ON DUALITY AND SPECTRAL PROPERTIES OF (a,k)-REGULARIZED RESOLVENTS

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Abstract. We construct a duality theory for (a,k) regularized resolvents, extending some of the known theorems for dual semigroups. We present several classes of spaces, which in the semigroup case correspond to the Favard class and the sun-dual space. By duality arguments spectral inclusions theorems for regularized resolvents are also obtained.

1. Introduction

Duality theory for semigroups was first developed by R.E. Phillips and more recently the subject has been studied by van Neerven and de Pagter [16] and by Clement et al. [5]. The duality theory for resolvent families of operators have been investigated by Jung [8] to study Volterra integral equations of convolution type.

In this paper, we give a first approach to some results on duality theory in the general context of $(a,k)-regularized$ families of operators. Applications can be found in different fields such as age-dependent population dynamics, and transport theory, this will be included in a forthcoming paper.

The $(a,k)-regularized$ resolvent families of operators, introduced in [12] is a notion which includes, that of r-times integrated solution family as well as k-convoluted semigroups, r-times integrated cosine families and integral resolvents. It allows us to study existence of solutions for the integral Volterra equation

$$u(t)=\int_0^t a(t-s)Au(s)ds+f(t), \quad t\geq 0,$$

by means of the analysis of the convolution transform (see [12, Theorem 2.7]). Here A is a linear unbounded operator on a Banach space X, f is an X valued function defined on $\mathbb{R}_+$, and $a\in L^1_{loc}([0,\infty))$. Recently, several properties of this class of families has been studied; see [9, 13, 14, 17].

Throughout this paper we assume that X is a complex Banach space and let $\mathcal{B}(X)$ be the algebra of bounded and linear operators on X. Let $a,k\in L^1_{loc}([0,\infty))$.

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Let A be a closed operator with domain $D(A)$. Then a strongly continuous function $R:\mathbb{R}_+\to\mathcal{B}(X)$ is called $(a,k)-$ *regularized* resolvent family with generator A if

(i) $R(0)=k(0)I$

(ii) $R(t)x\in D(A)$ and $AR(t)x=R(t)Ax$ for all $x\in D(A)$ and $t>0$

(iii) $(a*R)(t)x\in D(A)$ and

$$R(t)x=k(t)x+A(a*R)(t)x \qquad (t\geq 0) \quad \text{for all} \quad x\in X. \tag{1.1}$$

Hereafter we assume that the kernels a, and k, are both positive and $\rho(A)$ the resolvent set of A is non empty.

We notice that the choice of the pair (a,k) classifies different families of strongly continuous solution operators in $\mathcal{B}(X)$. For instance when $k(t)=1$ and a is arbitrary, then $(R(t))_{t\geq 0}$ corresponds to a *resolvent family*. In particular, when $k(t)=1$ and $a(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ with $0<\alpha\leq 2$, they are the α- times resolvent families studied by Bazhlekova [1], and corresponds to the solution families for fractional evolution equations, i.e. evolution equations where the integer derivative with respect to time is replaced by a derivative of fractional order. If $\alpha>0$ and $k(t)=\dfrac{t^{\alpha}}{\Gamma(\alpha+1)}$, and $a(t)=1$, then $(R(t))_{t\geq 0}$ corresponds to an $\alpha-times$ integrated semigroup. More generally if k is arbitrary and $a(t)=1$ and then $R(t)$ is a *k-convoluted semigroup;* see [3, 4, 7].

2. The Domain of A

We characterize the domain of the generator A of a given (a,k) regularized resolvent $R(t)$, when a,k are both positive, and $D(A)$ is not necessarily dense.
We recall that if Y is a closed subspace of X and A is a linear operator on X, then the part of A in Y is the operator A_Y defined by

$$D(A_Y)=\{y\in Y\cap D(A): \quad Ay\in Y\}$$

and $A_Yy=Ay$. In what follows we let $Y:=\overline{D(A)}$.

Theorem 2.1. *Let $R(t)$ be an (a,k) regularized resolvent, with generator A. Then*

(i) *For all $x\in Y$*

$$\lim_{t\to 0}\frac{(a*R)(t)x}{(a*k)(t)}=x.$$

(ii) *Let $x\in Y$ and $y\in X$ such that*

$$\lim_{t\to 0^+}\Big\|\frac{R(t)x-k(t)x}{(a*k)(t)}-y\Big\|=0. \tag{2.1}$$

Then $x\in D(A)$ *and* $Ax=y$*. Moreover, if* $y=0$ *then* $R(t)x=k(t)x$ $(t\geq 0)$

(iii) *If* X *is reflexive, and*

$$\lim_{t\to 0^+}\Big\|\frac{R(t)x-k(t)x}{(a*k)(t)}\Big\|<\infty. \tag{2.2}$$

Then $x\in D(A)$.

