(0)\|^2_{L^2(\Omega)} + \|q(0)\|_{L^2(\Omega)} \le C\|\Delta(\xi p)\|^2_{L^2(\omega)},$$

for any $(p_0,p_1,q_0) \in X\times H^1_0(\Omega)\times X$.

Proof. The proof is similar to the one given in Proposition 3.3. □

The above improved observability inequality (3.35) implies, immediately, the observability estimate (3.16). That is, putting together Proposition 3.7 and Proposition 3.8 we have proved the following sequence of equivalences:

Proposition 3.9. *The following assertions are equivalent:*

(1) *The system* (3.5) *for any final data* $(p_0,p_1,q_0) \in X\times H^1_0(\Omega)\times X$ *is observable on* ω *with weight* ξ.
(2) *The system* (3.4) *is null controllable for any initial data* $(z_0,z_1,y_0) \in L^2(\Omega)\times H^{-1}(\Omega)\times L^2(\Omega)$.
(3) *The solution of* (3.5) *satisfy the following improved observability estimate*

$$\|p(0)\|^2_{H^1_0(\Omega)} + \|p_t(0)\|^2_{L^2(\Omega)} + \|q(0)\|_{L^2(\Omega)} \le C\|\Delta(\xi p)\|^2_{L^2(\omega)}, \tag{3.35}$$

for any $(p_0,p_1,q_0) \in X\times H^1_0(\Omega)\times X$.

Therefore, in order to obtain the null controllability in this new formulation, it is enough to prove the following Theorem.

Theorem 3.10. *Suppose that* ω *fulfills the MGCC and* $\gamma > 0$. *Then the system* (3.5) *with final data* (p_0,p_1,q_0) *in* $X\times H^1_0(\Omega)\times X$ *is initially observable on* ω *with weight* ξ.

Proof. From (3.5) we have that q satisfy

$$-q_t + \beta q = p,$$

this is

$$q(x,t) = e^{\beta(t-T)}q_0 - \int_T^t e^{-\beta(s-t)}p(x,s)ds.$$

Since $q(x,t) = e^{\beta(t-T)}q_0(x)$ is the solution of

$$-q_t + \beta q = 0,$$

and ω_{ϵ_0} fulfills the MGCC, we obtain that there exists a constant $C > 0$ such that (see [10])

$$\|q_0\|^2_{L^2(\Omega)} \leq C\|q\|^2_{L^2(\omega_{\epsilon_0})}.$$

By standard energy estimates (see e.g. [9]) and the above observability inequality we obtain that

$$\|q\|^2_{L^2(Q)} \leq C(\|q\|^2_{L^2(\omega_{\epsilon_0})} + \|p\|^2_{L^2(Q)}). \tag{3.36}$$

On the other hand, since ω_{ϵ_0} satisfies MGCC from [43] we obtain that

$$\|p\|^2_{H^1(Q)} \leq C\left(\int_{\omega_{\epsilon_0}} p_t^2 dxdt + \|q\|^2_{L^2(Q)}\right). \tag{3.37}$$

Combining (3.36) and (3.37) we find that

$$\|p\|^2_{H^1(Q)} + \|q\|^2_{L^2(Q)} \leq C(\|p_t\|^2_{L^2(\omega_{\epsilon_0})} + \|q\|^2_{L^2(\omega_{\epsilon_0})} + \|p\|^2_{L^2(Q)}). \tag{3.38}$$

Using a compactness uniqueness argument, we claim that

$$\|p\|^2_{H^1(Q)} + \|q\|^2_{L^2(Q)} \leq C(\|p_t\|^2_{L^2(\omega_{\epsilon_0})} + \|q\|^2_{L^2(\omega_{\epsilon_0})}). \tag{3.39}$$

Indeed, if it was not true, then there would exists a sequence $\{p^i, q^i\}_{i\in\mathbb{N}}$ belongs to $H^1(Q) \times L^2(Q)$ such that for all $i \in \mathbb{N}$

$$\|(p^i, q^i)\|_{H^1(Q)\times L^2(Q)} = 1, \tag{3.40}$$

$$\|p^i_t\|^2_{L^2(\omega_{\epsilon_0})} + \|q^i\|^2_{L^2(\omega_{\epsilon_0})} \leq \frac{1}{i}. \tag{3.41}$$

Using (3.38), (3.40) and (3.41) we have that

(3.42)

$$1 = \|p^i\|^2_{H^1(Q)} + \|q^i\|^2_{L^2(Q)} \leq C(\|p^i_t\|^2_{L^2(\omega_{\epsilon_0})} + \|q^i\|^2_{L^2(\omega_{\epsilon_0})} + \|p^i\|^2_{L^2(Q)}) \leq C\left(\frac{1}{i} + \|p^i\|^2_{L^2(Q)}\right).$$

On the other hand, from (3.40) we can extract a subsequence, denoted in the same way, of $\{p^i, q^i\}_{i\in\mathbb{N}}$ such that (p^i, q^i) converges weakly to $(\overline{p}, \overline{q})$ in $H^1(Q) \times L^2(Q)$. Thus, it is easy to see that $(\overline{p}, \overline{q})$ is a weak solution of (3.5). Then, by the weak convergence in $H^1(Q)$ we obtain that p^i converges strongly in $L^2(Q)$. Combining this with (3.36) and (3.41) we have q^i converges strongly in $L^2(Q)$. Also, from (3.42) we deduce that $\|\overline{p}\|_{L^2(Q)} > 0$.

