EXISTENCE OF MILD SOLUTIONS FOR A SEMILINEAR INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL INITIAL CONDITIONS

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Abstract. In this paper, using Hausdorff measure of noncompactness and a fixed–point argument we prove the existence of mild solutions for the semilinear integrodifferential equation submitted to nonlocal initial conditions

$$\left.\begin{array}{rcl} u'(t) & = & Au(t)+\displaystyle\int_0^t B(t-s)u(s)ds+f(t,u(t)), \quad t\in[0,1], \\ u(0) & = & g(u), \end{array}\right\}$$

where $A : D(A) \subseteq X \to X$ and for every $t \in [0, 1]$ the maps $B(t) : D(B(t)) \subseteq X \to X$ are linear closed operators defined in a Banach space X. We assume further that $D(A) \subseteq D(B(t))$ for every $t \in [0, 1]$, and the functions $f : [0, 1]\times X \to X$ and $g : C([0, 1]; X) \to X$ are X–valued functions which satisfy appropriate conditions.

1. Introduction

The concept of *nonlocal initial condition* has been introduced to extend the study of classical initial value problems. This notion is more precise for describing nature phenomena than the classical notion because additional information is taken into account. For the importance of nonlocal conditions in different fields, the reader is referred to [11, 12, 33] and the references cited therein.

The earliest works related with problems submitted to nonlocal initial conditions were made by Byszewski [6, 7, 8, 9]. In these works, using methods of semigroup theory and the Banach fixed point theorem the author has proved the existence of mild and strong solutions for the first order Cauchy problem

$$\left.\begin{array}{rcl} u'(t) & = & Au(t)+f(t,u(t)), \quad t\in[0,1], \\ u(0) & = & g(u). \end{array}\right\} \tag{1.1}$$

where A is an operator defined in a Banach space X which generates a semigroup $\{T(t)\}_{t\geqslant 0}$, and the maps f and g are suitable X–valued functions.

Henceforth, the equation (1.1) has been extensively studied by many authors. We just mention a few of these works. Byszewski and Lakshmikantham [10] have studied the existence and uniqueness of mild solutions whenever f and g satisfy Lipschitz–type conditions. Ntouyas and Tsamatos [25, 26] have studied this problem under conditions of compactness for the semigroup generated by A and the function g. Recently, Zhu, Song and Li [37], have investigated this problem without conditions of compactness on the semigroup generated by A, or the function f.

Key words and phrases. Mild solutions; Integrodifferential equations; Resolvent operators; Nonlocal initial conditions.

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On the other hand, the study of abstract integrodifferential equations has been an active topic of research in recent years because it has many applications in different areas. In consequence, there exists an extensive literature about integrodifferential equations with nonlocal initial conditions, (cf. e.g., [1, 2, 4, 14, 17, 20, 21, 29, 30, 31, 32, 34, 35, 36]). Our work is a contribution to this theory. Indeed, this paper is devoted to study the existence of mild solutions for the following semilinear integrodifferential evolution equation

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u(t)), \quad t\in[0,1], \\ u(0) &= g(u), \end{aligned}\right\} \tag{1.2}$$

where $A : D(A) \subseteq X \to X$ and for every $t \in [0,1]$ the mappings $B(t) : D(B(t)) \subseteq X \to X$ are linear closed operators defined in a Banach space X. We assume further that $D(A) \subseteq D(B(t))$ for every $t \in [0,1]$, and the functions $f : [0,1] \times X \to X$ and $g : C([0,1];X) \to X$ are X–valued functions that satisfy appropriate conditions which we will describe later. In order to abbreviate the text of this paper, henceforth we will denote by I the interval $[0,1]$, and $C(I;X)$ is the space of all continuous functions from I to X endowed with the uniform convergence norm.

The classical initial value version of the equation (1.2), i.e. $u(0) = u_0$ for some $u_0 \in X$, has been extensively studied by many researchers because has many important applications in different fields of natural sciences such as thermodynamics, electrodynamics, heat conduction in materials with memory, continuum mechanics and population biology, among others. For more information see [19, 23, 27]. For this reason the study of existence and another properties of the solutions for the equation (1.2) is a very important problem. However, to the best of our knowledge, the existence of mild solutions for the nonlocal initial value problem (1.2) has not been addressed in the existing literature. Most of the authors obtain the existence of solutions and well–posedness for the equation (1.2) by establishing the existence of a resolvent operator $\{R(t)\}_{t\in I}$ and a variation of parameters formula (see [15, 28]). Using and adaptation of the methods described in [37], we are able to prove the existence of mild solutions of the equation (1.2) under conditions of compactness of the function g and continuity of the function $t \mapsto R(t)$ for $t > 0$. Furthermore, in the particular case $B(t) = b(t)A$ for all $t \in [0,1]$, where the operator A is the infinitesimal generator of a C_0–semigroup defined in a Hilbert space H, and the kernel b is a scalar map which satisfies appropriate hypotheses, we are able to give sufficient conditions for the existence of mild solutions only in terms of spectral properties of the operator A and regularity properties of the kernel b. We show that our abstract results can be applied to concrete situations. Indeed, we consider an example with a particular choice of the function b and the operator A is defined by

