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The semi-discrete diffusion convection equation with decay

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ABSTRACT

In this paper we find the fundamental solution of the semi-discrete diffusion convection equation with decay, and we show that under a certain combination of the parameters of the equation said fundamental solution constitutes a uniformly continuous semigroup of operators in the Lebesgue spaces $\ell^p(\mathbb{Z}), 1 \leq p \leq \infty$.

1. Introduction

In this article we study a semi-discrete version of the one-dimensional equation

$$\begin{cases} u_t(t,x) - \alpha u_{xx}(t,x) + c u_x(t,x) + \lambda u(t,x) = f(t,u(t,x)), & t \geq 0, \quad x \in \mathbb{R}, \\ u(0,x) = \phi(x) & \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

given by

$$\begin{cases} u_t(t,n) - \alpha \Delta_d u(t,n) + c \nabla_d u(t,n) + \lambda u(t,n) = F(t,u(t,n)), & t \geq 0, \quad n \in \mathbb{Z}, \\ u(0,n) = \varphi(n), & \quad n \in \mathbb{Z}. \end{cases} \tag{1.2}$$

Here, $\Delta_d f(n) := f(n+1) - 2f(n) + f(n-1)$ denotes the discrete Laplacian and $\nabla_d f(n) := f(n) - f(n-1)$ is the discrete nabla operator. We assume that $\alpha > 0$ and $c, \lambda \in \mathbb{R}$ are arbitrary real numbers.

For the case $c, \lambda > 0$ Eq. (1.1) is known as the one-dimensional diffusion convection equation with decay [1] and models the transport, dispersion and decay of a chemical solute with concentration $u(t,x)$. In the case $c = 0$, with $f(t,u) = -ru^2$ and $\lambda = -r,\ r > 0$, it is known as the Fisher–KPP equation and was originally studied by R. Fisher [2] in connection with population dynamics. If $c, \lambda < 0$ satisfy $-\lambda < \frac{c^2}{2\alpha} < 1$ and $f \equiv 0$, then the system (1.1) appears as a non-trivial example of PDE whose semigroup associated with the solution is chaotic [3, Example 4.12].

For $c = \lambda = 0$ and $F \equiv 0$ the *semi-discrete diffusion equation* (1.2) has been studied by many authors (see [4] and the references therein) and it is well known, see e.g. [5], that it admits an explicit solution that has the form $u(t,n) = e^{\alpha t \Delta_d} \varphi(n)$ where

$$e^{t\Delta_d} \varphi(n) := \sum_{m \in \mathbb{Z}} e^{-2t} I_{n-m}(2t) \varphi(m), \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{1.3}$$

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and I_k, $k \in \mathbb{Z}$, denotes the modified Bessel function.

We observe that if $\lambda \neq 0$ but $c = 0$, then it is easy to see that the unique solution of (1.2) is given by

$$u(t,n) = e^{-\lambda t} e^{\alpha t \Delta_d} \varphi(n), \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{1.4}$$

because it constitutes a simple translation by λI of the discrete Laplace operator Δ_d.

Explicit solutions for semi-discrete equations are an interesting object of study and have received increasing interest in recent years [6–12]. For instance, Slavik [13] studied the asymptotic behavior of bounded solutions to the one-dimensional diffusion Eq. (1.2) ($\lambda = c = 0, F \equiv 0$) and Lizama and Roncal [4] proved existence and uniqueness results of almost periodic solutions for (1.2) with $c = \lambda = 0$ and, for $\lambda < -4, c = 0$ well-posedness on periodic Hölder spaces [4, Theorem 1.7]. The study of semilinear versions of (1.2) for $c = 0$ has an older date. To name a few, it appears in the semi-discrete version of the Fisher's equation [14] with $\lambda = -r < 0$ and $c = 0$ or the Nagumo equation with $0 < \lambda < 1/2$ and $c = 0$, see [15].

