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Exterior controllability properties for a fractional Moore–Gibson–Thompson equation

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Abstract

The three concepts of *exact, null and approximate* controllabilities are analyzed from the exterior of the Moore–Gibson–Thompson equation associated with the fractional Laplace operator subject to the nonhomogeneous Dirichlet type exterior condition. Assuming that $b > 0$ and $\alpha - \frac{\tau c^2}{b} > 0$, we show that if $0 < s < 1$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$, then there is no control function g such that the following system

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} + c^2(-\Delta)^s u + b(-\Delta)^s u_t = 0 & \text{in } \Omega \times (0,T), \\ u = g\chi_{\mathcal{O}} & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(\cdot,0) = u_0, u_t(\cdot,0) = u_1, u_{tt}(\cdot,0) = u_2 & \text{in } \Omega, \end{cases}$$

is exactly or null controllable in time $T > 0$. However, we prove that for $0 < s < 1$, the system is approximately controllable for every $g \in H^1((0,T); L^2(\mathcal{O}))$, where $\mathcal{O} \subset \mathbb{R}^N \setminus \overline{\Omega}$ is an arbitrary non-empty open set.

Keywords Fractional Laplace operator · Moore–Gibson–Thompson Equation · Exterior control problem · Exact and null controllabilities · Approximate controllability

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1 Introduction

In the present work we investigate the controllability properties of the following third order nonlocal partial differential equation:

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} + c^2(-\Delta)^s u + b(-\Delta)^s u_t = 0 & \text{in } \Omega \times (0,T), \\ u = g\chi_{\mathcal{O}} & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad u_{tt}(\cdot,0) = u_2 & \text{in } \Omega, \end{cases} \tag{1.1}$$

that we call a *nonlocal* version of the so called Moore-Gibson-Thompson (MGT) equation [34–36]. In (1.1), $\Omega \subset \mathbb{R}^N$ is a bounded open set with a Lipschitz continuous boundary $\partial\Omega$, $\tau > 0$, $\alpha, b > 0$ and c are real numbers, $(-\Delta)^s$ $(0 < s < 1)$ is the fractional Laplace operator (see (2.2)), $u = u(x,t)$ is the state to be controlled, $\chi_{\mathcal{O}}$ stands for the characteristic function for $\mathcal{O}$, and $g = g(x,t)$ is the control function which is localized in a nonempty open set $\mathcal{O} \subset \mathbb{R}^N \setminus \overline{\Omega}$. Beyond the possible applications, our main reason for studying this model comes mainly from a mathematical interest, in order to analyze a possible dependence of the fractional order s with the controllability of the equation.

The *local* Jordan–Moore–Gibson-Thompson equation, i.e. (1.1) with $s = 1$ and with source nonlinear term $f(u_t, u_{tt}, \nabla u, \nabla u_t)$, arises from modeling high amplitude sound waves (in that case g is prescribed at the boundary $\partial\Omega$). The classical nonlinear acoustics models include Kuznetsov's equation, the Westervelt equation, and the Kokhlov - Zabolotskaya - Kuznetsov equation. We refer to [8, 11, 30–32, 34, 35, 42] and the references therein for the derivation of the local version of the MGT equation. A complete analysis concerning well-posedness, regularity, stability and asymptotic behavior of solutions has been established in the above mentioned references. The physical meaning of the parameters τ, c and b are the following: $c > 0$ is the speed of sound, $b = \delta + \tau c^2$, where $\delta \geq 0$ is the diffusivity of sound, and τ is a positive constant accounting for relaxation.

Despite the wide range of applications of the local MGT equation, such as the medical and industrial use of high intensity ultrasound in lithotripsy, thermotherapy, ultra-sound cleaning, etc., there is only one work about their controllability properties [39]. In that work, the authors proved that the local MGT equation can be controlled using an interior control function supported on a moving subset of the domain Ω, in such a way that it can visit all the domain. In other words, it is impossible to get an interior null controllability to the local MGT equation when the control function is localized in a fixed subset of the domain. This poor control property is closely related to the fact that the damping term $b\Delta u_t$, in the local case, generates accumulation points in the spectrum. The boundary control problem will have the same issues. Consequently, and due to the nature of the applications, it is reasonable to ask if the dynamics of the model can be controlled by means of external forces.

For external controls, the fractional case seems to be more suitable to handle because, on the one hand, for the fractional case the associated stationary (time independent) system, hence, the evolution equation (1.1), is ill-posed if the control function g is prescribed at the boundary $\partial\Omega$ and, on the other hand, it has been very recently shown by Warma [40] that for nonlocal PDEs associated with the fractional Laplacian, the exterior control, as in (1.1), is the right notion that replaces the classical boundary control problems, that is, when the control function is localized on a subset ω of the boundary $\partial\Omega$.

It is worthwhile to recall that fractional order operators have emerged as a modeling alternative in various branches of science. They usually model anomalous phenomena. For instance, a number of stochastic models explaining anomalous diffusion have been introduced in the literature; among them we quote the fractional Brownian motion, the continuous time random walk, the Lévy flights, the Schneider grey Brownian motion, and more generally, random walk models based on evolution equations of single and distributed fractional order in space (see e.g. [14, 24, 41, 50, 59]). In general, a fractional diffusion operator corresponds to a diverging jump length variance in the random walk. We refer to [12, 52] and the references therein for a complete analysis, the derivation, and applications of the fractional Laplace operator. For further details we also refer to [17, 19] and their references.

Mathematical control theory is a broad topic which has been widely investigated during the past decades. The foundations of modern control theory date back among others, to the early works of Bellman in the context of dynamical programming [3], Kalman in filtering techniques and the algebraic approach to linear systems [28, 29], and Pontryagin with the maximum principle for nonlinear optimal control problems [45].

The field of controllability of Partial Differential Equations (PDEs) started to evolve with seminal contributions like those by Fattorini and Russell [15], who introduced the use of biorthogonal sequences to control one-dimensional heat equations. The survey paper [49] by Russell that collected a wide spectrum of methods and results, describes rather precisely the evolution of the field until then.

The area flourished again with the contributions by J.-L. Lions, gathered in [38]. There, it was shown, in a systematic manner, that PDEs controllability problems can be reduced to the dual problem of observability of the corresponding adjoint system, which amounts to prove quantitative versions of unique continuation results. This opened the opportunity to apply and further develop different techniques that became some of the key tools in the area, such as multipliers, microlocal analysis, Carleman inequalities, non-harmonic Fourier series, etc. This opened also a path towards the development of novel numerical methods to tackle these issues [22, 23, 60]. For further information about basic facts on control theory we refer to [37, 61] and their references.

In the present paper, we shall show that if $b > 0$ and (u_0, u_1, u_2) belongs to a suitable Banach space, then for every function $g \in H^1((0,T); L^2(\mathcal{O}))$, the system (1.1) has a unique weak solution (u, u_t, u_{tt}) satisfying the regularity $u \in C([0,T]; L^2(\Omega)) \cap C^2([0,T]; W^{-s,2}(\overline{\Omega}))$. In that case, the set of reachable states can be defined as follows:

$$\mathcal{R}((u_0, u_1, u_2), T) := \Big\{(u(\cdot, T), u_t(\cdot, T), u_{tt}(\cdot, T)) : \ g \in H^1((0, T); L^2(\mathcal{O}))\Big\}.$$

The classical three notions of controllability for this system can then be defined as follows [37, 61]:

- We shall say that the system (1.1) is null controllable in time $T > 0$ if

$$(0, 0, 0) \in \mathcal{R}((u_0, u_1, u_2), T).$$

- The system will be said exactly controllable in time $T > 0$ if

$$\mathcal{R}((u_0, u_1, u_2), T) = L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega}).$$

- Finally, we will say that the system is approximately controllable in time $T > 0$ if

$$\mathcal{R}((u_0, u_1, u_2), T) \text{ is dense in } L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega}).$$

From the above definitions, it is easy to see that null or exact controllability implies the approximate controllability. But as usual the converse is not always true. We refer to Section 2 for the definition of the function spaces involved.

Our first main result states that if $b > 0$ and $\alpha - \frac{\tau c^2}{b} > 0$, then the system (1.1) is not exactly or null controllable in time $T > 0$. However, we obtain that the system is indeed approximately controllable in any time $T > 0$ and for every $g \in H^1((0, T); L^2(\mathcal{O}))$ where $\mathcal{O}$ is an arbitrary nonempty open subset of $\mathbb{R}^N \setminus \overline{\Omega}$, which is indeed our second main contribution. We remark that this is the best possible conclusion that can be obtained regarding the controllability property of the system (1.1). These two results are stated in Theorems 7 and 9.

We observe that in our study of the controllability properties, we shall always assume that $b > 0$, otherwise if $b = 0$, then the system (1.1) is ill-posed (see Section 3 for more details).

As far as we know, the present work is the first one that provides new insights about the exterior controllability properties for the fractional MGT equation.

Let us mention that the well-posedness of an abstract version of (1.1) (with $g = 0$) where $(-\Delta)^s$ is replaced with a generic self-adjoint operator A with dense domain $D(A)$ in a Hilbert space H, has been completely examined in several papers by using semigroups method (see e.g. [8, 11, 30–32, 34–36, 42] and the references therein). An interesting result provided in [31] is the fact that if $b = 0$, then the local MGT equation is ill–posed. Another important fact proved in [10], is that, if $\alpha - \frac{\tau c^2}{b} < 0$, then the local MGT equation exhibits a chaotic behavior. These two facts also occur in the fractional case investigated in the present paper. For that reason, we shall assume that $b > 0$ and $\alpha - \frac{\tau c^2}{b} > 0$.

We note that nonlinear models and some versions including memory terms have been also intensively studied by Lasiecka and Wang [34–36] where they have obtained some fundamental and beautiful results.

In spite that $(-\Delta)^s$, with zero exterior condition, is a self-adjoint operator in $L^2(\Omega)$ with dense domain and has a compact resolvent (see Sect. 2), and hence it enters in the framework of semigroups theory, it should be noted that in (1.1) we have a non zero exterior condition which does not satisfy the conditions contained in the above references. For this reason, in the present article we shall also include new results of existence and regularity of solutions to our nonhomogeneous system (1.1). To do this, we shall exploit a new technique which has been developed in [40, 57, 58] to solve fractional wave equations and strong damping wave equations. This original method shall allow us not only to prove well-posedness but also to have an explicit representation of solutions in terms of series which is crucial for the analysis of the controllability properties of the system.

To summarize, the main novelties of the present paper can be formulated as follows.

(1) For the first time, a *nonlocal* version of the MGT equation associated with the fractional Laplace operator with a non-zero exterior condition has been studied. Some well-posedness results and an explicit representation of solutions in terms of series of the nonhomogeneous exterior value nonlocal evolution system (1.1) have been established.
(2) We have shown that the system is not null or exactly controllable in any time $T > 0$.
(3) The unique continuation property of solutions to the adjoint system associated with (1.1) has been established. This result can be obtained by using the recently and technical result about the strong unique continuation property of the elliptic problem associated with the fractional Laplace operator contained in [20, 21]. However, we give a second alternative proof which is interesting on its own, and is based by a careful exploitation of the unique continuation property for eigenvalues problems associated with the operator $(-\Delta)_D^s$ recently obtained in [53], and by using some powerful tools of complex analysis.
(4) The final important result is the approximate controllability of the system which is a direct consequence of the unique continuation property of the dual system. In fact, we have shown that the two notions are equivalent as in the classical case of the heat and wave equations.
(5) Notice that even if we have a third order evolution equation, our control function g belongs only to $H^1((0,T); L^2(\mathcal{O}))$. For that reason a notion of very weak or solutions by transposition has been introduced in order to deal with the existence and regularity of solutions to the system (1.1).

The rest of the paper is structured as follows. In Sect. 2 we introduce the function spaces needed throughout the paper, give a rigorous definition of the fractional Laplacian and recall some known results that will be used in the proofs of our main results. The well-posedness and the explicit representation of solutions to the system (1.1) and the associated dual system are contained in Sect. 3. Finally, in Sect. 4 we state and prove our main results, namely, Theorems 7 and 9.

2 Preliminaries

In this section we give some notations and recall some well-known results that are needed throughout the paper.

We start with the main tools used to study the controllability properties of PDEs. As we said in the introduction, the null controllability of the system (1.1) consists to study if there is a control function g such that the associated unique solution u of the system satisfies $u(\cdot, T) = u_t(\cdot, T) = u_{tt}(\cdot, T) = 0$ in Ω. By [38], this problems is related to the dual problem of observability of the corresponding adjoint system.

The approximate controllability of the system can be also characterized by using the associated adjoint system (3.30). More precisely, one can show that the system is approximately controllable if and only if the associated adjoint system satisfies the unique continuation property. We refer to the monographs [37, 61] and their references for more specific and abstract results in this direction. For the special case investigated here, the above mentioned result will be proved in Remark 7.

Next, we introduce the fractional order Sobolev spaces. Given $0 < s < 1$ and $\Omega \subset \mathbb{R}^N$ an arbitrary open set, we let

$$W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dxdy < \infty \right\},$$

and we endow it with the norm

$$\|u\|_{W^{s,2}(\Omega)} := \left(\int_\Omega |u(x)|^2\, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{\frac{1}{2}}.$$

We set

$$W_0^{s,2}(\overline{\Omega}) := \left\{ u \in W^{s,2}(\mathbb{R}^N) : \ u = 0 \ \text{ in } \ \mathbb{R}^N \setminus \Omega \right\},$$

and denote by $W^{-s,2}(\overline{\Omega})$ its dual with respect to the pivot space $L^2(\Omega)$. We notice that if Ω is a bounded open set with a Lipschitz continuous boundary and $0 < s \neq 1/2 < 1$, then $W_0^{s,2}(\overline{\Omega}) = \overline{\mathcal{D}(\Omega)}^{W^{s,2}(\Omega)}$ with equivalent norms. But if $s = 1/2$, even for smooth domains, then $W_0^{s,2}(\overline{\Omega})$ is a proper subspace of $\overline{\mathcal{D}(\Omega)}^{W^{s,2}(\Omega)}$ (see e.g. [25,Chapter 1]) and their references. The corresponding result for general bounded open sets is contained in [53].