$$(Aw)(t,\xi) = a_1(\xi)\frac{\partial^2}{\partial\xi^2}w(t,\xi) + b_1(\xi)\frac{\partial}{\partial\xi}w(t,\xi) + \overline{c}(\xi)w(t,\xi),$$

where the given coefficients $a_1,\ b_1,\ \overline{c}$ satisfy the usual uniform ellipticity conditions.

2. Preliminaries

Most of the notations used throughout this paper are standard. So, $\mathbb{N},\mathbb{Z},\mathbb{R}$ and $\mathbb{C}$ denote the set of natural, integers, real and complex numbers respectively, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{R}_0^+ = [0,\infty)$.

In this work X and Y always are complex Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$; the subscript will be dropped when there is no danger of confusion. We denote the space of all bounded linear operators from X to Y by $\mathcal{L}(X,Y)$. In the case $X = Y$, we will write briefly $\mathcal{L}(X)$. Let A be an operator defined in X. We will denote its domain by $D(A)$, its domain endowed with the graph norm by $[D(A)]$, its resolvent set by $\rho(A)$, and its spectrum by $\sigma(A) = \mathbb{C} \setminus \rho(A)$.

As we have already mentioned $C(I; X)$ is the vector space of all continuous functions $f : I \to X$. This space is a Banach space endowed with the norm

$$\|f\|_\infty = \sup_{t\in I} \|f(t)\|_X .$$

In the same manner, for $n \in \mathbb{N}$ we write $C^n(I; X)$ for denoting the space of all function from I to X which are n–times differentiable. Further, $C^\infty(I; X)$ represents the space of all infinitely differentiable from I to X.

We denote by $L^1(I; X)$ the space of all (equivalent classes of) Bochner–measurable functions $f : I \mapsto X$ such that $\|f(t)\|_X$ is integrable for $t \in I$. It is well known that this space is a Banach space with the norm

$$\|f\|_{L^1(I;X)} = \int_I \|f(s)\|_X ds.$$

We next include some preliminaries concerning to the theory of resolvent operator $\{R(t)\}_{t\in I}$ for the equation (1.2).

Definition 2.1. *Let X a complex Banach space. A family $\{R(t)\}_{t\in I}$ of bounded linear operators defined in X is called a resolvent operator for the equation* (1.2) *if the following conditions are fulfilled.*

(R1) *For each $x \in X$, $R(0)x = x$ and $R(\cdot)x \in C(I; X)$.*
(R2) *The map $R : I \to \mathcal{L}([D(A)])$ is strongly continuous.*
(R3) *For each $y \in D(A)$, the function $t \mapsto R(t)y$ is continuously differentiable and*

$$\begin{aligned} \frac{d}{dt}R(t)y &= AR(t)y + \int_0^t B(t-s)R(s)yds = \\ &= R(t)Ay + \int_0^t R(t-s)B(s)yds, \quad t \in I. \end{aligned} \tag{2.1}$$

In what follows we assume that there exists a resolvent operator $\{R(t)\}_{t\in I}$ for the equation (1.2) satisfying the following property:

(P) The function $t \mapsto R(t)$ is continuous from $(0, 1]$ to $\mathcal{L}(X)$ endowed with the uniform operator norm $\|\cdot\|_{\mathcal{L}(X)}$.

Note that property ***(P)*** is also named in different ways in the existing literature on the subject, mainly the theory of C_0-semigroups, namely: *norm continuity for $t > 0$, eventually norm continuity,* or *equicontinuity.*

The existence of solutions of the *linear* problem

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t), \quad t \geqslant 0, \\ u(0) &= u_0 \in X, \end{aligned}\right\} \tag{2.2}$$

has been studied by many authors. Assuming that $f : [0, +\infty) \to X$ is locally integrable, it follows from [15] that the function u given by

$$u(t) = R(t)u_0 + \int_0^t R(t-s)f(s)ds, \quad \text{for } t \geqslant 0, \tag{2.3}$$

is a mild solution of the problem (2.2). Motivated by this result, we adopt the following concept of solution.

Definition 2.2. *A continuous function $u \in C(I;X)$ is called a* mild solution *of the equation* (1.2) *if the equation*

$$u(t) = R(t)g(u) + \int_0^t R(t-s)f(s,u(s))ds, \qquad t \in I, \tag{2.4}$$

is verified.