Proof. Since $(a*k)(t)$ is an increasing function it follow that

$$\frac{(a*a*k)(t)}{(a*k)(t)}\leq\int_0^t a(s)ds\to 0 \quad \text{as} \quad t\to 0. \tag{2.3}$$

On the other hand by (1.1) we have that

$$\frac{(a*R)(t)x}{(a*k)(t)}-x=\frac{(a*a*R)(t)Ax}{(a*k)(t)} \qquad \text{for} \quad x\in D(A).$$

Thus by (2.3)

$$\Big\|\frac{(a*a*R)(t)Ax}{(a*k)(t)}\Big\|\leq M\frac{(a*a*k)(t)\|Ax\|}{(a*k)(t)}\to 0 \quad \text{when} \quad t\to 0.$$

Now statement (i) holds for all $x\in\overline{D(A)}$, since $\Big\|\frac{(a*R)(t)}{(a*k)(t)}\Big\|\leq M$ for $t>0$. To show (ii) we recall that $(a*R)(t)x\in D(A)$ for all $x\in X$ (see [12, Lemma 2.2]). Now by the hypothesis and the resolvent equation (1.1) follows that

$$A\Big(\frac{(a*R)(t)}{(a*k)(t)}\Big)x=\Big(\frac{R(t)-k(t)}{(a*k)(t)}\Big)x\to y \quad \text{when} \quad t\to 0.$$

Since, $\Big(\frac{(a*R)(t)}{(a*k)(t)}\Big)x\to x$ as $t\to 0$ by (i). Hence by the closeness of A we obtain that $x\in D(A)$ and $Ax=y$. To show (iii) we let

$$U(s):=\frac{R(s)-k(s)}{(a*k)(s)},$$

and then we claim that

$$U(s)(a*R)(t)=(R(t)-k(t))\Big[\frac{(a*R)(s)}{(a*k)(s)}\Big]. \tag{2.4}$$

To show (2.4) we first recall that $R(s)R(t)=R(t)R(s)$ for $s,t\geq 0$. Then

$$A(a*R)(s)(a*R)(t)=A(a*R)(t)(a*R)(s). \tag{2.5}$$

Now applying (1.1) it follows that(2.5) equals

$$(R(s) - k(s))(a * R)(t) = (R(t) - k(t))(a * R)(s)$$

and hence

$$\frac{R(s) - k(s)}{(a * k)(s)}(a * R)(t) = (R(t) - k(t))\frac{(a * R)(s)}{(a * k)(s)} \tag{2.6}$$

and (2.4) follows. Now (2.2) implies that there exists a sequence $(s_n) \subset \mathbb{R}_+$ such that

$$\sup_n \|U(s_n)x\| = \sup_n \left\| \frac{R(s_n)x - k(s_n)x}{(a * k)(s_n)} \right\| < \infty.$$

Hence by the reflexivity of X there is a subsequence, say (s'_n), such that for every $x^* \in X^*$

$$\langle x^*, U(s'_n)x \rangle \to \langle x^*, y \rangle \quad \text{when} \quad n \to \infty$$

for some $y \in X$. Hence for $t \geq 0$

$$\langle x^*, U(s'_n)(a * R)(t)x \rangle \to \langle x^*, (a * R)(t)y \rangle \quad \text{when} \quad n \to \infty. \tag{2.7}$$

On the other side, by (2.4) we also have that

$$\langle x^*, U(s'_n)(a * R)(t)x \rangle = \langle x^*, (R(t) - k(t))\frac{(a * R)(s'_n)}{(a * k)(s'_n)}x \rangle. \tag{2.8}$$

Now the right hand side of (2.8) approaches to $\langle x^*, (R(t) - k(t))x \rangle$ as $n \to \infty$. Then by (2.7) and (2.8) we get

$$\langle x^*, (a * R)(t)y \rangle = \langle x^*, (R(t) - k(t))x \rangle.$$

for all $x^* \in X^*$. Thus $\frac{(R(t) - k(t))}{(a * k)(t)}x = \frac{(a * R)(t)y}{(a * k)(t)}$. Hence we obtain that

$$\frac{(R(t) - k(t))}{(a * k)(t)}x \to y \quad \text{when} \quad t \to 0$$

and the proof of (*iii*) follows by (*ii*).

□

The following corollary is an extension of [14, Theorem 2.1] when the kernels k and a are positive. We remark that in [14] the characterization of the domain of A was given under the hypothesis that k is increasing, and $D(A)$ is dense. Here this two conditions have been removed.

