SINGULAR PERTURBATIONS FOR INTEGRO-DIFFERENTIAL EQUATIONS.

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ABSTRACT. We study the singular perturbation problem

$$(E_\epsilon) \qquad \epsilon^2 u''_\epsilon(t) + u'_\epsilon(t) = Au_\epsilon(t) + (K * Au_\epsilon)(t) + f_\epsilon(t), t \geq 0, \epsilon > 0,$$

for the integrodifferential equation

$$(E) \qquad w'(t) = Aw(t) + (K * Aw)(t) + f(t), t \geq 0,$$

in a Banach space, when $\epsilon \to 0^+$. Under the assumption that A is the generator of a strongly continuous cosine family and under some regularity conditions on the scalar-valued kernel K we show that problem (E_ϵ) has a unique solution $u_\epsilon(t)$ for each small $\epsilon > 0$. Moreover $u_\epsilon(t)$ converges to $u(t)$ as $\epsilon \to 0^+$, the unique solution of equation (E).

1. INTRODUCTION

The purpose of this paper is the study of linear integro differential equations of convolution type given by

$$(1.1) \qquad \begin{cases} \epsilon^2 u''(t,\epsilon) + u'(t,\epsilon) = Au(t,\epsilon) + \int_0^t K(t-s)Au(s,\epsilon)ds + f(t,\epsilon), & t \geq 0 \\ u(0,\epsilon) = u_0(\epsilon), \\ u'(0,\epsilon) = u_1(\epsilon). \end{cases}$$

and

$$(1.2) \qquad \begin{cases} w'(t) = Aw(t) + \int_0^t K(t-s)Aw(s)ds + f(t), & t \geq 0 \\ w(0) = w_0. \end{cases}$$

on arbitrary Banach space X. The problem to study the behavior of (1.1) as $\epsilon \to 0$ is called a *singular perturbation problem* .

In the above formulation A is a closed linear operator with densely defined domain in X and K is a real-valued function.

Questions concerning the singular perturbation problem occur frequently in linear viscoelasticity theory, for instance when $A = \Delta$ is the Laplace operator. In this case, $\epsilon = \rho$ represents the material density as $\rho \to 0$. For more information on the subject of viscoelasticity theory we refer to the monographs of Christensen [3], Mainardi [16] and Renardy, Hrusa and Nohel [19].

The singular perturbation problem for (1.1) with $K \equiv 0$ was first considered by Kisyński [8] in the case where A is a self adjoint, positive definite operator on a Hilbert space. Latter, Sova [20] study the problem under the assumptions that A is the generator of a strongly continuous cosine function. The most precise results for the homogeneous problem are those by Kisyński [9] who applied the theory of monotonic functions and gave explicit solutions of (1.1). See also [4] and [2] for others

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developments. The treatment of the nonhomogeneous equation is due to Fattorini [5, Chapter VI]; see also the references therein.

When $K(t)$ is vector-valued and different from zero, James H. Liu [12] (see also [6] and [10]) has treated the problem by considering the integral term, $(K * Au)(t)$, as a perturbation of the main term $Au(t)$ and under the assumption of existence of solutions. Then the problem is treated as in the case $K \equiv 0$.

In this article we solve the singular perturbation problem by a new method. In constrast with the previous works on the subject we do not need to assume the existence of solutions beforehand. The strategy is to apply the notion of resolvent families to prove existence and uniqueness of solution when A is the generator of a cosine family.

The key to solve the singular perturbation problem relies on result on convergence of resolvent families due to Prüss [18, Corollary 6.5], which extends the classical Trotter-Kato theorem on convergence and approximation of C_0-semigroups.

The conditions that we impose on the kernel $K(t)$ are taken from the known examples which occur in several of the concrete applications. For instance, the important class $K(t) = be^{-at}$ with $a > 0$ and $a + b > 0$ is shown to satisfy our assumptions. For this and more examples, see the monograph [18].

We give an explicit representation of the solution. In case that A generates a cosine family we prove in section 2 that there exists a resolvent family $R_\epsilon(t)$ and a sequence of functions $a_\epsilon(t)$ $(\epsilon \geq 0)$ such that $u(t, \epsilon)$ can be written as

$$u(t,\epsilon) \;=\; R_\epsilon(t)u_0(\epsilon) + \int_0^t e^{(-1/\epsilon^2)(t-s)} R_\epsilon(s)u_1(\epsilon)ds + \frac{1}{\epsilon^2}\int_0^t e^{(-1/\epsilon^2)(t-s)}(R_\epsilon * f_\epsilon)(s)ds.$$

2. Preliminaries

We assume that X is a complex Banach space, A a closed linear unbounded operator in X with dense domain $D(A)$, and $a \in L^1_{loc}(\mathbb{R}_+)$ a scalar kernel $\neq 0$. We consider the Volterra equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \geq 0, \tag{2.1}$$

where $f \in C(\mathbb{R}; X)$. We recall the following definition which is a fundamental notion in the theory of linear Volterra equations, see Prüss [18]. Here the symbol $\widehat{\ }$ denotes the Laplace transform and $\rho(A)$ stands for the resolvent set of the operator A.