The main results of this paper are based on the concept of measure of noncompactness. For general information the reader can see [3]. In this paper, we use the notion of Hausdorff measure of noncompactness. For this reason we recall a few properties related with this concept.

Definition 2.3. *Let S be a bounded subset of a normed space Y. The Hausdorff measure of noncompactness of S is defined by*

$$\eta(S) = \inf\{\varepsilon > 0 : S \text{ has a finite cover by balls of radius } \varepsilon\}.$$

***Remark* 2.4.** *Let S_1, S_2 be bounded sets of a normed space Y. The Hausdorff measure of noncompactness has the following properties.*

- *If $S_1 \subseteq S_2$ then $\eta(S_1) \leqslant \eta(S_2)$.*
- *$\eta(S_1) = \eta(\overline{S_1})$, where $\overline{S_1}$ denotes the closure of A.*
- *$\eta(S_1) = 0$ if and only if S_1 is totally bounded.*
- *$\eta(\lambda S_1) = |\lambda|\eta(S_1)$ with $\lambda \in \mathbb{R}$.*
- *$\eta(S_1 \cup S_2) = \max\{\eta(S_1), \eta(S_2)\}$.*
- *$\eta(S_1 + S_2) \leqslant \eta(S_1) + \eta(S_2)$, where $S_1 + S_2 = \{s_1 + s_2 : \ s_1 \in S_1, \ s_2 \in S_2\}$.*
- *$\eta(S_1) = \eta(\overline{co}(S_1))$ where $\overline{co}(S_1)$ is the closed convex hull of S_1.*

We next collect some specific properties of the Hausdorff measure of noncompactness which are needed to establish our results. Henceforth, when we need to compare the measures of noncompactness in X and $C(I;X)$, we will use ζ to denote the Hausdorff measure of noncompactness defined in X and γ to denote the Hausdorff measure of noncompactness on $C(I;X)$. Moreover, we will use η for the Hausdorff measure of noncompactness for general Banach spaces Y.

Lemma 2.5. *Let $W \subseteq C(I;X)$ be a subset of continuous functions. If W is bounded and equicontinuous, then the set $\overline{co}(W)$ is also bounded and equicontinuous.*

For the rest of the paper we will use the following notation. Let W be a set of functions from I to X and $t \in I$ fixed, we denote by $W(t) = \{w(t) : w \in W\}$. The proof of Lemma 2.6 can be found in [3].

Lemma 2.6. *Let $W \subseteq C(I;X)$ be a bounded set. Then $\zeta(W(t)) \leqslant \gamma(W)$ for all $t \in I$. Furthermore, if W is equicontinuous on I, then $\zeta(W(t))$ is continuous on I, and*

$$\gamma(W) = \sup\{\zeta(W(t)) \ : \ t \in I\}.$$

A set of functions $W \subseteq L^1(I;X)$ is said to be uniformly integrable if there exists a positive function $\kappa \in L^1(I;\mathbb{R}^+)$ such that $\|w(t)\| \leqslant \kappa(t)$ a.e. for all $w \in W$.

The next property has been studied by several authors, the reader can see [24] for more details.

Lemma 2.7. *If $\{u_n\}_{n\in\mathbb{N}} \subseteq L^1(I;X)$ is uniformly integrable, then for each $n \in \mathbb{N}$ the function $t \mapsto \zeta(\{u_n(t)\}_{n\in\mathbb{N}})$ is measurable and*

$$\zeta\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \leqslant 2\int_0^t \zeta\big(\{u_n(s)\}_{n=1}^{\infty}\big)ds.$$

The next result is crucial for our work, the reader can see its proof in [5, Theorem 2].

Lemma 2.8. *Let Y be a Banach space. If $W \subseteq Y$ is a bounded subset, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W$ such that*

$$\eta(W) \leqslant 2\eta(\{u_n\}_{n=1}^{\infty}) + \varepsilon$$

The following Lemma is essential for the proof of Theorem 3.2, which is the main result of this paper. For more details of its proof see [16, Theorem 3.1].

Lemma 2.9. *For all $0 \leqslant m \leqslant n$, denote by $C_m^n = \binom{n}{m}$. If $0 < \epsilon < 1$ and $h > 0$ and let*

$$S_n = \epsilon^n + C_1^n \epsilon^{n-1} h + C_2^n \epsilon^{n-2} \frac{h^2}{2!} + \cdots + \frac{h^n}{n!}, \quad n \in \mathbb{N},$$

then $\lim_{n\to\infty} S_n = 0$.

Clearly, a manner for proving the existence of mild solutions for the equation (1.2) is using fixed–point arguments. The fixed–point theorem which we will apply has been established in [16, Lemma 2.4].

