COSINE AND GAUSSIAN TRANSFORMS

CARLOS LIZAMA AND ROLANDO REBOLLEDO

Abstract. This paper introduces and investigates probabilistic properties of a class of gaussian processes connected with cosine transforms, which have been used to describe non Markovian classical open systems in Physics. Several examples of these processes are considered, namely, the Fractional Brownian Motion with Hurst coefficient $> 1/2$ and other processes appearing in viscoelasticity models.

1. Introduction

This paper concerns the construction of Gaussian non Markovian processes obtained from integration with respect to a Brownian motion on the "frequency domain". These processes are motivated by the non Markovian approach to Open System Dynamics.

Indeed, consider first a classical mechanical system which can be observed and will be referred as the *main system*. This system is immersed in a reservoir or heat bath composed of a great number of harmonic oscillators which collide with the main system, where the frequencies vary over the positive real line. To describe the action of the reservoir on the main system one supposes first that the interaction is produced at discrete frequencies jh, where $j \in \mathbb{N}$ and $h > 0$. Variations on the momentum of the main system are described by a memory kernel and a random force. The latter depends on the variation of initial conditions of each harmonic oscillator. This is the setup used in a number of recent papers in Physics (see eg. [5]).

This paper is aimed at investigating probabilistic properties of non Markovian gaussian processes generated by the above approach to open system dynamics. Moreover, we enlarge the class of such processes by tracing a parallel with the theory of cosine transform. Cosine transforms appear naturally when considering classical heat bath made of an infinite number of harmonic oscillators.

In the next section we start studying positive cosine transforms, to follow in section 3 with the definition of a gaussian transform. In section 4, these gaussian transforms are represented as stochastic integrals. The case of discrete gaussian transforms and their limits are considered in section 5. We end the paper by

2000 *Mathematics Subject Classification.* 60G15; 60G22.

Key words and phrases. cosine transform, non Markovian Gaussian processes, fractional Brownian motion.

This research has been supported by the Bicentennial Foundation grant PBCT-ADI13 "Laboratorio de Análisis Estocástico" and FONDECYT project 1100485.

providing a number of examples. Applications to open systems in Physics will be considered in a separate paper.

2. Positive cosine transforms

Let $v \in L^1(\mathbb{R}^+)$ be a continuous positive convex function, therefore its second derivative in distribution's sense is positive.
Define

$$R(t) = \int_0^t (t-u)v(u)du, \tag{2.1}$$

for all $t \geq 0$. Note that $R(t)$ is a positive function. We begin with the following result.

Proposition 2.1. *The function*

$$K(t,s) = R(t) + R(s) - R(t-s), \tag{2.2}$$

is positive definite.

Proof. Since v is convex, the function

$$\frac{2}{\pi}\int_0^\infty v(t)\cos(xt)dt, \tag{2.3}$$

is positive as proved in [8]. Therefore, define

$$f(x) = \sqrt{\frac{2}{\pi}\int_0^\infty v(t)\cos(xt)dt}, \tag{2.4}$$

for all $x \geq 0$. This is a function in $L^2(\mathbb{R}^+)$. Moreover, $v(t)$ is the cosine transform of the function f^2, that is

$$v(t) = \int_0^\infty f^2(x)\cos(xt)dx.$$

As a result, R can be written

$$R(t) = \int_0^\infty \left(\frac{f(x)}{x}\right)^2 (1-\cos(xt))dt.$$

Therefore, K can be written as

$$K(s,t) = \int_0^\infty \left(\frac{f(x)}{x}\right)^2 \left[(1-\cos(xt))(1-\cos(xs)) + \sin(xt)\sin(xs)\right]dx. \tag{2.5}$$

Now, take any finite set I of positive numbers and $z(t) \in \mathbb{C}$, $t \in I$, then

$$\begin{aligned}
\sum_{s,t\in I} z(t)\overline{z(s)}K(s,t) &= \int_0^\infty \left(\frac{f(x)}{x}\right)^2 \left|\sum_{t\in I} z(t)(1-\cos(xt))\right|^2 dx \\
&+ \int_0^\infty \left(\frac{f(x)}{x}\right)^2 \left|\sum_{t\in I} z(t)\sin(xt)\right|^2 dx \\
&\geq 0.
\end{aligned}$$

