On a class of Non Markovian Langevin Equations*

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(Received)

Abstract. This paper proposes a generalized Langevin's equation for a small classical mechanical system embedded in a reservoir. The interaction of the main system with the reservoir is given by a *gaussian transform* as introduced in our previous paper [8]. Thus, a first result proves the existence of a strong solution to this equation in the space where the gaussian transform (or non Markovian noise) is defined. The interpretation of the noise is obtained by considering a finite number n of oscillating particles with discrete frequencies in the reservoir. The action of this discrete reservoir on the small system is described by a memory kernel and a sequence of zero-mean gaussian processes. So, an integro-differential equation for the evolution of a generic particle in the main system arises for each n. This equation has a unique solution X_n which converges in distribution towards the solution of the initial non-Markovian Langevin's equation.

1. Introduction

In 1908, Paul Langevin [7] proposed a successful description of Brownian motion, inspired by the previous work of Einstein published in 1905. His approach applied Newton's second law to a representative Brownian particle. This was considered as a more friendly set up than the one proposed by Einstein, who derived and solved a Fokker-Planck partial differential equation. However, Langevin was conscious that his equation introduced a number of non well defined mathematical objects like "white noise" together with its unusual properties. He arranged to handle that cautiously and intuitively, so opening a vast terrain of research to both, Physics and

*Research supported by FONDECYT grants 1100485 and 1120063, and CONICYT-PIA ACT1112 *Stochastic Analysis Research Network*

Mathematics.

Since then, the equation bearing his name has appeared as a guesstimate in a number of branches of Physics, Chemistry, Engineering and many other sciences (see eg. [3] and references therein, as well as [4], [14], [13]). It is perhaps a prototype of a cross-disciplinary paradigm of dynamics.

A crucial assumption in Langevin's derivation of his equation was the absence of memory in the dynamics. It is well-known that the mathematical model of a Brownian motion enjoys two main properties: it is both, a Gaussian and Markovian stochastic process. However, a number of open systems include memory effects in the main dynamics, which determine a more complex interaction with the heath bath or environment. Therefore, a number of authors have been studying new versions of Langevin's approach to capture different types of memory effects. For instance, fractional noises (or dynamics) appears in [1], [6], [9], [10]. Other researches have focused on generalized versions of the main dynamics of the open system ([2], [5], [11]).

Within this paper, we introduce first a general nonlinear dynamics in the main system, adding a memory kernel which determines uniquely a non-Markovian interaction of it with the environment. The environment action is described via a class of Gaussian non-Markovian noises previously studied in [8].

The paper is organized as follows. The next section introduces notations and states the class of Langevin's equations we study. In section 3, we recall the main properties of introduced noises and prove the existence and uniqueness of solutions to our equation. Section 4 is devoted to approach the solution by a family of processes indexed by discrete frequencies. The final section shows some applications.

2. Preliminaries

Consider a closed 1-dimensional mechanical system, and use the customary notations for the state variables of the system: the position q, and the momentum p. So that, the space of states $x = (q, p)$ is supposed to be a Borel set Σ of $\mathbb{R}^2$. Moreover, in $\mathbb{R}^2$ we denote $(e_i)_{i\in\{1,2\}}$ the canonical basis and $E_q : \mathbb{R}^2 \to \mathbb{R}$ (resp. $E_p : \mathbb{R}^2 \to \mathbb{R}$) the canonical projection $(x_1, x_2) \mapsto (x_1, 0)$, (resp. $(x_1, x_2) \mapsto (0, x_2)$).

Let be given an autonomous Hamiltonian $H : \Sigma \to \mathbb{R}$ which defines the dynamics of the main system through the unperturbed equation of motion in Hamilton's version of Newtonian Mechanics:

$$dx(t) = J\nabla H(x(t))dt,\ x(0) = x_0, \tag{1}$$

where J is the symplectic matrix

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \tag{2}$$

The above Hamiltonian is supposed to be a C^1–function of the form

$$H(q,p) = \frac{1}{2m}p^2 + V(q), \tag{3}$$

where, V is the potential energy.

Equation (1) is a short way of writing the integral equation which is its rigorous interpretation:

$$x(t) = x_0 + \int_0^t J\nabla H(x(t))ds. \tag{4}$$

This system is immersed in a reservoir or heat bath composed of a great number of harmonic oscillators which collide with the main system, where the frequencies vary over the positive real line. The open system dynamics requires the construction of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which has to satisfy the following conditions:

(A1) The main dynamics is characterized by a stochastic evolution equation for the process $X(\omega,t) = (Q(\omega,t), P(\omega,t))$, where $Q(\omega,t)$ (resp. $P(\omega,t)$) denotes the position (resp. the momentum) of a particle following the trajectory $\omega \in \Omega$ at time $t \geq 0$.

(A2) The variations on the momentum of the main system are described by a memory kernel and a random force. The latter depends on the choice of initial conditions of each harmonic oscillator.

(A3) The random force will be assumed to be Gaussian, but not Markovian.