For more information on fractional order Sobolev spaces, we refer to [12, 25, 27, 53].

Next, we give a rigorous definition of the fractional Laplace operator. Let

$$\mathcal{L}_s^1(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \ \text{ measurable and } \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}}\, dx < \infty \right\}.$$

For $u \in \mathcal{L}_s^1(\mathbb{R}^N)$ and $\varepsilon > 0$ we set

$$(-\Delta)_\varepsilon^s u(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N:\, |x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N,$$

where $C_{N,s}$ is a normalization constant given by

$$C_{N,s} := \frac{s 2^{2s} \Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}. \tag{2.1}$$

The *fractional Laplacian* $(-\Delta)^s u$ is defined for $u \in \mathcal{L}_s^1(\mathbb{R}^N)$ by the following singular integral:

$$(-\Delta)^s u(x) := C_{N,s} \,\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s u(x), \quad x \in \mathbb{R}^N, \tag{2.2}$$

provided that the limit exists for a.e. $x \in \mathbb{R}^N$. The fractional Laplacian can be also defined as the pseudo-differential operator with symbol $|\xi|^{2s}$. For more details on the fractional Laplace operator we refer to [6, 7, 12, 16–18, 53, 54] and their references.

Next, let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary, and consider the following Dirichlet problem:

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.3}$$

Definition 1 Let $g \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ and $\widetilde{g} \in W^{s,2}(\mathbb{R}^N)$ be such that $\widetilde{g}|_{\mathbb{R}^N \setminus \Omega} = g$. A function $u \in W^{s,2}(\mathbb{R}^N)$ is said to be a weak solution of (2.3) if $u - \widetilde{g} \in W_0^{s,2}(\overline{\Omega})$ and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy = 0, \quad \forall\, v \in W_0^{s,2}(\overline{\Omega}).$$

The following existence result is taken from [26] (see also [21]).

Proposition 1 *For every $g \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$, there is a unique $u \in W^{s,2}(\mathbb{R}^N)$ satisfying (2.3) in the sense of Definition 1. In addition, there is a constant $C > 0$ such that*

$$\|u\|_{W^{s,2}(\mathbb{R}^N)} \le C \|g\|_{W^{s,2}(\mathbb{R}^N \setminus \Omega)}. \tag{2.4}$$

Now, we consider the selfadjoint operator in $L^2(\Omega)$ given by

$$D((-\Delta)_D^s) := \left\{ u \in W_0^{s,2}(\overline{\Omega}) : \ (-\Delta)^s u \in L^2(\Omega) \right\}, \quad (-\Delta)_D^s u := ((-\Delta)^s u)|_{\Omega}. \tag{2.5}$$

Then $(-\Delta)_D^s$ is the realization of $(-\Delta)^s$ in $L^2(\Omega)$ with the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$. It is well-known (see e.g. [9, 52, 56]) that $(-\Delta)_D^s$ has a compact resolvent and its eigenvalues form a non-decreasing sequence of real numbers $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots$ satisfying $\lim_{n\to\infty} \mu_n = \infty$. In addition, the eigenvalues are of finite multiplicity. Let $(\varphi_n)_{n\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions associated with

$(\mu_n)_{n\in\mathbb{N}}$. Then $\varphi_n \in D((-\Delta)^s_D)$ for every $n \in \mathbb{N}$, $(\varphi_n)_{n\in\mathbb{N}}$ is total in $L^2(\Omega)$ and satisfies

$$\begin{cases} (-\Delta)^s\varphi_n = \mu_n\varphi_n & \text{in } \Omega, \\ \varphi_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.6}$$

With this setting, for every real number $\gamma \geq 0$, we can define the γ-powers of $(-\Delta)^s_D$ as follows:

$$\begin{cases} D([(-\Delta)^s_D]^\gamma) := \left\{ u \in L^2(\Omega) : \sum_{n=1}^{\infty} \left|\mu_n^\gamma (u, \varphi_n)_{L^2(\Omega)}\right|^2 < \infty \right\}, \\ [(-\Delta)^s_D]^\gamma u := \sum_{n=1}^{\infty} \mu_n^\gamma (u, \varphi_n)_{L^2(\Omega)}\varphi_n. \end{cases} \tag{2.7}$$

Using (2.7), we can easily show that $D([(-\Delta)^s_D]^{\frac{1}{2}}) = W_0^{s,2}(\overline{\Omega})$ and for $u \in W_0^{s,2}(\overline{\Omega})$ we have that

$$\|u\|^2_{W_0^{s,2}(\overline{\Omega})} = \sum_{n=1}^{\infty} \left|\mu_n^{\frac{1}{2}} (u, \varphi_n)_{L^2(\Omega)}\right|^2, \tag{2.8}$$

defines an equivalent norm on $W_0^{s,2}(\overline{\Omega})$. If $u \in D((-\Delta)^s_D)$, then

$$\|u\|^2_{D((-\Delta)^s_D)} = \|(-\Delta)^s_D u\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} \left|\mu_n (u, \varphi_n)_{L^2(\Omega)}\right|^2.$$

In addition, for $u \in W^{-s,2}(\overline{\Omega})$, we have that

$$\|u\|^2_{W^{-s,2}(\overline{\Omega})} = \sum_{n=1}^{\infty} \left|\mu_n^{-\frac{1}{2}} (u, \varphi_n)_{L^2(\Omega)}\right|^2. \tag{2.9}$$

In that case, using the so called Gelfand triple (see e.g. [2]) we have the following continuous embeddings $W_0^{s,2}(\overline{\Omega}) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-s,2}(\overline{\Omega})$.

Next, for $u \in W^{s,2}(\mathbb{R}^N)$ we introduce the *nonlocal normal derivative* $\mathcal{N}_s u$ of u defined by

$$\mathcal{N}_s u(x) := C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{2.10}$$

where $C_{N,s}$ is the constant given in (2.1). By [21,Lemma 3.2], for every $u \in W^{s,2}(\mathbb{R}^N)$, we have that $\mathcal{N}_s u \in W^{s,2}_{\text{loc}}(\mathbb{R}^N \setminus \Omega)$. If in addition $(-\Delta)^s u \in L^2(\Omega)$, then $\mathcal{N}_s u \in L^2(\mathbb{R}^N \setminus \Omega)$.

The following unique continuation property, recently obtained in [56,Theorem 3.10], shall play an important role in the proof of our main results.

Lemma 1 *Let $\mu > 0$ be a real number and $\mathcal{O} \subset \mathbb{R}^N \setminus \overline{\Omega}$ an arbitrary non-empty open set. If $\varphi \in D((-\Delta)^s_D)$ satisfies*

$$(-\Delta)^s_D \varphi = \mu\varphi \ \text{ in } \ \Omega \ \text{ and } \ \mathcal{N}_s\varphi = 0 \ \text{ in } \ \mathcal{O},$$

then $\varphi = 0$ in $\mathbb{R}^N$.

Remark 1 The following important identity has been recently proved in [56,Remark 3.9]. Let $g \in W^{s,2}(\mathbb{R}^N \setminus \Omega)$ and $U_g \in W^{s,2}(\mathbb{R}^N)$ the associated unique weak solution of the Dirichlet problem (2.3). Then

$$\int_{\mathbb{R}^N \setminus \Omega} g\mathcal{N}_s\varphi_n \, dx = -\mu_n \int_\Omega \varphi_n U_g \, dx, \tag{2.11}$$

where we recall that (φ_n) and (μ_n) denote the eigenfunctions and eigenvalues of the operator $(-\Delta)^s_D$.

For more details on the Dirichlet problem associated with the fractional Laplace operator we refer the interested reader to [4–7, 26, 46, 47, 53, 56] and their references.

The following integration by parts formula is contained in [13,Lemma 3.3] for smooth functions. The version given here can be obtained by using a simple density argument (see e.g. [56]).

Proposition 2 *Let $u \in W^{s,2}(\mathbb{R}^N)$ be such that $(-\Delta)^s u \in L^2(\Omega)$. Then for every $v \in W^{s,2}(\mathbb{R}^N)$,*

$$\begin{aligned} &\frac{C_{N,s}}{2} \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy \\ &= \int_\Omega v(-\Delta)^s u \, dx + \int_{\mathbb{R}^N \setminus \Omega} v\mathcal{N}_s u \, dx. \end{aligned} \tag{2.12}$$

We conclude this section with the following observation.

Remark 2 If $u = 0$ in $\mathbb{R}^N \setminus \Omega$ or $v = 0$ in $\mathbb{R}^N \setminus \Omega$, then

$$\begin{aligned} &\int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy, \end{aligned}$$

so that for such functions, the identity (2.12) becomes

$$\begin{aligned} &\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy \\ &= \int_\Omega v(-\Delta)^s u \, dx + \int_{\mathbb{R}^N \setminus \Omega} v\mathcal{N}_s u \, dx. \end{aligned} \tag{2.13}$$

3 Series representation of solutions

In this section we prove the existence of solutions (very weak) to the system (1.1) and obtain a representation in terms of series of weak solutions to the associated dual system. Evolution equations with non-homogeneous boundary or exterior conditions are in general not so easy to solve, since one cannot apply directly semigroups method due to the fact that the associated operator is in general not a generator of a semigroup. For this reason, we shall give more details in the proofs. The representation of solutions of the dual system in terms of series will be used in the proofs of our main results.

Throughout the remainder of the paper, without any mention, $\tau > 0$, $\alpha, b > 0$, c and $0 < s < 1$ are real numbers and $\Omega \subset \mathbb{R}^N$ denotes a bounded open set with a Lipschitz continuous boundary. Given a measurable set $E \subset \mathbb{R}^N$, we shall denote by $(\cdot,\cdot)_{L^2(E)}$ the scalar product in $L^2(E)$. We shall denote by $\langle\cdot,\cdot\rangle_{-\frac{1}{2},\frac{1}{2}}$ the duality pairing between $W^{-s,2}(\overline{\Omega})$ and $W_0^{s,2}(\overline{\Omega})$.

We also recall that we have assumed that $b > 0$ and $\alpha - \frac{\tau c^2}{b} > 0$.

3.1 Existence of solutions to the system (1.1)

Before we start the study of solutions to the system (1.1) we give an important result taken from [31].

Let A be a self-adjoint positive operator on a Hilbert space H such that $D(A) \subset H$ is dense in H. Consider the operator matrix $\mathcal{P}$ with domain $D(\mathcal{P}) = D(A)\times D(A)\times H$ given by

$$\mathcal{P} := \begin{pmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ \frac{c^2}{\tau}A & \frac{b}{\tau}A & \frac{\alpha}{\tau}I \end{pmatrix}. \tag{3.1}$$

Let

$$\mathcal{H} := D(A^{\frac{1}{2}}) \times D(A^{\frac{1}{2}}) \times H$$

be endowed with the graph norm. The following result is taken from [31,Theorem 1.2].

Theorem 1 *The operator $-\mathcal{P}$ defined in (3.1) generates a strongly continuous group in $\mathcal{H}$.*

Next, let $(u_0, u_1, u_2) \in W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$ and consider the following two systems:

$$\begin{cases} \tau v_{ttt} + \alpha v_{tt} + c^2(-\Delta)^s v + b(-\Delta)^s v_t = 0 & \text{in } \Omega \times (0,T), \\ v = g & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ v(\cdot,0) = 0, \quad v_t(\cdot,0) = 0, \quad v_{tt}(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \tau w_{ttt} + \alpha w_{tt} + c^2(-\Delta)^s w + b(-\Delta)^s w_t = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ w(\cdot,0) = u_0, \quad w_t(\cdot,0) = u_1, \quad w_{tt}(\cdot,0) = u_2 & \text{in } \Omega. \end{cases} \tag{3.3}$$

Then, it is clear that $u := v + w$ solves the system (1.1). In addition, let

$$W := \begin{pmatrix} w \\ w_t \\ w_{tt} \end{pmatrix} \quad \text{and} \quad W_0 := \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}.$$

Then the system associated to w can be rewritten as the following first order Cauchy problem:

$$\begin{cases} W_t + \mathcal{A}W = 0 & \text{in } \Omega \times (0,T), \\ W(\cdot,0) = W_0 & \text{in } \Omega, \end{cases} \tag{3.4}$$

where the operator matrix $\mathcal{A}$ with domain $D(\mathcal{A}) = D((-\Delta)^s_D) \times D((-\Delta)^s_D) \times L^2(\Omega)$ is given by

$$\mathcal{A} := \begin{pmatrix} 0 & -I & 0 \\ 0 & 0 & -I \\ \frac{c^2}{\tau}(-\Delta)^s_D & \frac{b}{\tau}(-\Delta)^s_D & \frac{\alpha}{\tau} I \end{pmatrix}. \tag{3.5}$$

Let

$$\mathcal{H} := D([(-\Delta)^s_D]^{\frac{1}{2}}) \times D([(-\Delta)^s_D]^{\frac{1}{2}}) \times L^2(\Omega) = W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$$

be endowed with the graph norm

$$\|W_0\|^2_{\mathcal{H}} = \|[(-\Delta)^s_D]^{\frac{1}{2}} u_0\|^2_{L^2(\Omega)} + \|[(-\Delta)^s_D]^{\frac{1}{2}} u_1\|^2_{L^2(\Omega)} + \|u_2\|^2_{L^2(\Omega)}.$$

Notice that the operator $(-\Delta)^s_D$ enters in the framework of [31]. Therefore, we have the following result which is a direct application of Theorem 1.