Definition 2.1. A family $\{R(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ of bounded linear operators in X is called a resolvent family for (2.1) if the following three conditions are satisfied.

(R1) $R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = I$;

(R2) $R(t)$ commutes with A.

(R3) $R(t)x = x + \int_0^t a(t-s)AR(s)xds$ for all $x \in D(A)$, $\quad t \geq 0$.

A resolvent family is called exponentially bounded if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$||R(t)|| \leq Me^{\omega t}, \quad \text{for all } t \geq 0;$$

the pair (M, ω) is called a type of $R(t)$.

Remark 2.2.

If $a(t) \equiv 1$ we obtain the concept of C_0 semigroup and when $a(t) = t$, we obtain the theory of strongly continuous cosine families, related to the abstract Cauchy problem of second order.

A very important tool in the theory of resolvent families is the generation Theorem, which give us necessary and sufficient conditions in order to have existence of a such family. The result is as follows; see [18, Theorem 1.3].

Theorem 2.3. *Let A be a closed linear unbounded operator with densely defined domain $D(A)$, defined on a Banach space X and let $a \in L^1_{loc}(\mathbb{R}_+)$ satisfying $\int_0^\infty e^{-\omega t}\,|a(t)|\,dt < \infty$. Then (2.1) admits a resolvent family $\{R(t)\}_{t\geq 0}$ of type (M,ω) if and only if the following two conditions are satisfied*

(K1) $\hat{a}(\lambda) \neq 0$ *and* $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$, *for all* $\lambda > \omega$.

(K2) $H(\lambda) := \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}$ *satisfy the inequality*

$$||H^{(n)}(\lambda)|| \leq M\, n!\, (\lambda-\omega)^{-(n+1)}, \; \lambda > \omega, \; n \in \mathbb{N}_0 .$$

Throughout this paper, the symbol $*$ always denotes the finite convolution. We will follow the same notations as those given in the book [18].

3. Convergence of resolvents and singular perturbation

We will consider the problems

$$u_\epsilon(t) = f_\epsilon(t) + \int_0^t a_\epsilon(t-s)Au_\epsilon(s)ds, \quad t \geq 0, \quad \epsilon \geq 0, \tag{3.1}$$

where $f_\epsilon \to f_0$ and $a_\epsilon \to a_0$ in an appropriate sense as $\epsilon \to 0$. Assuming the existence of resolvents $R_\epsilon(t)$ for (3.1) as well as the stability condition

$$\sup_{\epsilon>0} ||R_\epsilon(t)|| \leq Me^{\omega t}, \quad t \in \mathbb{R}_+.$$

It can be shown as in the case of the Trotter-Kato theorem on convergence of C_0-semigroups, the strong convergence $R_\epsilon(t) \to R_0(t)$ in X.

Theorem 3.1. *Let $\{a_\epsilon\}_{\epsilon\geq 0} \in L^1_{loc}(\mathbb{R}_+)$, and let A be a closed linear and densely defined operator on X, such that $\int_0^\infty e^{-\omega s}|a_\epsilon(s) - a_0(s)|ds \to 0$ as $\epsilon \to 0$. Assume (3.1) admits a resolvent $\{R_\epsilon(t)\}_{t\geq 0}$ in X for each $\epsilon > 0$ and such that the stability condition*

$$||R_\epsilon(t)|| \leq Me^{\omega t}, \quad t \in \mathbb{R}_+, \quad \epsilon > 0 \tag{3.2}$$

holds. Then there is a resolvent $R_0(t)$ of type (M,ω) for (3.1) with $\epsilon = 0$ and

$$R_\epsilon(t)x \to R_0(t)x \tag{3.3}$$

as $\epsilon \to 0$ uniformly on compact subsets of $\mathbb{R}_+ \times X$.

For a proof, see Prüss [18, Corollary 6.5]. Observe that existence of $R_0(t)$ need not to be assumed, but can be proved.

For $\epsilon > 0$ we define:

$$a_\epsilon(t) = 1 + (1*K)(t) - e_\epsilon(t) - (e_\epsilon * K)(t), \quad t \geq 0, \tag{3.4}$$

where $e_\epsilon(t) := e^{-\frac{1}{\epsilon^2}t}$ and for $\epsilon = 0$

$$a_0(t) = 1 + (1*K)(t), \quad t \geq 0. \tag{3.5}$$

We always assume that the Laplace transform $\hat{a}_\epsilon(\lambda)$ exists and is non-zero for all $\lambda > \omega, \; \epsilon > 0$ and some $\omega \in \mathbb{R}$. Then from (3.4) and (3.5) we easily see that

$$\hat{a}_\epsilon(\lambda) \to \hat{a}_0(\lambda), \tag{3.6}$$

as $\epsilon \to 0$ for all λ sufficiently large.