Lemma 2.10. *Let S be a closed and convex subset of a complex Banach space Y, let $F : S \to S$ be a continuous operator such that $F(S)$ is a bounded set. Define*

$$F^1(S) = F(S) \quad \text{and} \quad F^n(S) = F(\overline{co}(F^{n-1}(S))), \qquad n = 2, 3, \ldots$$

If there exist a constant $0 \leqslant r < 1$ and $n_0 \in \mathbb{N}$ such that

$$\eta(F^{n_0}(S)) \leqslant r\eta(S),$$

then F has a fixed point in the set S.

3. Main Results

In this section we will present our main results. Henceforth, we assume that the following assertions hold:

- ***(H1)*** There exists a resolvent operator $\{R(t)\}_{t\in I}$ for the equation (1.2) having the property ***(P)***.
- ***(H2)*** The function $g : C(I; X) \to X$ is a compact map.
- ***(H3)*** The function $f : I \times X \to X$ satisfies the Carathéodory type conditions, that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in I$.
- ***(H4)*** There exist a function $m \in L^1(I; \mathbb{R}^+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, x)\| \leqslant m(t)\Phi(\|x\|)$$

  for all $x \in X$ and almost all $t \in I$.
- ***(H5)*** There exists a function $H \in L^1(I; \mathbb{R}^+)$ such that for any bounded $S \subseteq X$

$$\zeta(f(t, S)) \leqslant H(t)\zeta(S)$$

  for almost all $t \in I$.

***Remark* 3.1.** *Assuming that the function g satisfies the hypothesis **(H2)**, it is clear that g takes bounded set into bounded sets. For this reason, for each $R \geqslant 0$ we will denote by g_R the number $g_R = \sup\{\|g(u)\| : \|u\|_\infty \leqslant R\}$.*

The following theorem is the main result of this paper.

Theorem 3.2. *If the hipotheses **(H1)–(H5)** are satisfied and there exists a constant $R \geqslant 0$ such that*

$$Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R,$$

where $K = \sup\{\|R(t)\| : t \in I\}$, then the problem (1.2) *has at least one mild solution.*

Proof. Define $F : C(I;X) \to C(I;X)$ by

$$(Fu)(t) = R(t)g(u) + \int_0^t R(t-s)f(s,u(s))ds, \quad t \in I,$$

for all $u \in C(I;X)$.

We begin showing that F is a continuous map. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq C(I;X)$ such that $u_n \to u$ as $n \to \infty$ (in the norm of $C(I;X)$). Note that

$$\|F(u_n) - F(u)\| \leqslant K\|g(u_n) - g(u)\| + K\int_0^1 \|f(s,u_n(s)) - f(s,u(s))\|ds,$$

by hypotheses ***(H2)*** and ***(H3)*** and the Dominated Convergence Theorem we have that $\|F(u_n) - F(u)\| \to 0$ when $n \to \infty$.

Let $R \geqslant 0$ and denote by $B_R = \{u \in C(I;X) : \|u(t)\| \leqslant R$ for all $t \in I\}$ and note that for any $u \in B_R$ we have

$$\begin{aligned}\|(Fu)(t)\| &\leqslant \|R(t)g(u)\| + \left\|\int_0^t R(t-s)f(s,u(s))ds\right\| \\ &\leqslant Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R.\end{aligned}$$

Therefore $F : B_R \to B_R$ and $F(B_R)$ is a bounded set. Moreover, by continuity of the function $t \mapsto R(t)$ on $(0,1]$, we have that the set $F(B_R)$ is an equicontinuous set of functions.

Define $\mathscr{B} = \overline{co}(F(B_R))$. It follows from Lemma 2.5 that the set $\mathscr{B}$ is equicontinuous. In addition, the operator $F : \mathscr{B} \to \mathscr{B}$ is continuous and $F(\mathscr{B})$ is a bounded set of functions.

Let $\varepsilon > 0$. Since the function g is a compact map, by Lemma 2.8 there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset F(\mathscr{B})$ such that

$$\zeta(F(\mathscr{B})(t)) \leqslant 2\zeta(\{v_n(t)\}_{n=1}^{\infty} + \varepsilon \leqslant 2\zeta\left(\int_0^t \{R(t-s)f(s,u_n(s))\}_{n=1}^{\infty}ds\right) + \varepsilon.$$

By the hypothesis ***(H4)***, for each $t \in I$ we have $\|R(t-s)f(s,u_n(s))\| \leqslant K\Phi(R)m(s)$. Therefore, by the condition ***(H5)*** we have

$$\begin{aligned}\zeta(F(\mathscr{B})(t)) &\leqslant 4K\int_0^t \zeta(\{f(s,u_n(s))\}_{n=1}^{\infty}ds + \varepsilon \\ &\leqslant 4K\int_0^t H(s)\zeta(\{u_n(s)\}_{n\in\mathbb{N}})ds + \varepsilon \\ &\leqslant 4K\gamma(\mathscr{B})\int_0^t H(s)ds + \varepsilon.\end{aligned}$$