Now, as (p^i, q^i) converges weakly in $H^1(Q) \times L^2(Q)$, by the definition of weak convergence we obtain that

$$\|\overline{p}_t\|^2_{L^2(\omega_{\epsilon_0})} + \|\overline{q}\|^2_{L^2(\omega_{\epsilon_0})} \leq \liminf_{i\to\infty}(\|p^i_t\|^2_{L^2(\omega_{\epsilon_0})} + \|q^i\|^2_{L^2(\omega_{\epsilon_0})}) = 0. \tag{3.43}$$

Namely, we have that $\overline{p}_t = \overline{q} = 0$ in ω_{ϵ_0} and, by (3.38),

$$\|\overline{p}\|^2_{H^1(Q)} + \|\overline{q}\|^2_{L^2(Q)} \leq C\|\overline{p}\|^2_{L^2(Q)}. \tag{3.44}$$

Since $\|\overline{p}\|_{L^2(Q)}$ is strictly positive, by (3.44) we conclude that $(\overline{p},\overline{q})$ is not zero.

Now, let V be a linear subspace of $H^1(Q)\times L^2(Q)$ defined as

$$V := \{(p,q)\in H^1(Q)\times L^2(Q):\ (p,q) \text{ satisfies the two first equation in (3.5)}, \\ p|_\Gamma = 0,\ p_t = q = 0 \text{ in } \omega_{\epsilon_0}\}.$$

It is easy to prove, following the ideas of [34], that V satisfies

$$V\subset H^4(Q)\times H^3(Q), \tag{3.45}$$

and

$$dim(V)<\infty. \tag{3.46}$$

From the definition of V we have that $q=0$ in ω_{ϵ_0}. Since ω_{ϵ_0} fulfills the MGCC, we obtain that $q=0$ on Γ. Besides, from (3.45) the pair $(\Delta p,\Delta q)$ is also solution of (3.5), namely

$$\begin{cases} (\Delta p)_{tt} - b\Delta(\Delta p) - \gamma(\Delta p)_t - \gamma\beta\Delta p + \gamma\beta^2\Delta q &= 0 \quad , \text{ in } Q,\\ -(\Delta q)_t + \beta\Delta q &= \Delta p \quad , \text{ in } Q,\\ \Delta p \ = \ \Delta q &= 0 \quad , \text{ on } \Gamma. \end{cases} \tag{3.47}$$

Even more, since $p_t = q = 0$ in ω_{ϵ_0}, we have that $(\Delta p)_t = \Delta q = 0$ in ω_{ϵ_0}. This implies that $(\Delta p,\Delta q)\in V$.

As Δp belongs to V and V is a finite dimensional space, there exists $\lambda\in\mathbb{C}$ and $(\breve{p},\breve{q})\in V\setminus\{0\}$ such that for any $t\in(0,T)$

$$\begin{cases} \Delta\breve{p}(t) &= \lambda\breve{p}(t) \quad , \text{ in } \Omega,\\ \breve{p}(t) &= 0 \quad , \text{ on } \partial\Omega. \end{cases} \tag{3.48}$$

Now, as $(\breve{p},\breve{q})$ solves (3.5) and from (3.48), we get that

$$\begin{cases} \breve{p}_{tt} - b\lambda\breve{p} - \gamma\breve{p}_t - \gamma\beta\breve{p} + \gamma\beta^2\breve{q} &= 0 \quad , \text{ in } Q,\\ -\breve{q}_t + \beta\breve{q} &= \breve{p} \quad , \text{ in } Q,\\ \breve{p} \ = \ \breve{p} &= 0 \quad , \text{ on } \Gamma. \end{cases} \tag{3.49}$$

Since $\breve{p}_t = \breve{q} = 0$ in ω_{ϵ_0}, from the first equation in (3.49) we obtain that

$$\breve{p} = \frac{\gamma\beta^2}{b\lambda+\gamma\beta}\breve{q} = 0, \qquad \text{in } \omega_{\epsilon_0}. \tag{3.50}$$

Let $t_0\in(0,T)$ be fixed and $\overline{x}\in\omega_{\epsilon_0}$. Then from (3.49) we obtain that $(\breve{p}(\cdot,\overline{x}),\breve{q}(\cdot,\overline{x}))$ solves

$$\begin{cases} \breve{p}_{tt}(t,\overline{x}) - (b\lambda+\gamma\beta)\breve{p}(t,\overline{x}) - \gamma\breve{p}_t(t,\overline{x}) + \gamma\beta^2\breve{q}(t,\overline{x}) &= 0 \quad , \text{ in } (0,T),\\ -\breve{q}_t(t,\overline{x}) + \beta\breve{q}(t,\overline{x}) &= \breve{p}(t,\overline{x}) \quad , \text{ in } (0,T),\\ \breve{p}(t_0,\overline{x}) = 0,\ \ \breve{p}_t(t_0,\overline{x}) = 0,\ \ \breve{q}(t_0,\overline{x}) &= 0. \end{cases} \tag{3.51}$$