However, in the case $c \neq 0$ the analysis of the semi-discrete diffusion convection equation with decay (1.2) as well as the description of an explicit solution together with a study of its qualitative properties seems to be an open problem.

In this article, we solve this problem by proving that an explicit solution of the homogeneous problem (1.2), that is, $F \equiv 0$, has the explicit form

$$u(t,n) = \sum_{m \in \mathbb{Z}} \left(1 + \frac{c}{\alpha}\right)^{(n-m)/2} e^{-\left(2\alpha + c + \lambda\right)t} I_{n-m}(2t\sqrt{\alpha(\alpha+c)})\varphi(m), \ t \geq 0, \ n \in \mathbb{Z}. \tag{1.5}$$

As an interesting consequence, we note that different combinations of the parameters $\alpha > 0$ and $c, \lambda \in \mathbb{R}$ provide new insights into the qualitative and dynamic behavior of the solution. In particular, we find a set of parameters where we have stability (in time t) given by

$$\Omega_\alpha := \left\{(c,\lambda) \ : \ \alpha + c > 0, \ 2\alpha + c + \lambda > 0, \ c^2 + 2c\lambda + \lambda^2 + 4\alpha\lambda > 0\right\}, \quad \alpha > 0.$$

We also find that under the conditions $\alpha + c > 0$ and $\lambda \geq 0$ the formula (1.5) constitutes a generalization of the semigroup (1.3), and has $\alpha\Delta_d - c\nabla_d - \lambda I$ as its infinitesimal generator.

It should be noted that Slavík and Stehlík [16] already investigated an explicit solution form for the linear counterpart of Eq. (1.2) in the case of general time scales. In this reference, continuous time is considered as a special case, and in this special case (and with a unitary initial condition) the form of the explicit solution is given, and under conditions similar as those given in the set Ω_α, some properties have been obtained.

2. Main results

By a solution of (1.2) we understand the existence of a sequence $u(t,n)$ that satisfies (1.2). Our first main result in this article is the following theorem.

Theorem 2.1. *A solution of the semi-discrete diffusion convection equation with decay*

$$\begin{cases} u_t(t,n) = \alpha\Delta_d u(t,n) - c\nabla_d u(t,n) - \lambda u(t,n), \quad t \geq 0, \ n \in \mathbb{Z}; \\ u(0,n) = \varphi(n), \quad n \in \mathbb{Z}, \end{cases} \tag{2.1}$$

has the form

$$u(t,n) = w(t,n)\left(1 + \frac{c}{\alpha}\right)^{n/2} e^{-\left(2\alpha + c + \lambda - 2\sqrt{\alpha(\alpha+c)}\right)t} \tag{2.2}$$

where $w(t,n)$ is a solution of the diffusion equation

$$\begin{cases} w_t(t,n) = \sqrt{\alpha(\alpha+c)}\Delta_d w(t,n), \quad t \geq 0, \ n \in \mathbb{Z}; \\ w(0,n) = \left(1 + \frac{c}{\alpha}\right)^{-n/2} \varphi(n), \quad n \in \mathbb{Z}. \end{cases} \tag{2.3}$$

Proof. Define $X(t) := e^{-\gamma t}$ where $\gamma := 2\alpha + c + \lambda - 2\sqrt{\alpha(\alpha+c)}$, and $Y(n) := \mu^n$ where $\mu := \sqrt{\frac{\alpha+c}{\alpha}}$. Then, from (2.2) we obtain that

$$u_t(t,n) = w_t(t,n)Y(n)X(t) - \gamma w(t,n)Y(n)X(t). \tag{2.4}$$

Now, it is easy to see that given two arbitrary sequences $f(n)$ and $g(n)$, the following identities hold:

$$\nabla_d(fg)(n) = \nabla_d f(n)g(n) + f(n-1)\nabla_d g(n),$$

and

$$\Delta_d(fg)(n) = (\Delta_d f)(n)g(n+1) + 2(\nabla_d f)(n)(\nabla_d g)(n+1) + f(n-1)(\Delta_d g)(n).$$