Since the function $H \in L^1(I;\mathbb{R}^+)$, for $\alpha < \dfrac{1}{4K}$ there exists $\varphi \in C(I;\mathbb{R}^+)$ satisfying $\displaystyle\int_0^1 |H(s)-\varphi(s)|ds < \alpha$. Hence,

$$\begin{aligned}\zeta(F(\mathscr{B})(t)) &\leqslant 4K\gamma(\mathscr{B})\left[\int_0^t |H(s)-\varphi(s)|ds + \int_0^t \varphi(s)ds\right] + \varepsilon\\ &\leqslant 4K\gamma(\mathscr{B})\left[\alpha + Nt\right] + \varepsilon,\end{aligned}$$

where $N = \|\varphi\|_\infty$. Since $\varepsilon > 0$ is arbitrary, we have

$$\zeta(F(\mathscr{B})(t)) \leqslant (a+bt)\gamma(\mathscr{B}) \quad \text{where} \quad a = 4\alpha K \text{ and } b = 4KN. \tag{3.1}$$

Let $\varepsilon > 0$. Since the function g is a compact map and applying the Lemma 2.8 there exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq \overline{co}(F(\mathscr{B}))$ such that

$$\begin{aligned}\zeta(F^2(\mathscr{B})(t)) &\leqslant 2\zeta\left(\int_0^t \{R(t-s)f(s,w_n(s))\}_{n=1}^\infty ds\right) + \varepsilon\\ &\leqslant 4K\int_0^t \zeta\{f(s,w_n(s))\}_{n=1}^\infty ds + \varepsilon\\ &\leqslant 4K\int_0^t H(s)\zeta(\overline{co}(F^1(\mathscr{B})(s))) + \varepsilon = 4K\int_0^t H(s)\zeta(F^1(\mathscr{B})(s)) + \varepsilon.\end{aligned}$$

Using the inequality (3.1) we have that

$$\begin{aligned}\zeta(F^2(\mathscr{B})(t)) &\leqslant 4K\int_0^t \big[|H(s)-\varphi(s)| + |\varphi(s)|\big](a+bs)\gamma(\mathscr{B})ds + \varepsilon\\ &\leqslant 4K(a+bt)\gamma(\mathscr{B})\int_0^t |H(s)-\varphi(s)|ds + 4KN\gamma(\mathscr{B})\Big(at + \frac{bt^2}{2}\Big) + \varepsilon\\ &\leqslant a(a+bt) + b(at + \frac{bt^2}{2}) + \varepsilon \leqslant \big(a^2 + 2bt + \frac{(bt)^2}{2}\big)\gamma(\mathscr{B}) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\zeta(F^2(\mathscr{B})(t)) \leqslant \left(a^2 + 2bt + \frac{(bt)^2}{2}\right)\gamma(\mathscr{B}).$$

By an inductive process, for all $n \in \mathbb{N}$, it holds

$$\zeta(F^n(\mathscr{B})(t)) \leqslant \left(a^n + C_1^n a^{n-1}bt + C_2^n a^{n-2}\frac{(bt)^2}{2!} + \cdots + \frac{(bt)^n}{n!}\right)\gamma(\mathscr{B}),$$

where for $0 \leqslant m \leqslant n$, the symbol C_m^n denotes the binomial coefficient $\binom{n}{m}$.

In addition, for all $n \in \mathbb{N}$ the set $F^n(\mathscr{B})$ is an equicontinuous set of functions. Therefore, using the Lemma 2.6 we conclude that

$$\gamma(F^n(\mathscr{B})) \leqslant \left(a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}\frac{b^2}{2!} + \cdots + \frac{b^n}{n!}\right)\gamma(\mathscr{B}).$$

Since $0 \leqslant a < 1$ and $b > 0$, it follows from Lemma 2.7 that there exists $n_0 \in \mathbb{N}$ such that

$$\left(a^{n_0} + C_1^{n_0} a^{n_0-1}b + C_2^{n_0} a^{n_0-2}\frac{b^2}{2!} + \cdots + \frac{b^{n_0}}{n_0!}\right) = r < 1.$$

Consequently, $\gamma(F^{n_0}(\mathscr{B})) \leqslant r\gamma(\mathscr{B})$. It follows from Lemma 2.9 that F has a fixed point in $\mathscr{B}$, and this fixed point is a mild solution of the equation (1.2). ☐

Our next result is related with a particular case of the equation (1.2). Consider the following Volterra equation of convolution type

$$\left.\begin{aligned} u'(t) &= Au(t) + \int_0^t b(t-s)Au(s)ds + f(t,u(t)), \quad t \in I, \\ u(0) &= g(u), \end{aligned}\right\} \tag{3.2}$$

where A is a closed linear operator defined on a Hilbert space H, the kernel $b \in L^1_{loc}(\mathbb{R}^+;\mathbb{R})$, and the function f is an appropriate H–valued map.