A continuous positive and convex function $v \in L^1(\mathbb{R}^+)$ is given to characterize both, the memory kernel and the random force. Indeed, as it was proved in [8], a Gaussian Transform is associated to the function v. This corresponds to a Gaussian process $(\mathbf{B}_v(\omega,t);\ t \geq 0)$ with covariance

$$K(t,s) = R(t) + R(s) - R(t-s),\ (t,s \geq 0), \tag{5}$$

where

$$R(t) = \int_0^t (t-u)v(u)du,\ (t \geq 0). \tag{6}$$

Define $\phi(t) = \int_0^t v(s)ds$, for all $t \geq 0$. Then (6) is equivalent to

$$R(t) = \int_0^t \phi(s)ds.$$

The evolution equation with which we will be concerned within this paper can be stated in the following differential form:

$$\begin{cases} dQ(\omega,t) = P(\omega,t)dt, \\ dP(\omega,t) = -\left(\partial_q V(Q(\omega,t)) + (v \star P(\omega,\cdot))(t)\right)dt + d\mathbf{B}_v(\omega,t). \end{cases} \tag{7}$$

Or, equivalently, in integral form:

$$\left\{\begin{array}{l} Q(\omega,t) = Q(\omega,0) + \int_0^t P(\omega,s)ds, \\ P(\omega,t) = P(\omega,0) - \int_0^t \left(\partial_q V(Q(\omega,s)) + (\phi \star P(\omega,\cdot))(s)\right) ds + \mathbf{B}_v(\omega,t), \end{array}\right. \tag{8}$$

Or, in a more condensed form:

$$X(\omega,t) = X(\omega,0) + \int_0^t \left(J\nabla H(X(\omega,s)) - \phi \star |e_2\rangle\langle e_2|X(\omega,s)\right) ds + \mathbf{B}_v(\omega,t)e_2, \tag{9}$$

where $|e_2\rangle\langle e_2|$ is the projection on the momentum component:

$$|e_2\rangle\langle e_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the equations before we used the customary symbol $\star$ for the convolution product of two functions. To alleviate notations, the reference to $\omega \in \Omega$ in stochastic processes will be omitted in the sequel, unless there is a risk of confusion.

This paper is aimed at constructing a solution to equation (8) and investigating its main properties. In section 2 we recall the Gaussian Transform $\mathbf{B}_v$ constructed in [8] and construct a trajectorial solution to (8) on a probability space supporting $\mathbf{B}_v$. In section 3, we provide a method for numerically solve the above stochastic differential equation. That method is based on the approximation of $\mathbf{B}_v$, as $h \to 0$, by a family $\mathbf{B}_v^h(\omega,\nu,t)$ of a two-parameter gaussian processes. We end the paper in section 4 by considering a number of examples.

3. Solving the Langevin equation

We start by recalling the key features of $\mathbf{B}_v$.

Since v is a continuous, positive, convex integrable function on the real line, it is the cosine transform of a positive function which we denote $f^2(x)$ (cf. [15]). That is,

$$f(x) = \sqrt{\frac{2}{\pi}\int_0^\infty v(t)\cos(xt)dt}, \tag{10}$$

f is then continuous, square integrable, and

$$v(t) = \int_0^\infty f^2(x)\cos(xt)dx,\ (t \geq 0). \tag{11}$$

It is important to notice that the hypothesis on integrability of v on $[0,\infty[$ can be relaxed to local integrability in some applications, if in addition the positive integral in (10) is finite.

We keep the above notations throughout the remains of the paper. The two following results have been proved in [8].

Theorem 1 *The variance of the gaussian transform is given by*

$$\mathbb{E}\left(|\mathbf{B}_v(t) - \mathbf{B}_v(s)|^2\right) = 2R(|t-s|).$$

For each fixed $T > 0$, the Gaussian transform process $(\mathbf{B}_v(t);\ t \in [0,T])$ admits a uniformly continuous version and its modulus of continuity is $w(\mathbf{B}_v, u) \leq C\sqrt{\log\left(\frac{1}{u}\right) R(u)}$, $(0 < u < 1)$, where $C > 0$ is a constant.

Moreover, $\mathbf{B}_v$ has a representation as a stochastic integral on the frequency domain. Consider two independent Brownian motions $W_1 = (W_1(x);\ x \geq 0)$, $W_2 = (W_2(x);\ x \geq 0)$ defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_x)_{x\in\mathbb{R}^+}, \mathbb{P})$, be given.

Notice that the functions $x \mapsto \psi_1(x,t) = \left(\frac{f(x)}{x}\right)(1-\cos(xt))$ and $x \mapsto \psi_2(x,t) = \left(\frac{f(x)}{x}\right)\sin(xt)$ are in $L^2([0,\infty[)$, they are deterministic, so trivially predictable, therefore, they can be integrated with respect to W_1 and W_2 respectively. In this way we obtain a process $\mathbf{B}_v(x,t)$, where x is frequency variable, and t is in the time domain, given by

$$\mathbf{B}_v(x,t) = \int_0^x \psi_1(y,t)dW_1(y) + \int_0^x \psi_2(y,t)dW_2(y), \tag{12}$$

$(x,t) \in \mathbb{R}^+ \times [0,T]$.