Thus, we have that $\breve{p}(t,\overline{x}) = \breve{q}(t,\overline{x}) = 0$ for any $t\in(0,T)$. Since ω_{ϵ_0} fulfills the MGCC, we deduce that for any $x\in\Omega$ and any $t\in(0,T)$

$$\breve{p}(t,x) = \breve{q}(t,x) = 0.$$

Namely, $\breve{p} = \breve{q} = 0$ in Q. This implies that $V=\{0\}$ which is a contradiction with the fact that $(\overline{p},\overline{q})$ is not zero. Therefore, we obtain (3.39).

We observe that from the first equation in (3.5) and as $\gamma,\beta>0$, q satisfies

$$q = -\frac{1}{\gamma\beta^2}(p_{tt} - b\Delta p - \gamma p_t - \gamma\beta p). \tag{3.52}$$

Replacing (3.52) in (3.39), it follows that

$$\|p\|^2_{H^1(Q)} + \|q\|_{L^2(Q)} \le C\|p\|_{H^2(\omega_{\epsilon_0})}. \tag{3.53}$$

Finally, form the energy estimates for (3.5) and (3.53) we obtain

$$\|p(0)\|^2_{H^1_0(\Omega)} + \|p_t(0)\|^2_{L^2(\Omega)} + \|q(0)\|_{L^2(\Omega)} \leq C\|p\|^2_{H^2(\omega_{\epsilon_0})} \leq C\|\xi p\|^2_{H^2(\omega)}.$$

□

Finally, as we have already announced, we need to prove Theorem 3.4.

Proof of Theorem 3.4. For any $(p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$, we take

$$(P_0, P_1, Q_0) = (\mathcal{A}^{-1}p_0, \mathcal{A}^{-1}p_1, \mathcal{A}^{-1}q_0), \tag{3.54}$$

where $\mathcal{A}$ is the Dirichlet Laplace operator defined in Section 2.

Then, we obtain that $(P_0, P_1, Q_0) \in X \times H^1_0(\Omega) \times X$. Since (p, q) is the solution of (3.5) with final data (p_0, p_1, q_0), we obtain

$$\begin{cases} (\mathcal{B}p)_{tt} - b\Delta(\mathcal{B}p) - \gamma(\mathcal{B}p)_t - \gamma\beta(\mathcal{B}p) + \gamma\beta^2(\mathcal{B}q) &= 0 \quad , \text{ in } Q, \\ -(\mathcal{B}q)_t + \beta(\mathcal{B}q) &= (\mathcal{B}p) \quad , \text{ in } Q, \\ \mathcal{B}p \; = \; \mathcal{B}q &= 0 \quad , \text{ on } \Gamma, \\ (\mathcal{B}p)(T) = \mathcal{B}p_0, \;\; (\mathcal{B}p)_t(T) = \mathcal{B}p_1, \;\; (\mathcal{B}q)(T) &= \mathcal{B}q_0 \quad , \text{ in } \Omega, \end{cases} \tag{3.55}$$

where $\mathcal{B} = \mathcal{A}^{-1}$. From (3.55), we have that $(P, Q) = (\mathcal{A}^{-1}p, \mathcal{A}^{-1}q)$ is in Ω.

Now, by Theorem 3.10 the system (3.55) is initially observable on ω with weight ξ and final data (P_0, P_1, Q_0). From Proposition 3.7 we have that the system (3.4) is null controllable and, by Proposition 3.8, we have that the solution (P, Q) satisfy the observability inequality (3.35), namely

$$\|\mathcal{A}^{-1}p(0)\|^2_{H^1_0(\Omega)} + \|\mathcal{A}^{-1}p_t(0)\|^2_{L^2(\Omega)} + \|\mathcal{A}^{-1}q(0)\|_{L^2(\Omega)} \leq C\|\Delta(\xi\mathcal{A}^{-1}p)\|^2_{L^2(\omega)}, \tag{3.56}$$

which implies that

$$\|p(0)\|^2_{H^{-1}(\Omega)} + \|p_t(0)\|^2_{X'} + \|q(0)\|_{X'} \leq C\|\xi p\|^2_{L^2(\omega)}.$$

□

3.2. **Case $\gamma = 0$.** In this section we give a proof of Theorem 1.3 in the special situation when $\gamma = 0$. The analysis is particularly simple in this case. Indeed, when $\gamma = 0$ the system (3.4) takes the following cascade form

$$\begin{cases} z_{tt} - b\Delta z &= \chi_{\omega(t)}u \quad , \text{ in } Q, \\ y_t + \alpha y &= z \quad , \text{ in } Q, \\ y \; = \; z &= 0 \quad , \text{ on } \Gamma, \\ z(0) = z_0, \;\; z_t(0) = z_1, \;\; y(0) &= y_0 \quad , \text{ in } \Omega, \end{cases} \tag{3.57}$$

where the first equation of (3.57) is uncoupled.