Since the equation (3.2) is a convolution type equation, it is natural to employ the Laplace transform for its study.

Let X be a Banach space and $a \in L^1_{loc}(\mathbb{R}^+;\mathbb{R})$. We say that the function a is Laplace transformable if there is $\omega \in \mathbb{R}$ such that $\displaystyle\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$. In addition, we denote by $\widehat{a}(\lambda) = \int_0^\infty e^{-\lambda t}a(t)dt$, for $Re\lambda > \omega$, the Laplace transform of the function a.

We need the following definitions for proving the existence of a resolvent operator for the equation (3.2). These concepts have been introduced by J. Prüss in [27].

Definition 3.3. *Let $a \in L^1_{loc}(\mathbb{R}^+;\mathbb{R})$ be Laplace transformable and $k \in \mathbb{N}$. We say that the function a is k–regular if there exists a constant $C > 0$ such that*

$$|\lambda^n \widehat{a}^{(n)}(\lambda)| \leqslant C|\widehat{a}(\lambda)|$$

for all $Re\lambda \geqslant \omega$ and $0 < n \leqslant k$.

Convolutions of $k-$regular functions are again $k-$regular. Moreover, integration and differentiation are operations which preserve $k-$regularity as well. See [27, p.70].

Definition 3.4. *Let $f \in C^\infty(\mathbb{R}^+;\mathbb{R})$. We say that f is a completely monotone function if and only if $(-1)^n f^{(n)}(\lambda) \geqslant 0$ for all $\lambda > 0$ and $n \in \mathbb{N}$.*

Definition 3.5. *Let $a \in L^1_{loc}(\mathbb{R}^+;\mathbb{R})$ such that a is Laplace transformable. We say that a is completely positive function if and only if*

$$\frac{1}{\lambda\widehat{a}(\lambda)} \quad \text{and} \quad \frac{-\widehat{a}'(\lambda)}{(\widehat{a}(\lambda))^2}$$

are completely monotone functions.

Finally, we recall that a one-parameter family $\{T(t)\}_{t\geqslant 0}$ of bounded and linear operators is said to be exponentially bounded of type (M,ω) if there are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leqslant Me^{\omega t}, \quad \text{for all } t \geqslant 0.$$

The next proposition guarantees the existence of a resolvent operator fot the equation (3.2) satisfying the property ***(P)***. With this purpose we will introduce the conditions ***(C1)*** and ***(C2)***.

(C1) The kernel a defined by $a(t) = 1 + \displaystyle\int_0^t b(s)ds$, for all $t \geqslant 0$, is 2–regular and completely positive.

(C2) The operator A is a the generator of a semigroup of type (M,ω) and there exists $\mu_0 > \omega$ such that

$$\lim_{|\mu|\to\infty} \left\| \frac{1}{\widehat{b}(\mu_0+i\mu)+1}\left(\frac{\mu_0+i\mu}{\widehat{b}(\mu_0+i\mu)+1} - A\right)^{-1} \right\| = 0$$

Proposition 3.6. *Suppose that A is the generator of a C_0–semigroup of type (M,ω) in a Hilbert space H. If the conditions **(C1)–(C2)** are satisfied, then there exists a resolvent operator $\{R(t)\}_{t\in I}$ for the equation* (3.2) *having the property **(P)**.*

Proof. Integrating in time the equation (3.2) we get

$$u(t) = \int_0^t a(t-s)Au(s)ds + \int_0^t f(s,u(s)) + g(u). \tag{3.3}$$

Since the scalar kernel a is completely positive and A generates a C_0–semigroup, it follows from [27, Theorem 4.2] that there exists a family of operators $\{R(t)\}_{t\in I}$ strongly continuous, exponentially bounded that commutes with A, satisfying

$$R(t)x = x + \int_0^t a(t-s)AR(s)xds, \qquad \text{for all } x \in D(A). \tag{3.4}$$

On the other hand, using the condition ***(C2)*** and since the scalar kernel a is 2–regular, it follows from [22, Theorem 2.2] that the function $t \mapsto R(t)$ is continuous for $t > 0$. Further, since $a \in C^1(\mathbb{R}^+;\mathbb{R})$, it follows from equation (3.4), that for all $x \in D(A)$ the map $R(\cdot)x$ is differentiable for all $t \geqslant 0$ and satisfies

$$\frac{d}{dt}R(t)x = AR(t)x + \int_0^t b(t-s)AR(s)xds, \quad t \in I. \tag{3.5}$$

From the quality (3.5), we conclude that $\{R(t)\}_{t\in I}$ is a resolvent operator for the equation (3.2) having the property ***(P)***. □