Theorem 2 *The operator $-\mathcal{A}$ defined in (3.5) generates a strongly continuous group in $\mathcal{H}$. As a consequence, for every $W_0 := (u_0, u_1, u_2) \in \mathcal{H}$, the system (3.4) has a unique strong solution W given by $W(t) = e^{-t\mathcal{A}} W_0$, where $(e^{-t\mathcal{A}})_{t \geq 0}$ is the strongly continuous semigroup on $\mathcal{H}$ generated by $-\mathcal{A}$.*

Knowing that the system (3.4) is well-posed (by Theorem 2), we are interested to have an explicit representation of solutions. We define the real numbers m_1, m_2 as follows:

$$m_1 := \tau \frac{-C_1 - \sqrt{C_2}}{8b^3}, \quad m_2 := \tau \frac{-C_1 + \sqrt{C_2}}{8b^3}, \tag{3.6}$$

with

$$C_1 := 27 - 18\left(\frac{b}{\tau}\right) - \left(\frac{b}{\tau}\right)^2, \quad C_2 := C_1 - 64\left(\frac{b}{\tau}\right)^3.$$

Each pair $\{\mu_n, \varphi_n\}$ of eigenvalues and eigenfunctions of $(-\Delta)^s_D$ generates a system of eigenvalues $\{\lambda_{n,j}\}_{n\in\mathbb{N}}$, $j = 1, 2, 3$, of $\mathcal{A}$ given as the roots of the following cubic equation:

$$\tau\lambda_{n,j}^3 + \alpha\lambda_{n,j}^2 + (\mu_n b)\lambda_{n,j} + \mu_n c^2 = 0. \tag{3.7}$$

From the work of Pellicer and Solà–Morales [44,Proposition 2], where the spectral properties of $\mathcal{A}$ was derived, we can obtain the representation of solutions in terms of series. The proof is based on the spectral theorem of selfadjoint operators. Therefore, for sake of brevity, the proof is omitted.

Proposition 3 *Let $(u_0, u_1, u_2) \in W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$. Then the unique weak solution w of the system (3.4) can be written in a series representation as follows:*

(a) If $\frac{1}{9} < \frac{\tau}{b} < 1$, then w is given by

$$w(x,t) = \sum_{n=1}^{\infty}\Big(A_n(t)(u_0,\varphi_n)_{L^2(\Omega)} + B_n(t)(u_1,\varphi_n)_{L^2(\Omega)} + C_n(t)(u_2,\varphi_n)_{L^2(\Omega)}\Big)\varphi_n(x), \tag{3.8}$$

where

$$A_n(t) := \frac{\lambda_{n,2}\lambda_{n,3}}{\xi_{n,1}}e^{\lambda_{n,1}t} - \frac{\lambda_{n,1}\lambda_{n,3}}{\xi_{n,2}}e^{\lambda_{n,2}t} + \frac{\lambda_{n,1}\lambda_{n,2}}{\xi_{n,3}}e^{\lambda_{n,3}t}, \tag{3.9}$$

$$B_n(t) := -\frac{\lambda_{n,2}+\lambda_{n,3}}{\xi_{n,1}}e^{\lambda_{n,1}t} + \frac{\lambda_{n,1}+\lambda_{n,3}}{\xi_{n,2}}e^{\lambda_{n,2}t} - \frac{\lambda_{n,1}+\lambda_{n,2}}{\xi_{n,3}}e^{\lambda_{n,3}t}, \tag{3.10}$$

$$C_n(t) := \frac{1}{\xi_{n,1}}e^{\lambda_{n,1}t} - \frac{1}{\xi_{n,2}}e^{\lambda_{n,2}t} + \frac{1}{\xi_{n,3}}e^{\lambda_{n,3}t}, \tag{3.11}$$

and

$$\begin{aligned} \xi_{n,1} &:= (\lambda_{n,1}-\lambda_{n,2})(\lambda_{n,1}-\lambda_{n,3}) \\ \xi_{n,2} &:= (\lambda_{n,1}-\lambda_{n,2})(\lambda_{n,2}-\lambda_{n,3}) \\ \xi_{n,3} &:= (\lambda_{n,1}-\lambda_{n,3})(\lambda_{n,2}-\lambda_{n,3}). \end{aligned} \tag{3.12}$$

(b) If $0 < \frac{\tau}{b} < \frac{1}{9}$ and there exists n_1 or n_2 such that $\mu_{n_1} = m_1$ or $\mu_{n_2} = m_2$, then the solution w can be written as

$$w(x,t) = \sum_{n\in\mathbb{M}}\Big(A_n(t)(u_0,\varphi_n)_{L^2(\Omega)} + B_n(t)(u_1,\varphi_n)_{L^2(\Omega)} + C_n(t)(u_2,\varphi_n)_{L^2(\Omega)}\Big)\varphi_n(x)$$

$$+\Big(a_{f(n_1,n_2),1}e^{\lambda_{f(n_1,n_2),1}t}+a_{f(n_1,n_2),2}e^{\lambda_{f(n_1,n_2),2}t}+ a_{f(n_1,n_2),3}te^{\lambda_{f(n_1,n_2),2}t}\Big)\varphi_{n_{f(n_1,n_2)}}(x), \tag{3.13}$$

where $\mathbb{M}=\mathbb{N}\setminus\{n_1,n_2\}$,

$$f(n_1,n_2)=\begin{cases} n_1 & \text{if } \mu_{n_1}=m_1,\\ n_2 & \text{if } \mu_{n_2}=m_2,\end{cases}$$

and $(a_{f(n_1,n_2),1},a_{f(n_1,n_2),2},a_{f(n_1,n_2),3})$ *are given by*

$$\begin{pmatrix} 1 & 1 & 0\\ \lambda_{f(n_1,n_2),1} & \lambda_{f(n_1,n_2),2} & 1\\ \lambda^2_{f(n_1,n_2),1} & \lambda^2_{f(n_1,n_2),2} & 2\lambda_{f(n_1,n_2),2}\end{pmatrix}\begin{pmatrix} a_{f(n_1,n_2),1}\\ a_{f(n_1,n_2),2}\\ a_{f(n_1,n_2),3}\end{pmatrix}=\begin{pmatrix} u_{0,f(n_1,n_2)}\\ u_{1,f(n_1,n_2)}\\ u_{2,f(n_1,n_2)}\end{pmatrix},$$

and A_n, B_n, C_n *are given by (3.9), (3.10) and (3.11), respectively.*

(c) *If* $\frac{\tau}{b}=\frac{1}{9}$ *and there exists* n_1 *such that* $\mu_{n_1}=\frac{3c^4}{b^2}$, *then we obtain*

$$\begin{aligned} w(x,t) &= \sum_{n\in\mathbb{L}}\Big(A_n(t)(u_0,\varphi_n)_{L^2(\Omega)}+B_n(t)(u_1,\varphi_n)_{L^2(\Omega)}+C_n(t)(u_2,\varphi_n)_{L^2(\Omega)}\Big)\varphi_n(x)\\ &\quad+\Big(a_{n_1,1}e^{\lambda_{n_1,1}t}+a_{n_1,2}te^{\lambda_{n_1,1}t}+a_{n_1,1}t^2e^{\lambda_{n_1,1}t}\Big)\varphi_{n_1}(x), \end{aligned} \tag{3.14}$$

where $\mathbb{L}=\mathbb{N}\setminus\{n_1\}$, $(a_{n_1,1},a_{n_1,2},a_{n_1,3})$ *are given by*

$$\begin{pmatrix} 1 & 0 & 0\\ \lambda_{n_1,1} & 1 & 0\\ \lambda^2_{n_1,1} & 2\lambda_{n_1,1} & 2\end{pmatrix}\begin{pmatrix} a_{n_1,1}\\ a_{n_1,2}\\ a_{n_1,3}\end{pmatrix}=\begin{pmatrix} u_{0,n_1}\\ u_{1,n_1}\\ u_{2,n_1}\end{pmatrix}.$$

For the sake of simplicity, throughout the remainder of the paper, we assume that $\frac{1}{9}<\frac{\tau}{b}<1$. The other two cases given in [44,Proposition 2] can be naturally extended in the context of the representation of solutions.

Now, since our control function g belongs to $H^1((0,T);L^2(\mathbb{R}^N\setminus\Omega))$, we do not have enough regularity to have weak solutions. For this reason we need a new notion of solutions to the system (3.2), that we shall call very weak solutions or solutions by transposition.

To do this, let z be the weak solution of the following backward problem:

$$\begin{cases} -\tau z_{ttt}+\alpha z_{tt}+c^2(-\Delta)^s z-b(-\Delta)^s z_t=f & \text{in } \Omega\times(0,T),\\ z=0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T),\\ z(\cdot,T)=0,\quad z_t(\cdot,T)=0,\quad z_{tt}(\cdot,T)=0 & \text{in } \Omega, \end{cases} \tag{3.15}$$

where $f\in L^2((0,T);L^2(\Omega))$. Following the arguments in Theorem 2, we have that (3.15) has a unique weak solution $(z,z_t,z_{tt})\in C([0,T];W_0^{s,2}(\overline{\Omega})\times W_0^{s,2}(\overline{\Omega})\times$

$L^2(\Omega))$. Multiplying (3.2) by z and using the integration by parts formula (2.13), we obtain

$$\int_0^T\int_\Omega vf\,dxdt = -\int_0^T\int_{\mathbb{R}^N\setminus\Omega}\Big(c^2g+bg_t\Big)\mathcal{N}_s z\,dxdt.$$

Since $z \in W_0^{s,2}(\overline{\Omega})$ and $(-\Delta)^s z \in L^2(\Omega)$, it follows that $\mathcal{N}_s z \in L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$.

Definition 2 Let $g \in H^1((0,T);L^2(\mathbb{R}^N\setminus\Omega))$. The solution by transposition (or very weak solution) of (3.2) is defined as the unique function $v \in L^2(\mathbb{R}^N\times(0,T))$ satisfying the identity

$$\int_0^T\int_\Omega vf\,dxdt = -\int_0^T\int_{\mathbb{R}^N\setminus\Omega}(c^2g+bg_t)\mathcal{N}_s z\,dxdt, \tag{3.16}$$

for every $f \in L^2((0,T)\times\Omega)$, where z is the unique weak solution of (3.15) with the source term f.

The following result shows the existence and uniqueness of solutions.

Theorem 3 *Let $g \in H^1((0,T);L^2(\mathbb{R}^N\setminus\Omega))$. Then there exists a unique solution by transposition to (3.2) according to Definition 2.*

Proof We prove the result in three steps.

Step 1 Let $\eta \in L^2(\Omega\times(0,T))$ be a given function and consider the following dual problem:

$$\begin{cases}-\tau w_{ttt}+\alpha w_{tt}+c^2(-\Delta)^s w-b(-\Delta)^s w_t=\eta & \text{in } \Omega\times(0,T),\\ w=0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T),\\ w(\cdot,T)=0,\quad w_t(\cdot,T)=0,\quad w_{tt}(\cdot,T)=0 & \text{in } \Omega.\end{cases} \tag{3.17}$$

From Theorem 2, one can easily deduce that the system (3.17) has a unique weak solution $(w,w_t,w_{tt}) \in C([0,T];W_0^{s,2}(\overline{\Omega})\times W_0^{s,2}(\overline{\Omega})\times L^2(\Omega))$. In addition, since $w \in L^2((0,T);W_0^{s,2}(\overline{\Omega}))$ and $(-\Delta)^s w \in L^2(\Omega)$, we have that $\mathcal{N}_s w \in L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$.

Step 2 Now, we consider the map

$$\Lambda : L^2(\Omega\times(0,T)) \to L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega)),\quad \eta\mapsto \Lambda\eta := -\mathcal{N}_s w.$$

By definition, we have that Λ is linear and continuous. Since g belongs to $H^1(0,T;L^2(\mathbb{R}^N\setminus\Omega))$, we have that $(c^2g+bg_t) \in L^2((0,T);L^2(\mathbb{R}^N\setminus\Omega))$. Therefore, letting $v := \Lambda^*(c^2g+bg_t)$, we obtain

$$\int_0^T\int_\Omega v\eta\,dxdt = \int_0^T\int_\Omega(\Lambda^*(c_2g+bg_t))\eta\,dxdt$$

$$= -\int_0^T \int_{\mathbb{R}^N\setminus\Omega} (c^2 g + b g_t)\mathcal{N}_s w dx dt.$$

We have constructed a solution by transposition $v \in L^2(\Omega \times (0,T))$ to the system (3.2).

Step 3 Now we show the uniqueness of solutions. Let us assume that (3.2) has two solutions by transposition v_1, v_2 with the same exterior datum g. Then, it follows from (3.16) that

$$\int_0^T \int_\Omega (v_1 - v_2)\eta \, dx dt = 0,$$

for every $\eta \in L^2(\Omega \times (0,T))$. It follows from the fundamental lemma of the calculus of variations that $v_1 = v_2$ a.e. in $(0,T) \times \Omega$. Since $v_1 = v_2 = g$ in $(0,T) \times \mathbb{R}^N \setminus \Omega$, we can conclude that $v_1 = v_2$ a.e. in $(0,T) \times \mathbb{R}^N$. The proof is complete.

□

It follows from Proposition 3 that the system (1.1) has a unique solution by transposition u given by $u := v + w$ where w is the unique weak solution of (3.4) and v the unique solution by transposition of (3.2).

We conclude this section by giving a representation of u. To do this, we need some preparation. We rely on the book by Tucsnak and Weiss [51,Chapter 2] to do that.

Remark 3 We have the following observations.