Theorem 2 *The process defined by (12) is a centered Gaussian process with trajectories in the space $C(\mathbb{R}^+ \times [0,T])$ and covariance given by*

$$\mathbf{K}((y,s),(x,t)) = \int_0^{x\wedge y} \left(\frac{f(r)}{r}\right)^2 [(1-\cos(rt))(1-\cos(rs)) + \sin(rt)\sin(rs)]\, dr. \tag{13}$$

Moreover, the limit $\mathbf{B}_v(x,t)$ converges in L^2 to a limit $\mathbf{B}_v(\infty,t)$, and this convergence is uniform in $t \in [0,T]$. The distribution of the process $(\mathbf{B}_v(\infty,t);\ t \in [0,T])$ coincides with that of $(\mathbf{B}_v(t);\ t \in [0,T])$.

From now on, we assume at the outset a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbf{B}_v$ is defined. We will prove the existence of solutions $X(\omega,t)$ $(\omega \in \Omega,\ t \in [0,T])$ to (8), where $T > 0$ is an arbitrary fixed time.

Theorem 3 *Suppose that the Hamiltonian H is of class C^2 with bounded Hessian, and that $v \in L^1(\mathbb{R}^+)$ is positive, continuous and convex.*

Let $x \in \mathbb{R}^2$. Then there exists a unique stochastic process $X = (X(t))_{t\in[0,T]}$ which satisfies (8), with $X(0) = x$, almost surely. Moreover, X has continuous trajectories almost surely.

Proof. Given a vector $x \in \mathbb{R}^{2d}$, we construct a Picard's style sequence as follows:

$$\begin{aligned} X^0(t) &= x \qquad (14)\\ X^{n+1}(t) &= x + \int_0^t J\nabla H(X^n(s))ds - \int_0^t \phi(t-s)|e_2\rangle\langle e_2|X^n(s)ds + \mathbf{B}_v(t)e_2, \end{aligned}$$

for all $n \in \mathbb{N}$ and all $t \in [0,T]$.

Then, for all $t \in [0,T]$,

$$\begin{aligned} \mathbb{E}\left(\sup_{t\le T}\left\|X^1(t)-X^0(t)\right\|^2\right) &\le 3\sup_{t\le T}\left\{\left\|\int_0^t J\nabla H(x)ds\right\|^2 + \left(\int_0^t \phi(t-s)ds\right)^2\|x\|^2 + \mathbb{E}\left(\mathbf{B}_v^2(t)\right)\right\}\\ &\le 3\left\{T\int_0^T \|J\nabla H(x)\|^2 ds + R(T)^2\|x\|^2 + 2R(T)\right\}\\ &\le 3\left\{T^2L^2\|x\|^2 + R(T)^2\|x\|^2 + 2R(T)\right\}\\ &\le 3\left\{T^2(L^2+K^2)\|x\|^2 + 2KT\right\}\\ &\le 3M^2\|x\|^2T^2 + 6KT, \end{aligned}$$

where L is a Lipschitz constant for ∇H, $K = \sup_{0<t\le T}\phi(t)$ and $M^2 = L^2 + K^2$.

Denote $B := 3M^2\|x\|^2T^2 + 6KT$ and $\Delta_n(t) = \sup_{s\le t}\|X^n(s) - X^{n-1}(s)\|^2$. An elementary computation shows that

$$\begin{aligned} \left\|X^{n+1}(t)-X^n(t)\right\|^2 &\le 2\left(\int_0^t L\left\|X^n(s)-X^{n-1}(s)\right\|ds\right)^2\\ &+ 2\left\|\int_0^t \phi(t-s)|e_2\rangle\langle e_2|\left(X^n(s)-X^{n-1}(s)\right)ds\right\|^2\\ &\le 2\left\{L^2t\int_0^t \Delta_n(s)ds + K^2t\int_0^t \Delta_n(s)ds\right\}, \end{aligned}$$

so that

$$\mathbb{E}\left(\Delta_{n+1}(t)\right) \le 2M^2T\int_0^t \mathbb{E}\left(\Delta_n(s)\right)ds, \qquad (15)$$

for all $0 \le t \le T$. Finally,

$$\mathbb{E}\left(\Delta_n(t)\right) \le B\frac{(2M^2T)^{n-1}t^{n-1}}{(n-1)!},\ (n\ge 1), \qquad (16)$$

and

$$\sum_{n=1}^{\infty}\mathbb{E}\left(\sup_{t\le T}\left\|X^n(t)-X^{n-1}(t)\right\|^2\right) < \infty, \qquad (17)$$

so that $(X^n)_{n\in\mathbb{N}}$ converges in L^2 uniformly in $t \in [0,T]$ for any $T > 0$. In particular one obtains that for each $T > 0$,

$$\sum_{n=1}^{\infty} \sup_{t\in[0,T]} \left\| X^n(t) - X^{n-1}(t) \right\| < \infty, \tag{18}$$

almost surely. Thus, the sequence $(X^n)_{n\in\mathbb{N}}$ converges uniformly on each compact subinterval of $[0,\infty[$ almost surely, and since each term of the sequence is continuous with probability one, so is the limit.