Corollary 2.2. *Let $R(t)$ be an (a,k) regularized resolvent family with generator A and such that $\|R(t)\| \le Mk(t)$. Then*

$$D(A_Y) = \{x \in Y : \quad \lim_{t\to 0} \frac{(R(t)-k(t))}{(a*k)(t)}x \quad \text{exists}\}.$$

Moreover

$$A_Y x = \lim_{t\to 0} \frac{(R(t)-k(t))}{(a*k)(t)}x \quad \text{for all} \quad x \in D(A_Y). \tag{2.9}$$

Proof. Let

$$D = \{x \in Y : \quad \lim_{t\to 0} \frac{(R(t)-k(t))}{(a*k)(t)}x \quad \text{exists}\}.$$

Then $D \subseteq D(A_Y)$ by (ii) of Theorem 2.1. Now if $x \in D(A_Y)$ then

$$\begin{aligned} \lim_{t\to 0} \frac{(R(t)-k(t))}{(a*k)(t)}x &= \lim_{t\to 0} \frac{(a*R)(t)}{(a*k)(t))}Ax \\ &= Ax. \end{aligned}$$

by (i) of Theorem 2.1. Hence $x \in D$.

□

Remark 2.3. We remark that (2.9) was proved for resolvent families by J.-C.Chang and S.-Y.Shaw [6, Proposition 2.2(i)] and by H. Liu and S.-Y.Shaw for n-times integrated solution families, [10, Proposition 2.2 (c)]; see also [8, Theorem 2.5].

A direct consequence of Corollary 2.2 and (1.1) is the following Proposition. We assume that $R(t)$ is defined for all $t \in \mathbb{R}$ by allowing $t < 0$ in (1.1).

Proposition 2.4. *Let $(R(t))_{t\in\mathbb{R}}$ be an (a,k) regularized resolvent with generator A on a Hilbert space X . Suppose that a and k are positive even functions defined on $\mathbb{R}$. Then A is skew adjoint if and only if $R(-t) = R^*(t)$ for all $t \in \mathbb{R}$.*

Proof. Since a and k are even functions, then the convolution $a*k$ is an odd function defined on $\mathbb{R}$. Suppose now that $R(-t) = R^*(t)$ for all $t \in \mathbb{R}$. Thus by Corollary 2.2 we conclude that $A = -A^*$. Conversely, assume that A is skew adjoint. Then define

$S(t) = R^*(-t)$. Hence by taking the adjoint in (1.1), it follows

$$\begin{aligned} S(t) = R^*(-t) &= k(-t)I + A^* \int_0^{-t} a(-t-s)R^*(s)ds \\ &= k(t)I + \int_0^t a(-\tau)R^*(-t+\tau)d\tau \\ &= k(t)I + \int_0^t a(\tau)S(t-\tau)d\tau. \end{aligned}$$

But $R(t)$ the unique solution of (1.1)(see [12, Remark 2.4]); thus $S(t) = R(t)$ for all $t \in \mathbb{R}$. □

3. A Favard class with kernels

The following corresponds to a natural extension of the Favard class frequently used in approximation theory for semigroups; see [2].

Definition 3.1. *Let a and k be continuous and positive. Let A be the generator of an (a,k) regularized resolvent $\{R(t)\}_{t\geq 0}$ on X. We define the Favard class of A with kernels a and k as*

$$F_{a,k} = \{\ x \in X \ : \ \sup_{t>0} \frac{||R(t)x - k(t)x||}{(a*k)(t)} < \infty\ \} \tag{3.1}$$

Remark 3.2.

(1) By the definition it follows that $D(A) \subset F_{a,k}$. Thus for different pairs of functions $a(t)$ and $k(t)$ we obtain different Favard classes which may be considered as interpolation spaces between $D(A)$ and X.

(2) When $a(t) \equiv 1$, and $k(t) \equiv 1$ we recall that $R(t)$ corresponds to a bounded C_0-semigroup generated by A. Then the Favard class is;

$$F_{1,1} = \{\ x \in X \ : \ \sup_{t>0} \frac{||R(t)x - x||}{t} < \infty\ \}, \tag{3.2}$$

see e.g., [2]. The proof of the following is immediate.

Proposition 3.3. *The Favard class $F_{a,k}$, is a Banach space with respect to the norm $|x|_{F_{a,k}} = ||x|| + \sup_{t>0} \dfrac{||R(t)x - k(t)x||}{(a*k)(t)}$.*

We now characterize the Favard class $F_{a,k}$.