(a) We consider the following elliptic problem

$$(-\Delta)^s \varphi = 0 \text{ in } \Omega, \quad \varphi = g \text{ in } \mathbb{R}^N \setminus \Omega. \tag{3.18}$$

By a solution φ by transposition (or very weak solution) of (3.18) we mean a function $\varphi \in L^2(\mathbb{R}^N)$ such that $\varphi = g$ in $\mathbb{R}^N \setminus \Omega$ and the equality

$$\int_\Omega \varphi (-\Delta)^s \phi \, dx = \int_{\mathbb{R}^N\setminus\Omega} g\mathcal{N}_s \phi \, dx,$$

holds, for every $\phi \in D((-\Delta)^s_D)$. It has been shown in [1,Theorem 3.5] that for every $g \in L^2(\mathbb{R}^N \setminus \Omega)$, equation (3.18) has a unique solution φ defined by transposition. Therefore, the Dirichlet map $\mathbb{D} : L^2(\mathbb{R}^N \setminus \Omega) \to L^2(\Omega)$ given by

$$\mathbb{D}g = u \Longleftrightarrow (-\Delta)^s u = 0 \text{ in } \Omega \text{ and } u = g \text{ in } \mathbb{R}^N \setminus \Omega, \tag{3.19}$$

is well–defined.

(b) Next, let us denote by $\mathbb{A} := (-\Delta)^s_D$, with $D(\mathbb{A}) := D((-\Delta)^s_D)$. That is, $\mathbb{A}$ is the realization in $L^2(\Omega)$ of the fractional Laplace operator with zero Dirichlet exterior condition defined in (2.5). The operator $\mathbb{A}$ can be extended to a bounded operator from $W_0^{s,2}(\overline{\Omega})$ into $W^{-s,2}(\overline{\Omega})$. Its extension is a self-adjoint positive operator on $W^{-s,2}(\overline{\Omega})$. If there is no confusion we use the same notation $\mathbb{A}$. Then, the operator $-\mathbb{A} : D(\mathbb{A}) \subset W_0^{s,2}(\overline{\Omega}) \to W^{-s,2}(\overline{\Omega})$ generates a strongly continuous submarkovian semigroup $(\mathbb{S}(t))_{t\geq 0}$ on $W^{-s,2}(\overline{\Omega})$ which coincides with the semigroup $(e^{-t(-\Delta)_D^s})_{t\geq 0}$ on $L^2(\Omega)$ (see e.g. [9]).

(c) Let $(D(\mathbb{A}))^\star$ denote the dual space of $D(\mathbb{A})$ so that we have the following continuous and dense embeddings:

$$D(\mathbb{A}) \hookrightarrow W^{-s,2}(\overline{\Omega}) \hookrightarrow (D(\mathbb{A}))^\star.$$

The semigroup $\mathbb{S}$ can be also extended to $(D(\mathbb{A}))^\star$ and its generator is an extension of $\mathbb{A}$ which is also a self-adjoint positive operator on $(D(\mathbb{A}))^\star$.

The following definitions are inspired by [51,Section 4.2 and 4.3].

Definition 3 (a) An operator $\mathbb{B} \in \mathcal{L}(L^2(\mathbb{R}^N \setminus \Omega); (D(\mathbb{A}))^\star)$ is called an admissible control operator for the semigroup $(\mathbb{S}(t))_{t\geq 0}$, if for some $\tau > 0$, $\mathrm{Rang}(\Phi_\tau) \subset W^{-s,2}(\overline{\Omega})$, where for $g \in L^2((0,T); L^2(\mathbb{R}^N \setminus \Omega))$ we have set

$$\Phi_\tau g(t) := \int_0^t \mathbb{S}(t-\tau)\mathbb{B}g(\tau)\, d\tau.$$

(b) An operator $\mathbb{E} \in \mathcal{L}(D(\mathbb{A}), L^2(\mathbb{R}^N \setminus \Omega))$ is called an admissible observation operator for the semigroup $(\mathbb{S}(t))_{t\geq 0}$, if for some $\tau > 0$, Ψ_τ has a continuous extension to $W^{-s,2}(\overline{\Omega})$, where for $u_0 \in D(\mathbb{A})$,

$$(\Psi_\tau u_0)(t) := \begin{cases} \mathbb{E}\mathbb{S}(t)u_0 & \text{if } t \in [0,\tau], \\ 0 & \text{if } t > \tau. \end{cases}$$

Equivalently, $\mathbb{E} \in \mathcal{L}(D(\mathbb{A}), L^2(\mathbb{R}^N \setminus \Omega))$ is an admissible observation operator for the semigroup $(\mathbb{S}(t))_{t\geq 0}$, if and only if for some $\tau > 0$, there exists a constant $K_\tau \geq 0$ such that

$$\int_0^\tau \|\mathbb{E}\mathbb{S}(t)z_0\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} dt \leq K_\tau^2 \|z_0\|^2_{D(\mathbb{A})}. \tag{3.20}$$

With the previous notations, we consider the control operator $\mathbb{B} \in \mathcal{L}(L^2(\mathbb{R}^N \setminus \Omega); (D(\mathbb{A}))^\star)$ defined by

$$\mathbb{B}g := \mathbb{A}\mathbb{D}g,$$

where $\mathbb{D} : L^2(\mathbb{R}^N \setminus \Omega) \to L^2(\Omega)$ is the nonlocal Dirichlet map defined in (3.19). Then, $\mathbb{B}$ is an admissible control operator for the semigroup $(\mathbb{S}(t))_{t\geq 0}$ in the sense of Definition 3. In addition, the operator $\mathbb{B}^* : \mathcal{L}(D(\mathbb{A}); L^2(\mathbb{R}^N \setminus \Omega))$ is given by

$$\mathbb{B}^*\varphi = -\mathcal{N}_s(\mathbb{A}^{-1}\varphi), \quad \forall \varphi \in L^2(\Omega). \tag{3.21}$$

Besides, it follows from [51,Theorem 4.4.3] that $\mathbb{B}^*$ is an admissible observation operator for the semigroup $(\mathbb{S}(t))_{t\geq 0}$.

Remark 4 We observe the following facts.

(a) Recall that from Theorem 2 we have that the operator $-\mathcal{A}$ defined in (3.5), with $D(\mathcal{A}) := D((-\Delta)^s_D) \times D((-\Delta)^s_D) \times L^2(\Omega)$, generates a strongly continuous semigroup $(\mathbb{T}(t))_{t\geq 0} := (e^{-t\mathcal{A}})_{t\geq 0}$ in $\mathcal{H} := W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$.
(b) Let $\mathbb{A}$ be the extension of the operator $(-\Delta)^s_D$ given in Remark 3(b) and let $\mathcal{A}$ be the associated operator where $(-\Delta)^s_D$ is replaced with $\mathbb{A}$. It follows from Theorem 1 that the operator $-\mathcal{A} : D(\mathcal{A}) \to D(\mathbb{A}^{\frac{1}{2}}) \times D(\mathbb{A}^{\frac{1}{2}}) \times W^{-s,2}(\overline{\Omega})$ generates a strongly continuous semigroup $(\mathbb{T}(t))_{t\geq 0}$ on $D(\mathbb{A}^{\frac{1}{2}}) \times D(\mathbb{A}^{\frac{1}{2}}) \times W^{-s,2}(\overline{\Omega})$ which coincides with the semigroup $(e^{-t(-\mathcal{A}})_{t\geq 0}$ on $\mathcal{H}$.
(c) Let $(D(\mathbb{A}))^\star$ denote the dual space of $D((-\Delta)^s_D)$. Let $\mathbb{A}$ be the extension of the operator $(-\Delta)^s_D$ given in Remark 3(c) and let $\mathcal{A}$ be the associated operator where $(-\Delta)^s_D$ is replaced with $\mathbb{A}$. It follows from Theorem 1 that the operator $-\mathcal{A} : D(\mathcal{A}) \to D(\mathbb{A}^{\frac{1}{2}}) \times D(\mathbb{A}^{\frac{1}{2}}) \times (D(\mathbb{A}))^\star$ generates a strongly continuous semigroup $(\mathbb{T}(t))_{t\geq 0}$ on $D(\mathbb{A}^{\frac{1}{2}}) \times D(\mathbb{A}^{\frac{1}{2}}) \times (D(\mathbb{A}))^\star$ which coincides with the above semigroups on each subspace of $\mathcal{H}$.

Throughout the following, if there is no confusion, we shall only denote by $\mathbb{T}$ any of the above mentioned three semigroups.

Finally, we introduce the notion of an exterior control system, which is inspired from the local case (see [51,Chapter 10]). Let V, Z, X be three Hilbert spaces such that $Z \hookrightarrow X$, with continuous embedding. We shall call V the input space, Z the solution space, and X the state space. Let us consider the system

$$z'(t) = Lz(t), \quad Gz(t) = w(t), \tag{3.22}$$

where L and G are appropriate operators that we shall introduce below.

Definition 4 An exterior control system on U, X and Z is a pair of operators (L, G), where

$$L \in \mathcal{L}(Z, X), \quad G \in \mathcal{L}(Z, V), \tag{3.23}$$

such that the following conditions holds. There exists a constant $\beta \in \mathbb{C}$ such that

(i) G is onto;
(ii) $\ker(G)$ is dense in X;
(iii) $\beta I - L$ restricted to $\ker(G)$ is onto; and
(iv) $\ker(\beta I - L) \cap \ker(G) = \{0\}$.

From the previous considerations, for the MGT equation we consider as control space $V := L^2(\mathbb{R}^N \setminus \Omega)$, the state space will be $X := L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega})$ and the solution space $Z := W \times W_0^{s,2}(\overline{\Omega}) \times V$, where $W = W_0^{s,2}(\overline{\Omega}) + \mathbb{D}V$. We recall that $\mathbb{D}$ is the nonlocal Dirichlet operator defined in (3.19). We observe that if $f \in W$, then f can be written as $f = f_0 + v$, where $f_0 \in W_0^{s,2}(\overline{\Omega})$ and $v \in L^2(\mathbb{R}^N \setminus \Omega)$. Therefore, from the definition of the Dirichlet map $\mathbb{D}$, we have that $(-\Delta)^s f = (-\Delta)^s f_0 \in W^{-s,2}(\overline{\Omega})$. Thus, the operators $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$ given by

$$L \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} := \begin{bmatrix} f_2 \\ f_3 \\ -\frac{c^2}{\tau}(-\Delta)^s f_1 - \frac{b}{\tau}(-\Delta)^s f_2 - \frac{\alpha}{\tau} f_3 \end{bmatrix}, \quad G \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} := f_1\Big|_{\mathbb{R}^N \setminus \Omega}, \tag{3.24}$$

for every $(f_1, f_2, f_3) \in Z$ are well defined.

We have the following result.

Theorem 4 *The pair (L, G) is a well–posed exterior control system on V, Z and X. The associated control adjoint operators are given by*

$$\mathcal{B}g = \begin{pmatrix} 0 \\ 0 \\ c^2 \mathbb{B}g \end{pmatrix}, \quad \forall g \in V, \tag{3.25}$$

and

$$\mathcal{B}^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = -c^2 \mathcal{N}_s(\mathbb{A}^{-1} f_3), \quad \forall (\varphi_1, \varphi_2, \varphi_3) \in D(\mathcal{A}), \tag{3.26}$$

respectively. Moreover, the control operator $\mathcal{B}$ is an admissible control operator for the semigroup generates by $-\mathcal{A}$, in the sense of Definition 3.

Proof Let $f_1 \in W$, that is, $f_1 = f_0 + \mathbb{D}v$ where $f_0 \in W_0^{s,2}(\overline{\Omega})$ and $v \in L^2(\mathbb{R}^N \setminus \Omega)$. Then,

$$G \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix} = (f_0 + \mathbb{D}v)\Big|_{\mathbb{R}^N \setminus \Omega} = v.$$

Thus, we obtain that G is onto in V. Besides, $\ker(G) = D(\mathcal{A})$ and $L\Big|_{\ker(G)} = \mathcal{A}$. To prove the conditions (ii)-(iv) in Definition 4, we observe that they are satisfied for $\beta = 0$, since $0 \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$. This shows that the pair (L, G) defines an exterior control system.

Now, to prove the formula for $\mathcal{B}$, we need to solve the fractional exterior elliptic system for $(f_1, f_2, f_3) \in Z$. Namely,

$$L \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0, \quad G \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = v, \quad \forall v \in L^2(\mathbb{R}^N \setminus \Omega).$$

It is immediate that $f_2 = f_3 = 0$ and $f_1 \in W_0^{s,2}(\overline{\Omega})$ satisfies

$$(-\Delta)^s f_1 = 0 \ \text{ in } \ \Omega, \quad f_1 = v \ \text{ in } \ \mathbb{R}^N \setminus \Omega.$$

From the well–posedness of the elliptic problem, we have that $f_1 = \mathbb{D}v$. From [51,Proposition 10.1.2], we have that for every $\beta \in \rho(\mathcal{A})$,

$$(\beta I - \mathcal{A})^{-1}\mathcal{B}v = \begin{bmatrix} \mathbb{D}v \\ 0 \\ 0 \end{bmatrix}.$$

Now, taking $\beta = 0$ and applying $\mathcal{A}$ in both sides, we obtain the formula for $\mathcal{B}$.

The formula for the adjoint $\mathcal{B}^*$ of $\mathcal{B}$, follows from the definition of $\mathcal{B}$ and (3.21).

Finally, we prove that $\mathcal{B}$ is an admissible control operator. We observe that from the definition of $\mathcal{B}^*$ and $\mathcal{A}$, we have that

$$\mathcal{B}^*\mathcal{A}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = -\frac{c^2}{\tau}\mathcal{N}_s(c^2 f_1 + bf_2) + \frac{c^2\alpha}{\tau}\mathbb{B}^* f_3, \quad \forall (f_1, f_2, f_3) \in D(\mathcal{A}).$$

On the other hand, it is not difficult to show that (see (3.53)) the solution W of (3.4) satisfies the estimate

$$\int_0^T \|\mathcal{N}_s W(\cdot,t)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} dt \leq C\|(f_1, f_2, f_3\|^2_{D(\mathcal{A})} \tag{3.27}$$

for every $W_0 = (f_1, f_2, f_3) \in D(\mathcal{A})$.