Therefore, we have that

$$\Delta_d u(t,n) = X(t)\Big[\Delta_d w(t,n)Y(n+1) + 2\nabla_d w(t,n)\nabla_d Y(n+1) + w(t,n-1)\Delta_d Y(n)\Big] \tag{2.5}$$

and

$$\nabla_d u(t,n) = X(t)\Big[\nabla w_d(t,n)Y(n) + w(t,n-1)\nabla_d Y(n)\Big] \tag{2.6}$$

where

$$\Delta_d Y(n) = Y(n-1)(\mu-1)^2 \text{ and } \nabla_d Y(n) = Y(n-1)(\mu-1). \tag{2.7}$$

Replacing (2.7) into (2.5) and (2.6), and using (2.4), we obtain that

$$\begin{aligned}
&u_t(t,n) - \alpha\Delta_d u(t,n) + c\nabla_d u(t,n) + \lambda u(t,n)\\
&\quad = X(t)\Big[w_t(t,n)Y(n) - \gamma w(t,n)Y(n) - \alpha\Delta_d w(t,n)Y(n+1)\\
&\qquad - 2\alpha\nabla_d w(t,n)Y(n)(\mu-1) - \alpha w(t,n-1)Y(n-1)(\mu-1)^2 + c\nabla_d w(t,n)Y(n) + cw(t,n-1)Y(n-1)(\mu-1) + \lambda w(t,n)Y(n)\Big]\\
&\quad = X(t)Y(n)\Big[w_t(t,n) - \alpha\mu\Delta_d w(t,n) - w(t,n)[\gamma + 2\alpha(\mu-1) - c - \lambda] + w(t,n-1)[2\alpha(\mu-1) - \alpha\mu^{-1}(\mu-1)^2 - c + c\mu^{-1}(\mu-1)]\Big]
\end{aligned}$$

where using (2.3) and the identity $\alpha\mu = \sqrt{\alpha(\alpha+c)}$ we obtain that

$$w_t(t,n) - \alpha\mu\Delta_d w(t,n) = 0, \qquad \gamma + 2\alpha(\mu-1) - c - \lambda = 0,$$

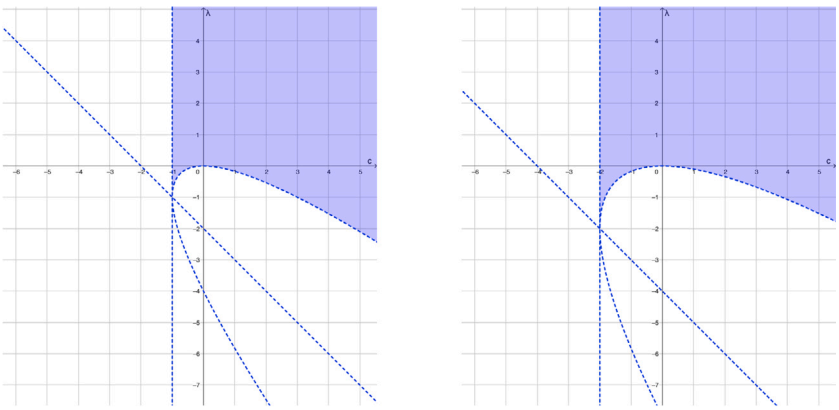
and

$$2\alpha(\mu-1) - \alpha\mu^{-1}(\mu-1)^2 - c + c\mu^{-1}(\mu-1) = \frac{1}{\mu}[2\alpha\mu(\mu-1) - \alpha(\mu-1)^2 - c] = \frac{\alpha}{\mu}[\mu^2 - \frac{\alpha+c}{\alpha}] = 0,$$

proving the first part of (2.1). Finally, setting $t = 0$ in (2.2) we obtain by (2.3) that $u(0,n) = \varphi(n)$. It proves the theorem. □

Remark 2.2. Note that $2\alpha + c + \lambda - 2\sqrt{\alpha(\alpha+c)} > 0$ if and only if the pair (c,λ) belongs to the set $\Omega_\alpha := \Big\{(c,\lambda) \;:\; \alpha + c > 0,\ 2\alpha + c + \lambda > 0,\ c^2 + 2c\lambda + \lambda^2 + 4\alpha\lambda > 0\Big\}$.