In what follows we denote by $\mathcal{K}(\mathbb{R}, \epsilon_0)$ the set of all functions $K \in C^1(\mathbb{R})$ which satisfies the following conditions

1. $K(t) \geq 0$ for all $t > 0$ and $K(0) > 0$.
2. $K'(t) \leq 0$ for all $t \geq 0$.
3. $\lim_{t\to\infty} K(t) = 0$.
4. There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ the function

$$(\frac{1}{\epsilon^2}K(t) + K'(t))(e_\epsilon(t) + (e_\epsilon * K)(t)) - K(t)^2 \quad \text{is nonnegative.} \tag{3.7}$$

Example.

A calculation shows that the kernel $K(t) = be^{-at}$ where $a > 0, b > 0$ belongs to the class $\mathcal{K}(\mathbb{R}, \epsilon_0)$. In particular, condition (3.7) is satisfied for all $0 < \epsilon \leq \frac{1}{\sqrt{a+b}}$.

The following definition is due to Prüss [18, Definition 4.4, p.94].

Definition 3.2. A function $a : (0, \infty) \to \mathbb{R}$ is called a creep function if $a(t)$ is nonnegative, nondecreasing and concave.

A creep function $a(t)$ has the standard form

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(s)ds, \quad t > 0, \tag{3.8}$$

where $a_0 = a(0+) \geq 0, a_\infty = \lim_{t\to\infty} \frac{a(t)}{t} \geq 0$, and $a_1(t) = \dot{a}(t) - a_\infty$ is nonnegative, nonincreasing, $\lim_{t\to\infty} a_1(t) = 0$.

Lemma 3.3. *Suppose $K \in \mathcal{K}(\mathbb{R}, \epsilon_1)$. Then, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ we have that $a_\epsilon(t)$ is a creep function with $a_1^\epsilon(t)$ log- convex.*

Proof. Since $a_\epsilon(t) = (1 - e_\epsilon(t)) + ((1 - e_\epsilon) * K)(t)$, we obtain $a'_\epsilon(t) = \frac{1}{\epsilon^2}e_\epsilon(t) + \frac{1}{\epsilon^2}(e_\epsilon * K)(t)$. It follows that $a_\epsilon(t)$ is positive and nondecreasing. Moreover

$$a''_\epsilon(t) = (\frac{K(0)}{\epsilon^2} - \frac{1}{\epsilon^4})e_\epsilon(t) + \frac{1}{\epsilon^2}(e_\epsilon * K')(t) < 0$$

for all ϵ such that $0 < \epsilon < \frac{1}{\sqrt{K(0)}}$, and hence a_ϵ is creep for small $\epsilon > 0$. To show that $a_1^\epsilon(t)$ is log-convex, we recall that a_ϵ can be represented as

$$a_\epsilon(t) = a_0^\epsilon + a_\infty^\epsilon t + (1 * a_1^\epsilon)(t),$$

where $a_0^\epsilon = a_\epsilon(0) = 0$ for all $\epsilon > 0$ and $0 \leq a_\infty^\epsilon = \lim_{t\to\infty} \frac{a_\epsilon(t)}{t} = \lim_{t\to\infty}[\frac{a_0(t)}{t} - \frac{e_\epsilon}{t} - \frac{(e_\epsilon * K)(t)}{t}] \leq \lim_{t\to\infty} \frac{a_0(t)}{t} = \lim_{t\to\infty}[\frac{1}{t} + \frac{1}{t}(1 * K)(t)] = \lim_{t\to\infty} K(t) = 0$, by hypothesis. Thus $a_\epsilon(t) = (1 * a_1^\epsilon)(t)$ and hence

$$a_1^\epsilon(t) = a'_\epsilon(t). \tag{3.9}$$

Now we prove that $a'_\epsilon(t)$ is log-convex. Let $f_\epsilon(t) = log(a'_\epsilon(t))$, then

$$f''_\epsilon(t) = \frac{a'''_\epsilon(t)a'_\epsilon(t) - (a''_\epsilon(t))^2}{(a'_\epsilon(t))^2}.$$

Since

$$\epsilon^2 a''_\epsilon(t) = -a'_\epsilon(t) + K(t)$$

and

$$\epsilon^2 a'''_\epsilon(t) = \frac{1}{\epsilon^2}a'_\epsilon(t) - \frac{1}{\epsilon^2}K(t) + K'(t),$$

we obtain

$$\begin{aligned} \epsilon^2 a'''_\epsilon(t)\epsilon^2 a'_\epsilon(t) - (\epsilon^2 a''_\epsilon(t))^2 &= K(t)a'_\epsilon(t) + \epsilon^2 K'(t)a'_\epsilon(t) - K(t)^2 \\ &= (\frac{1}{\epsilon^2}K(t) + K'(t))(e_\epsilon(t) + (e_\epsilon * K)(t)) - K(t)^2. \end{aligned}$$

Thus, by hypothesis, $f_\epsilon''(t) \geq 0$ for sufficiently small $\epsilon > 0$. This finishes with the proof.