□

Remark 2.2. Notice that K is decomposed as a sum of two positive definite kernels $K = K_1 + K_2$, where

$$\begin{aligned} K_1(s,t) &= \int_0^\infty \left(\frac{f(x)}{x}\right)^2 (1-\cos(xt))(1-\cos(xs))dx & (2.6)\\ K_2(s,t) &= \int_0^\infty \left(\frac{f(x)}{x}\right)^2 \sin(xt)\sin(xs)dx. & (2.7)\end{aligned}$$

To each positive definite function one can associate a Hilbert space and a scalar product. We recall briefly the construction of a self-reproducing Hilbert space $\mathfrak{h}(K)$ associated to a positive definite kernel K due to Aroszajn (see for instance [6]).

Consider the space $\mathcal{V}(K)$ of all finite linear combinations $\sum_{s\in\mathbb{R}} a(s)K(s,\cdot)$, where $a : \mathbb{R} \to \mathbb{C}$ is a function with finite support. Define,

$$\langle g, h\rangle = \sum_{s,t} a(s)\overline{b(t)}K(s,t), \tag{2.8}$$

where $g = \sum_s a(s)K(s,\cdot)$ and $h = \sum_t b(t)K(t,\cdot)$, a and b with finite support. So that,

$$\langle g, K(t,\cdot)\rangle = g(t),\ (t \in \mathbb{R}).$$

Thus, in particular, $\langle K(s,\cdot), K(t,\cdot)\rangle = K(s,t)$, $s,t \in \mathbb{R}^+$. The space $\mathfrak{h}(K)$ is then the completion of $\mathcal{V}(K)$ for the scalar product (2.8).

Notice that, $\langle K_1(s,\cdot), K(t,\cdot)\rangle = K_1(s,t) = \langle K_1(s,\cdot), K_1(t,\cdot)\rangle$. Therefore,

$$\begin{aligned} \langle K_1(s,\cdot), K_2(t,\cdot)\rangle &= \langle K_1(s,\cdot), K(t,\cdot) - K_1(t,\cdot)\rangle\\ &= \langle K_1(s,\cdot), K(t,\cdot)\rangle - \langle K_1(s,\cdot), K_1(t,\cdot)\rangle\\ &= K_1(s,t) - K_1(s,t)\\ &= 0.\end{aligned}$$

As a result we have the following easy consequence.

Corollary 2.3. *Under the above notations, it holds:*

$$\mathfrak{h}(K) = \mathfrak{h}(K_1) \oplus \mathfrak{h}(K_2). \tag{2.9}$$

3. Gaussian transforms

We keep the notations of the previous section. So that to each $v \in L^1(\mathbb{R}^+)$ continuous, positive and convex function, we associate a positive definite kernel K which is decomposed in two kernels K_1 and K_2.

Applying Thm. 2.4 in [6] one easily obtains the following Proposition.

Proposition 3.1. *There exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two independent, centered Gaussian processes $X_i = (X_i(t);\, t \geq 0)$ with covariances K_i, $i = 1, 2$.*

Moreover, a Hilbert space $\mathfrak{h}(X_i) \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is associated to each process X_i by means of an isomorphism of Hilbert spaces defined through $\sum_s a(s)K_i(s,\cdot) \mapsto \sum_s a(s)X_i(s)$, where a has finite support, $i = 1, 2$.

As a result, if one defines $X = (X_1, X_2)$ and $Z = X_1 + X_2$, they are centered gaussian processes as well and it holds:

$$\mathfrak{h}(X) = \mathfrak{h}(X_1) \otimes \mathfrak{h}(X_2) \tag{3.1}$$

$$\mathfrak{h}(Z) = \mathfrak{h}(X_1) \oplus \mathfrak{h}(X_2). \tag{3.2}$$

Moreover, $\mathfrak{h}(X)$ is isomorphic to $\mathfrak{h}(K_1) \otimes \mathfrak{h}(K_2)$ and $\mathfrak{h}(Z)$ is isomorphic to $\mathfrak{h}(K)$.

The above processes are all continuous as applications from $[0, \infty[$ into the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$. The continuity of their trajectories will be obtained in Theorem 3.4.