Let us now denote by $\mathcal{C} := \mathcal{B}^*\mathcal{A} \in \mathcal{L}(D(\mathcal{A}), L^2(\mathbb{R}^N \setminus \Omega))$. From (3.27), (3.20) and the admissibility for $\mathbb{B}^*$, we obtain that $\mathcal{C}$ is an admissible observation operator for $(\mathbb{T}(t))_{t\geq 0}$, and hence also for the associated inverse semigroup. Since $(\mathbb{T}(t))_{t\geq 0}$ is a group, it follows that $\mathcal{C}$ is admissible for $(\mathbb{T}^*(t))_{t\geq 0}$. The previous fact implies that $\mathcal{B}^* := \mathcal{C}\mathcal{A}^{-1}$ is an admissible observation operator for $(\mathbb{T}^*(t))_{t\geq 0}$ in $L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega})$. Using [51,Theorem 4.4.3], we can deduce that $\mathcal{B}$ is an admissible control operator for $(\mathbb{T}(t))_{t\geq 0}$. The proof is finished. □

We can deduce from the results above that the nonlocal Dirichlet exterior control problem (1.1) can be rewritten as the following first order Cauchy problem:

$$\begin{cases} U_t + \mathcal{A}U = \mathcal{B}g, \quad t > 0, \\ U(0) = U_0. \end{cases} \tag{3.28}$$

Finally, we have the following exterior control semigroup formula for the solution U of (3.28).

Theorem 5 *Let $\mathcal{B} \in \mathcal{L}(L^2(\mathbb{R}^N \setminus \Omega), D(\mathcal{A}))$ be the operator given in (3.25). Then, the Cauchy problem (3.28) (hence, (1.1)) has a unique (very weak solution (or a solution* *by transposition)* $U \in L^2(\Omega \times (0,T)) \cap C([0,T]; L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega}))$ *given for every* $U_0 \in L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega})$ *and* $g \in L^2((0,T); L^2(\mathbb{R}^N \setminus \Omega))$ *by*

$$U(\cdot,t) = \mathbb{T}(t)U_0 + \int_0^t \mathbb{T}(t-\tau)\mathcal{B}g(\cdot,\tau)\, d\tau. \tag{3.29}$$

Remark 5 If the exterior data enjoy more time–regularity, then we obtain more regular solutions. That is, from [51,Lemma 4.2.8], if g belongs to $H^1((0,T); L^2(\mathbb{R}^N \setminus \Omega))$ and satisfies $g(0) = 0$, then $U \in L^2(\Omega \times (0,T)) \cap C([0,T]; Z) \cap C^1([0,T]; L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega}))$, where the Hilbert space Z is given by

$$Z = D(\mathcal{A}) + (\beta I - \mathcal{A})^{-1}\mathcal{B}\Big(L^2(\mathbb{R}^N \setminus \Omega)\Big),$$

with $\beta \in \rho(\mathcal{A})$.

3.2 Series solutions of the dual system

Now, we consider the dual system associated to (3.3). That is, the backward system

$$\begin{cases} -\tau\psi_{ttt} + \alpha\psi_{tt} + c^2(-\Delta)^s\psi - b(-\Delta)^s\psi_t = 0 & \text{in } \Omega \times (0,T), \\ \psi = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \psi(\cdot,T) = \psi_0, \quad -\psi_t(\cdot,T) = \psi_1, \quad \psi_{tt}(\cdot,T) = \psi_2 & \text{in } \Omega. \end{cases} \tag{3.30}$$

Let

$$\psi_{0,n} := (\psi_0, \varphi_n)_{L^2(\Omega)}, \quad \psi_{1,n} := (\psi_1, \varphi_n)_{L^2(\Omega)}, \quad \text{and} \quad \psi_{2,n} := (\psi_2, \varphi_n)_{L^2(\Omega)}.$$

Throughout this subsection we will denote

$$D_n(t) := A_n(t), \;\; E_n(t) := -B_n(t) \;\; \text{and} \;\; F_n(t) := C_n(t),$$

where $A_n(t)$, $B_n(t)$ and $C_n(t)$ are given in (3.9), (3.10) and (3.11), respectively.

Our notion of weak solutions to (3.30) is as follows.

Definition 5 Let $(\psi_0, \psi_1, \psi_2) \in W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$. A function $(\psi, \psi_t, \psi_{tt})$ is said to be a weak solution of (3.30), if for a.e. $t \in (0,T)$, the following properties hold:

- Regularity and final data:

$$\psi \in C^1([0,T]; W_0^{s,2}(\overline{\Omega})) \cap C^2([0,T]; L^2(\Omega)) \cap C^3((0,T); W^{-s,2}(\overline{\Omega})), \tag{3.31}$$

$\psi(\cdot,T) = \psi_0$, $\psi_t(\cdot,T) = \psi_1$ and $\psi_{tt}(\cdot,T) = \psi_2$ in Ω.

– Variational identity: For every $w \in W_0^{s,2}(\overline{\Omega})$ and a.e. $t \in (0, T)$, we have

$$\langle \tau\psi_{ttt} + \alpha\psi_{tt} + (-\Delta)^s(c^2\psi - b\psi_t), w\rangle_{-\frac{1}{2},\frac{1}{2}} = 0.$$

We begin with the following technical lemma, which is consequence of $\lambda_{n,1} < 0$ and $\mathrm{Re}(\lambda_{n,j}) < 0$, for $j = 2, 3$, implying that $\left|e^{\lambda_{n,j}t}\right| \le 1$ for $j = 1, 2, 3$. We omit the proof for brevity.

Lemma 2 *There is a constant $C > 0$ (independent of n) such that for every $t \in [0, T]$ and $n \in \mathbb{N}$,*

$$\max\left\{|D_n(t)|^2, |D_n'(t)|^2, \left|\frac{D_n''(t)}{\mu_n^{\frac{1}{2}}}\right|^2, \left|\frac{D_n''(t)}{\mu_n^{\frac{3}{2}}}\right|^2\right\} \le C, \tag{3.32}$$

$$\max\left\{|E_n(t)|^2, |E_n'(t)|^2, \left|\frac{E_n''(t)}{\mu_n^{\frac{1}{2}}}\right|^2, \left|\frac{E_n''(t)}{\mu_n^{\frac{3}{2}}}\right|^2\right\} \le C, \tag{3.33}$$

$$\max\left\{\left|\mu_n^{\frac{1}{2}}F_n(t)\right|^2, |\mu_n F_n(t)|^2, |\mu_n^{\frac{1}{2}}F_n'(t)|^2, |F_n''(t)|^2, \left|\frac{F_n''(t)}{\mu_n^{\frac{1}{2}}}\right|^2\right\} \le C. \tag{3.34}$$

Using the previous lemma, we obtain the following existence result.

Theorem 6 *For every $(\psi_0, \psi_1, \psi_2) \in W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$, the dual system (3.30) has a unique weak solution $(\psi, \psi_t, \psi_{tt})$ given by*

$$\psi(x,t) = \sum_{n=1}^{\infty}\Big(\psi_{0,n}D_n(T-t) + \psi_{1,n}E_n(T-t) + \psi_{2,n}F_n(T-t)\Big)\varphi_n(x), \tag{3.35}$$

where $D_n(t)$, $-E_n(t)$ and $F_n(t)$ are given in (3.9), (3.10) and (3.11), respectively. In addition the following assertions hold.

(a) There is a constant $C > 0$ such that for all $t \in [0, T]$,

$$\begin{aligned}&\|\psi(\cdot,t)\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_t(\cdot,t)\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_{tt}(\cdot,t)\|^2_{L^2(\Omega)}\\ &\le C\left(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_2\|^2_{L^2(\Omega)}\right),\end{aligned} \tag{3.36}$$

and

$$\|\psi_{ttt}(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})} \le C\left(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{W_0^{s,2}(\overline{\Omega})} + \|\psi_2\|^2_{L^2(\Omega)}\right). \tag{3.37}$$

(b) We have that $\psi \in L^\infty((0,T); D((-\Delta)^s_D))$.

(c) The mapping

$$[0,T)\ni t\mapsto \mathcal{N}_s\psi(\cdot,t)\in L^2(\mathbb{R}^N\setminus\Omega),$$

can be analytically extended to the half-plane $\Sigma_T := \{z\in \mathbb{C}C:\ \mathrm{Re}(z)<T\}$. *Here,* $\mathcal{N}_s\psi$ *is the nonlocal normal derivative of* ψ *defined in* (2.10).

Proof Let

$$\psi_0=\sum_{n=1}^{\infty}\psi_{0,n}\varphi_n,\quad \psi_1=\sum_{n=1}^{\infty}\psi_{1,n}\varphi_n,\quad \psi_2=\sum_{n=1}^{\infty}\psi_{2,n}\varphi_n. \tag{3.38}$$

The proof of the theorem is divided in several steps.

Step 1 Proceeding as in the proof of Proposition 3, we easily get that

$$\psi(x,t)=\sum_{n=1}^{\infty}\Big[D_n(T-t)\psi_{0,n}+E_n(T-t)\psi_{1,n}+F_n(T-t)\psi_{2,n}\Big]\varphi_n(x), \tag{3.39}$$

where $D_n(t)=A_n(t)$, $E_n(t)=-B_n(t)$ and $F_n(t)=C_n(t)$. In addition, a simple calculation gives $\psi(x,T)=\psi_0(x)$, $\psi_t(x,T)=-\psi_1(x)$ and $\psi_{tt}(x,T)=\psi_2(x)$ for a.e. $x\in\Omega$.
Let us show that ψ satisfies the regularity and variational identity requirements. Let $1\le n\le m$ and set

$$\psi_m(x,t):=\sum_{n=1}^{m}\Big[D_n(T-t)\psi_{0,n}+E_n(T-t)\psi_{1,n}+F_n(T-t)\psi_{2,n}\Big]\varphi_n(x).$$

For every $m,\tilde{m}\in\mathbb{N}$ with $m>\tilde{m}$ and $t\in[0,T]$, we have that

$$\begin{aligned}\|\psi_m(x,t)-&\psi_{\tilde{m}}(x,t)\|^2_{W_0^{s,2}(\overline{\Omega})}\\ &\le 2\sum_{n=\tilde{m}+1}^{m}\left|\mu_n^{\frac12}D_n(T-t)\psi_{0,n}\right|^2+2\sum_{n=\tilde{m}+1}^{m}\left|\mu_n^{\frac12}E_n(T-t)\psi_{1,n}\right|^2\\ &\quad+2\sum_{n=\tilde{m}+1}^{m}\left|\mu_n^{\frac12}F_n(T-t)\psi_{2,n}\right|^2.\end{aligned} \tag{3.40}$$

Using (3.32), (3.33) and (3.34) we get from (3.40) that, for every $m,\tilde{m}\in\mathbb{N}$ with $m>\tilde{m}$, and $t\in[0,T]$,

$$\|\psi_m(x,t)-\psi_{\tilde{m}}(x,t)\|^2_{W_0^{s,2}(\overline{\Omega})}$$

$$\leq C\left(\sum_{n=\tilde{m}+1}^{m}\left|\mu_n^{\frac{1}{2}}\psi_{0,n}\right|^2+\sum_{n=\tilde{m}+1}^{m}\left|\mu_n^{\frac{1}{2}}\psi_{1,n}\right|^2+\sum_{n=\tilde{m}+1}^{m}\left|\psi_{2,n}\right|^2\right)\longrightarrow 0,$$

as $m,\tilde{m}\to\infty$. We have shown that

$$\sum_{n=1}^{\infty}\Big[D_n(T-t)\psi_{0,n}+E_n(T-t)\psi_{1,n}+F_n(T-t)\psi_{2,n}\Big]\varphi_n\longrightarrow\psi(\cdot,t)$$

in $W_0^{s,2}(\overline{\Omega})$ and that the convergence is uniform in $t\in[0,T]$. Hence, $\psi\in C([0,T];W_0^{s,2}(\overline{\Omega}))$. Using (3.32), (3.33) and (3.34) again, we get that there is a constant $C>0$ such that for every $t\in[0,T]$,

$$\|\psi(\cdot,t)\|_{W_0^{s,2}(\overline{\Omega})}\leq C\Big(\|\psi_0\|_{W_0^{s,2}(\overline{\Omega})}+\|\psi_1\|_{W_0^{s,2}(\overline{\Omega})}+\|\psi_2\|_{L^2(\Omega)}\Big). \tag{3.41}$$

Step 2 Next, we show that $\psi_t\in C([0,T];W_0^{s,2}(\overline{\Omega}))$. Indeed, we have

$$(\psi_m)_t(x,t)=-\sum_{n=1}^{m}\Big[D_n'(T-t)\psi_{0,n}+E_n'(T-t)\psi_{1,n}+F_n'(T-t)\psi_{2,n}\Big]\varphi_n(x).$$

Proceeding as above, we obtain that

$$\sum_{n=1}^{\infty}\Big[D_n'(T-t)\psi_{0,n}+E_n'(T-t)\psi_{1,n}+F_n'(T-t)\psi_{2,n}\Big]\varphi_n\longrightarrow\psi_t(\cdot,t)$$

in $W_0^{s,2}(\overline{\Omega})$ and the convergence is uniform in $t\in[0,T]$. As in the previous case, using (3.32), (3.33) and (3.34), we get that there is a constant $C>0$ such that for every $t\in[0,T]$,

$$\|\psi_t(\cdot,t)\|^2_{L^2(\Omega)}\leq C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_1\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_2\|^2_{L^2(\Omega)}\Big). \tag{3.42}$$

Step 3 Next, we claim that $\psi_{tt}\in C([0,T];L^2(\Omega))$. A simple calculation shows that

$$\partial_{tt}\psi_m(x,t)=\sum_{n=1}^{m}\Big[D_n''(T-t)\psi_{0,n}+E_n''(T-t)\psi_{1,n}+F_n''(T-t)\psi_{2,n}\Big]\varphi_n(x).$$

As in Step 1, we obtain that for every $m,\tilde{m}\in\mathbb{N}$ with $m>\tilde{m}$ and $t\in[0,T]$, we have

$$\|\partial_{tt}\psi_m(x,t)-\partial_{tt}\psi_{\tilde{m}}(x,t)\|^2_{L^2(\Omega)}$$

$$\leq C\left(\sum_{n=\tilde{m}+1}^{m}\left|\mu_n^{\frac{1}{2}}\psi_{0,n}\right|^2+\sum_{n=\tilde{m}+1}^{m}\left|\mu_n^{\frac{1}{2}}\psi_{1,n}\right|^2+\sum_{n=\tilde{m}+1}^{m}\left|\psi_{2,n}\right|^2\right)$$
$$\longrightarrow 0 \text{ as } m,\tilde{m}\to\infty. \tag{3.43}$$

Again, we can deduce that

$$\sum_{n=1}^{\infty}\Big[D_n''(T-t)\psi_{0,n}+E_n''(T-t)\psi_{1,n}+F_n''(T-t)\psi_{2,n}\Big]\varphi_n\longrightarrow\psi_{tt}(\cdot,t)$$

in $L^2(\Omega)$ and the convergence is uniform in $t\in[0,T]$. In addition, using (3.32), (3.33) and (3.34), we get that there is a constant $C>0$ such that for every $t\in[0,T]$,

$$\|\psi_{tt}(\cdot,t)\|^2_{L^2(\Omega)}\leq C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_1\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_2\|^2_{L^2(\Omega)}\Big). \tag{3.44}$$

Finally, the estimate (3.36) follows from (3.41), (3.42) and (3.44).