Theorem 3.4. *Let A be a linear and closed operator with dense domain $D(A)$ in a Banach space X. Suppose that A generates a uniformly bounded (a,k) regularized resolvent $\{R(t)\}_{t\geq 0}$. Assume, that the Laplace transform $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$ exists for $\lambda > 0$ and satisfy $\sup_{t>0} \dfrac{(1*a)(t)}{(k*a)(t)} < \infty$ and $\lim_{\lambda\to 0^+} \hat{a}(0) = \infty$. Then*

$$F_{a,k} = \{x \in X \quad : \quad \sup_{\lambda>0} || \frac{1}{\hat{a}(\lambda)} A\, (\frac{1}{\hat{a}(\lambda)} - A)^{-1} x|| < \infty\ \}$$

In particular, $F_{a,k}$ does not depend on k.

Proof. Since $\|R(t)\| \le M$, then $\widehat{R}(\lambda)$ exists for all $\lambda > 0$. Moreover,

$$\widehat{R}(\lambda) = \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}. \tag{3.3}$$

Let $x \in F_{a,k}$ then there is $J_x > 0$ such that $\dfrac{||R(t)x - k(t)x||}{(k * a)(t)} \le J_x$ for $t > 0$. Now for all $\lambda > 0$ we have that, $A\widehat{R}(\lambda) = \dfrac{1}{\widehat{a}(\lambda)}(\widehat{R}(\lambda) - \widehat{k}(\lambda))$ by (3.3). But

$$\frac{1}{\widehat{a}(\lambda)}(\widehat{R}(\lambda) - \widehat{k}(\lambda))x = \frac{1}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda s} \frac{(R(s) - k(s))}{(a * k)(s)}\, (a * k)(s)\, x ds.$$

Hence $\|A\widehat{R}(\lambda)x\| \le J_x \hat{a}(\lambda)^{-1}\widehat{(a * k)}(\lambda) = J_x \widehat{k}(\lambda)$. Therefore by (3.3)

$$\sup_{\lambda>0} ||A(I - \widehat{a}(\lambda)A)^{-1}x|| \le J_x < \infty\,.$$

Conversely, let $x \in X$ be such that $\sup_{\lambda>0} ||(I - \widehat{a}(\lambda)A)^{-1}x|| =: N_x$. Now, from the identity

$$\hat{k}(\lambda)^{-1}\widehat{R}(\lambda) - \hat{a}(\lambda)\hat{k}(\lambda)^{-1}A\widehat{R}(\lambda) = I,$$

we obtain $x = \hat{k}(\lambda)^{-1}\widehat{R}(\lambda)x - \hat{a}(\lambda)\hat{k}(\lambda)^{-1}A\widehat{R}(\lambda)x =: x_\lambda - y_\lambda$. But $x_\lambda = \hat{k}(\lambda)^{-1}\widehat{R}(\lambda)x$ is in $D(A)$ and $\|Ax_\lambda\| = \|\widehat{k}(\lambda)^{-1}A\widehat{R}(\lambda)x\| \le N_x$. By the resolvent identity (1.1) follows that

$$||R(t)x_\lambda - k(t)x_\lambda|| \le (a * ||R(\cdot)||)(t)\, ||Ax_\lambda||.$$

But $R(t)$ is uniformly bounded, hence

$$\begin{aligned}(a * \|R(\cdot)\|(t)\|Ax_\lambda\| &\le M\|Ax_\lambda\|(1 * a)(t)\\ &= M\, ||\hat{k}(\lambda)^{-1}A\widehat{R}(\lambda)x||(1 * a)(t)\\ &= M\, N_x\, (1 * a)(t).\end{aligned}$$

On the other hand,

$$||R(t)y_\lambda - k(t)y_\lambda|| \le ||R(t)y_\lambda|| + ||k(t)y_\lambda|| \le (M + k(t))N_x \hat{a}(\lambda)\,.$$

Dividing by $(a * k)(t)$ we have that, for all $\lambda > 0$,

$$\frac{||R(t)x - k(t)x||}{(a * k)(t)} \le MN_x \frac{(1 * a)(t)}{(a * k)(t)} + \frac{M + k(t)}{(a * k)(t)} N_x \hat{a}(\lambda)\,.$$

Since $\lim_{\lambda\to 0^+}\hat{a}(0)=\infty$ we obtain that $\hat{a}(\lambda)$ is surjective, hence there exists $\lambda_t>0$ so that $(\hat{a}(\lambda_t))^{-1}=\dfrac{M+k(t)}{(k*a)(t)}$. Since $\dfrac{(1*a)(t)}{(k*a)(t)}$ is bounded. Then it follows that there exits $C_x>0$ such that

$$\frac{||R(t)x-k(t)x||}{(a*k)(t)}\le C_x,$$

for all $t>0$.