Definition 3.2. Let $\mathcal{V}$ denote the class of all functions $v \in L^1(\mathbb{R}^+)$ which are continuous, positive and convex. To each $v \in \mathcal{V}$ the previous Proposition associates the centered Gaussian process Z, denoted from now on $\mathbf{B}_v$, whose covariance is K given by (2.2) (equivalently by (2.5)). We say that the (distribution) of $\mathbf{B}_v$ is the Gaussian transform of the function v.

It is worth noticing that the process Z constructed in Proposition 3.1 is not unique. However, its distribution is uniquely determined by K, so that the Gaussian transform is defined as a probability distribution on the space of trajectories of the process Z. To alleviate the notations we will abuse the language speaking of the process $\mathbf{B}_v$ as the Gaussian transform of v. We need the following Lemma.

Lemma 3.3. *The function R given by* (2.1) *satisfies*

$$\int_1^\infty \sqrt{2R(e^{-y^2})}dy \le (2\,\|v\|_1)^{1/2} \int_1^\infty e^{-x^2/2}dx < \infty. \tag{3.3}$$

Proof. Let $x \in [1, \infty[$, then

$$\begin{aligned} R(e^{-x^2}) &= \int_0^{e^{-x^2}} (e^{-x^2} - u)v(u)du \\ &\le e^{-x^2} \int_0^{e^{-x^2}} v(u)du \\ &\le e^{-x^2}\,\|v\|_1 . \end{aligned}$$

Therefore,

$$\int_1^\infty \sqrt{2R(e^{-y^2})}dy \le (2\,\|v\|_1)^{1/2} \int_1^\infty e^{-x^2/2}dx < \infty.$$

□

The following Theorem is the main result of this section.

Theorem 3.4. *For each fixed $T > 0$, the Gaussian transform process $(\mathbf{B}_v(t);\ t \in [0, T])$ admits a uniformly continuous version and its modulus of continuity is $w(\mathbf{B}_v, u) \le C\sqrt{\log\left(\frac{1}{u}\right) R(u)}$, $(0 < u < 1)$, where $C > 0$ is a constant.*

Proof. A straightforward computation gives

$$\mathbb{E}\left(|\mathbf{B}_v(t) - \mathbf{B}_v(s)|^2\right) = 2R(|t-s|).$$

So that $\|\mathbf{B}_v(t) - \mathbf{B}_v(s)\|_2 = \sqrt{2R(|t-s|)}$. The conclusion follows from a straightforward application of a result due to Fernique, see for instance Prop. 3.4 in [6] (see also [1],[2],[3],[4]) . □

Remark 3.5. As a result, note that there is a unique probability distribution $\mathbb{P}_v$ on the space C of continuous functions, endowed with its Borel σ–algebra $\mathcal{B}(C)$, which corresponds to any continuous version of $\mathbf{B}_v$.

4. A representation of B_v as a stochastic integral

Within this section we construct a canonical version of the process $(\mathbf{B}_v(t);\ t \in [0,T])$, where $T > 0$ is fixed. To this end, let two independent Brownian motions $W_1 = (W_1(x);\ x \geq 0)$, $W_2 = (W_2(x);\ x \geq 0)$ defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_x)_{x\in\mathbb{R}^+}, \mathbb{P})$, be given.

Notice that the functions $x \mapsto \psi_1(x,t) = \left(\frac{f(x)}{x}\right)(1-\cos(xt))$ and $x \mapsto \psi_2(x,t) = \left(\frac{f(x)}{x}\right)\sin(xt)$ are in $L^2([0,\infty[)$, they are deterministic, so trivially predictable, therefore, they can be integrated with respect to W_1 and W_2 respectively. In this way we obtain a process $Z(x,t)$ given by

$$Z(x,t) = \int_0^x \psi_1(y,t)dW_1(y) + \int_0^x \psi_2(y,t)dW_2(y), \tag{4.1}$$

$(x,t) \in \mathbb{R}^+ \times [0,T]$.

The following is one of the main results of this paper.