Borrowing the ideas of [10], we investigate the null controllability of the following system

$$\begin{cases} z_{tt} - b\Delta z &= \chi_{\omega(t)}\tilde{u} \quad , \text{ in } Q, \\ y_t + \alpha y &= \chi_{\omega(t)}\tilde{v} + z \quad , \text{ in } Q, \\ y \; = \; z &= 0 \quad , \text{ on } \Gamma, \\ z(0) = z_0, \;\; z_t(0) = z_1, \;\; y(0) &= y_0 \quad , \text{ in } \Omega, \end{cases} \tag{3.58}$$

where $(z_0, z_1, y_0) \in X \times H^1_0(\Omega) \times X$.

Note that, roughly speaking, one can first control the wave equation by a suitable control $\tilde{u}$ and then, once this is done, and viewing z as a given source term, we can control the transport equation by a convenient $\tilde{v}$.

Proof of Theorem 1.3 when $\boldsymbol{\gamma} = \mathbf{0}$. Denote by T_0 the infimum of $T > 0$ such that ω satisfies Definition 1.1. Let ω_0 be an open set such that

$$\omega_0 = \bigcup_{t\in(0,T')} \omega(t),$$

for a time $T' > T_0$.

Then, It is well know that the wave equation with this geometrical condition is null controllable (see [43]). Namely there exists $u \in L^2(0,T'; L^2(\Omega))$ such that the solution $z = z(x,t)$ of

$$\begin{cases} z_{tt} - b\Delta z &= \chi_{\omega_0} u \quad, \text{ in } \Omega \times (0,T'), \\ z &= 0 \quad, \text{ on } \partial\Omega \times (0,T'), \\ z(0) = z_0, \;\; z_t(0) &= z_1 \quad, \text{ in } \Omega, \end{cases} \tag{3.59}$$

satisfies

$$z(x,T') = z_t(x,T') = 0, \quad x \in \Omega.$$

Now, let $T > T'$ and consider

$$\begin{aligned} \tilde{u} = \chi_{\omega_0} u, &\quad \text{in } \Omega \times (0,T') \\ \tilde{u} = 0, &\quad \text{in } \Omega \times (T',T) \\ \tilde{v} = 0, &\quad \text{in } \Omega \times (0,T'). \end{aligned}$$

We conclude that the solution of

$$\begin{cases} z_{tt} - b\Delta z &= \chi_{\omega(t)} \tilde{u} \quad, \text{ in } \Omega \times (0,T), \\ z &= 0 \quad, \text{ on } \partial\Omega \times (0,T), \\ z(0) = z_0, \;\; z_t(0) &= z_1 \quad, \text{ in } \Omega, \end{cases} \tag{3.60}$$

satisfies

$$z(x,t) = 0, \quad \forall t \in [T',T]. \tag{3.61}$$

Next, we have to prove the exactly controllability of the following problem

$$\begin{cases} y_t + \alpha y &= \chi_{\omega(t)} v \quad, \text{ in } \Omega \times (T',T), \\ y &= 0 \quad, \text{ on } \partial\Omega \times (T',T), \\ y(T') &= y_0 \quad, \text{ in } \Omega. \end{cases} \tag{3.62}$$

Again, by a duality argument, this is equivalent to prove that the solution $q = q(x,t)$ of

$$\begin{cases} -q_t + \alpha q &= 0 \quad, \text{ in } \Omega \times (T',T), \\ q &= 0 \quad, \text{ on } \partial\Omega \times (T',T), \\ q(T) &= q_0 \quad, \text{ in } \Omega, \end{cases} \tag{3.63}$$

satisfies the following observability estimate

$$\int_\Omega |q_0(x)|^2 dx \le C \int_{T'}^T \int_\Omega \chi_{\omega(t)}(x) |q(x,t)|^2 dx dt. \tag{3.64}$$

The solution of (3.63) is given by

$$q(x,t) = e^{\alpha(t-T)} q_0(x),$$

which implies that

$$\int_{T'}^T \int_\Omega \chi_{\omega(t)}(x) |q(x,t)|^2 dx dt \ge e^{2\alpha(T'-T)} \int_\Omega |q_0(x)|^2 \left(\int_{T'}^T \chi_{\omega(t)}(x) dt \right) dx. \tag{3.65}$$

From Definition 1.1, we can see that for every $x \in \overline{\Omega}$, there is $t_0 \in (T', T)$ and some $\eta_0 > 0$ such that for any $s \in B(x, \eta_0)$ and any $t \in (T', T) \cap (t_0 - \eta_0, t_0 + \eta_0)$ we have $s \in \omega(t)$. This means that there exists some $\eta > 0$ such that

$$\int_{T'}^{T} \chi_{\omega(t)}(x)dt > \eta > 0, \forall x \in \overline{\Omega}. \tag{3.66}$$