This set is represented by the shadow sector below



(A) Set Ω_1. (B) Set Ω_2.

Substituting the solution of (2.3) given by means of the formula (1.3) into (2.2) we obtain the following explicit formula for the solution of (2.1)

$$u(t,n) = \sum_{m\in\mathbb{Z}} \left(1 + \frac{c}{\alpha}\right)^{(n-m)/2} e^{-\left(2\alpha+c+\lambda\right)t} I_{n-m}(2t\sqrt{\alpha(\alpha+c)})\varphi(m),\ t \geq 0,\ n \in \mathbb{Z}. \tag{2.8}$$

Define

$$S_t(n) := e^{-\left(2\alpha+c+\lambda-2\sqrt{\alpha(\alpha+c)}\right)t} T_t(n), \quad t \geq 0, \quad n \in \mathbb{Z}, \tag{2.9}$$

where

$$T_t(n) := e^{-2t\sqrt{\alpha(\alpha+c)}} I_n(2t\sqrt{\alpha(\alpha+c)}). \tag{2.10}$$

Given $\varphi \in \ell^p(\mathbb{Z})$ where $1 \le p \le \infty$, by Young's convolution inequality we know that $\psi * \varphi \in \ell^p(\mathbb{Z})$ whenever $\psi \in \ell^1(\mathbb{Z})$. Using [17, Formula 5.8.3 (2)] we obtain for $\mu := (1+\frac{c}{\alpha})^{1/2} > 0$ the following identity

$$\sum_{m\in\mathbb{Z}} \mu^m I_m(2t\sqrt{\alpha(\alpha+c)}) = e^{(2\alpha+c)t}, \tag{2.11}$$

where $I_k(t) \ge 0$. Hence, the sequence $\mu^n S_t(n)$ belongs to $\ell^1(\mathbb{Z})$ and therefore, for any $t \ge 0$, we can define a family of operators $\mathcal{S}_t : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ by

$$\mathcal{S}_t\varphi(n) := \sum_{m\in\mathbb{Z}} \mu^{n-m} S_t(n-m)\varphi(m), \quad n \in \mathbb{Z}. \tag{2.12}$$

We recall from [4, Theorem 1.1] that in the case $\lambda = c = 0$ the family $\{\mathcal{S}_t\}_{t\ge 0}$ is a C_0-semigroup (in fact, an analytic semigroup) on $\ell_p(\mathbb{Z})$ for all $1 \le p \le \infty$ with bounded generator $\alpha\Delta_d$. We observe that in such a case the one parameter family $\{\mathcal{S}_t\}_{t\ge 0}$ is even a uniformly continuous semigroup (see e.g., [18]). We broadly generalize this classical result in the following theorem.

Theorem 2.3. *Suppose that*

$$\alpha + c > 0 \quad \text{and} \quad \lambda \ge 0. \tag{2.13}$$

Then $\{\mathcal{S}_t\}_{t\ge 0}$ defined in (2.12) is a uniformly continuous semigroup on $\ell_p(\mathbb{Z}), 1 \le p \le \infty$, with bounded generator $A := \alpha\Delta_d - c\nabla_d - \lambda I$