Step 4: We show that $\psi_{ttt}\in C([0,T);W^{-s,2}(\overline{\Omega}))$. Using (2.9), (3.32), (3.33) and (3.34), we get that for every $t\in[0,T]$, the following inequality hold:

$$\|(-\Delta)_D^s\psi(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})}\leq C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_1\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_2\|^2_{L^2(\Omega)}\Big). \tag{3.45}$$

Using (2.9), (3.32), (3.33) and (3.34) again, we get that there is a constant $C>0$ such that for every $t\in[0,T]$,

$$\|(-\Delta)_D^s\psi_t(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})}\leq C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_1\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_2\|^2_{L^2(\Omega)}\Big). \tag{3.46}$$

Finally, using (2.9), (3.32), (3.33) and (3.34), we get that

$$\|\psi_{tt}(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})}\leq C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_1\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_2\|^2_{L^2(\Omega)}\Big). \tag{3.47}$$

Proceeding as above, we can deduce that the series converges in $W^{-s,2}(\overline{\Omega})$ and the convergence is uniform in any compact subset of $[0,T)$. This shows that $\psi_{ttt}\in C([0,T);W^{-s,2}(\overline{\Omega}))$.
Since $\psi_{ttt}(\cdot,t)=-\alpha\psi_{tt}(\cdot,t)-c^2(-\Delta)_D^s\psi(\cdot,t)+b(-\Delta)_D^s\psi_t(\cdot,t)$, it follows from (3.45), (3.46) and (3.47) that

$$\|\psi_{ttt}(\cdot,t)\|^2_{W^{-s,2}(\overline{\Omega})}\leq C\Big(\|\psi_0\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_1\|^2_{W_0^{s,2}(\overline{\Omega})}+\|\psi_2\|^2_{L^2(\Omega)}\Big).$$

We have also shown (3.37).

Step 5 We claim that $\psi \in L^\infty((0,T); D((-\Delta)^s_D))$. It follows from the estimate (3.36) that $\psi \in L^\infty((0,T); L^2(\Omega))$. Since $D((-\Delta)^s_D)\times D((-\Delta)^s_D)\times L^2(\Omega)$ is dense in $W_0^{s,2}(\overline{\Omega})\times W_0^{s,2}(\overline{\Omega})\times L^2(\Omega)$, it suffices to consider $(\psi_0,\psi_1,\psi_2)\in D((-\Delta)^s_D)\times D((-\Delta)^s_D)\times L^2(\Omega)$. Proceeding as above we get

$$\begin{aligned}&\|\psi(\cdot,t)\|^2_{D((-\Delta)^s_D)} = \|(-\Delta)^s_D\psi(\cdot,t)\|^2_{L^2(\Omega)}\\ \le&2\sum_{n=1}^{\infty}\left(\left|D_n(T-t)\mu_n\psi_{0,n}\right|^2+\left|E_n(T-t)\mu_n\psi_{1,n}\right|^2+\left|\mu_nF_n(T-t)\psi_{2,n}\right|^2\right).\end{aligned} \tag{3.48}$$

It follows from (3.48), (3.32), (3.33) and (3.34) that

$$\|\psi(\cdot,t)\|^2_{D((-\Delta)^s_D)}\le C\Big(\|\psi_0\|^2_{D((-\Delta)^s_D)}+\|\psi_1\|^2_{D((-\Delta)^s_D)}+\|\psi_2\|^2_{L^2(\Omega)}\Big).$$

Thus, $\psi\in L^\infty((0,T); D((-\Delta)^s_D))$ and we have shown the claim.

Step 6 It is not difficult to see that the mapping $[0,T)\ni t\to\psi(\cdot,t)\in L^2(\mathbb{R}^N\setminus\Omega)$ can be analytically extended to Σ_T. We also recall that for every $t\in[0,T)$ fixed, we have that $\psi(\cdot,t)\in D((-\Delta)^s_D)$. Therefore, $\mathcal{N}_s\psi(\cdot,t)$ exists and belongs to $L^2(\mathbb{R}^N\setminus\Omega)$.

We claim that

$$\mathcal{N}_s\psi(x,t)=\sum_{n=1}^{\infty}\Big(D_n(T-t)\psi_{0,n}+E_n(T-t)\psi_{1,n}+F_n(T-t)\psi_{2,n}\Big)\mathcal{N}_s\varphi_n(x), \tag{3.49}$$

and that the series is convergent in $L^2(\mathbb{R}^N\setminus\Omega)$ and the convergence is uniform in $t\in[0,T)$. Indeed, let $\eta>0$ be fixed but arbitrary and let $t\in[0,T-\eta]$. Let $n,m\in\mathbb{N}$ with $n>m$. Since $\mathcal{N}_s: D((-\Delta)^s_D)\to L^2(\mathbb{R}^N\setminus\Omega)$ is bounded, using (3.32), (3.33) and (3.34), we have that there is a constant $C>0$ such that

$$\begin{aligned}&\left\|\sum_{n=m+1}^{\infty}\Big(D_n(T-t)\psi_{0,n}+E_n(T-t)\psi_{1,n}+F_n(T-t)\psi_{2,n}\Big)\mathcal{N}_s\varphi_n\right\|^2_{L^2(\mathbb{R}^N\setminus\Omega)}\\ \le&C\left(\sum_{n=m+1}^{\infty}|\psi_{0,n}|^2+\sum_{n=m+1}^{\infty}|\psi_{1,n}|^2+\sum_{n=m+1}^{\infty}|\psi_{2,n}|^2\right)\longrightarrow 0 \text{ as } m\to\infty.\end{aligned} \tag{3.50}$$

Thus, $\mathcal{N}_s$ is given by (3.49) and the series is convergent in $L^2(\mathbb{R}^N\setminus\Omega)$ uniformly in any compact subset of $[0,T)$.

Besides, we obtain the following continuous dependence on the data for the nonlocal normal derivative. Let $m \in \mathbb{N}$ and consider

$$\psi_m(x,t) = \sum_{n=1}^{m} \Big(D_n(T-t)\psi_{0,n} + E_n(T-t)\psi_{1,n} + F_n(T-t)\psi_{2,n} \Big) \mathcal{N}_s \varphi_n(x). \tag{3.51}$$

Using the fact that the operator $\mathcal{N}_s : D((-\Delta)_D^s) \to L^2(\mathbb{R}^N \setminus \Omega)$ is bounded, the continuous embedding $W_0^{s,2}(\overline{\Omega}) \hookrightarrow L^2(\Omega)$, (3.32), (3.33) and (3.32), we get that there is a constant $C > 0$ such that for every $t \in (0,T)$,

$$\|\psi_m(\cdot,t)\|_{L^2(\mathbb{R}^N\setminus\Omega)} \le C \left(\|\psi_0\|^2_{W^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{W^{s,2}(\overline{\Omega})} + \|\psi_2\|^2_{L^2(\Omega)} \right). \tag{3.52}$$

It follows from (3.52) and (3.50) that

$$\|\mathcal{N}_s\psi(\cdot,t)\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} \le C\Big(\|\psi_0\|^2_{W^{s,2}(\overline{\Omega})} + \|\psi_1\|^2_{W^{s,2}(\overline{\Omega})} + \|\psi_2\|^2_{L^2(\Omega)} \Big). \tag{3.53}$$

Next, since the functions $D_n(z)$, $E_n(z)$ and $F_n(z)$ are entire functions, it follows that the function

$$\sum_{n=1}^{m} \Big[D_n(T-z)\psi_{0,n} + E_n(T-z)\psi_{1,n} + F_n(T-z)\psi_{2,n} \Big] \mathcal{N}_s\varphi_n$$

is analytic in Σ_T.

Let $\sigma > 0$ be fixed but arbitrary. Let $z \in \mathbb{C}C$ satisfy $\mathrm{Re}(z) \le T - \sigma$. Then, proceeding as above by using (3.32), (3.33) and (3.34), we get

$$\begin{aligned}
&\left\| \sum_{n=m+1}^{\infty} \psi_{0,n} D_n(T-z)\mathcal{N}_s\varphi_n \right\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} \\
&+ \left\| \sum_{n=m+1}^{\infty} \psi_{1,n} E_n(T-z)\mathcal{N}_s\varphi_n \right\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} \\
&+ \left\| \sum_{n=m+1}^{\infty} \psi_{2,n} F_n(T-z)\mathcal{N}_s\varphi_n \right\|^2_{L^2(\mathbb{R}^N\setminus\Omega)} \\
\le& C \sum_{n=m+1}^{\infty} \left| \mu_n^{\frac{1}{2}} \psi_{0,n} \right|^2 + \sum_{n=m+1}^{\infty} \left| \mu_n^{\frac{1}{2}} \psi_{1,n} \right|^2 + C \sum_{n=m+1}^{\infty} |\psi_{2,n}|^2 \longrightarrow 0 \text{ as } m \to \infty.
\end{aligned}$$

We have shown that

$$\mathcal{N}_s\psi(\cdot,z) = \sum_{n=1}^{\infty} \psi_{0,n} D_n(T-z)\mathcal{N}_s\varphi_n + \sum_{n=1}^{\infty} \psi_{1,n} E_n(T-z)\mathcal{N}_s\varphi_n$$

$$+\sum_{n=1}^{\infty}\psi_{2,n}F_n(T-z)\mathcal{N}_s\varphi_n, \tag{3.54}$$

and that the series is convergent in $L^2(\mathbb{R}^N\setminus\Omega)$ uniformly in any compact subset of Σ_T. Thus, $\mathcal{N}_s\psi$ given in (3.54) is also analytic in Σ_T. The proof is finished. □

4 Controllability results

In this section we state and prove the main results of the article. We begin with the proof of the lack of null/exact controllability of the system (1.1). For this purpose, we will use the following concept of controllability.

Definition 6 The system (1.1) is said to be *spectrally controllable*, if any finite linear combination of eigenfunctions

$$u_0=\sum_{n=1}^{M}u_{0,n}\varphi_n,\quad u_1=\sum_{n=1}^{M}u_{1,n}\varphi_n,\quad u_2=\sum_{n=1}^{M}u_{2,n}\varphi_n,$$

can be steered to zero by a control function g.

Next, let (u,u_t,u_{tt}) and (ψ,ψ_t,ψ_{tt}) be the very weak and weak solutions of (1.1) and (3.30), respectively. Multiplying the first equation in (1.1) by ψ, then integrating by parts over $(0,T)\times\Omega$, and using the formulas (2.12)-(2.13), we get

$$\begin{aligned}&\int_\Omega\Big(\tau(u_{tt}\psi-u_t\psi_t+u\psi_{tt})+\alpha(u_t\psi-u\psi_t)+bu(-\Delta)^s\psi\Big)\Big|_{t=0}^{t=T}dx\\&\quad=-\int_0^T\int_{\mathcal{O}}\Big(c^2g(x,t)+bg_t(x,t)\Big)\mathcal{N}_s\psi(x,t)\,dxdt.\end{aligned}\tag{4.1}$$

Using the identity (4.1) and a density argument to pass to the limit, we obtain the following criteria of null controllability (see for instance [61] for an abstract version).

Lemma 3 *The system (1.1) is null controllable in time $T>0$, if and only if for each initial condition $(u_0,u_1,u_2)\in L^2(\Omega)\times W^{-s,2}(\overline{\Omega})\times W^{-s,2}(\overline{\Omega})$, there exists a control function $g\in H^1((0,T);L^2(\mathcal{O}))$ such that the unique weak solution (ψ,ψ_t,ψ_{tt}) of the dual system (3.30) satisfies*

$$\begin{aligned}&-\tau\langle u_2,\psi(\cdot,0)\rangle_{-\frac12,\frac12}+\tau\langle u_1,\psi_t(\cdot,0)\rangle_{-\frac12,\frac12}-\tau\langle u_0,\psi_{tt}(\cdot,0)\rangle_{-\frac12,\frac12}\\&-\alpha\langle u_1,\psi(\cdot,0)\rangle_{-\frac12,\frac12}+\alpha(u_0,\psi_t(\cdot,0))_{L^2(\Omega)}-b(u_0,(-\Delta)^s\psi(\cdot,0))_{L^2(\Omega)}\\&=-\int_0^T\int_{\mathcal{O}}\Big(c^2g(x,t)+bg_t(x,t)\Big)\mathcal{N}_s\psi(x,t)dxdt,\end{aligned}\tag{4.2}$$

for every $(\psi_0,\psi_1,\psi_2)\in D((-\Delta)_D^s)\times D((-\Delta)_D^s)\times L^2(\Omega)\hookrightarrow W_0^{s,2}(\overline{\Omega})\times W_0^{s,2}(\overline{\Omega})\times L^2(\Omega)$.