Then, we obtain the desired inequality (3.64). Therefore, the equation (3.62) is exactly controllable on (T', T) with some controls $v \in C([T', T]; L^2(\Omega))$. Now, let $y_1(x) = e^{-T'}y_0(x) + \int_0^{T'} e^{s-T'}z(x,s)ds$. Since $\tilde{v} = 0$ in $\Omega \times (0, T')$, we extend $\tilde{v}$ to $(0, T)$ such that $\tilde{v} \in L^2(0, T; L^2(\Omega))$ and the corresponding solution of

$$\begin{cases} y_t + \alpha y &= \chi_{\omega(t)}\tilde{v} \quad, \text{ in } \Omega \times (T', T), \\ y &= 0 \quad, \text{ on } \partial\Omega \times (T', T), \\ y(T') &= y_1 \quad, \text{ in } \Omega, \end{cases} \tag{3.67}$$

satisfies

$$y(T) = 0. \tag{3.68}$$

Namely, from (3.61) and (3.68) we obtain that the pair $(\tilde{u}, \tilde{v})$ is a control function such that the solution of system (3.58) satisfy $(z(T), z_t(T), y(T)) = (0, 0, 0)$. However, if we want to return to the original problem (3.57), we need to apply the operator $\partial_{tt} - b\Delta$ in each side of the second equation of (3.58), which implies that

$$y_{ttt} + \alpha y_{tt} - b\Delta y_t - c^2\Delta y = \chi_{\omega(t)}\tilde{u} + (\partial_{tt} - b\Delta)(\chi_{\omega(t)}\tilde{v}). \tag{3.69}$$

So, the control function does not belong to $L^2(\omega)$. Under the MGCC condition, specifically from the definition of ξ, from [10, 41] we have that

$$\|q_0\|_{X'} \leq C \int_{T'}^{T} \|(t - T')\xi q(\cdot, t)\|_{X'}^2 dt. \tag{3.70}$$

Then, using the HUM operator [32] we obtain that the system (3.62) is exactly controllable on (T', T) with some controls $\xi v \in C^1([T', T]; X)$ (since $q \in C^1([T', T]; X')$ for any $q_0 \in X'$). Moreover, differentiating the first equation in (3.63) with respect to time, we obtain that

$$\frac{d}{dt}q_t = \alpha^2 q \in C^1([T', T]; X).$$

Then, in particular, we deduce that $\xi v \in C^2([T', T]; X)$.

Finally, since $\xi\tilde{v} \in C^2([0, T]; X)$, we have that $(\partial_{tt} - b\Delta)(\xi\tilde{v}) \in L^2(\omega)$. Therefore, the right hand side of (3.69) belongs to $L^2(\omega)$ and this completes the proof.

□

4. Controllability of the nonlinear MGT equation

The classical argument to prove local null controllability results is the combination of the Implicit Function Theorem with an appropriate result of controllability in the linear case. In order to prove Theorem 1.4, we need the following definitions and technical Lemmas given in [15].

We denote by $\mathcal{L}(X, Y)$ the space of continuous linear mappings from X to Y, where X and Y are Banach spaces.

Definition 4.1. (1) *An operator $G : X \to Y$ is said to be Hadamard differentiable at $a \in X$ if there exists $M \in \mathcal{L}(X, Y)$ such that, for any continuous function $r : [0, 1] \to$*

X for which $r'(0^+)$ exists and $r(0)=a$, the operator $F=G\circ r$ is differentiable at 0^+, with $F'(0^+)=Mr'(0^+)$, thus

$$G(r(t))-G(r(0))-Mr'(0^+)t=o(t) \quad \textit{as } t\downarrow 0,$$

where M is the Hadamard derivative.

(2) *An operator $G:X\to Y$ is called strongly Hadamard differentiable at $a\in X$ if $F=G\circ r$ is strongly differentiable at 0^+.*

Lemma 4.2. *Let $G:X\to Y$ be an operator such that has a Gâteaux variation $\delta G(x;h)$ at all points in a convex neighborhood Ω of $x_0\in X$ and all $h\in X$. If $\delta G(\cdot\,;\cdot)$ is continuous at $(x_0,0)$, then G is strongly Hadamard differentiable at x_0.*

Definition 4.3. *The linear mapping $M:D\to Y$, where D is a dense subset of X, is called approximately outer invertible if, for each $\mu\in(0,1)$, there exists a bounded linear mapping $B_\mu:Y\to X$ and a bound $\sigma(\mu)$ such that*

$$\|(B_\mu M B_\mu - B_\mu)y\|\le \mu\|B_\mu y\| \quad \textit{and} \quad \|B_\mu y\|\le\sigma(\mu)\|y\|, \quad \forall y\in Y.$$

Then each B_μ is called an approximately outer inverse of M, with bound function $\sigma(\mu)$.