□

Remark 3.5. We notice that the spaces $F_{a,k}$ are also independent of the kernel a, since a is positive and $\widehat{a}(\lambda)\to 0$ as $\lambda\to\infty$. Hence for the type of kernels (a,k) we have considered, the Favard classes as introduced above coincides with $F_{1,1}$ which corresponds to the semigroup case. This property for the Favards classes has been already observed by Jung in the resolvent case that is $k(t)\equiv 1$; see [8, Proposition 3.3].

4. The sun-dual

We define $X^{\odot}:=\overline{D(A^*)}^{X^*}$, and $A^{\odot}$ the part of A^* in $X^{\odot}$. Furthermore we let $R^{\odot}(t)=R^*(t)|_{X^{\odot}}$. Henceforth we assume that A has dense domain in X.

The following is the main result of this section.

Theorem 4.6. *Let A be the generator of an (a,k) regularized resolvent $R(t)$ such that $\|R(t)\|\le Mk(t)$. Assume that A is densely defined. Then $R^{\odot}(t)$ is a strongly continuous (a,k) regularized resolvent with generator $A^{\odot}$.*

Proof. Let $H(\lambda):=\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda)-A)^{-1}$, for $\lambda>\omega$. Since $\rho(A)\subseteq\rho(A^*)$ and $\langle x^*,H(\lambda)x\rangle=\langle\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda)-A^*)^{-1}x^*,x\rangle$. Then for $\|x\|=1$, $\|x^*\|=1$ we have that

$$\begin{aligned}|\langle\Big[\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda)-A^*)^{-1}\Big]^{(n)}x^*,x\rangle| &\le \|H^{(n)}(\lambda)\|\\ &\le \frac{Mn!}{(\lambda-\omega)^{n+1}}\qquad n\in\mathbb{N}_0\end{aligned}\tag{4.4}$$

Moreover

$$|\langle\Big[\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda)-A^*)^{-1}\Big]x^*,x\rangle|\le\|H(\lambda)\|\|x\|\|x^*\|\tag{4.5}$$

On the other hand

$$\|H(\lambda)\| = \|\widehat{R}(\lambda)\| \leq M\widehat{k}(\lambda) \qquad \text{for} \quad (\lambda > \omega)$$

since $\|R(t)\| \leq Mk(t)$. Hence, for $\lambda > \omega$ we have $\|H(\lambda)\| \leq M\widehat{k}(\lambda)$. Thus from (4.5) we obtain

$$|\langle \Big[\frac{1}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda) - A^*)^{-1}\Big] x^*, x\rangle| \leq M\|x\|\|x^*\| \quad (\lambda > \omega) \tag{4.6}$$

Let $J(\mu)$ be the restriction of $(\mu - A^*)^{-1}$ to $X^{\odot}$. Then $J(\lambda)$ has dense range since $\text{Ran}(J(\lambda)) = D(A^*)$ which is dense in $X^{\odot}$. Now let $1/\mu =: \widehat{a}(\lambda)$, then (4.6) yields

$$\|\mu J(\mu)\| \leq M \tag{4.7}$$

Hence by (4.7) there is μ_0 such that $\sup_{\mu > \mu_0} \|\mu J(\mu)\| < \infty$. Since A is densely defined then $\rho(A^*) = \rho(A)$, thus A^* is closed since $\rho(A) \neq \emptyset$. Furthermore, $J(\lambda)$ is a pseudo resolvent since

$$J(\mu) - J(\nu) = (\nu - \mu)J(\mu)J(\nu).$$

Thus $J(\mu)$ is the resolvent of a unique closed and densely defined operator $A^{\odot}$ on $X^{\odot}$. Since for every $y^* \in D(A^*_{X^{\odot}})$ we have $(\mu - A^{\odot})^{-1}(\mu - A^*)y^* = y^*$ then $A^{\odot}$ is the part of A^* in $X^{\odot}$, that is $A^{\odot} = A^*_{X^{\odot}}$. Hence $A^{\odot}$ is the generator of an (a, k)-regularized resolvent $R^{\dagger}(t)$ on $X^{\odot}$, by (4.4) and the generation Theorem for regularized resolvent families of [12]. Since the Laplace transform of $R^{\dagger}(t)$ is given by

$$\widehat{R^{\dagger}}(\lambda)y^* = \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda) - A^{\odot})^{-1}y^*.$$

Then,

$$\begin{aligned}
\langle \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda) - A^{\odot})^{-1}y^*, x\rangle &= \langle \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda) - A^*_{X^{\odot}})^{-1}y^*, x\rangle \\
&= \langle y^*, \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda) - A)^{-1}x\rangle.
\end{aligned}$$