Corollary 3.7. *Suppose that A generates a C_0–semigroup of type (M,ω) in a Hilbert space H. Assume further that the conditions **(C1)–(C2)** are fulfilled. If the hypotheses **(H2)–(H5)** are satisfied and there exists $R \geqslant 0$ such that*

$$Kg_R + K\Phi(R)\int_0^1 m(s)ds \leqslant R, \quad \textit{where} \quad K = \sup\{\|R(t)\| : t \in I\},$$

then equation (3.2) *has at least one mild solution.*

Proof. It follows from Proposition 3.6 that there exists a resolvent operator $\{R(t)\}_{t\in I}$ for the equation and this resolvent operator has the property ***(P)***. Since the hypotheses ***(H2)–(H5)*** are satisfied, we apply the Theorem 3.2 and conclude that the equation (3.2) has at least one mild solution. □

4. Applications

In this section we apply the abstract results which we have obtained in the preceding section to study the existence of solutions for a partial differential equation submitted to nonlocal initial conditions. This type of equations arises in the study of heat conduction in materials with memory (see [19, 23]). Specifically, we will study the following problem

$$\left.\begin{array}{rcl} \dfrac{\partial w(t,\xi)}{\partial t} &=& Aw(t,\xi) + \displaystyle\int_0^t \beta e^{-\alpha(t-s)}Aw(s,\xi)ds + p_1(t)p_2(w(t,\xi)), \quad t \in I, \\ w(t,0) &=& w(t,2\pi), \quad \text{for } t \in I, \\ w(0,\xi) &=& \displaystyle\int_0^1\int_0^\xi qk(s,\xi)w(s,y)dsdy, \quad 0 \leqslant \xi \leqslant 2\pi, \end{array}\right\} \tag{4.1}$$

where $k : I \times [0,2\pi] \to \mathbb{R}^+$ is a continuous function such that $k(t,2\pi) = 0$ for all $t \in I$, the constant $q \in \mathbb{R}^+$ and the constants α, β satisfy the relation $-\alpha \leqslant \beta \leqslant 0 \leqslant \alpha$. The operator A is defined by

$$(Aw)(t,\xi) = a_1(\xi)\frac{\partial^2}{\partial \xi^2}w(t,\xi) + b_1(\xi)\frac{\partial}{\partial \xi}w(t,\xi) + \overline{c}(\xi)w(t,\xi),$$

where the coefficients a_1 , b_1 , $\overline{c}$ satisfy the usual uniformly ellipticity conditions, and $D(A) = \{v \in L^2([0,2\pi];\mathbb{R}) : v'' \in L^2([0,2\pi];\mathbb{R})\}$. The functions $p_1 : I \to \mathbb{R}^+$ and $p_2 : \mathbb{R} \to \mathbb{R}$ satisfy appropriate conditions which will be specified later.

Identifying $u(t) = w(t,\cdot)$ we model this problem in the space $X = L^2(\mathbb{T};\mathbb{R})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. We will use the identification between functions on $\mathbb{T}$ and 2π–periodic functions on $\mathbb{R}$. Specifically, in what follows we denote by $L^2(\mathbb{T};\mathbb{R})$ the space of 2π–periodic and square integrable functions from $\mathbb{R}$ into $\mathbb{R}$. Consequently, the equation (4.1) is rewritten as

$$\left.\begin{array}{rcl} u'(t) &=& Au(t) + \displaystyle\int_0^t b(t-s)Au(s)ds + f(t,u(t)), \quad t \in I, \\ u(0) &=& g(u), \end{array}\right\} \tag{4.2}$$

where the function $g : C(I;X) \to X$ is defined by $g(w)(\xi) = \displaystyle\int_0^1 \int_0^\xi qk(s,\xi)w(s,y)dsdy$, and $f(t,u(t)) = p_1(t)p_2(u(t))$ where p_1 is integrable on I, and p_2 is a bounded function satisfying a Lipschitz type condition with Lipschitz constant L.

We will prove that there exists $q > 0$ sufficiently small such that equation (4.2) has a mild solution on $L^2(\mathbb{T};\mathbb{R})$.

With this purpose, we begin noting that $\|g\| \leqslant q(2\pi)^{1/2}\left(\displaystyle\int_0^{2\pi}\int_0^1 k(s,\xi)^2 dsd\xi\right)^{1/2}$. Moreover, it is well known fact that the g is a compact map.

Further, the function f satisfies $\|f(t,u(t))\| \leqslant p_1(t)\Phi(\|u(t)\|)$, with $\Phi(\|u(t)\|) \equiv \|p_2\|$ and $\|f(t,u_1(t)) - f(t,u_2(t))\| \leqslant Lp_1(t)\|u_1 - u_2\|$ Thus, the conditions ***(H2)–(H5)*** are fulfilled.