Lemma 4.4. *Let H_1 and H_2 be two real Hilbert spaces and $M:H_1\to H_2$ be a compact linear operator. Then M is approximately outer invertible.*

Lemma 4.5 (Implicit Function Theorem)**.** *Let X and Y be real Banach spaces, with $a\in X$. Let S be a closed convex cone in Y. Let $G:X\to Y$ be an operator strongly Hadamard differentiable at a. Let $b=G(a)$ and assume $b\in S$. Let the Hadamard derivative $M=G'(a):X\to Y$ be a bounded linear operator with approximate outer inverse B_μ and bound function $\sigma(\mu)=k_0\mu^{-k}$, with $k<1$. Then for a sufficiently small μ, whenever c satisfies $-(G(a)+G'(a)c)\in S$, and $\|c\|=1$, there exists a solution $x=a+yc+\eta(t)\in X$ to $-G(x)\in S$, valid for all $t<0$ sufficiently small. with $x\neq a$, with an appropriate choice of $\mu=\mu(t)\downarrow 0$ as $t\downarrow 0$, $\|\eta(t)\|_{\mu(t)=o(t)}$ as $t\downarrow 0$.*

Proof of Theorem 1.4. We denote by $H_1=X\times X\times H_0^1(\Omega)$ and by $\mathcal{D}=X\times H_0^1(\Omega)\times L^2(\Omega)$. Define the mapping $G:H_1\times L^2(\omega)\to H_1$, $G(y^0,u)=(y(T),y_t(T),y_{tt}(T))$, where $y^0=(y_0,y_1,y_2)$ and y is the solution of the nonlinear MGT equation

$$\left\{\begin{array}{rcll} \tau y_{ttt}+\alpha y_{tt}-c^2\Delta y-b\Delta y_t &=& f(y)+\chi_{\omega(t)}u & ,\ \text{in } Q,\\ y &=& 0 & ,\ \text{on } \Gamma,\\ y(0)=y_0,\ \ y_t(0)=y_1,\ \ y_{tt}(0) &=& y_2 & ,\ \text{in } \Omega, \end{array}\right. \tag{4.1}$$

where $f(y)=k(y^2)_{tt}$.

We claim that G has a Gâteaux derivative $\delta G((y^0,u);(h^0,v))$ at all points $(y^0,u)\in H_1\times L^2(\omega)$ and all $(h^0,v)\in\mathcal{D}\times L^2(\omega)$, where $h^0=(h_0,h_1,h_2)$. Indeed, for any $(y^0,u)\in H_1\times L^2(\omega)$, $(h^0,v)\in\mathcal{D}\times L^2(\omega)$ and $\varepsilon>0$, we have that

$$G(y^0+\varepsilon h^0,u+\varepsilon v)=(y^\varepsilon(T),y_t^\varepsilon(T),y_{tt}^\varepsilon(T)), \tag{4.2}$$

where y^ε satisfy the following system

$$\left\{\begin{array}{rcll} \tau y_{ttt}^\varepsilon+\alpha y_{tt}^\varepsilon-c^2\Delta y^\varepsilon-b\Delta y_t^\varepsilon &=& f(y^\varepsilon)+\chi_{\omega(t)}(u+\varepsilon v) & ,\ \text{in } Q,\\ y^\varepsilon &=& 0 & ,\ \text{on } \Gamma,\\ y^\varepsilon(0)=y_0+\varepsilon h_0,\ \ y_t^\varepsilon(0)=y_1+\varepsilon h_1,\ \ y_{tt}^\varepsilon(0) &=& y_2+\varepsilon h_2 & ,\ \text{in } \Omega. \end{array}\right. \tag{4.3}$$

Let $z^\varepsilon = \dfrac{y^\varepsilon - y}{\varepsilon}$ be given. From (4.1) and (4.3) we deduce that

$$\text{(4.4)}\qquad \left\{\begin{array}{rcll} \tau z^\varepsilon_{ttt} + \alpha z^\varepsilon_{tt} - c^2\Delta z^\varepsilon - b\Delta z^\varepsilon_t &=& \dfrac{f(y+\varepsilon z^\varepsilon)-f(y)}{\varepsilon} + \chi_{\omega(t)} v & , \text{ in } Q,\\ z^\varepsilon &=& 0 & , \text{ on } \Gamma,\\ z^\varepsilon(0)=h_0,\ \ z^\varepsilon_t(0)=h_1,\ \ z^\varepsilon_{tt}(0) &=& h_2 & , \text{ in } \Omega. \end{array}\right.$$

Taking ε to zero, the solution z^ε of (4.4) converges to z, which solves the system

$$\text{(4.5)}\qquad \left\{\begin{array}{rcll} \tau z_{ttt} + \alpha z_{tt} - c^2\Delta z - b\Delta z_t &=& f'(y)z + \chi_{\omega(t)} v & , \text{ in } Q,\\ z &=& 0 & , \text{ on } \Gamma,\\ z(0)=h_0,\ \ z_t(0)=h_1,\ \ z_{tt}(0) &=& h_2 & , \text{ in } \Omega, \end{array}\right.$$

where y is the solution of (4.1) with control u and initial data y^0.