Proof. First, we observe that $I_0(0) = 1$ and $I_k(0) = 0$ for $k \ne 0$ proving that $\mathcal{S}_0\varphi = \varphi$. Second, from the identity

$$I_n(t+s) = \sum_{\ell\in\mathbb{Z}} I_{n-\ell}(t)I_\ell(s), \quad n \in \mathbb{Z}, \ t, s \in \mathbb{R},$$

it follows that

$$\begin{aligned}
\mathcal{S}_t(\mathcal{S}_s\varphi)(n) &= \sum_{m\in\mathbb{Z}} \mu^{n-m} S_t(n-m)(\mathcal{S}_s\varphi)(m) = \sum_{m\in\mathbb{Z}} \mu^{n-m} S_t(n-m) \sum_{k\in\mathbb{Z}} \mu^{m-k} S_s(m-k)\varphi(k) = \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \mu^{n-k} S_t(n-m)S_s(m-k)\varphi(k) \\
&= \sum_{k\in\mathbb{Z}} \mu^{n-k}\Big(\sum_{m\in\mathbb{Z}} S_t(n-m)S_s(m-k)\Big)\varphi(k) = \sum_{k\in\mathbb{Z}} \mu^{n-k}\Big(\sum_{m\in\mathbb{Z}} S_t((n-k)-(m-k))S_s(m-k)\Big)\varphi(k) \\
&= \sum_{k\in\mathbb{Z}} \mu^{n-k} S_{t+s}(n-k)\varphi(k) = (\mathcal{S}_{t+s}\varphi)(n).
\end{aligned}$$

It shows that the family $\{\mathcal{S}_t\}_{t\ge 0}$ is a semigroup (in fact, a group). We next show the uniform continuity. Note that $S_t(m) \ge 0$ for each $m \in \mathbb{Z}$, and $\mu > 0$ by the hypothesis (2.13). Moreover, the following identity

$$\sum_{m\in\mathbb{Z}} \mu^m S_t(m) = \sum_{m\in\mathbb{Z}} \mu^m e^{-(2\alpha+c+\lambda)t} I_m(2t\sqrt{\alpha(\alpha+c)}) = e^{-\lambda t} \le 1, \tag{2.14}$$

holds (see (2.11)). In particular, $1 - S_t(0) \ge \sum_{n\in\mathbb{Z}\setminus\{0\}} \mu^n S_t(n) \ge 0$. Also, observe that for any $\varphi \in \ell^p(\mathbb{Z})$,

$$\mathcal{S}_t\varphi(n) - \varphi(n) = \sum_{m\in\mathbb{Z}} \mu^{n-m}[S_t(n-m) - S_0(n-m)]\varphi(m) = (K_t * \varphi)(n)$$

with $K_t(j) := \mu^j[S_t(j) - S_0(j)]$, $j \in \mathbb{Z}$. Then, by Young's convolution inequality we obtain

$$\|\mathcal{S}_t\varphi - \varphi\|_p \le \|K_t\|_1\|\varphi\|_p, \quad 1 \le p \le \infty, \tag{2.15}$$

where, using (2.14), we obtain

$$\|K_t\|_1 = \sum_{n\in\mathbb{Z}} |\mu^n(S_t(n) - S_0(n))| = \sum_{n\in\mathbb{Z}\setminus\{0\}} |\mu^n S_t(n)| + |S_t(0) - 1| \le 1 - S_t(0) + |S_t(0) - 1| = 2(1 - S_t(0)).$$

Therefore, for any $\varphi \in \ell^p(\mathbb{Z})$ satisfying $\|\varphi\|_p \le 1$ we obtain from (2.15) that

$$\|\mathcal{S}_t - I\| \le 2(1 - S_t(0)) = 2\Big(1 - e^{-(2\alpha+c+\lambda)t} I_0(2t\sqrt{\alpha(\alpha+c)})\Big) \to 0 \text{ as } t \to 0, \tag{2.16}$$

proving the claim.