As a result, the limit X satisfies equation (8).

To prove uniqueness, suppose be given two solutions X and Y. Then, for all $T > 0$, and any $t \in [0,T]$

$$\|X(t) - Y(t)\| \leq \|X(0) - Y(0)\| + (L+K) \int_0^t \|X(s) - Y(s)\| \, ds. \tag{19}$$

So that, uniqueness follows by a straightforward application of Gronwall's inequality to $u(t) = \|X(t) - Y(t)\|$:

$$0 \leq u(t) \leq \|X(0) - Y(0)\| \, e^{(L+K)t}, \; (0 \leq t \leq T)$$

That is $X(0) = Y(0)$ implies $X(t) = Y(t)$ for all $t \in [0,T]$ almost surely. Since $T > 0$ is arbitrary and both processes have continuous trajectories almost surely, the conclusion follows. □

4. Frequency-time processes and related equations

An approach to the solution of (8) is proposed within this section. This approach is based on the representation (12) of $\mathbf{B}_v$. Indeed, (12) allows to introduce a discretization of Wiener processes on the frequency domain as well as an interpretation of the action of the reservoir on the main system. The method will consists in introducing processes indexed by both, time and frequency variables.

To start with, consider $\nu > h > 0$ and frequencies varying on the set

$$\{kh : \; k = 1, 2, \ldots, [\nu/h]\} .$$

Let be given a convex positive function $v \in L^1([0,\infty[)$, define $f \in L^2([0,\infty[)$ through (10), and

$$v^h(\nu, t) = \sum_{k=1}^{[\nu/h]} f(kh)^2 \cos(kht)h. \tag{20}$$

This family of functions will be used to represent a combination of oscillatory friction forces acting on a given particle.

If $h \to 0$ one gets a function $v(\nu,t) = \int_0^\infty 1_{[0,\nu]}(x) f^2(x)\cos(xt)dx$, and $v(t)$ is recovered as $v(t) = \lim_{\nu\to\infty} v(\nu,t)$ by a simple application of the Dominated Convergence Theorem.

The discretized version of (8) to be considered is

$$\begin{cases} Q_t(\omega) = q_0 + \int_0^t P_s(\omega)ds \\ P_t(\omega) = p_0 - \int_0^t \nabla V(Q_s(\omega))ds - \int_0^t \phi^h(\nu, t-s)P_s ds + \mathbf{B}_v^h(\nu,t). \end{cases} \tag{21}$$

Here $\phi^h(\nu,t)$ is given by

$$\phi^h(\nu,t) = \sum_{k=1}^{[\nu/h]} \frac{f(kh)^2}{kh}\sin(kht)h, \tag{22}$$

and this family of functions converges to

$$\phi(\nu,t) = \int_0^\nu \frac{f(x)^2}{x}\sin(xt)dx = \int_0^t v(\nu,s)ds,$$

which is a continuous function in both variables and a simple application of Lebesgue's Dominated Convergence Theorem yields

$$\int_0^\infty \left|\phi^h(\nu,t) - \phi(\nu,t)\right| dt \to 0,$$

as $h \to 0$.

Furthermore, the stochastic process $\mathbf{B}_v^h(\nu,\cdot)$, corresponds to the discrete Gaussian transform ([8], section 5),

$$\mathbf{B}_v^h(\nu,t) = \sqrt{h}\sum_{k=1}^{[\nu/h]} \frac{f(kh)}{kh}\left(\sin(kht)\xi_k + (1-\cos(kht))\zeta_k\right), \tag{23}$$

where $f \in L^2(\mathbb{R}^+)$ is the positive continuous function given by (10), $(\xi_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ are two mutually independent sequences of identically distributed real random variables. The law of each variable ξ_n and ζ_n is normal with mean 0 and variance 1.

Notice that $\mathbf{B}_v^h(\nu,t)$ is the integral of the process

$$b^h(\nu,t) = \sum_{j=1}^{[\nu/h]} \sqrt{h} f(jh)\left(\cos(jht)\xi_j + \sin(jht)\zeta_j\right), \tag{24}$$

which can be interpreted as an oscillatory random force exerted on the main system by the reservoir composed by $[\nu/h]$ harmonic oscillators (see for instance [6]). That is, to describe the action of the reservoir on the main system one supposes here that the interaction is produced at discrete frequencies jh, where $j \in \mathbb{N}$ and $h > 0$.

Let us describe the construction of a canonical probability space where the discrete Langevin equation (21) makes sense.