□

We recall that an infinitely differentiable function $f : (0, \infty) \to \mathbb{R}$ is called completely monotonic if

$$(-1)^n f^{(n)}(\lambda) \geq 0$$

for all $\lambda > 0, n = 0, 1, 2....$

Lemma 3.4. *Suppose $K \in C^1(\mathbb{R})$ satisfies $K' \leq 0, K(0) > 0$. Then for all $0 < \epsilon < \frac{1}{\sqrt{K(0)}}$ the function $\phi(\lambda) = \frac{1}{\sqrt{\lambda \hat{a}_\epsilon(\lambda)}}$ is completely monotonic.*

Proof. Since $a_1^\epsilon(0) = \frac{1}{\epsilon^2}$ and $a_0^\epsilon = 0, a_\infty^\epsilon = 0$, from (3.9) we have the identity $\lambda^2 \hat{a}_\epsilon(\lambda) = \lambda \hat{a}_1^\epsilon(\lambda) = \widehat{\dot{a_1}^\epsilon}(\lambda) + a_1^\epsilon(0) = \widehat{\dot{a_1}^\epsilon}(\lambda) + \frac{1}{\epsilon^2}$ and we can write

$$\frac{1}{\lambda\sqrt{\hat{a}_\epsilon(\lambda)}} = \frac{\epsilon}{[1 - (-\epsilon^2 \widehat{\dot{a_1}^\epsilon}(\lambda))]^{1/2}}.$$

Note that the right hand side is the composition of the absolutely monotonic function $\frac{1}{(1-x)^{1/2}}$ and the function $-\epsilon^2 \widehat{\dot{a_1}^\epsilon}(\lambda)$. But under our hypotheses $\dot{a_1}^\epsilon(t) = (\frac{K(0)}{\epsilon^2} - \frac{1}{\epsilon^4})e_\epsilon(t) + \frac{1}{\epsilon^2}(e_\epsilon * K')(t) < 0$, and hence $-\epsilon^2 \widehat{\dot{a_1}^\epsilon}(\lambda)$ is completely monotonic. Then, from [18, Proposition 4.1, p.91] the assertion follows.

□

Proposition 3.5. *If $K \in \mathcal{K}(\mathbb{R}, \epsilon_1)$ then the function $h_\epsilon(\lambda, t) := \frac{1}{\lambda\sqrt{\hat{a}_\epsilon(\lambda)}} e^{-\frac{1}{\sqrt{\hat{a}_\epsilon(\lambda)}}t}$ $(t > 0, \lambda > 0)$ is completely monotonic in λ for each $0 < \epsilon < \epsilon_1$.*

Proof. From Lemma 3.3 there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ we have that $a_\epsilon(t)$ is a creep function with $a_1^\epsilon(t)$ log- convex, it follows from [18, Lemma 4.2] that the function

$$\phi(\lambda) := \frac{1}{\sqrt{\hat{a}_\epsilon(\lambda)}}$$

is positive with $\phi'(\lambda)$ completely monotonic (i.e a Bernstein function, see [18, Definition 4.3, p.91]). Hence, applying [18, Proposition 4.5, p.96] (see also [5, Lemma 5.2, p.195]) we obtain that the function $\psi_t(\lambda) = e^{\frac{-1}{\sqrt{\hat{a}_\epsilon(\lambda)}}t}$ is completely monotonic, for every $t > 0, \lambda > 0$ and $0 < \epsilon < \epsilon_0$.

Finally from Lemma 3.4 and the product theorem we obtain that the function $h_\epsilon(\lambda, t)$ is completely monotonic for every $t > 0,\ \lambda > 0$ and $0 < \epsilon < \epsilon_0$.

□

We state the following main result concerning existence of resolvents satisfying the condition (3.2). Except for stability, the proof is much the same as that of [17, Theorem 5] (see also, [18, Theorem 4.1]).

Theorem 3.6. *Let A be the generator of a strongly continuous cosine function on a Banach space X. Assume $K \in \mathcal{K}(\mathbb{R}, \epsilon_0)$ is exponentially bounded. Then, (3.1) admits a resolvent $\{R_\epsilon(t)\}_{t \geq 0}$ in X for each $0 \leq \epsilon < \epsilon_0$ and the stability condition*

$$||R_\epsilon(t)|| \leq Me^{\omega t}, \quad t \in \mathbb{R}_+, \quad \epsilon > 0,$$

holds, where M, ω are independent of ϵ.