Finally, we show that the generator of $\{\mathcal{S}_t\}_{t\ge 0}$ is A. Indeed, from the identity $I_k'(z) = \frac{1}{2}[I_{k+1}(z) + I_{k-1}(z)]$ combined with (2.9) and (2.10) it follows that

$$S_t'(k) = -\gamma e^{-\gamma t} T_t(k) + e^{-\gamma t}\sqrt{\alpha(\alpha+c)}\Big[I_{k+1}(2t\sqrt{\alpha(\alpha+c)}) - 2I_k(2t\sqrt{\alpha(\alpha+c)}) + I_{k-1}(2t\sqrt{\alpha(\alpha+c)})\Big],$$

where $\gamma := 2\alpha + c + \lambda - 2\sqrt{\alpha(\alpha+c)}$. Hence, using the property that $I_k(0) = \delta_0(k)$, the Kronecker delta, and recalling that $\mu = (1+\frac{c}{\alpha})^{1/2}$ we obtain for each $\varphi \in \ell^p(\mathbb{Z})$

$$\begin{aligned}
A\varphi(n) = \mathcal{S}_t'\varphi(n)|_{t=0} &= \sum_{m\in\mathbb{Z}} \mu^{n-m} S_t'(n-m)|_{t=0}\varphi(m) \\
&= \sum_{m\in\mathbb{Z}} \mu^{n-m}\Big[-\gamma\delta_0(n-m) + \sqrt{\alpha(\alpha+c)}\Big(\delta_0(n-m+1) - 2\delta_0(n-m) + \delta_0(n-m-1)\Big)\Big]\varphi(m)
\end{aligned}$$

$$= -\gamma\varphi(n) + \frac{\sqrt{\alpha(\alpha+c)}}{\mu}\varphi(n+1) - 2\sqrt{\alpha(\alpha+c)}\varphi(n) + \mu\sqrt{\alpha(\alpha+c)}\varphi(n-1)$$
$$= -(2\alpha+c+\lambda)\varphi(n) + \alpha\varphi(n+1) + (\alpha+c)\varphi(n-1) = \alpha\Delta_d\varphi(n) - c\nabla_d\varphi(n) - \lambda\varphi(n).$$

This completes the proof. □

From semigroups theory, we immediately obtain the following corollary that gives more precise information about the regularity of the solutions in the discrete variable.

Corollary 2.4. *Under the assumption (2.13), for every $\varphi \in \ell_p(\mathbb{Z})$, $1 \le p \le \infty$, there exists a unique solution $u(t,\cdot) \in \ell_p(\mathbb{Z})$ of (2.1) given by $u(t,n) = \mathcal{S}_t\varphi(n)$, $t \ge 0$, $n \in \mathbb{Z}$.*

Combining Remark 2.2 with [5, Proposition 2] we obtain the following result that gives an interesting information about the regularity of the kernel of the semigroup $\{\mathcal{S}_t\}_{t\ge 0}$ or, in other words, the discrete Green function $\mathcal{S}_t(n)$ defined in (2.9).

Theorem 2.5. *For any pair $(c,\lambda) \in \overline{\Omega}_\alpha$ with $\lambda \ge 0$ and $m \in \mathbb{Z}$, the following estimates hold:*

$$\|\mathcal{S}_\bullet(m)\|_{L^\infty(0,\infty)} \le \frac{C_1}{|m|+1} \quad \text{and} \quad \|\nabla_d\mathcal{S}_\bullet(m)\|_{L^\infty(0,\infty)} \le \frac{C_2}{|m|^2+1},$$

where C_1 and C_2 are positive constants independent of $m \in \mathbb{Z}$.

Remark 2.6. From semigroups theory, it follows that under the hypothesis (2.13), Theorem 2.3 allows finding the solution of (1.2) as a fixed point in $\ell^p(\mathbb{Z})$ of the mapping

$$\mathcal{K}(u)(t) := \mathcal{S}_t\varphi + \int_0^t \mathcal{S}_{t-s}\mathcal{F}_s(u)ds,$$

where $\mathcal{F}_s(u)(n) := F(s,u(s,n))$.

Data availability

No data was used for the research described in the article.

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