One considers first the space $G = \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ of all sequences $g = (g_n)_{n\in\mathbb{N}}$ which have values in $\mathbb{R} \times \mathbb{R}$, so that $g_n = (g_{n,1}, g_{n,2})$. Let denote $\xi_n(g) = g_{n,1} \in \mathbb{R}$ and $\zeta_n(g) = g_{n,2} \in \mathbb{R}$, $n \in \mathbb{N}$. By means of Kolmogorov Extension Theorem, one can built up a probability measure P_G on $(G, \mathcal{G})$, where $\mathcal{G}$ is the product of Borel σ–fields $\mathcal{B}(\mathbb{R})^{\otimes\mathbb{N}} \otimes \mathcal{B}(\mathbb{R})^{\otimes\mathbb{N}}$ such that $(\xi_n)_{n\in\mathbb{N}}$, $(\zeta_n)_{n\in\mathbb{N}}$ be mutually independent identically distributed sequences of random variables each of them with distribution $\mathcal{N}(0,1)$. It suffices to consider the consistent projective family of measures

$$\mu_{1,\dots,k}(A_1 \times \dots A_k) = \frac{1}{(2\pi)^k} \int_{A_1} \cdots \int_{A_k} \exp\left(-\frac{1}{2}\|x\|^2\right) dx, \tag{25}$$

on each space $(\mathbb{R}\times\mathbb{R})^k$, $k \in \mathbb{N}$, where $A_1, \dots A_k \in \mathcal{B}(\mathbb{R}^2)$; $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$, and dx the Lebesgue measure on that space.

Let denote $D = D([0,\infty[, \Sigma)$ the space of functions $w = (w(t);\ t \geq 0)$ defined in $[0,\infty[$ with values in Σ, which have left-hand limits $(w(t-) = \lim_{s\to t,\, s<t} w(s))$ and are right-continuous $(w(t+) = \lim_{s\to t,\, s>t} w(s) = w(t))$ on each $t \geq 0$ (with the convention $w(0-) = w(0)$). Thus, D represents all possible trajectories followed by states of particles of the main system. The space D is endowed with the σ-algebra $\mathcal{D}$ generated by the maps $w \mapsto w(t)$ for all $t \geq 0$.

Now, let define $\Omega = D \times G$, and $\mathcal{F} = \mathcal{D} \otimes \mathcal{G}$. We extend the definition of the variables ξ_n, ζ_n to Ω by writing $\xi_n(\omega) = \xi_n(g)$, $\zeta_n(\omega) = \zeta_n(g)$, for each $\omega = (w, g) \in \Omega$. The probability P_G is extended as well to the space Ω through $\mathbb{P}(A\times B) = P_G(B)$, $A \in \mathcal{D}$, $B \in \mathcal{G}$. This corresponds to the initial probability, before the interaction system-reservoir starts.

Moreover, define $X(\omega, t) = w(t) \in \Sigma$ for all $t \geq 0$. Notice that $w(t)$, as a vector in Σ, has 2 components corresponding to position and momentum. So that, for each $\omega \in \Omega$, and any time $t \geq 0$, $X(\omega, t)$ is the state of the main system at time $t \geq 0$ when it follows the trajectory ω of the interaction system-reservoir. We can write the canonical process $X(\omega, t) = (Q(\omega, t), P(\omega, t))$, where $Q(\cdot, t), P(\cdot, t) : \Omega \to \mathbb{R}$ represent respectively, the position and momentum applications. Our aim is to solve (8) in distribution. That means to find a probability distribution $\mathbb{P}$ on Ω, such that the canonical process satisfies the above Langevin equation. We first construct a family of probabilities $(\mathbb{P}^h;\ h > 0)$ such that the canonical process satisfies the discrete version (21) of the Langevin equation precised below. Later, $\mathbb{P}$ is obtained as the limit of the family $\mathbb{P}^h$ when $h \to 0$.

For each $\nu > 0$, the process $\mathbf{B}_v^h(\nu, \cdot)$ has continuous trajectories, while for each $t \geq 0$, the process $\mathbf{B}_v^h(\cdot, t)$ is discontinuous. Thus, $\nu \mapsto \mathbf{B}_v^h(\nu, \cdot)$ is a process with discontinuous trajectories, taking values on the space of continuous functions of the positive real line.

Moreover $\mathbb{E}\left(\mathbf{B}_v^h(\nu,t)\right)=0$ and the covariance of the process $\mathbf{B}_v^h(\nu,\cdot)$ is

$$\begin{aligned}K^h(t,s) &= \mathbb{E}\left(\mathbf{B}_v^h(\nu,t)\mathbf{B}_v^h(\nu,s)\right)\\ &= h\sum_{k=1}^{[\nu/h]}\frac{f(kh)^2}{(kh)^2}\left(\cos(kh(t-s))+1-\cos(kht)-\cos(khs)\right)\\ &= R^h(t)+R^h(s)-R^h(t-s),\end{aligned}$$

where $R^h(t)=\int_0^t\phi^h(\nu,s)ds$, $t\geq 0$.