Proof. The proof of existence for resolvent families in case $\epsilon > 0$ follows by the general subordination principle for resolvents (see the proof of [18, Theorem 4.1]). Since A is the generator of a strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ in X, there is $M \geq 1$ and $\omega_0 > 0$ such that

$$||C(t)|| \leq M\cosh(\omega_0 t),$$

for all $t \in \mathbb{R}$.

For all $\mu > \omega_0$ and all $x \in X$ we have

$$(\mu - A)^{-1}x = \frac{1}{\sqrt{\mu}}\int_0^\infty e^{-\sqrt{\mu}t}C(t)xdt. \tag{3.10}$$

For all λ sufficiently large we have that $\frac{1}{\hat{a}_\epsilon(\lambda)} > \omega_0$, since $\hat{a}_\epsilon(\lambda) \to 0$ as $\lambda \to \infty$. Then, from (3.10) we obtain that $\frac{1}{\hat{a}_\epsilon(\lambda)} \in \rho(A)$ and

$$\begin{aligned}(\lambda - \lambda\hat{a}_\epsilon(\lambda)A)^{-1}x &= \frac{1}{\lambda a_\epsilon(\lambda)}[\frac{1}{\hat{a}_\epsilon(\lambda)} - A]^{-1}x \\ &= \frac{1}{\lambda\sqrt{\hat{a}_\epsilon(\lambda)}}\int_0^\infty e^{-\frac{1}{\sqrt{\hat{a}_\epsilon(\lambda)}}t}C(t)xdt \\ &= \int_0^\infty h_\epsilon(\lambda,t)C(t)xdt\end{aligned}$$

for all $x \in X$ and λ sufficiently large, say for $\lambda > \omega_1$.

In what follows, we will apply Theorem 2.3 to show that problem (3.1) admits a resolvent $\{R_\epsilon(t)\}_{t\geq 0}$ in X for all $\epsilon > 0$. Then we will prove that under our hypothesis the stability condition (3.2) is verified. The existence of a resolvent $R_0(t)$ for (3.1) with $\epsilon = 0$ then follows by Theorem 3.1.

Let $L_\lambda^n = (-1)^n(d/d\lambda)^n/n!$, for all $n = 0, 1, 2, ...$ By Proposition 3.5 we have that the function $h_\epsilon(\lambda,t)$ is completely monotonic for all $0 < \epsilon < \epsilon_0$. Hence, defining

$$H_\epsilon(\lambda) = (\lambda - \lambda\hat{a}_\epsilon(\lambda)A)^{-1},$$

we have

$$L_\lambda^n H_\epsilon(\lambda) = \int_0^\infty C(t)L_\lambda^n h_\epsilon(\lambda,t)dt.$$

In this way we obtain the following estimate

$$\begin{aligned}||L_\lambda^n H_\epsilon(\lambda)|| &\leq M\int_0^\infty \cosh(\omega_1 t)L_\lambda^n h_\epsilon(\lambda,t)dt \\ &= ML_\lambda^n\int_0^\infty \cosh(\omega_1 t)h_\epsilon(\lambda,t)dt \\ &= ML_\lambda^n\{(\lambda - \lambda\hat{a}_\epsilon(\lambda)\omega_1^2)^{-1}\} \\ &= ML_\lambda^n\hat{s}_\epsilon(\lambda; -\omega_1^2),\end{aligned}$$

where $s_\epsilon(t; -\omega_1^2)$ is the positive solution (see [18, Proposition 4.5]) of the equation

$$s_\epsilon(t; -\omega_1^2) = 1 + \omega_1^2\int_0^t s_\epsilon(\tau - t; -\omega_1^2)a_\epsilon(\tau)d\tau, \quad t > 0. \tag{3.11}$$

Next, we claim that there are constants $c > 0$ and $\omega_0 \in \mathbb{R}$ which are not depending on ϵ, such that

$$s_\epsilon(t; -\omega_1^2) \leq ce^{\omega_0 t}, \quad t > 0. \tag{3.12}$$

In fact, since $K(t) \geq 0$ we have from (3.4) and (3.5) the estimate

$$a_\epsilon(t) \leq a_0(t),$$

for all $t \geq 0$ and $\epsilon > 0$. Hence

$$s_\epsilon(t; -\omega_1^2) = 1 + \omega_1^2 (s_\epsilon * a_\epsilon)(t) \leq 1 + \omega_1^2 (s_\epsilon * a_0)(t), \quad t \geq 0.$$

Therefore, there exists a continuous and non negative function $g_\epsilon(t)$ such that

$$s_\epsilon(t; -\omega_1^2) = 1 + \omega_1^2 (s_\epsilon * a_0)(t) - g_\epsilon(t), \quad t \geq 0. \tag{3.13}$$