Using Lemma 3, we obtain the following negative controllability result.

Theorem 7 *Let $b > 0$ and $\alpha - \frac{\tau c^2}{b} > 0$. Then, the system (1.1) is not exactly or null controllable in any time $T > 0$.*

Proof Using Definition 6, we shall prove that no non-trivial finite linear combination of eigenfunctions can be driven to zero in finite time.

We first write the initial data in Fourier series

$$u_0 = \sum_{n=1}^{\infty} u_{0,n}\varphi_n, \quad u_1 = \sum_{n=1}^{\infty} u_{1,n}\varphi_n, \quad u_2 = \sum_{n=1}^{\infty} u_{2,n}\varphi_n, \tag{4.3}$$

and we suppose that there exists $M \in \mathbb{N}$ such that

$$u_{0,n} = u_{1,n} = u_{2,n} = 0, \quad \forall\, n \geq M. \tag{4.4}$$

Assume that the system (1.1) is spectrally controllable. Then, there exists a control function g such that the very weak solution (u, u_t, u_{tt}) of (1.1) with u_0, u_1, u_2 given by (4.3)–(4.4) satisfy $u(\cdot, T) = u_t(\cdot, T) = u_{tt}(\cdot, T) = 0$ in Ω. From Lemma 3 we have that

$$\begin{aligned}
&- \tau\langle u_2, \psi(\cdot,0)\rangle_{-\frac{1}{2},\frac{1}{2}} + \tau\langle u_1, \psi_t(\cdot,0)\rangle_{-\frac{1}{2},\frac{1}{2}} - \tau\langle u_0, \psi_{tt}(\cdot,0)\rangle_{-\frac{1}{2},\frac{1}{2}}\\
&- \alpha\langle u_1, \psi(\cdot,0)\rangle_{-\frac{1}{2},\frac{1}{2}} + \alpha(u_0, \psi_t(\cdot,0))_{L^2(\Omega)} - b(u_0, (-\Delta)^s\psi(\cdot,0))_{L^2(\Omega)}\\
= &- \int_0^T \int_{\mathcal{O}} \Big(c^2 g(x,t) + b g_t(x,t)\Big)\mathcal{N}_s\psi(x,t)\,dx\,dt,
\end{aligned} \tag{4.5}$$

for every weak solution $(\psi, \psi_t, \psi_{tt})$ of the dual system (3.30).

We consider the following trajectories:

$$\psi(x,t) = e^{\lambda_{n,j}(T-t)}\varphi_n(x), \quad j = 1, 2, 3. \tag{4.6}$$

Replacing (4.6) in (4.5) we obtain, for any $n \in [1, M-1]$, the following system:

$$\begin{aligned}
&-\tau u_{2,n} e^{\lambda_{n,j}T} + \tau u_{1,n}\lambda_{n,j} e^{\lambda_{n,j}T} - \tau u_{0,n}\lambda_{n,j}^2 e^{\lambda_{n,j}T} - \alpha e^{\lambda_{n,j}T}\Big(u_{1,n} - u_{0,n}\lambda_{n,j}\Big)\\
&\quad - b u_{0,n}\mu_n e^{\lambda_{n,j}T}\\
&\quad = -\int_0^T \int_{\mathcal{O}} (c^2 g(x,t) + b g_t(x,t)) e^{\lambda_{n,j}(T-t)}\mathcal{N}_s\varphi_n(x)\,dx\,dt.
\end{aligned} \tag{4.7}$$

Multiplying (4.7) with $e^{-\lambda_{n,j}T}$, for each $j = 1, 2, 3$, we obtain a moment problem which consists to find a function g that satisfies

$$-\tau u_{2,n} + \tau u_{1,n}\lambda_{n,j} - \tau u_{0,n}\lambda_{n,j}^2 - \alpha\Big(u_{1,n} - u_{0,n}\lambda_{n,j}\Big) - b u_{0,n}\mu_n$$

$$= -\int_0^T \int_{\mathcal{O}} (c^2 g(x,t) + b g_t(x,t)) e^{-\lambda_{n,j} t} \mathcal{N}_s \varphi_n(x) dx dt. \tag{4.8}$$

Next, inspired by the works [43, 58], we define the complex-valued function

$$F(z) = \int_0^T \left(\int_{\mathcal{O}} (c^2 g(x,t) + b g_t(x,t)) \mathcal{N}_s \varphi_n(x) dx \right) e^{izt} dt. \tag{4.9}$$

According to the Paley–Wiener theorem, F is an entire function. Due to (4.4), from (4.8) we obtain that F satisfies $F(i\lambda_{n,j}) = 0$, for all $n \geq M$. Besides, we know that $\lambda_{n,1} \to -\frac{\tau c^2}{b}$ as $n \to \infty$ (see [44,Proposition 2]). Then, F is zero in a set with a finite accumulation point. This implies that $F \equiv 0$. It follows from (4.8) and (4.9) that

$$\underbrace{\begin{pmatrix} \alpha\lambda_{n,1} - \tau\lambda_{n,1}^2 - b\mu_n & \tau\lambda_{n,1} - \alpha & -\tau \\ \alpha\lambda_{n,2} - \tau\lambda_{n,2}^2 - b\mu_n & \tau\lambda_{n,2} - \alpha & -\tau \\ \alpha\lambda_{n,3} - \tau\lambda_{n,3}^2 - b\mu_n & \tau\lambda_{n,3} - \alpha & -\tau \end{pmatrix}}_{B} \begin{pmatrix} u_{0,n} \\ u_{1,n} \\ u_{2,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Calculating, we get that

$$\det(B) = \tau^3 (\lambda_{n,1} - \lambda_{n,2})(\lambda_{n,1} - \lambda_{n,3})(\lambda_{n,2} - \lambda_{n,3}) \neq 0.$$

Hence, the matrix B is invertible and we can then conclude that $u_{0,n} = u_{1,n} = u_{2,n} = 0$, for $n < M$. Thus, the trivial state is the only one that can be steered to zero. That is, the system is not spectrally controllable. Therefore, we can conclude that the null controllability also fails. Besides, since the exact controllability implies the null controllability, we obtain that the system is not exactly controllable. The proof is finished. □

Since (1.1) is not exactly or null controllable if $b > 0$ and $\alpha - \frac{\tau c^2}{b} > 0$, we shall study if it can be approximately controllable. It is straightforward to verify that the study of the approximate controllability of (1.1) can be reduced to the case $u_0 = u_1 = u_2 = 0$. We refer to [33, 40, 48, 55, 61] for more details.

It is a well–known result that the approximate controllability is a direct consequence of the *unique continuation property* of solutions to the adjoint system (3.30).

Remark 6 Let us mention that it has been recently proved in [20, 21] that the fractional Laplacian satisfies the elliptic strong unique continuation property, that is, if $u \in W^{-r,2}(\mathbb{R}^N) := (W^{r,2}(\mathbb{R}^N))^\star$, for some $r > 0$, is such that $u = (-\Delta)^s u = 0$ in some nonempty open set $\mathcal{O} \subset \mathbb{R}^N$, then $u \equiv 0$ in $\mathbb{R}^N$.

Now, we can state and prove the unique continuation property of solutions to the adjoint system (3.30).

Theorem 8 *Let* $(\psi_0, \psi_1, \psi_2) \in W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$ *and let* $(\psi, \psi_t, \psi_{tt})$ *be the unique weak solution of (3.30). Let* $\mathcal{O} \subset \mathbb{R}^N \setminus \overline{\Omega}$ *be an arbitrary nonempty open set. If* $\mathcal{N}_s\psi = 0$ *in* $\mathcal{O} \times (0,T)$*, then* $\psi = 0$ *in* $\Omega \times (0,T)$.

Proof Let $\mathcal{O} \subset \mathbb{R}^N \setminus \overline{\Omega}$ be an arbitrary nonempty open set. We give two alternative proofs.

Alternative 1: Suppose that $\mathcal{N}_s\psi = 0$ in $\mathcal{O} \times (0, T)$. Since $\psi = 0$ in $(\mathbb{R}^N \setminus \Omega) \times (0, T))$, using the definition of $\mathcal{N}_s$ (see (2.10)), we have that

$$\mathcal{N}_s\psi = (-\Delta)^s\psi = 0 \text{ in } \mathcal{O} \times (0, T). \tag{4.10}$$

It follows from Remark 6 that $\psi \equiv 0$ in $\mathbb{R}^N \times (0, T)$. The proof is finished. Notice that the proof of the result given in Remark 6 is very technical. For that reason we shall give a second alternative which is a direct proof of our result and it also avoids any technicality.

Alternative 2: Suppose that $\mathcal{N}_s\psi = 0$ in $\mathcal{O} \times (0, T)$. Then, from Theorem 6, for all $(x, t) \in \mathcal{O} \times (0, T)$, we have that

$$\begin{aligned}\mathcal{N}_s\psi(x,t) &= \sum_{n=1}^{\infty}\Big(D_n(T-t)\psi_{0,n} + E_n(T-t)\psi_{1,n} + F_n(T-t)\psi_{2,n}\Big)\mathcal{N}_s\varphi_n(x)\\ &= 0.\end{aligned} \tag{4.11}$$

Since $\mathcal{N}_s\psi$ can be analytically extended to Σ_T, it follows that for all $(x, t) \in \mathcal{O} \times (-\infty, T)$,

$$\begin{aligned}\mathcal{N}_s\psi(x,t) &= \sum_{n=1}^{\infty}\Big(D_n(T-t)\psi_{0,n} + E_n(T-t)\psi_{1,n} + F_n(T-t)\psi_{2,n}\Big)\mathcal{N}_s\varphi_n(x)\\ &= 0.\end{aligned} \tag{4.12}$$

Let $\{\mu_k\}_{k\in\mathbb{N}}$ be the set of all eigenvalues of the operator $(-\Delta)^s_D$ and let $\{\varphi_{k_j}\}_{1\le j\le m_k}$ be the orthonormal basis for $\ker(\mu_k - (-\Delta)^s_D)$, where m_k is the multiplicity of μ_k. Then, (4.12) can be rewritten as

$$\begin{aligned}\mathcal{N}_s\psi(x,t) = &\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{0,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)D_k(T-t)\\ &+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{1,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)E_k(T-t) \qquad (4.13)\\ &+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{2,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)F_k(T-t) = 0, \qquad (4.14)\end{aligned}$$

for every $(x, t) \in \mathcal{O} \times (-\infty, T)$.

Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) = \eta > 0$ and let $m \in \mathbb{N}$. Using the fact that $\{\varphi_{k_j}\}_{j\in\mathbb{N}}$ is an orthonormal system, the operator $\mathcal{N}_s : D((-\Delta)^s_D) \subset W^{s,2}(\mathbb{R}^N) \to L^2(\mathbb{R}^N \setminus \Omega)$ is bounded, and the continuous dependence on the data of $\mathcal{N}_s$ (see (3.53)), and letting

$$\begin{aligned}\phi_m(\cdot,t) :=& \sum_{k=1}^{m}\left(\sum_{j=1}^{m_k}\psi_{0,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)e^{z(t-T)}D_k(T-t)\\&+\sum_{k=1}^{m}\left(\sum_{j=1}^{m_k}\psi_{1,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)e^{z(t-T)}E_k(T-t)\\&+\sum_{k=1}^{m}\left(\sum_{j=1}^{m_k}\psi_{2,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)e^{z(t-T)}F_k(T-t),\end{aligned}$$

we obtain that there is a constant $C > 0$ (independent of m) such that

$$\|\phi_m(\cdot,t)\|_{L^2(\mathbb{R}^N\setminus\Omega)} \le Ce^{\eta(t-T)}\left(\|\psi_0\|_{W^{s,2}(\overline{\Omega})}+\|\psi_1\|_{W^{s,2}(\overline{\Omega})}+\|\psi_2\|_{L^2(\Omega)}\right). \tag{4.15}$$

We note that the right hand side of (4.15) is integrable over $(-\infty, T)$ and

$$\begin{aligned}&\int_{-\infty}^{T} e^{\eta(t-T)}(\|\psi_0\|_{W^{s,2}(\overline{\Omega})}+\|\psi_1\|_{W^{s,2}(\overline{\Omega})}+\|\psi_2\|_{L^2(\Omega)})dt\\&\quad=\frac{1}{\eta}\left(\|\psi_0\|_{W^{s,2}(\overline{\Omega})}+\|\psi_1\|_{W^{s,2}(\overline{\Omega})}+\|\psi_2\|_{L^2(\Omega)}\right).\end{aligned}$$

By the Lebesgue dominated convergence theorem, we can deduce that for all $x \in \mathbb{R}^N \setminus \Omega$ and $\mathrm{Re}(z) > 0$

$$\begin{aligned}&\int_{-\infty}^{T} e^{z(t-T)}\left[\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{0,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)D_k(T-t)\right.\\&+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{1,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)E_k(T-t)\\&\left.+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{m_k}\psi_{2,k_j}\mathcal{N}_s\varphi_{k_j}(x)\right)F_k(T-t)\right]dt\\&=\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\left(G_k(z)\psi_{0,k_j}+H_k(z)\psi_{1,k_j}+I_k(z)\psi_{2,k_j}\right)\mathcal{N}_s\varphi_{k_j}(x),\end{aligned} \tag{4.16}$$

where

$$\begin{cases} G_k(z) = \dfrac{\lambda_{k,2}\lambda_{k,3}}{\xi_{k,1}(z-\lambda_{k,1})} - \dfrac{\lambda_{k,1}\lambda_{k,3}}{\xi_{k,2}(z-\lambda_{k,2})} + \dfrac{\lambda_{k,1}\lambda_{k,2}}{\xi_{k,3}(z-\lambda_{k,3})} \\ H_k(z) = -\dfrac{\lambda_{k,2}+\lambda_{k,3}}{\xi_{k,1}(z-\lambda_{k,1})} + \dfrac{\lambda_{k,1}+\lambda_{k,3}}{\xi_{k,2}(z-\lambda_{k,2})} - \dfrac{\lambda_{k,1}+\lambda_{k,2}}{\xi_{k,3}(z-\lambda_{k,3})} \\ I_k(z) = \dfrac{1}{\xi_{k,1}(z-\lambda_{k,1})} - \dfrac{1}{\xi_{k,2}(z-\lambda_{k,2})} + \dfrac{1}{\xi_{k,3}(z-\lambda_{k,3})}. \end{cases} \tag{4.17}$$

and $\xi_{k,1}$, $\xi_{k,2}$ and $\xi_{k,3}$ are given in (3.12). From (4.13) and (4.16) we obtain that for $\mathrm{Re}(z) > 0$ and a.e. $x \in \mathcal{O}$,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\Big(G_k(z)\psi_{0,k_j} + H_k(z)\psi_{1,k_j} + I_k(z)\psi_{2,k_j}\Big)\mathcal{N}_s\varphi_{k_j}(x) = 0. \tag{4.18}$$

Using the analytic continuation property in z, we obtain that (4.18) holds for every $z \in \mathbb{C} \setminus \{\lambda_{k,1}, \lambda_{k,2}, \lambda_{k,3}\}_{k\in\mathbb{N}}$.