Finally, we proved in [8] Proposition 5.2 and Corollary 5.5, that $\mathbf{B}_v^h(\nu,\cdot)$ converges in distribution to a process $\mathbf{B}_v(\nu,\cdot)$ as $h\to 0$, and in addition the distribution of $\mathbf{B}_v(\infty,\cdot)$ coincides with that of the Gaussian transform $\mathbf{B}_v$. It worths noticing that the above convergence in distribution takes place in the following precise sense. The distribution of each process $(\mathbf{B}_v^h(\nu,\cdot);\nu\in]0,\infty[)$ is a probability on the space $D(]0,\infty[,C(\mathbb{R}^+,\mathbb{R}))$. As $h\to 0$, the above family of distributions converges to a probability concentrated on $C(]0,\infty[,C(\mathbb{R}^+,\mathbb{R}))$.

To summarize, the equations of the discrete frequency open system dynamics can also be written in a more compact form as

$$X(t)=x_0+\int_0^t J\nabla H(X(s))ds-\int_0^t\phi^h(\nu,t-s)|e_2\rangle\langle e_2|X(s)ds+\mathbf{B}_v^h(\nu,t)e_2. \quad (26)$$

Mimicking the proof of Theorem 3 one easily obtains the following proposition

Proposition 1 *Suppose that the Hamiltonian H is of class C^2 with bounded Hessian, and that $v\in L^1(\mathbb{R}^+)$ is positive, continuous and convex.*

Then for any $h>0$, and $\nu>0$ fixed, there exists a unique stochastic process $X^h(\nu,\cdot)$ which almost surely satisfies equation (26). Moreover, the process $(\nu,t)\mapsto X^h(\nu,t)$ is continuous almost surely.

Thus, the distribution $\mathbf{P}^h$ of the couple $(X^h(\nu,\cdot),\mathbf{B}_v^h(\nu,\cdot))$, defined on the space $D([0,\infty[,\mathbb{R}^2\times\mathbb{R})$, is indeed concentrated on the subspace of continuous functions $\mathfrak{X}=C([0,\infty[,\mathbb{R}^2\times\mathbb{R})$. So that, in the sequel, we will assume that $\mathbf{P}^h$ is a probability measure on the space of trajectories $\mathfrak{X}$.

Theorem 4 *Suppose that the Hamiltonian H is of class C^2 with bounded Hessian, and that $v\in L^1(\mathbb{R}^+)$ is positive, continuous and convex.*

Then the family of processes $(X^h(\nu,\cdot),\mathbf{B}_v^h(\nu,\cdot))_{h>0}$ converges in distribution to $(X(\nu,\cdot),\mathbf{B}_v(\nu,\cdot))$ in $\mathfrak{X}$. Moreover, for each $\nu>0$ it satisfies

$$X(\nu,\cdot)=x_0+\int_0^\cdot J\nabla H(X(\nu,s))ds-\int_0^\cdot\phi(\nu,t-s)|e_2\rangle\langle e_2|X(\nu,s))ds+\mathbf{B}_v^h(\nu,\cdot)e_2. \quad (27)$$

Moreover, there exists $X(t) = X(\infty, t)$ *such that* $(X(\cdot), \mathbf{B}_v(\cdot))$ *is a solution of the Langevin equation* (8)*:*

$$X(t) = x_0 + \int_0^t J\nabla H(X(s))ds - \int_0^t \phi(t-s)|e_2\rangle\langle e_2|X(s)ds + \mathbf{B}_v(t)e_2,$$

for all $t \geq 0$.

Proof. Firstly, the family $(\mathbf{B}_v^h)$ converges in distribution to $\mathbf{B}_v$ as proved in [8], which is a continuous process in both variables (ν, t). So that, $(\mathbf{B}_v^h)$ is in particular a C-tight family of processes.

Each X^h is also continuous, so that tightness of the family of couples $(X^h, \mathbf{B}_v^h)$ will follow from tightness of the components. It remains to prove tightness of the family of processes $(X^h(\nu, \cdot); \nu \in]0, \infty[)$ with trajectories on $C(]0, \infty[, C)$, where C denotes the space $C(\mathbb{R}^+, \mathbb{R}^2)$. For that, since $X_0^h = x_0$ for all $h > 0$, it suffices to prove the convergence in probability

$$\sup_{0\leq t\leq N} \left\| X^h(T_h + \eta_h, t) - X^h(T_h, t) \right\| \to_{\mathbb{P}} 0, \tag{28}$$

as $h \to 0$, for any family $(T_h)_{h\in]0,\infty[}$ of stopping times on the frequency domain, bounded by $F > 0$ and any family $(\eta_h)_{h\in]0,\infty[}$ of real numbers decreasing to 0. Notice first that $\sup_h \sup_{0<\nu\leq F} \sup_{0\leq t\leq N} \left\| X^h(\nu, t) \right\| < \infty$ almost surely.

Indeed,

$$\begin{aligned} \left\| X^h(\nu, t) \right\| &\leq \|x_0\| + \sup_{0<\nu\leq F} \sup_{0\leq t\leq N} \left| \mathbf{B}_v^h(\nu, t) \right| \\ &+ L\left(\int_0^t \left\| X^h(\nu, s) \right\| ds \right). \end{aligned}$$

So that by Gronwall's lemma it follows that

$$\sup_{h>0} \sup_{0<\nu\leq F} \sup_{0\leq t\leq N} \left\| X^h(\nu, t) \right\| \leq C := \left(\|x_0\| + \sup_{h>0} \sup_{0<\nu\leq F} \sup_{0\leq t\leq N} \left| \mathbf{B}_v^h(\nu, t) \right| \right) e^{LN} < \infty, \tag{29}$$

almost surely.