Since $\widehat{R^*}(\lambda) = (\widehat{R}(\lambda))^*$, we obtain that,

$$\begin{aligned}
\langle \widehat{R^{\dagger}}(\lambda)y^*, x\rangle &= \langle y^*, \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}(1/\widehat{a}(\lambda) - A)^{-1}x\rangle \\
&= \langle y^*, \widehat{R}(t)x\rangle \\
&= \langle \widehat{R^*}(\lambda)y^*, x\rangle
\end{aligned}$$

for every $y^* \in X^{\odot}$ and $x \in X$. Hence by uniqueness of the Laplace transform follows that

$$R^{\dagger}(t) = R^*(t)_{X^{\odot}} = R^{\odot}(t).$$

□

Remark 4.7. Note that if $k \in C^1(R_+)$ and $k(0^+) < \infty$ then the hypothesis of the theorem are satisfied. Furthermore, A turns out to be the generator of a resolvent family; see [12, Proposition 2.5].

Corollary 4.8. *Let $R(t)$ be an (a,k) regularized resolvent with generator A and such that $\|R(t)\| \leq Mk(t)$. Then*

(a) $X^{\odot} = \{x^* \in X^* : \quad \lim_{t\to s} \|R^*(t)x^* - R^*(s)x^*\| = 0\}$,

(b) $D(A^{\odot}) \subset D(A^*) \subset X^{\odot}$,

(c) $\|R^*(t)x^* - k(t)x^*\| \leq (1 * a)(t) \sup_{0\leq\tau\leq t} \|R(\tau)\| \|A^*x^*\| \qquad (x^* \in D(A^*))$

(d) $\|R^*(t)x^* - k(t)x^*\| \leq M(a * k)(t)\|A^*x^*\| \qquad (x^* \in D(A^*))$

Proof. The proof of (a) and (b) follows by the definition of $X^{\odot}$. To show (c) let $\|x\| = 1$. Then for $x^* \in D(A^*)$

$$\begin{aligned} |\langle R^*(t)x^* - k(t)x^*, x\rangle| &\leq (a * |\langle R^*(\cdot)A^*x^*, x\rangle|)(t) \\ &= (a * |\langle A^*x^*, R(\cdot)x\rangle|)(t) \\ &\leq \|A^*x^*\|(a * \|R(\cdot)\|)(t) \\ &\leq \sup_{0\leq\tau\leq t} \|R(\tau)\| \|A^*x^*\|(1 * a)(t). \end{aligned}$$

Next to show (d) we recall that $R(t)$ satisfy $\|R(t)\| \leq Mk(t)$. Hence by the same reasoning as above

$$\begin{aligned} |\langle R^*(t)x^* - k(t)x^*, x\rangle| &\leq \|A^*x^*\|(a * \|R(\cdot)\|)(t) \\ &\leq M(a * k)(t)\|A^*x^*\|, \end{aligned}$$

and the proof follows.

□

Theorem 4.9. *Let $R(t)$ be an (a,k) regularized resolvent with generator A. Then A^* equals to the weak closure of $A^{\odot}$.*

Proof. Since A^* is weakly* closed, we will show that

$$graph(A^{\odot}) = \{(A^{\odot}f^*, f^*) \in X^* \times X^* : \quad f^* \in D(A^{\odot})\}$$

is weakly* dense in $graph(A^*)$. Let $(f,g)\in X\times X$ be such that

$$\langle A^{\odot}f^*, f\rangle - \langle f^*, g\rangle = 0, \qquad f^*\in D(A^{\odot}) \tag{4.8}$$

since $(X^*\times X^*)^{w^*}$ can be identified with $X\times X$. Thus, by (4.8)

$$\langle A^{\odot}(a*R^{\odot})(t)f^*, f\rangle = \langle (a*R^{\odot})(t)f^*, g\rangle.$$

It then follows

$$\begin{aligned}\langle f^*, (R(t)-k(t))f\rangle &= \langle R^{\odot}(t)f^* - k(t)f^*, f\rangle\\ &= \langle A^{\odot}(a*R^{\odot})(t)f^*, f\rangle\\ &= \langle (a*R^{\odot})(t)f^*, g\rangle.\end{aligned}$$

which implies that

$$\langle f^*, (R(t)-k(t))f\rangle = \langle f^*, (a*R)(t)g\rangle \qquad \text{for all} \quad f^*\in X^*.$$

Hence $(R(t)-k(t))f = (a*R)(t)g$. Thus

$$\frac{(R(t)-k(t))f}{(a*k)(t)} = \frac{(a*R)(t)g}{(a*k)(t)}. \tag{4.9}$$

Then by Theorem 2.1(i) and Corollary 2.2 it follows from (4.9) that $f\in D(A)$ and $Af=g$. Therefore the weak* continuous functional defined by (f,g) in (4.8) vanishes for all $f^*\in D(A^*)$.