Next, we take a small circle about $\lambda_{k,h}$, for some $h \in \{1, 2, 3\}$, but not including $\{\lambda_{l,j}\}_{l\neq k,\ j\neq h}$, with $j \in \{1, 2, 3\}$. Then, integrating over that circle we get the following for a.e. $x \in \mathcal{O}$:

$$\sum_{j=1}^{m_k}\left[\frac{\lambda_{k_j,2}\lambda_{k_j,3}}{\xi_{k_j,1}}\psi_{0,k_j} - \frac{\lambda_{k_j,2}+\lambda_{k_j,3}}{\xi_{k_j,1}}\psi_{1,k_j} + \frac{1}{\xi_{k_j,1}}\psi_{2,k_j}\right]\mathcal{N}_s\varphi_{k_j}(x) = 0, \tag{4.19}$$

$$\sum_{j=1}^{m_k}\left[\frac{-\lambda_{k_j,1}\lambda_{k_j,3}}{\xi_{k_j,2}}\psi_{0,k_j} + \frac{\lambda_{k_j,1}+\lambda_{k_j,3}}{\xi_{k_j,2}}\psi_{1,k_j} - \frac{1}{\xi_{k_j,2}}\psi_{2,k_j}\right]\mathcal{N}_s\varphi_{k_j}(x) = 0, \tag{4.20}$$

$$\sum_{j=1}^{m_k}\left[\frac{\lambda_{k_j,1}\lambda_{k_j,2}}{\xi_{k_j,3}}\psi_{0,k_j} - \frac{\lambda_{k_j,1}+\lambda_{k_j,2}}{\xi_{k_j,3}}\psi_{1,k_j} + \frac{1}{\xi_{k_j,3}}\psi_{2,k_j}\right]\mathcal{N}_s\varphi_{k_j}(x) = 0. \tag{4.21}$$

Let

$$\psi_k^1 := \sum_{j=1}^{m_k}\left[\frac{\lambda_{k_j,2}\lambda_{k_j,3}}{\xi_{k_j,1}}\psi_{0,k_j} - \frac{\lambda_{k_j,2}+\lambda_{k_j,3}}{\xi_{k_j,1}}\psi_{1,k_j} + \frac{1}{\xi_{k_j,1}}\psi_{2,k_j}\right]\varphi_{k_j},$$

$$\psi_k^2 = \sum_{j=1}^{m_k}\left[\frac{-\lambda_{k_j,1}\lambda_{k_j,3}}{\xi_{k_j,2}}\psi_{0,k_j} + \frac{\lambda_{k_j,1}+\lambda_{k_j,3}}{\xi_{k_j,1}}\psi_{1,k_j} - \frac{1}{\xi_{k_j,2}}\psi_{2,k_j}\right]\varphi_{k_j},$$

$$\psi_k^3 = \sum_{j=1}^{m_k}\left[\frac{\lambda_{k_j,1}\lambda_{k_j,2}}{\xi_{k_j,3}}\psi_{0,k_j} - \frac{\lambda_{k_j,1}+\lambda_{k_j,2}}{\xi_{k_j,3}}\psi_{1,k_j} + \frac{1}{\xi_{k_j,3}}\psi_{2,k_j}\right]\varphi_{k_j}.$$

It follows from (4.19), (4.20) and (4.21) that $\mathcal{N}_s\psi_k^1 = \mathcal{N}_s\psi_k^2 = \mathcal{N}_s\psi_k^3 = 0$ in $\mathcal{O}$. Since $\{\varphi_{k_j}\}_{j\in\mathbb{N}}$ satisfies $\mu_k\varphi_{k_j} = (-\Delta)_D^s\varphi_{k_j}$, for every $j \in \{1, \ldots, m_k\}$, it follows from the definition of ψ_k^l, with $l = 1, 2, 3$, that

$$(-\Delta)^s \psi_k^l = \mu_k \psi_k^l \quad \text{in } \Omega \quad \text{and} \quad \mathcal{N}_s \psi_k^l = 0 \quad \text{in } \mathcal{O},\ l = 1, 2, 3.$$

From Lemma (1), we can deduce that $\psi_k^l = 0$, for every $k \in \mathbb{N}$ and $l = 1, 2, 3$. Since the system $\{\varphi_{k_j}\}_{1 \le j \le m_k}$ is linearly independent in $L^2(\Omega)$, we have that

$$\underbrace{\begin{pmatrix} \dfrac{\lambda_{k,2}\lambda_{k,3}}{\xi_{k,1}} & -\dfrac{\lambda_{k,2}+\lambda_{k,3}}{\xi_{k,1}} & \dfrac{1}{\xi_{k,1}} \\ \dfrac{-\lambda_{k,1}\lambda_{k,3}}{\xi_{k,2}} & \dfrac{\lambda_{k,1}+\lambda_{k,3}}{\xi_{k,2}} & -\dfrac{1}{\xi_{k,2}} \\ \dfrac{\lambda_{k,1}\lambda_{k,2}}{\xi_{k,3}} & -\dfrac{\lambda_{k,1}+\lambda_{k,2}}{\xi_{k,3}} & \dfrac{1}{\xi_{k,3}} \end{pmatrix}}_{A} \begin{pmatrix} \psi_{0,k} \\ \psi_{1,k} \\ \psi_{2,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A simple calculation shows that the determinant of the matrix A is given by

$$\det(A) = \frac{i}{2\,\mathrm{Im}(\lambda_{k,2})\left[(\mathrm{Re}(\lambda_{k,2}) - \lambda_{k,1})^2 + (\mathrm{Im}(\lambda_{k,2}))^2\right]} \neq 0.$$

Since the matrix A is invertible, we can deduce that

$$\psi_{0,k} = \psi_{1,k} = \psi_{2,k} = 0,\ k \in \mathbb{N}.$$

This implies that $\psi_0 = \psi_1 = \psi_2 = 0$ a.e. in Ω. Since the solution $(\psi, \psi_t, \psi_{tt})$ of the adjoint system is unique, it follows that $\psi = 0$ in $\Omega \times (0, T)$. The proof is finished. □

The last main result concerns the approximate controllability of (1.1). This result is a direct consequence of the unique continuation property for the adjoint system (Theorem 8).

Theorem 9 *The system (1.1) is approximately controllable in any time $T > 0$ and $g \in H^1((0, T); L^2(\mathcal{O}))$, where $\mathcal{O} \subset \mathbb{R}^N \setminus \overline{\Omega}$ is an arbitrary nonempty open set. That is,*

$$\overline{\mathcal{R}((0,0,0), T)}^{L^2(\Omega)\times W^{-s,2}(\overline{\Omega})\times W^{-s,2}(\overline{\Omega})} = L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega}).$$

Proof Let (u, u_t, u_{tt}) be the unique weak solution of (1.1) with $u_0 = u_1 = u_2 = 0$. Let $(\psi, \psi_t, \psi_{tt})$ be the unique weak solution of (3.30) with final data $(\psi_0, \psi_1, \psi_2) \in D((-\Delta)_D^s) \times D((-\Delta)_D^s) \times L^2(\Omega) \hookrightarrow W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$. Using (4.1) we can deduce that

$$\begin{aligned}\tau\langle u_{tt}(\cdot, T), \psi_0\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle u_t(\cdot, T), \alpha\psi_0 + \tau\psi_1\rangle_{-\frac{1}{2},\frac{1}{2}} + \Big(u(\cdot, T), \alpha\psi_1 + \tau\psi_2 \\ +b(-\Delta)^s\psi_0\Big)_{L^2(\Omega)} = -\int_0^T\int_{\mathcal{O}}\Big(c^2 g(x,t) + b g_t(x,t)\Big)\mathcal{N}_s\psi(x,t)\,dx\,dt. \quad (4.22)\end{aligned}$$

Since $D((-\Delta)_D^s) \times D((-\Delta)_D^s) \times L^2(\Omega)$ is dense in $W_0^{s,2}(\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega)$, to prove that the set

$$\Big\{(u(\cdot,T), u_t(\cdot,T), u_{tt}(\cdot,T)) : \ g \in H^1((0,T); L^2(\mathcal{O}))\Big\}$$

is dense in $L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega})$, it suffices to show that if $(\psi_0, \psi_1, \psi_2) \in D((-\Delta)_D^s) \times D((-\Delta)_D^s) \times L^2(\Omega)$ is such that

$$\begin{aligned} &\tau\langle u_{tt}(\cdot,T), \psi_0\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle u_t(\cdot,T), \alpha\psi_0 + \tau\psi_1\rangle_{-\frac{1}{2},\frac{1}{2}} \\ &\quad + \Big(u(\cdot,T), \alpha\psi_1 + \tau\psi_2 + b(-\Delta)^s\psi_0\Big)_{L^2(\Omega)} = 0, \end{aligned} \tag{4.23}$$

for every $g \in H^1((0,T); L^2(\mathcal{O}))$, then $\psi_0 = \psi_1 = \psi_2 = 0$.

Indeed, let $(\psi_0, \psi_1, \psi_2) \in D((-\Delta)_D^s) \times D((-\Delta)_D^s) \times L^2(\Omega)$ satisfy (4.23). It follows from (4.22) and (4.23) that

$$\int_0^T \int_{\mathcal{O}} \Big(c^2 g(x,t) + b g_t(x,t)\Big)\mathcal{N}_s\psi(x,t)dxdt = 0,$$

for every $g \in H^1((0,T); L^2(\mathcal{O}))$. Recall that $b > 0$. By the fundamental lemma of the calculus of variations, we can deduce that

$$\mathcal{N}_s\psi = 0 \quad \text{in} \ \ \mathcal{O} \times (0,T).$$

It follows from Theorem 8 that $\psi = 0$ in $\Omega \times (0,T)$. Since the solution $(\psi, \psi_t, \psi_{tt})$ of (3.30) is unique, we can conclude that $\psi_0 = \psi_1 = \psi_2 = 0$ in Ω. The proof is finished. □

We conclude the paper by observing that from the proof of the previous theorem, we can show the equivalence between the approximate controllability of the system and the unique continuation property proved in Theorem 8 as in the classical case of the heat and wave equations.

Remark 7 The system (1.1) is approximately controllable in time $T > 0$ if and only if the solution ψ of the associated adjoint system (3.30) satisfies the following:

$$(\psi \ \ \text{solution of} \ \ (3.30), \ \mathcal{N}_s\psi\big|_{\mathcal{O}\times(0,T)} = 0) \Longrightarrow \ \psi_0 = \psi_1 = \psi_2 = 0 \ \ \text{in} \ \ \Omega.$$

Indeed, consider the following mapping:

$$\begin{aligned} F : H^1((0,T); L^2(\mathcal{O})) &\to L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega}), \\ g &\mapsto \Big(u(\cdot,T), u_t(\cdot,T), u_{tt}(\cdot,T)\Big), \end{aligned}$$

where u is the unique very weak solution of (1.1) with $u_0 = u_1 = u_2 = 0$. Then it is easy to see that the system (1.1) is approximately controllable in time $T > 0$ if and only if the range of F, that is, $\mathrm{Ran}(F)$ is dense in $L^2(\Omega) \times W^{-s,2}(\overline{\Omega}) \times W^{-s,2}(\overline{\Omega})$. This is equivalent to $\mathrm{Ker}(F^\star) = \{(0,0,0)\}$, where $F^\star$ is the adjoint of F. It follows from the proof of Theorem 9 that $F^\star$ is the mapping given by

$$F^\star : W_0^{s,2}\overline{\Omega}) \times W_0^{s,2}(\overline{\Omega}) \times L^2(\Omega) \to L^2((0,T) \times \mathcal{O}),$$
$$(\psi_0, \psi_1, \psi_2) \mapsto \mathcal{N}_s \psi\big|_{\omega \times (0,T)},$$

where ψ is the unique solution of the adjoint system (3.30). Again $\mathrm{Ker}(F^\star) = \{(0,0,0)\}$ is the unique continuation principle, namely,

$$(\psi \;\text{ solution of }\; (3.30),\; \mathcal{N}_s \psi\big|_{\mathcal{O} \times (0,T)} = 0) \Longrightarrow \;\psi_0 = \psi_1 = \psi_2 = 0 \;\text{ in }\; \Omega.$$

The proof is finished.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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