Theorem 4.1. *The process Z defined by (4.1) is a centered Gaussian process with trajectories in the space $C(\mathbb{R}^+ \times [0,T])$ and covariance given by*

$$\mathbf{K}((y,s),(x,t)) = \int_0^{x\wedge y} \left(\frac{f(r)}{r}\right)^2 [(1-\cos(rt))(1-\cos(rs)) + \sin(rt)\sin(rs)]\,dr. \tag{4.2}$$

Moreover, the limit $Z(x,t)$ converges in L^2 to a limit $Z(\infty,t)$, and this convergence is uniform in $t \in [0,T]$. The distribution of the process $(Z(\infty,t);\ t \in [0,T])$ coincides with that of $\mathbf{B}_v$.

Proof. Z is clearly a Gaussian process since it is the sum of two independent stochastic integrals of deterministic functions with respect to the Brownian motions W_1 and W_2.

The covariance is the sum of the covariances of each integral since they are independent. These covariances are, respectively:

$$\int_0^{x\wedge y} \psi_1(r,t)\psi_1(r,s)dr;\ \int_0^{x\wedge y} \psi_2(r,t)\psi(r,s)dr.$$

This yields formula (4.2).

Moreover, notice that for $k=1,2$, $\sup_{t\in[0,T]} \int_x^\infty \psi_k^2(r,t)dr$ tends to 0 as $x \to \infty$ since the functions $t \mapsto \psi_k^2(r,t)1_{[x,\infty[}(r)$ are continuous and decreasing to 0 on the compact interval $[0,T]$, thus they converge to 0 uniformly by Dini's Lemma. As a result, $Z(\infty, t)$ exists and $t \mapsto Z(\infty, t)$ is a continuous centered Gaussian process given by

$$Z(\infty, t) = \int_0^\infty \left(\frac{f(x)}{x}\right)(1-\cos(xt))dW_1(x) + \int_0^\infty \left(\frac{f(x)}{x}\right)\sin(xt)dW_2(x). \quad (4.3)$$

Finally, the covariance of $Z(\infty, \cdot)$ is

$$\mathbb{E}\left(Z(\infty,t)Z(\infty,s)\right) =$$
$$\int_0^\infty \left(\frac{f(x)}{x}\right)^2 \left[(1-\cos(xt))(1-\cos(xs)) + \sin(xt)\sin(xs)\right] dx,$$

so that it coincides with K, the covariance of the Gaussian transform $\mathbf{B}_v$. Therefore, the distribution of $Z(\infty, \cdot)$ coincides with that of $\mathbf{B}_v$. This completes the proof. □

The following two Corollaries, provide concrete examples. The first one, is concerned with Fractional Brownian motion. We recall that fractional Brownian motion was defined some years ago to model diffusive, subdifussive and superdifussive transport processes, also known as anomalous difussion processes.

Corollary 4.2. *The Fractional Brownian Motion with Hurst coefficient $H = 1 - \frac{\gamma}{2}$, where $0 < \gamma < 1$ corresponds to the Gaussian transform*

$$\sqrt{(2-\gamma)(1-\gamma)}\mathbf{B}_v = \sqrt{2H(2H-1)}\mathbf{B}_v$$

of the function $v(t) = t^{-\gamma}$, $(t > 0)$.

Proof. Notice that v is convex on $]0, \infty[$, and one obtains

$$(f(x))^2 = 2\pi^{-1}\sin\frac{\pi\gamma}{2}\Gamma(1-\gamma)x^{\gamma-1},$$

for all $x \in [0, \infty[$. Moreover,

$$R(t) = \int_0^t \int_0^s u^{-\gamma} du ds = \frac{1}{(2-\gamma)(1-\gamma)} t^{2-\gamma}.$$

As a result, for all $s, t \geq 0$,

$$K(s,t) = R(t) + R(s) - R(|t-s|) = \frac{1}{(2-\gamma)(1-\gamma)}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

where $H = 1 - \gamma/2$. □

Remark 4.3. The relationship between the cosine transform of $f^2(x)$ and $v(t)$ helps to explain, in a wide sense, the fundamental choice done by Kupfermann in [5, eq. (2.3), p.295]. In fact, consider the linear integral convolution equation

$$Q(t) = Q_0 - \int_0^t R(t-s)Q(s)ds. \quad (4.4)$$

In the case $H = 1/2$, we obtain $R(t) = 1$ and the above equation has solution $Q(t) = Q_0 e^{-t}$ which corresponds to the standard diffusive model. For general $H = 1 - \gamma/2$, the equation is equivalent to the fractional differential equation

$$D_t^\gamma Q(t) = -Q(t),$$

where D_t^γ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1)$ (equivalently, $H \in (1/2,1)$). The solution is given by:

$$Q(t) = Q_0 t^{\gamma-1} E_{\gamma,\gamma}(-t^\gamma),$$

where $E_{\gamma,\gamma}$ denotes the Mittag-Leffler function. Observe that the choice of the memory kernel $R(t)$ (equivalently, $v(t)$) is arbitrary (v convex) in our context.