□

5. Spectral properties of resolvent families

For a closed operator A we denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$, and $\sigma_a(A)$, the spectrum, the point spectrum, the residual spectrum, and the approximate spectrum of A, respectively. We recall that $\sigma_r(A)=\sigma_p(A^*)$ by the Hahn-Banach theorem, provided the adjoint A^* of A is well defined, i.e. A is densely defined.

Proposition 5.1. *Let $R(t)$ be an (a,k) regularized resolvent with generator A. Then $\sigma_p(R^*(t))=\sigma_p(R^{\odot}(t)), (t\geq 0)$ and $\sigma_p(A^*)=\sigma_p(A^{\odot})$.*

Proof. Since an eigenvector of $R^*(t)$ always belongs to $X^{\odot}$, the first identity follows from $R^{\odot}(t)=R^*(t)|_{X^{\odot}}$.

For the second identity we first recall that $D(A^*)\subset X^{\odot}$ by 4.8(b). Hence for all $x^*\in D(A^*)$ and $x\in X$ we have

$$\begin{aligned} < R^*(t)x^* - k(t)x^*, x > &= < x^*, R(t)x - k(t)x >\\ &= (a* < R^*(\cdot)A^*x^*, x >)(t),\end{aligned}$$

Now if $A^*x^* = \lambda x^*$, then by (1.1) and the above identity show that

$$\begin{aligned} < \frac{R^*(t)x^* - k(t)x^*}{(a*k)(t)} - \lambda x^*, x > &= \frac{1}{(a*k)(t)}(a* < R^*(\cdot)\lambda x^*, x >)(t) \\ &- < \lambda x^*, x > \\ &= \frac{\lambda}{(a*k)(t)}(a* < R^*(\cdot)x^* - k(\cdot)x^*, x >)(t). \end{aligned}$$

Hence

$$\|\frac{R^*(t)x^* - k(t)x^*}{(a*k)(t)} - \lambda x^*\| \le |\lambda|||x^*|| \sup_{0\le s\le t} ||R(s)x - k(s)x||.$$

Now, we let $t \to 0$, in the last inequality. Hence we obtain that $x^* \in D(A^\odot)$ and $A^\odot x^* = \lambda x^*$, thus $\lambda \in \sigma_p(A^\odot)$.

Conversely, if $\lambda \in \sigma_p(A^\odot)$ and $A^\odot x^* = \lambda x^*$ for some $x^\odot \in D(A^\odot)$, then for all $x \in D(A)$ we have by Theorem 2.1 and Theorem 4.6 that

$$\begin{aligned} < x^\odot, Ax > &= \lim_{t\to 0} < x^\odot, \frac{R(t)x - k(t)x}{(a*k)(t)} > \\ &= \lim_{t\to 0} \frac{1}{(a*k)(t)} < R^\odot(t)x^\odot - k(t)x^\odot, x > \\ &= < A^\odot x^\odot, x > \\ &= \lambda < x^\odot, x > . \end{aligned}$$

This shows that $x^\odot \in D(A^*)$ and $A^*x^\odot = \lambda x^\odot$, so $\lambda \in \sigma_p(A^*)$. □

Note if $k \equiv 1$ then $R(t)$ defines a resolvent family. Hence Proposition 5.1 has the following application.

Corollary 5.2. *Let $S(t)$ be a resolvent family with generator A. Then $\sigma_p(S^*(t)) = \sigma_p(S^\odot(t)), (t \ge 0)$ and $\sigma_p(A^*) = \sigma_p(A^\odot)$.*

To state the next the Theorem we take into account the following considerations. For each $\lambda \in \mathbb{C}$, we denote by $s(t, \mu)$, the unique solution of the scalar valued convolution equation

$$s(t, \lambda) = a(t) + \lambda \int_0^t a(t-\tau)s(\tau, \lambda)d\tau, \quad t \ge 0.$$