By the variation of parameters formula we get that

$$s_\epsilon(t; -\omega_1^2) = (1 - g_\epsilon(t)) + r * (1 - g_\epsilon)(t), \quad t \geq 0, \tag{3.14}$$

is a solution of (3.13), where $r(t)$ is the non negative solution of

$$r(t) - \omega_1^2 a_0(t) = (\omega_1^2 a_0 * r)(t), \quad t \geq 0.$$

We observe that the function $r(t)$ can be explicitly written as

$$r(t) = \sum_{k=1}^{\infty} (\omega_1^2)^k a_0^{*^k}(t), \quad t \geq 0,$$

since $K(t)$ is exponentially bounded implies that both $a_0(t)$ and that $r(t)$ are exponentially bounded independently of ϵ. Hence from (3.14) we obtain that

$$s_\epsilon(t; -\omega_1^2) \leq 1 + (1 * r)(t), \quad t \geq 0,$$

which implies that $s_\epsilon(t, -\omega_1^2)$ satisfies (3.12), proving the claim.

Using the above estimate it is not difficult to see that

$$L_\lambda^n \hat{s}_\epsilon(\lambda; -\omega_1^2) \leq c(\lambda - \omega_0)^{-n-1}$$

for all $\lambda > \omega_0$. Then,

$$||L_\lambda^n H_\epsilon(\lambda)|| \leq c(\lambda - \omega_0)^{-n-1} \tag{3.15}$$

for all $\lambda > \omega_0$ and all $n = 0, 1, 2, ...$

According to Theorem 2.3 we conclude that for all $0 < \epsilon < \epsilon_0$ the problem (3.1) admits a resolvent family $\{R_\epsilon(t)\}_{t \geq 0}$, which satisfy the stability condition (3.2). This finishes with the proof.

□

Definition 3.7. We say that $u : \mathbb{R}_+ \to X$ is a solution of (1.1) if $u \in C^2(\mathbb{R}_+; X), u(t) \in D(A)$ for $t \geq 0$ and (1.1) is satisfied on $\mathbb{R}_+$.

The following is the main result of this paper.

Theorem 3.8. *Let A be the generator of a strongly continuous cosine function on a Banach space X. Suppose $K \in \mathcal{K}(\mathbb{R}, \epsilon_1)$ is exponentially bounded, and*

(H1) $u_0(\epsilon), w_0 \in D(A), u_0(\epsilon) \to w_0, u_1(\epsilon) \to w_1 \in X,$ *as* $\epsilon \to 0$.

(H2) $f_\epsilon(\cdot) \in C(\mathbb{R}_+; D(A))$ *and* $f_\epsilon(t) \to f(t)$ *as* $\epsilon \to 0$.

Then for each $\epsilon > 0$ the solution $u_\epsilon(t)$ of problem (1.1) converges to the unique solution of problem (1.2) as $\epsilon \to 0^+$, and the convergence is uniform on compact intervals of $\mathbb{R}_+$.

Proof. Since $\hat{a}_\epsilon(\lambda) \to \hat{a}_0(\lambda)$ as $\epsilon \to 0$ it follows by Theorem 3.6 and Theorem 3.1 that for each $0 \leq \epsilon < \epsilon_0$, (3.1) admits a resolvent $\{R_\epsilon(t)\}_{t \geq 0}$ in X such that $R_\epsilon(t) \to R_0(t)$, as $\epsilon \to 0$.

We show next that for each $\epsilon > 0$, the solution $u_\epsilon(t)$ of equation (1.1) can be represented by means of the resolvent family as

$$u_\epsilon(t) = R_\epsilon(t) u_0(\epsilon) + (e_\epsilon * R_\epsilon)(t)(u_1(\epsilon)) + \frac{1}{\epsilon^2}(e_\epsilon * R_\epsilon * f_\epsilon)(t). \tag{3.16}$$

First, we consider the integrated equation which is obtained by integrating two times equation (1.1), and thus we get its equivalent form

$$\epsilon^2(u_\epsilon(t)-u_0(\epsilon)-u_1(\epsilon)t)+(1*u_\epsilon)(t)-u_0(\epsilon)t=((t+t*K)*Au_\epsilon)(t)+(t*f_\epsilon)(t). \tag{3.17}$$

Hence it suffices to verify that $u_\epsilon(t)$ defined by (3.16) satisfy (3.17). Towards this purpose we will make use of the resolvent equation (R3), written in abbreviate form as $R_\epsilon = I + a_\epsilon * AR_\epsilon$. First we note that $\frac{1}{\epsilon^2}(1*e_\epsilon)=1-e_\epsilon$, and hence from (3.4) we obtain

$$a_\epsilon(t)=\frac{1}{\epsilon^2}[(1*e_\epsilon)(t)+(1*e_\epsilon*K)(t)]. \tag{3.18}$$