The choice of an exponential function v yields another example.

Corollary 4.4. *Consider $v(t) = e^{-\alpha t}$, for $t \geq 0$. Then the Gaussian transform of v is represented as*

$$\begin{aligned} \mathbf{B}_v(t) &= \int_0^\infty \frac{1}{x}\sqrt{\frac{2\alpha}{\pi(\alpha^2+x^2)}}(1-\cos(xt))dW_1(x) \\ &+ \int_0^\infty \frac{1}{x}\sqrt{\frac{2\alpha}{\pi(\alpha^2+x^2)}}\sin(xt)dW_2(x). \end{aligned} \tag{4.5}$$

Proof. v is a convex function and,

$$(f(x))^2 = \frac{2}{\pi}\int_0^\infty e^{-\alpha t}\cos(xt)dx = \frac{2\alpha}{\pi(\alpha^2+x^2)}. \tag{4.6}$$

Consequently,

$$R(t) = \frac{1}{\alpha^2}e^{-\alpha t} + \frac{1}{\alpha}t - \frac{1}{\alpha^2}, \tag{4.7}$$

so that

$$K(s,t) = \frac{1}{\alpha^2}\left(e^{-\alpha t} + e^{-\alpha s} - e^{-\alpha|t-s|}\right) + \frac{1}{\alpha}(t+s-|t-s|) - \frac{1}{\alpha^2}. \tag{4.8}$$

Therefore, the Gaussian transform of v is represented as (4.5).

□

5. Discrete Gaussian transforms

The representation of $\mathbf{B}_v$ as a stochastic integral suggests to introduce the following family of processes, based on approximations of Brownian motions by means of random walks. We start by constructing a canonical probability space where two mutually independent sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$ of $\mathcal{N}(0,1)$–random variables are defined. This is done by a well-known standard procedure on the space $G = \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$ of all sequences $g = (g_n)_{n\in\mathbb{N}}$ which have values in $\mathbb{R}\times\mathbb{R}$, so that $g_n = (g_n^1, g_n^2)$ and call $\xi_n(g) = g_n^1 \in \mathbb{R}$ and $\zeta_n(g) = g_n^2 \in \mathbb{R}$, $n \in \mathbb{N}$. By means of Kolmogorov Extension Theorem, one can built up a probability measure P_G on $(G,\mathcal{G})$, where $\mathcal{G}$ is the product of Borel σ–fields $\mathcal{B}(\mathbb{R})^{\otimes\mathbb{N}} \otimes \mathcal{B}(\mathbb{R})^{\otimes\mathbb{N}}$ such that $(\xi_n)_{n\in\mathbb{N}}$, $(\zeta_n)_{n\in\mathbb{N}}$ be mutually independent identically distributed sequences of random variables each of them with distribution $\mathcal{N}(0,1)$.

Denote $D = D([0,\infty[,\mathbb{R})$ the space of functions $w = (w(t);\ t \geq 0)$ defined in $[0,\infty[$ with values in $\mathbb{R}$, which have left-hand limits ($w(t-) = \lim_{s\to t,\ s<t} w(s)$) and are right-continuous ($w(t+) = \lim_{s\to t,\ s>t} w(s) = w(t)$) at each $t \geq 0$ (with the convention $w(0-) = w(0)$). Thus D represents all possible trajectories followed by states of particles of the main system. D is endowed with the σ-algebra $\mathcal{D}$ generated by the maps $w \mapsto w(t)$ for all $t \geq 0$.