Define $a(t) = 1 + \displaystyle\int_0^t \beta e^{-\alpha s}ds$, for all $t \in \mathbb{R}_0^+$. Since the kernel b defined by $b(t) = \beta e^{-\alpha t}$ is 2–regular, it follows that a is 2–regular. Furthermore, we claim that a is completely positive. In fact, we have

$$\widehat{a}(\lambda) = \frac{\lambda+\alpha+\beta}{\lambda(\lambda+\alpha)}.$$

Define the functions f_1 and f_2 by $f_1(\lambda) = \dfrac{1}{\lambda\widehat{a}(\lambda)}$ and $f_2(\lambda) = \dfrac{-\widehat{a}\,'(\lambda)}{[\widehat{a}(\lambda)]^2}$ respectively. In another words

$$f_1(\lambda) = \frac{\lambda+\alpha}{\lambda+\alpha+\beta} \quad \text{and} \quad f_2(\lambda) = \frac{\lambda^2 + 2(\alpha+\beta)\lambda + \alpha\beta + \alpha^2}{(\lambda+\alpha+\beta)^2}.$$

A direct calculation shows that

$$f_1^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(n+1)!}{(\lambda+\alpha+\beta)^{n+1}} \quad \text{and} \quad f_2^{(n)}(\lambda) = \frac{(-1)^{n+1}\beta(\alpha+\beta)(n+1)!}{(\lambda+\alpha+\beta)^{n+2}} \quad \text{for } n \in \mathbb{N}.$$

Since $-\alpha \leqslant \beta \leqslant 0 \leqslant \alpha$, we have that f_1 and f_2 are completely monotone. Thus, the kernel a is completely positive.

On the other hand, it follows from [13] that A generates an analytic, non compact semigroup $\{T(t)\}_{t\geqslant 0}$ on $L^2(\mathbb{T};\mathbb{R})$. In addition, there exists a constant $M>0$ such that

$$M=\sup\{\|T(t)\| : t\geqslant 0\}<+\infty.$$

It follows from the preceding fact and the Hille–Yosida theorem that $z\in\rho(A)$ for all $z\in\mathbb{C}$ such that $Re(z)>0$. Let $z=\mu_0+i\mu$. By direct computation we have

$$Re\left(\frac{\mu_0+i\mu}{\widehat{b}(\mu_0+i\mu)+1}\right)=\frac{\mu_0^3+\mu_0^2\alpha+\mu_0^2(\alpha+\beta)+\mu_0\alpha(\alpha+\beta)+\mu_0\mu^2-\mu^2\beta}{(\alpha+\beta)^2+2\mu_0(\alpha+\beta)+\mu_0^2+\mu^2}.$$

Hence, $Re\left(\frac{\mu_0+i\mu}{\widetilde{b}(\mu_0+i\mu)+1}\right)>0$ for all $z=\mu_0+i\mu$, such that $\mu_0>0$. This implies that

$$\left(\frac{\mu_0+i\mu}{\widetilde{b}(\mu_0+i\mu)+1}-A\right)^{-1}\in\mathcal{L}(X),\ \text{for all}\ \ \mu_0>0.$$

Since the semigroup generated by A is an analytic semigroup we have

$$\left\|\frac{1}{\widehat{b}(\mu_0+i\mu)+1}\left(\frac{\mu_0+i\mu}{\widehat{b}(\mu_0+i\mu)+1}-A\right)^{-1}\right\|\leqslant\left\|\frac{M}{\mu_0+i\mu}\right\|$$

Therefore,

$$\lim_{|\mu|\to\infty}\left\|\frac{1}{\widehat{b}(\mu_0+i\mu)+1}\left(\frac{\mu_0+i\mu}{\widehat{b}(\mu_0+i\mu)+1}-A\right)^{-1}\right\|=0.$$

It follows from Proposition 3.6 that the equation (4.2) admits a resolvent operator $\{R(t)\}_{t\in I}$ satisfying property ***(P)***.

Let $K=\sup\{\|R(t)\| : t\in I\}$ and $c=(2\pi)^{1/2}\left(\int_0^{2\pi}\int_0^1 k(s,\xi)^2 ds d\xi\right)^{1/2}$.
A direct computation shows that for each $R\geqslant 0$ the number g_R is equal to $g_R=qcR$.

Therefore the expression $\left(Kg_R+K\Phi(R)\int_0^1 m(s)ds\right)$, is equivalent to $(qcKR+\|p_1\|_1 LK)$.

Since, there exists $q>0$ such that $qcK<1$, we have that there exists $R\geqslant 0$ such that

$$qcKR+\|p_1\|_1 LK\leqslant R.$$

From the Corollary 3.7 we conclude that there exists a mild solution of the equation (4.1).

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