We also define

$$r(t, \lambda) := k(t) + \lambda \int_0^t s(t-\tau, \lambda)k(\tau)d\tau.$$

Theorem 5.3. *Let $R(t)$ be an (a, k) regularized resolvent with generator A. Then*

$$\sigma(R(t)) \supset r(t, \sigma(A)), \quad t \ge 0.$$

Proof. Let $x \in D(A)$. Then identity (1.1) and

$$R(t)x = k(t)x + (a * AR)(t)x, \qquad (t \geq 0) \tag{5.1}$$

show that

$$\begin{aligned}(s * (\lambda - A)R)(t)x &= \lambda(s * R)(t)x - (s * AR)(t)x \\ &= \lambda(s * R)(t)x - ([a + \lambda(a * s)] * AR)(t)x \\ &= \lambda(s * R)(t)x - (a * AR)(t)x - \lambda(a * s * AR)(t)x \\ &= \lambda(s * R)(t)x - [R - k](t)x - \lambda(s * [R - k])(t)x \\ &= k(t)x + \lambda(s * k)(t)x - R(t)x \\ &= r(t, \lambda)x - R(t)x,\end{aligned}$$

for all $\lambda \in \mathbb{C}$ and $t \geq 0$. Analogously, using the closedness of A, we prove

$$(\lambda - A)\int_0^t s(t - \tau, \lambda)R(\tau)x d\tau = r(t, \lambda)x - R(t)x$$

for all $x \in X$.

Suppose $r(t, \lambda) \in \rho(R(t))$ for some $\lambda \in \mathbb{C}$ and $t \geq 0$, and denote the inverse of $r(t, \lambda) - R(t)$ by $L_{\lambda,t}$. Since $L_{\lambda,t}$ commutes with $R(t)$ and hence also with A, we have

$$(\lambda - A)\int_0^t s(t - \tau, \lambda)R(\tau)L_{\lambda,t}x d\tau = x$$

for all $x \in X$ and

$$\int_0^t s(t - \tau, \lambda)R(\tau)L_{\lambda,t}(\lambda - A)x d\tau = x$$

for all $x \in D(A)$. This shows that the bounded operator B_λ defined by

$$B_\lambda x = \int_0^t s(t - \tau, \lambda)R(\tau)L_{\lambda,t}x d\tau$$

is a two-sided inverse of $\lambda - A$. It follows that $\lambda \in \rho(A)$.

□

Remark 5.4. When $a \equiv 1$ and $k \equiv 1$, this is the well known spectral inclusion for C_0-semigroups. If $a(t) \equiv t$ and $k \equiv 1$ it gives the spectral inclusion for cosine families (cf.[15]). If $a(t) = t^{\alpha-1}(\alpha \geq 1)$ and $k \equiv 1$, it corresponds to the spectral inclusion for α-times resolvent families studied recently in [11, Theorem 3.2]. All the other cases, e.g. convoluted semigroups, or even resolvent families, including the case of α-times resolvent families, are new.

In the following we consider spectral inclusions for the point, residual and approximate point spectrum.

Theorem 5.5. *Let $R(t)$ be an (a,k) regularized resolvent with generator A. Then*

$$\sigma_p(R(t)) \supset r(t, \sigma_p(A)), \quad t \geq 0.$$

Proof. If $\lambda \in \sigma_p(A)$ and $x \in D(A)$ is an eigenvector corresponding to λ, the identity

$$\int_0^t s(t-\tau, \lambda) R(\tau)(\lambda - A) x d\tau = r(t, \lambda)x - R(t)x \tag{5.2}$$

valid for all $x \in D(A)$, shows that $R(t)x = r(t,\lambda)x$, i.e. $r(t,\lambda)$ is an eigenvalue of $R(t)$ with eigenvector x. This proves the inclusion.

□

The following result give information about the residual spectrum.

Theorem 5.6. *Let $R(t)$ be an (a,k) regularized resolvent with generator A. Assume that A is densely defined. Then*

$$\sigma_r(R(t)) \supset r(t, \sigma_r(A)), \quad t \geq 0.$$

Proof. We have $\sigma_r(R(t)) = \sigma_p(R^*(t)) = \sigma_p(R^{\odot}(t))$ and $\sigma_r(A) = \sigma_p(A^*) = \sigma_p(A^{\odot})$. The theorem now follows from Theorem 5.5 and Theorem 4.6 applied to the (a,k)-regularized resolvent $R^{\odot}(t)$.

□

We end this paper with the following result.

Theorem 5.7. *Let $R(t)$ be an (a,k) regularized resolvent with generator A. Then*

$$\sigma_a(R(t)) \supset r(t, \sigma_a(A)), \quad t \geq 0.$$

Proof. If $\lambda \in \sigma_a(A)$, then there is a sequence $(x_n) \subset D(A)$, $||x_n|| = 1$ such that $||(\lambda - A)x_n|| \to 0$ as $n \to \infty$, hence from (5.2) we obtain

$$||r(t,\lambda)x_n - R(t)x_n|| = ||(s * R)(t)(\lambda - A)x_n|| \leq M||(\lambda - A)x_n|| \to 0$$

as $n \to \infty$, so we have $r(t,\lambda) \in \sigma_a(R(t))$. □

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