Now, define $\Omega = D\times G$, and $\mathcal{F} = \mathcal{D}\otimes\mathcal{G}$. We extend the definition of the variables ξ_n, ζ_n to Ω by writing $\xi_n(\omega) = \xi_n(g)$, $\zeta_n(\omega) = \zeta_n(g)$, for each $\omega = (w, g) \in \Omega$. and the probability P_G is extended as well to the space Ω through $\mathbb{P}(A \times B) = P_G(B)$, $A \in \mathcal{D}$, $B \in \mathcal{G}$.

Consider $h > 0$ and let us define

$$Z^h(x,t) = \sqrt{h}\sum_{k=1}^{[x/h]} \frac{f(kh)}{kh} \left(\sin(kht)\xi_k + (1-\cos(kht))\zeta_k\right), \tag{5.1}$$

for $(x,t) \in \mathbb{R}^+ \times [0,T]$.

Introducing the σ-algebras $\mathcal{F}_x^h$ generated by $\{(\xi_j, \zeta_j)\ j \le [x/h]\}$, the processes

$$W_1^h(x) = \sqrt{h}\sum_{j=1}^{[x/h]} \xi_j \tag{5.2}$$

and

$$W_2^h(x) = \sqrt{h}\sum_{j=1}^{[x/h]} \zeta_j \tag{5.3}$$

become two independent martingales. It is well known that the distribution of $W^h = (W_1^h, W_2^h)$ weakly converges to the distribution of a two-dimensional Wiener process $W = (W_1, W_2)$.

We start by studying the convergence in distribution of the family of processes (Z^h).

Lemma 5.1. *Let a real separable Banach space $\mathfrak{b}$ endowed with its Borel σ-field, be given and let $\psi \in L^2([0,\infty[, \mathfrak{b})$ be a locally bounded function. Then, the process*

$$I^h(\cdot) := \int_0^{\cdot} \psi(u)dW_1^h(u) = \sqrt{h}\sum_{j=1}^{[\cdot/h]} \psi(jh)\xi_j,$$

with trajectories in $D(\mathbb{R}^+, \mathfrak{b})$, converges in distribution to

$$I(\cdot) := \int_0^{\cdot} \psi(u)dW_1(u),$$

as $h \to 0$. Moreover, the process $I(\cdot)$ is a square integrable martingale.

Proof. This Lemma is an easy extension to the Banach space setting of a well-known consequence of limit theorems for the convergence in distribution of stochastic integrals. Here we state a more direct and self-contained proof.

Since, $\mathfrak{b}$ is separable, the σ–fields of Borel and that generated by cylinders coincide when they are completed by null sets under any tight probability measure. On the other hand, the space $D([0,\infty[, \mathfrak{b})$ of discontinuous functions with no oscillatory discontinuities becomes a Polish space with the Skorokhod topology. Introduce the family of $D([0,\infty[, \mathfrak{b})$-modulus as follows:

For any element $\varphi \in D([0,\infty[, \mathfrak{b})$, $\delta > 0$ and each $N \in \mathbb{N}$, let

$$w_N(\varphi,\delta) := \inf_{\{x_i\}} \max_{0<i\le r} \sup\left\{\|\varphi(u) - \varphi(v)\| :\ u, v \in [x_i, x_{i+1}[\right\}, \tag{5.4}$$

where the infimum extends over the finite sets of points $\{x_i\}$ satisfying

$$\begin{cases} 0 = x_0 < x_1 < \ldots < x_r = N, \\ x_i - x_{i-1} > \delta, \ i = 1, 2, \ldots, r. \end{cases} \tag{5.5}$$

According to Prokhorov's Theorem, a family of processes $(\Phi^h)_{h>0}$ with trajectories in $D([0,\infty[,\mathfrak{b})$ is tight if and only if the following two conditions below are satisfied:

(T1) For all $N \in \mathbb{N}$,

$$\lim_{a\to\infty} \sup_{h>0} \mathbb{P}\left(\sup_{x\in[0,N]} \left\| \Phi^h(x) \right\| > a \right) = 0;$$

(T2) For all $N \in \mathbb{N}$, for all $\epsilon > 0$ it holds that

$$\lim_{\delta\to 0} \sup_{h>0} \mathbb{P}\left(w_N(\Phi^h, \delta) > \epsilon \right) = 0.$$