Now, consider the difference $\Delta_h = \sup_{0\le t\le N} \left\| X^h(T_h+\eta_h,t) - X^h(T_h,t)\right\|$:

$$\begin{array}{c}
\left\| X^h(T_h+\eta_h,t) - X^h(T_h,t)\right\| \le \\
\int_0^t \left\| J\nabla H(X^h(T_h+\eta_h,s) - J\nabla H(X^h(T_h,s)\right\| ds \\
+\int_0^t \left\| \left(\phi^h(T_h+\eta_h,t-s)|e_2\rangle\langle e_2|X^h(T_h+\eta_h,s) - \phi^h(T_h,t-s)|e_2\rangle\langle e_2|X^h(T_h,t-s)\right)\right\| ds \\
+\left|\mathbf{B}_v^h(T_h+\eta_h,t) - \mathbf{B}_v^h(T_h,t)\right| \\
\le L\int_0^t \left\| X^h(T_h+\eta_h,s) - X^h(T_h,s)\right\| ds \\
+\int_0^t \left\| \phi^h(T_h+\eta_h,t-s)|e_2\rangle\langle e_2| \left(X^h(T_h+\eta_h,s) - X^h(T_h,s)\right)\right\| ds \\
+\int_0^t \left\| \left(\phi^h(T_h+\eta_h,t-s) - \phi^h(T_h,t-s)\right)|e_2\rangle\langle e_2|X^h(T_h,s)\right\| ds \\
+\left|\mathbf{B}_v^h(T_h+\eta_h,t) - \mathbf{B}_v^h(T_h,t)\right| \\
\le (L+K)\int_0^t \left\| X^h(T_h+\eta_h,s) - X^h(T_h,s)\right\| ds \\
+C\int_0^t \left|\phi^h(T_h+\eta_h,s) - \phi^h(T_h,s)\right| ds \\
+\left|\mathbf{B}_v^h(T_h+\eta_h,t) - \mathbf{B}_v^h(T_h,t)\right|.
\end{array}$$

A new application of Gronwall's inequality yields

$$\Delta_h \le \varepsilon_h e^{(L+K)N},$$

where

$$\varepsilon_h = \sup_{0\le t\le N} \left|\mathbf{B}_v^h(T_h+\eta_h,t) - \mathbf{B}_v^h(T_h,t)\right| + C\int_0^N \left|\phi^h(T_h+\eta_h,s) - \phi^h(T_h,s)\right| ds$$

Since $(\mathbf{B}_v^h)_{h>0}$ is a C-tight family of processes, the term

$$\sup_{0\le t\le N} \left|\mathbf{B}_v^h(T_h+\eta_h,t) - \mathbf{B}_v^h(T_h,t)\right|$$

tends to 0 in probability as $h\to 0$. On the other hand, since $\phi^h(\nu,\cdot)$ converges to $\phi(\nu,\cdot)$ in L^1, which is a continuous function, it follows that

$$\int_0^N \left|\phi^h(T_h+\eta_h,s) - \phi^h(T_h,s)\right| ds \to 0$$

almost surely as $h\to 0$.

Therefore, $\Delta_h \to 0$ in probability. As a result, $(X^h)_{h>0}$ is a tight family and furthermore, $(X^h, \mathbf{B}_v^h)_{h>0}$ is tight as well.

It remains to prove that any limit point X of X^h (in distribution) satisfies the Generalized Langevin Equation.

Let fix $t>0$ and take a convergent subfamily of X^h which we denote again by X^h. Therefore $(X^h, \mathbf{B}_v^h)$ converges towards $(X, \mathbf{B}_v)$ in the weak product topology.

As a result, $\int_0^t J\nabla H(X^h(\nu,s))ds \to \int_0^t J\nabla H(X(\nu,s))ds$ due to the continuity of the map $\int_0^t J\nabla H(\cdot)ds$ on the space of continuous functions. Also,

$$\int_0^t \phi^h(t-s)|e_2\rangle\langle e_2|X^h(\nu,s)ds \to \int_0^t \phi(t-s)|e_2\rangle\langle e_2|X(s)ds,$$

since the family of maps $u \mapsto \int_0^t \phi^h(\nu,t-s)u(s)ds$, on the space of continuous functions u, converges to the continuous map $u \mapsto \int_0^t \phi(\nu,t-s)u(s)ds$.

So that the limit point X satisfies the equation

$$X(\nu,t) = x_0 + \int_0^t J\nabla H(X(\nu,s))ds - \int_0^t \phi(\nu,t-s)|e_2\rangle\langle e_2|X(\nu,s)ds + \mathbf{B}_v(\nu,t).$$

Since, this equation has a unique solution, this shows that the set of all limit points of the family $(X^h)_{h>0}$ is reduced to the singleton $\{X\}$, that is, the whole family converges in distribution to X in the space $C(]0,\infty[, C(\mathbb{R}^+,\mathbb{R}^2))$.