Therefore, it is easy to see that the solution z of (4.5) satisfies

$$\delta G((y^0,u);(h^0,v)) = (z(T), z_t(T), z_{tt}(T)).$$

Next, we prove that G is of class C^1 at $((0,0);(0,0))$. Indeed, to show that δG is continuous at $((0,0);(0,0))$ is equivalent to proving that whenever $(y^0_j,u_j)\in H_1\times L^2(\omega)$ and $(h^0_j,v_j)\in \mathcal{D}\times L^2(\omega)$ satisfies

$$\text{(4.6)}\qquad (y^0_j,u_j)\to(0,0), \quad \text{in } H_1\times L^2(\omega),$$

$$\text{(4.7)}\qquad (h^0_j,v_j)\to(0,0), \quad \text{in } \mathcal{D}\times L^2(\omega),$$

we have that the solution z^j of

$$\text{(4.8)}\qquad \left\{\begin{array}{rcll} \tau z^j_{ttt} + \alpha z^j_{tt} - c^2\Delta z^j - b\Delta z^j_t &=& f'(y^j)z^j + \chi_{\omega(t)} v_j & , \text{ in } Q,\\ z^j &=& 0 & , \text{ on } \Gamma,\\ z^j(0)=h^j_0,\ \ z^j_t(0)=h^j_1,\ \ z^j_{tt}(0) &=& h^j_2 & , \text{ in } \Omega, \end{array}\right.$$

where y^j is the solution of

$$\text{(4.9)}\qquad \left\{\begin{array}{rcll} \tau y^j_{ttt} + \alpha y^j_{tt} - c^2\Delta y^j - b\Delta y^j_t &=& f(y^j) + \chi_{\omega(t)} u_j & , \text{ in } Q,\\ y^j &=& 0 & , \text{ on } \Gamma,\\ y^j(0)=y^j_0,\ \ y^j_t(0)=y^j_1,\ \ y^j_{tt}(0) &=& y^j_2 & , \text{ in } \Omega, \end{array}\right.$$

satisfy

$$\text{(4.10)}\qquad (z^j(T), z^j_t(T), z^j_{tt}(T)) \stackrel{j\to\infty}{\longrightarrow} (0,0,0) = \delta G((0,0);(0,0)) \quad \text{in } \mathcal{D}.$$

For this purpose, we claim that when $(y^0_j,u_j)\to(y^0,u)$ in $H_1\times L^2(\omega)$, then the solutions y^j of (4.9) are such that

$$\text{(4.11)}\qquad f'(y^j)-f'(y)\to 0 \quad \text{in } C([0,T];L^2(\Omega)).$$

From the well–posedness of the nonlinear equation, see Section 2, we deduce that $r^j = y^j - y$ satisfies

$$\text{(4.12)}\qquad \|(r^j(t), r^j_t(t), r^j_{tt}(t))\|^2_{\mathcal{D}} \le C(\|y^0_j - y^0\|^2_{\mathcal{D}} + \|u_j-u\|^2_{L^2(\omega)}), \quad \forall t\in[0,T],$$

where $C>0$ is independent of t. Then, in view of (4.12) we have that

$$\text{(4.13)}\qquad (y^j, y^j_t, y^j_{tt}) \to (y, y_t, y_{tt}) \quad \text{in } C([0,T];(L^2(\Omega))^3).$$

Since the nonlinear term $f(y) = 2k(y^2)_{tt}$, we deduce that f' is a continuous function and $f'(y)\in C([0,T];L^2(\Omega))$. Then, from (4.13) we obtain

$$\text{(4.14)}\qquad f'(y^j(t))\to f'(y(t)) \quad \text{in } L^2(\Omega),\ \forall t\in[0,T].$$

In view of (4.14) and Arzela–Ascoli's Theorem, (4.11) holds if we prove that

$$\{f'(y^j(t))\}_j \quad \text{is equicontinuous in } C([0,T];L^2(\Omega)). \tag{4.15}$$

From (4.13) we have that the set $U = \{(y^j(t), y^j_t(t), y^j_{tt}(t)) : \ j \in \mathbb{N},\ t \in [0,T]\}$ is relatively compact in $(L^2(\Omega))^3$. On the other hand, since the mapping $f' : C([0,T];X) \to C([0,T];L^2(\Omega))$ that associates $f'(y) \in C([0,T];L^2(\Omega))$ to any $y \in C([0,T];X)$ is continuous, then is uniformly continuous in U. That is, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f'(y^j(t)) - f'(y^j(\tau))\|_{L^2(\Omega)} \le \varepsilon, \tag{4.16}$$

provided that

$$\|(y^j(t), y^j_t(t), y^j_{tt}(t)) - (y^j(\tau), y^j_t(\tau), y^j_{tt}(\tau))\|_{L^2(\Omega)} \le \delta. \tag{4.17}$$

Since $(y^j, y^j_t, y^j_{tt}) \to (y, y_t, y_{tt})$ in $C([0,T];(L^2(\Omega))^3)$ and y is uniformly continuous from $[0,T]$ to $L^2(\Omega)$, we obtain that there exists $n_0 \in \mathbb{N}$ and δ_0 such that (4.17) holds for any $n \ge n_0$ and $t, \tau \in [0,T]$ such that $|t-\tau| \le \delta_0$. From the uniformly continuity of y^j from $[0,T]$ to $L^2(\Omega)$ we deduce that for every $k \in \{i, \dots, n_0\}$ there exists δ_k such that (4.17) holds with $j = k$ if $|t - \tau| \le \delta_j$. Taking $\overline{\delta} = \min\{\delta_0, \delta_1, \dots, \delta_{n_0}\}$ we obtain that if $|t-\tau| \le \overline{\delta}$ then (4.17) holds. Therefore, we have that (4.15) holds, and finally we obtain (4.11), proving the claim.