A straightforward computation shows that

$$\mathbb{P}\left(\sup_{x\in[0,N]} \left\| I^h(x) \right\| > a \right) \leq \frac{1}{a^2} h \sum_{j=1}^{N/h} \left\| \psi(jh) \right\|^2 \leq \frac{N}{a^2} \int_0^\infty \left\| \psi(u) \right\|^2 du, \tag{5.6}$$

which implies (T1) and

$$w_N(I^h, \delta) \leq \sup_{u\in[0,N]} \left\| \psi(u) \right\| w_N(W_1^h, \delta). \tag{5.7}$$

Since $(W_1^h)_{h>0}$ is a $D([0,\infty[,\mathbb{R})$–tight (even better, it is $C([0,\infty[,\mathbb{R})$–tight), from the above inequality one obtains easily that $(I^h)_{h>0}$ satisfies the hypothesis (T2) and so is $D([0,\infty[,\mathfrak{b})$–tight.

Finally, to finish the proof, one needs to prove that for any functional $\langle \mu, \cdot \rangle$ in the dual $\mathfrak{b}^*$ of the Banach space $\mathfrak{b}$ and any finite family $x_1, \ldots, x_m$ of positive real numbers, the random variables in $\mathbb{R}^m$

$$\left(\langle \mu, I^h(x_1) \rangle, \ldots, \langle \mu, I^h(x_m) \rangle \right)$$

converge in distribution to

$$\left(\langle \mu, I(x_1) \rangle, \ldots, \langle \mu, I(x_m) \rangle \right).$$

This is equivalent to show that for any finite family $\langle \mu_1, \cdot \rangle, \ldots, \langle \mu_m, \cdot \rangle$ of functionals in $\mathfrak{b}^*$ the convergence in distribution of $\sum_{i=1}^m \langle \mu_i, I^h(x_i) \rangle$ to $\sum_{i=1}^m \langle \mu_i, I(x_i) \rangle$ holds (Cramer-Wold device). However,

$$\sum_{i=1}^m \langle \mu_i, I^h(x_i) \rangle = \int_0^{\max_i x_i} \left(\sum_{i=1}^\infty \langle \mu_i, \psi(u) \rangle 1_{[0,x_i[}(u) \right) dW_1^h(u), \tag{5.8}$$

so that it suffices to show that for any measurable and bounded real function g, one has the convergence in distribution of $(g \cdot W_1^h)_x = \int_0^x g(u) dW_1^h(u)$ towards $(g \cdot W_1)_x = \int_0^x g(u) dW_1(u)$ for any fixed $x \in [0,\infty[$. This is an easy consequence of the

Central Limit Theorem for Local Martingales. Indeed, the associated increasing processes of the martingales $g \cdot W_1^h$ are given by

$$x \mapsto h \sum_{j=1}^{[x/h]} (g(jh))^2,$$

and this family converges to

$$\int_0^x (g(u))^2 du,$$

which is the associated increasing process of the continuous gaussian martingale $g \cdot W_1$. Therefore, by the Martingale Central Limit Theorem [7], $g \cdot W_1^h$ converges in distribution, as a process, to $g \cdot W_1$. As a result $g \cdot W_1^h(x)$ converges to $g \cdot W_1(x)$ for all $x \in [0, \infty[$ and the proof is complete. Moreover, since $\psi \in L^2([0, \infty[, \mathfrak{b})$ the process I is a square-integrable martingale. As a result, there exists the terminal variable $I(\infty) \in \mathfrak{b}$.

$\square$

Consider the Banach space C of real-valued continuous functions defined on $[0, T]$, with the uniform norm $\|\cdot\|$. Denote $S_x(t) = \sin(xt)$, $C_x(t) = \cos(xt)$, for all $x \in \mathbb{R}^+$. Call $D(\mathbb{R}^+, C)$ the Polish space of all cadlag functions from $\mathbb{R}^+$ to C endowed with Skorokhod's topology.