Also from (3.4) we get the identity,

$$(1*a_\epsilon)(t)=(t+t*K)(t)-(1*e_\epsilon+1*e_\epsilon*K)(t)=(t+t*K)(t)-\epsilon^2 a_\epsilon(t). \tag{3.19}$$

Using the above identities we obtain that

$$\begin{aligned}
((t \;&+\; t*K)*Au_\epsilon)(t)=(t*Au_\epsilon)(t)+(t*K*Au_\epsilon)(t)\\
&=\; t*AR_\epsilon(t)u_0(\epsilon)+[(t*e_\epsilon)(t)u_1(\epsilon)\\
&+\; \frac{1}{\epsilon^2}(t*e_\epsilon*f_\epsilon)]*AR_\epsilon(t)+t*K*AR_\epsilon(t)u_0(\epsilon)\\
&+\; [(t*K*e_\epsilon)u_1(\epsilon)+\frac{1}{\epsilon^2}(t*K*e_\epsilon*f_\epsilon)]*AR_\epsilon(t)\\
&=\; [t+t*K]*AR_\epsilon(t)u_0(\epsilon)+[(t*e_\epsilon)+(t*K*e_\epsilon)]*AR_\epsilon(t)u_1(\epsilon)\\
&+\; \frac{1}{\epsilon^2}[(t*e_\epsilon*f_\epsilon)+(t*K*e_\epsilon*f_\epsilon)]*AR_\epsilon(t)\\
&=\; [\epsilon^2 a_\epsilon+1*a_\epsilon]*AR_\epsilon(t)u_0(\epsilon)\\
&+\; [(1*e_\epsilon)+(1*K*e_\epsilon)]*1*AR_\epsilon(t)u_1(\epsilon)\\
&+\; \frac{1}{\epsilon^2}[(1*e_\epsilon)+(1*K*e_\epsilon)]*1*f_\epsilon*AR_\epsilon(t)\\
&=\; \epsilon^2[R_\epsilon-I](t)u_0(\epsilon)+1*[R_\epsilon-I](t)u_0(\epsilon)\\
&+\; \epsilon^2 a_\epsilon*1*AR_\epsilon(t)u_1(\epsilon)+a_\epsilon*1*f_\epsilon*AR_\epsilon(t)\\
&=\; \epsilon^2 R_\epsilon(t)u_0(\epsilon)-\epsilon^2 u_0(\epsilon)+1*R_\epsilon(t)u_0(\epsilon)-tu_0(\epsilon)\\
&+\; \epsilon^2 1*[R_\epsilon-I](t)u_1(\epsilon)+1*f_\epsilon*[R_\epsilon-I](t)\\
&=\; \epsilon^2 R_\epsilon(t)u_0(\epsilon)-\epsilon^2 u_0(\epsilon)+1*R_\epsilon(t)u_0(\epsilon)-tu_0(\epsilon)\\
&+\; \epsilon^2(1*R_\epsilon)(t)u_1(\epsilon)-\epsilon^2 u_1(\epsilon)t+(1*f_\epsilon*R_\epsilon)(t)-(t*f_\epsilon)(t)
\end{aligned}$$

Hence,

$$\begin{aligned}
((t \;&+\; t*K)*Au_\epsilon)(t)+(t*f_\epsilon)(t)+\epsilon^2 u_1(\epsilon)t+tu_0(\epsilon)+\epsilon^2 u_0(\epsilon)\\
&=\; \epsilon^2 R_\epsilon(t)u_0(\epsilon)+1*R_\epsilon(t)u_0(\epsilon)+\epsilon^2(1*R_\epsilon)(t)u_1(\epsilon)+(1*f_\epsilon*R_\epsilon)(t)\\
&=\; (\epsilon^2 R_\epsilon(t)+1*R_\epsilon(t))u_0(\epsilon)+\epsilon^2(e_\epsilon+\frac{1}{\epsilon^2}(1*e_\epsilon))*R_\epsilon(t)u_1(\epsilon)\\
&+\; (e_\epsilon+\frac{1}{\epsilon^2}(1*e_\epsilon))*f_\epsilon*R_\epsilon(t)\\
&=\; \epsilon^2 u_\epsilon(t)+(1*u_\epsilon)(t)
\end{aligned}$$

which shows that $u_\epsilon(t)$ is the unique solution to problem (1.1) and thus it can be represented by (3.16).

Next we obtain from the resolvent equation that

$$R_\epsilon(t)x=x+\int_0^t a_\epsilon(t-s)AR_\epsilon(s)x\,ds,\ \text{for all } x\in D(A). \tag{3.20}$$

Now we notice that differentiability of a_ϵ implies the fact that the map $t \to R_\epsilon(t)x, x \in D(A)$ is also differentiable. Since $(H2)$ implies that $f_\epsilon(\cdot) \in D(A)$. Thus we have the identity

$$\frac{1}{\epsilon^2}((e_\epsilon * R_\epsilon) * f_\epsilon)(t) = (R_\epsilon * f_\epsilon)(t) - (e_\epsilon * R'_\epsilon * f_\epsilon)(t)$$

which follows by integration by parts.