Now, if (y^0_j, u_j) converges to $(0,0)$ in $H_1 \times L^2(\omega)$, from (4.11), (4.12) and (4.13) we deduce that

$$\|f'(y^j)\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{4.18}$$

where $C > 0$ is independent of j. Then, from the well posednees of the equation (4.8), we have that

$$\|(z^j, z^j_t, z^j_{tt})\|^2_{C([0,T];\mathcal{D})} \le C(\|h^0_j\|^2_{\mathcal{D}} + \|v_j\|^2_{L^2(\omega)}). \tag{4.19}$$

Then, by (4.6), (4.7), (4.18) and (4.19) we obtain that

$$(z^j(T), z^j_t(T), z^j_{tt})(T) \to (0,0,0) \quad \text{as } j \to \infty, \text{ in } \mathcal{D}. \tag{4.20}$$

Moreover, if (h^0_j, v_j) converges to $(h^0, v) \in U$, with U a bounded subset of $\mathcal{D} \times L^2(\omega)$, then the convergence in (4.20) is uniform. That is, the operator $G'(0,0) : H_1 \times L^2(\omega) \to H_1$ is compact.

Finally, since $G(0,0) = (0,0,0)$, from the null controllability of the linear MGT equation (see Theorem 1.3), for any $h^0 \in \mathcal{D}$ there exists a control $v \in L^2(\omega)$ such that $G'((0,0);(h^0,v)) = (0,0,0)$. Let $a = (0,0)$, $c = (h^0, v)$ and $S = \{0\} \subset L^2(\omega)$. Assume that $\|(h^0,v)\|_{H_1 \times L^2(\omega)} = 1$. Then, we obtain that $-(G(0,0) + G'((0,0);c)) = 0 \in S$. Thus, G satisfies all the conditions of the Implicit Function Theorem (Lemma 4.5). That is, there exists $\rho > 0$ sufficiently small such that for any $y^0 \in H_1$ with $\|y^0\|_{H_1} \le \rho$, there exists a control $u \in L^2(\omega)$ such that the corresponding solution of (4.1) satisfies

$$(y(T), y_t(T), y_{tt}(T)) = (0,0,0).$$

□

5. Conclusions and future work

The paper was devoted to the study of the null controllability property of the MGT equation. It has been proved that the linear equation fulfills the rest condition at time $T > 0$ using a distributed moving control domain. The reason why we use a domain that moves is due to the existence of the damping term $-b\Delta y_t$, which implies that the spectrum has an accumulation point. Using a generalized Implicit Function Theorem and the control property of the linear equation, we prove local null controllability property for the nonlinear MGT equation.

As far as we know, this work is the first control study for the MGT equation and present a novel way to deal with nonlinear control problem. The are still many questions to consider in connection with the control properties considered in this paper. One of these is the proof of the observability inequality (3.16). In our case this could be obtained from the regularity assumptions on the initial conditions and the hypothesis on the control domain ω. Another approach which can be used is a suitable Carleman estimates for the coupled adjoint system (3.5). The key is to use the same weight function both for the Carleman inequality of the wave equation with viscous damping and the ODE. In this context, we refer to [2] and [10] for Carleman estimates for a coupled parabolic–hyperbolic system and for a heat equation coupled with an ODE, respectively.

An interesting work would be the study of controllability problems for the MGT equation with memory terms. For example, the following MGT equation with a viscoelastic term (see [29])

$$\tau y_{ttt} + \alpha y_{tt} - c^2\Delta y - b\Delta y_t + \int_0^t g(t-s)\Delta y(s)ds = 0. \tag{5.1}$$

In this case, the correct control property to study is the called *memory–type null controllability*, see [34, 11]. The reason is because, for example, it is well known that the heat equation without memory is null controllable. However, if we add a memory term the null controllability does not hold for all initial conditions [20].

As we mentioned in the introduction, when $b = 0$ and $F(t, y, y_t, y_{tt}) = \nu(y^2)_t$ we obtain the Westervelt equation. The analysis developed in this paper cannot be applied for this equation, even if we consider the linear case ($F = 0$), because we have used the critical coefficient $\gamma = \alpha - \frac{c^2}{b} \geq 0$ in all our proofs. An interesting problem would be the analysis of the controllability of this equation. In the one–dimensional case, the controllability of the Westervelt equation is part of our forthcoming work.

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