Writing $Z^h(x, \cdot) = \sqrt{h} \sum_{k=1}^{[x/h]} \frac{f(kh)}{kh} (\xi_k S_{kh} + \zeta_k(1 - C_{kh}))$ one notices that each process $x \mapsto Z_x^h = Z^h(x, \cdot)$ has trajectories in $D(\mathbb{R}^+, C)$. Moreover,

$$Z_x^h = M_1^h(x) + M_2^h(x), \tag{5.9}$$

where $M_1^h(x) = \sqrt{h} \sum_{k=1}^{[x/h]} \frac{f(kh)}{kh} \xi_k S_{kh}$ and $M_2^h(x) = \sqrt{h} \sum_{k=1}^{[x/h]} \frac{f(kh)}{kh} \zeta_k (1 - C_{kh})$ are two independent C–valued martingales.

Proposition 5.2. *If f is defined by (2.4), the family $((M_1^h, M_2^h))_{h>0}$ converges in distribution to a Gaussian $C \times C$-valued martingale $M = (M_1, M_2)$, which can be represented as*

$$M(x) = \left(\int_0^x \frac{f(u)}{u} S_u dW_1(u), \int_0^x \frac{f(u)}{u} (1 - C_u) dW_2(u) \right).$$

And Z^h converges in distribution as $h \to 0$ to the C-valued process Z given by

$$Z(x, \cdot) = \int_0^x \frac{f(u)}{u} S_u dW_1(u) + \int_0^x \frac{f(u)}{u} (1 - C_u) dW_2(u). \tag{5.10}$$

Proof. The proof follows from Lemma 5.1: take $\psi_1(u) = \frac{f(u)}{u} S_u$ and $\psi_2(u) = \frac{f(u)}{u}(1 - C_u)$, so that

$$M^h = \left(\int_0^\cdot \psi_1(u) dW_1^h(u), \int_0^\cdot \psi_2(u) dW_2^h(u) \right).$$

Each component having a limit in distribution with continuous trajectories, the couple converges in distribution to

$$M = \left(\int_0^\cdot \psi_1(u) dW_1(u), \int_0^\cdot \psi_2(u) dW_2(u) \right),$$

and the process Z^h converges to $Z = \int_0^{\cdot} \psi_1(u)dW_1(u) + \int_0^{\cdot} \psi_2(u)dW_2(u)$. □

Corollary 5.3. *Under the above hypotheses, each martingale M_j $(j = 1, 2)$ has a final variable $M_j(\infty)$, and*

$$Z(\infty, \cdot) = \int_0^\infty \frac{f(u)}{u} S_u dW_1(u) + \int_0^\infty \frac{f(u)}{u}(1 - C_u)dW_2(u) \tag{5.11}$$

coincides with the canonical version of $\mathbf{B}_v$.

References

[1] Dudley, R.: Gaussian processes on several parameters. *The Annals of Mathematical Statistics*, 36(3):771–788, Jun 1965.

[2] Fernique, X.: Des résultats nouveaux sur les processus gaussiens. *Séminaire de probabilités de Strasbourg*, Jan 1975.

[3] Fernique, X.: Sur la régularité de certaines fonctions aléatoires d'Ornstein-Uhlenbeck. *Annales de l'institut Henri Poincaré (B) Probabilités et ...*, Jan 1990.

[4] Fernique, X.: Régularité de fonctions aléatoires gaussiennes stationnaires. *Probability Theory and Related Fields*, Jan 1991.

[5] Kupferman, R.: Fractional kinetics in kac–zwanzig heat bath models. *Journal of Statistical Physics*, Jan 2004.

[6] Neveu, J.: *Processus Aléatoires Gaussiens.* Séminaire de Mathématiques Supérieures. Presses de l'Université de Montréal, Jan 1968 (Ed. polycopié. Cours Université de Paris VI, 1972).

[7] Rebolledo, R.: Central limit theorems for local martingales. *Z. Wahrsch. Verw. Gebiete*, 51(3):269–286, 1980.

[8] Tuck, E.O.: On positivity of Fourier transforms. *Bull. Austral. Math. Soc.*, 74(1):133–138, 2006.

Centro de Análisis Estocástico y Aplicaciones and Departamento de Matemática, Universidad de Santiago de Chile, Casilla 304, Santiago, Chile

E-mail address: carlos.lizama@usach.cl

Centro de Análisis Estocástico y Aplicaciones, Facultad de Matemáticas, Universidad Católica de Chile, Casilla 306, Santiago 22, Chile

E-mail address: rrebolle@uc.cl