As a result, since $\phi(\infty,t) = \int_0^t v(s)ds = \phi(t)$, and $\mathbf{B}_v(\infty,t) = \mathbf{B}_v(t)$, the process $X(\infty,t)$ has the same distribution than the process $X(t)$ which is the unique solution of the equation

$$X(t) = x_0 + \int_0^t J\nabla H(X(s))ds - \int_0^t \phi(t-s)|e_2\rangle\langle e_2|X(s)ds + \mathbf{B}_v(t).$$

□

5. Applications

5.1. Langevin equations with fractional Brownian motion

Consider the locally integrable function v defined by

$$v(t) = t^{-\gamma},\ 0<\gamma<1,\ t>0. \tag{30}$$

Note that v is the cosine transform of the function

$$\frac{\pi}{2}f^2(x) = \sin(\frac{1}{2}\pi\gamma)\Gamma(1-\gamma)x^{\gamma-1}.$$

Its Gaussian transform $\mathbf{B}_v$ coincides with a multiple of the fractional Brownian motion $B^{\mathtt{H}}$ with Hurst coefficient $\mathtt{H} = 1-\frac{1}{2}\gamma$:

$$\mathbf{B}_v(t) = \frac{2}{(1-\gamma)(2-\gamma)}B^{\mathtt{H}}(t). \tag{31}$$

Note that our restriction on γ implies $\frac{1}{2} < \mathtt{H} < 1$. On the other hand, since $R(t) = \int_0^t (t-s)v(s)ds = \frac{t^{2-\gamma}}{(2-\gamma)(1-\gamma)}$ we obtain by Theorem 1 that the variance of the Gaussian transform is given explicitly by

$$\mathbb{E}(|\mathbf{B}_v(t) - \mathbf{B}_v(s)|^2) = \frac{2}{(2-\gamma)(1-\gamma)}|t-s|^{2-\gamma}.$$

And the covariance of $\mathbf{B}_v$ is

$$K(t,s) = \frac{t^{2-\gamma}}{(2-\gamma)(1-\gamma)} + \frac{s^{2-\gamma}}{(2-\gamma)(1-\gamma)} - \frac{(t-s)^{2-\gamma}}{(2-\gamma)(1-\gamma)}. \tag{32}$$

So that Theorem 3 provides a solution to the equation

$$\begin{aligned} X(t) &= x_0 + \int_0^t J\nabla H(X(s))ds \\ &- \int_0^t \frac{1}{1-\gamma}(t-s)^{1-\gamma}|e_2\rangle\langle e_2|X(s)ds \\ &+ \frac{2}{(1-\gamma)(2-\gamma)}B^{\mathtt{H}}(t). \end{aligned} \tag{33}$$

Finally, note that incidentally the function $\phi(t)$, in this case given by

$$\phi(t) = \frac{t^{1-\gamma}}{1-\gamma},$$

describes the material function (creep compliance) of a classical type of mechanical models in viscoelasticity theory. In fact, it corresponds to a fractional Newton model (cf. [12]). See also the example below.

5.2. A model in viscoelasticity

Consider this time the function

$$v(t) = e^{-\alpha t}, \quad \alpha > 0, \tag{34}$$

Then $\phi(t) = \frac{1}{\alpha}[1 - e^{-\alpha t}]$ and corresponds exactly to the standard Voigt model in the theory of viscoelasticity (see [12]) with material function $\phi(t)$. In this example, we have

$$f^2(x) = \frac{\alpha}{\alpha^2 + x^2}.$$

Moreover, according to our notations,

$$R(t) = \frac{1}{\alpha^2}e^{-\alpha t} + \frac{1}{\alpha}t - \frac{1}{\alpha^2}.$$

Hence, the process $\mathbf{B}_v$ has the covariance

$$K(t,s) = \frac{1}{\alpha^2}[e^{-\alpha t} + e^{-\alpha s} - e^{-\alpha(t-s)}] + \frac{2s}{\alpha} - \frac{1}{\alpha^2}. \tag{35}$$

Finally, it is interesting to note that for a free particle, that is $V(q) = 0$ in equation (8), $Q(t) = Q_0 + \int_0^t P(s)ds$ and P satisfies the following stochastic integral equation of Volterra type

$$P(t) + \int_0^t \phi(t-s)P(s)ds = P_0 + \mathbf{B}_v(t), \tag{36}$$

whose solution is guaranteed by Theorem 3.

6. Conclusions and future work

We introduced a general class of open systems with memory, ruled by equations of Langevin's type driven by a non-Markovian gaussian noise. The method consisted of representing the environment by stochastic integrals in the frequency domain.

In a forthcoming paper, the authors will consider microscopic models extending the above class of noises to a quantum setting. Moreover, applications to ion channel dynamics with memory are under study.

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