Hence, we can also represent u_ϵ as follows

$$u_\epsilon(t) \quad = \quad R_\epsilon(t)u_0(\epsilon) + (R_\epsilon * f_\epsilon)(t) + (e_\epsilon * R_\epsilon)(t)u_1(\epsilon) - (e_\epsilon * R'_\epsilon * f_\epsilon)(t)$$

We now compute the limit of $u_\epsilon(t)$ as $\epsilon \to 0^+$, and we show that the limit is uniform for t in an arbitrary interval $[0, b]$. First by $(H1)$ and Theorem 3.1 we notice that $\lim_{\epsilon\to 0^+} R_\epsilon(t)u_0(\epsilon) = R_0(t)w_0$. On the other hand $e_\epsilon(t) \to 0$ as $\epsilon \to 0^+$ and $\|R_\epsilon(t)\| \leq Me^{\omega t}(\omega > 0)$ by hypothesis and since by $(H1)$ the sequence $u_1(\epsilon)$ is convergent. It then follows that there is some constant $C > 0$ such that

$$\|(e_\epsilon * R_\epsilon)(t)u_1(\epsilon)\| \leq Ce^{\omega b}\int_0^b e_\epsilon(s)ds \to 0$$

as $\epsilon \to 0^+$. Thus $(e_\epsilon * R_\epsilon)(t)u_1(\epsilon) \to 0$ as $\epsilon \to 0^+$.

Analogously,

$$\begin{aligned} ||e_\epsilon * (R'_\epsilon * f_\epsilon)(t)|| &\leq \int_0^b e_\epsilon(s)||(R'_\epsilon * f_\epsilon)(t-s)||ds \\ &\leq \sup_{t-b\leq r\leq t} ||(R'_\epsilon * f_\epsilon)(r)|| \int_0^b e_\epsilon(s)ds \\ &\leq \sup_{|r|\leq b} ||(R'_\epsilon * f_\epsilon)(r)|| \int_0^b e_\epsilon(s)ds \to 0 \end{aligned}$$

as $\epsilon \to 0^+$. Thus $e_\epsilon * R'_\epsilon * f_\epsilon(t) \to 0$ as $\epsilon \to 0^+$.

Hence we have showed that

$$\lim_{\epsilon\to 0^+} u_\epsilon(t) = R_0(t)w_0 + \int_0^t R_0(t-s)f(s)ds.$$

Now let

$$u(t) := R_0(t)w_0 + \int_0^t R_0(t-s)f(s)ds. \tag{3.21}$$

We will prove that $u(t)$ satisfy equation (1.2) solving the singular perturbation problem. From the fact that $R_0(t)$ is a resolvent family it satisfy the equation

$$R_0(t)x = x + \int_0^t a_0(t-s)AR_0(s)xds,$$

for all $x \in D(A)$.

Since $w_0 \in D(A)$ by $(H2)$, from (3.21) we get $u(0) = w_0$ and

$$\begin{aligned} (a_0 * Au)(t) &= a_0 * A[R_0(t)w_0 + (R_0 * f)(t)] \\ &= (a_0 * AR_0)(t)w_0 + [(a_0 * AR_0) * f](t) \\ &= (R_0(t) - I)w_0 + [(R_0 - I) * f](t) \\ &= R_0(t)w_0 - w_0 + (R_0 * f)(t) - (1 * f)(t) \\ &= u(t) - w_0 - (1 * f)(t) \end{aligned}$$

Since $a_0(t) = 1 + (1 * K)(t)$ we get from the above equality that $u'(t)$ exists and equation (1.2) is satisfied.

□

Remark 3.9.

As was mentioned in the introduction, James H. Liu [12] has treated the singular perturbation problem (1.1) - (1.2) under the assumption of existence of solutions and also imposing the following set of hypotheses:

(L1) A is the generator of a strongly continuous semigroup and a cosine family of operators in X.
(L2) $K \in C^2(\mathbb{R}_+)$.
(L3) $f_\epsilon(\cdot), f(\cdot) \in C^1(\mathbb{R}_+; X)$
(L4) $u_0(\epsilon), w_0 \in D(A), u_0(\epsilon) \to w_0, \epsilon^2 u_1(\epsilon) \to 0$, as $\epsilon \to 0$.
(L5) For any $T > 0, f(\cdot;\epsilon) \to f(\cdot)$ in $L^1(0,T;X)$ as $\epsilon \to 0$.

In opposition to the condition $\epsilon^2 u_1(\epsilon) \to 0$ the above theorem requires only the convergence of $u_1(\epsilon)$.

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