MAXIMAL REGULARITY FOR PERTURBED INTEGRAL EQUATIONS ON THE LINE

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Abstract. We characterize the existence and uniqueness of solutions for a perturbed linear integral equation with infinite delay in Hölder spaces. The method is based on the theory of operator-valued Fourier multipliers.

1. Introduction

The theory of maximal regularity for linear evolution equations in abstract spaces is a very useful tool in the study of solutions to nonlinear partial differential equations. See for instance, Amann [1, 2] and Denk-Hieber-Prüss [12], for more details. In the study of this theory, the main tool to obtain characterizations of maximal regularity is based in some recent results in vector-valued Fourier multipliers in abstract spaces. See Arendt, Batty, Bu and Kim [3, 4, 5, 9] for further information. Indeed, several results that characterize the existence and uniqueness of solutions for several classes of equations in Hölder spaces, have been obtained in the last years. See [3, 8, 11, 15, 18] for more details.

In this paper we the study of the existence, uniqueness and maximal regularity of solutions in Hölder spaces for the following perturbed integral equation with infinite delay

$$u(t) = A\int_{-\infty}^{t} a(t-s)u(s)ds + B\int_{-\infty}^{t} b(t-s)u(s)ds + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where $(A, D(A))$ and $(B, D(B))$ are two closed linear operators defined on a Banach space X, with $D(A)\cap D(B) \neq \{0\}$, $a, b \in L^1_{\text{loc}}(\mathbb{R}_+)$ are scalar-valued kernels and $f \in C^\alpha(\mathbb{R}; X)$, $0 < \alpha < 1$. Problems in the form of (1.1) has been motived by [20, Section 4] and [19, p.235].

In the particular case when $B = A^\varepsilon$, $a(t) = -te^{-\gamma t}$ and $b(t) = -e^{-\gamma t}$ (where $\gamma, \varepsilon > 0$) the equation (1.1) can be considered as a integral version of the second order equation

$$u''(t) + A^\varepsilon u'(t) + Au(t) = f(t), \tag{1.2}$$

since, formally, it corresponds to the second derivative of (1.1) in the limit case $\gamma = 0$. L^p-maximal regularity for equation (1.2) with initial conditions $u(0) = u'(0) = 0$ has been studied in [10] in the case when A is a sectorial operator and admits a bounded RH^∞ functional calculus of angle less than $\pi/2\varepsilon$, $\frac{1}{2} < \varepsilon \leq 1$.

2000 *Mathematics Subject Classification.* 45N05, 47N20, 34G10.

Key words and phrases. Integral equations; Operator-valued Fourier multipliers; Maximal regularity.

The first author is partially supported by FONDECYT 1100485.

Maximal regularity on periodic vector-valued Lebesgue spaces, $L^p_{2\pi}(\mathbb{R};X)$, $1<p<\infty$, (where X is a UMD space) to equation (1.1) have been studied in [16], whereas that on periodic Besov spaces (and therefore on periodic Hölder spaces), the problem (1.1) have been recently considered in [7].

We apply the method of operator-valued Fourier multipliers (see [3]) to obtain a characterization that ensure the existence and uniqueness of solutions to equation (1.1) in $C^\alpha(\mathbb{R};X)$, the vector-valued Hölder spaces, where $0<\alpha<1$. We notice that in this characterization, there are no conditions in the commutativity of operators A,B or in the existence of bounded inverse of A or B.

The plan of the paper is the following: In Section 2 we collects the definitions and some results about vector-valued Fourier multipliers. The Section 3 is devoted to our main result, where a characterization that ensure the well-posedness of the problem (1.1) in $C^\alpha(\mathbb{R};X)$ is obtained. We remark that this characterization is solely in terms of the boundedness of operators $(I-\tilde{a}(\eta)A-\tilde{b}(\eta)B)^{-1}$ and $\tilde{b}(\eta)B(I-\tilde{a}(\eta)A-\tilde{b}(\eta)B)^{-1}$, $\eta\in\mathbb{R}$, where $\tilde{a},\tilde{b}$ denote the Fourier transform of a,b (more precisely of their extension to $\mathbb{R}$ by setting them equal to 0 on $(-\infty,0)$). Finally, some examples are examined in Section 4. In particular, we show that problem (1.1) with $a(t)=-te^{-\gamma t}$, $b(t)=-e^{-\gamma t}$ $(\gamma>0)$ and $B=A^{1/2}$ is C^α-well posed whenever A is a sectorial operator which admits a bounded H^∞-functional calculus of angle $\beta\in(0,\pi/3)$ (see Proposition 4.13 below).

2. Preliminaries

Let X and Y be Banach spaces. We denote by $\mathcal{B}(X,Y)$ be the space of all bounded linear operators from X to Y. If $X=Y$, we write simply $\mathcal{B}(X)$. Let $0<\alpha<1$. We denote by $C^\alpha(\mathbb{R};X)$ the space of all X-valued functions f on $\mathbb{R}$, such that

$$||f||_\alpha=\sup_{t\neq s}\frac{||f(t)-f(s)||}{|t-s|^\alpha}<\infty.$$

If we define $||f||_{C^\alpha}:=||f||_\alpha+||f(0)||$, then $C^\alpha(\mathbb{R};X)$ is a Banach space under the norm $||\cdot||_{C^\alpha}$.

The kernel of the seminorm $||\cdot||_\alpha$ on $C^\alpha(\mathbb{R};X)$ is the space of all constant functions and the corresponding quotient space $\dot{C}^\alpha(\mathbb{R};X)$ is a Banach space in the induced norm. We identify a function $f\in C^\alpha(\mathbb{R};X)$ with its equivalence class

$$\dot{f}:=\{g\in C^\alpha(\mathbb{R};X): f-g\equiv \text{ constant}\}.$$

In this way, $\dot{C}^\alpha(\mathbb{R};X)$ may be identified with the space of all $f\in C^\alpha(\mathbb{R};X)$ such that $f(0)=0$. See [3, Section 5].

We denote by $\mathcal{F}f$ the Fourier transform of f, that is

$$(\mathcal{F}f)(s):=\tilde{f}(s):=\int_{\mathbb{R}}e^{-ist}f(t)dt,$$

for $s\in\mathbb{R}$ and $f\in L^1(\mathbb{R};X)$. Let $\Omega\subset\mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all $C^\infty-$functions in Ω having compact support in Ω.

Definition 2.1. *Let $M : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X,Y)$ be continuous. We say that M is a $\dot{C}^\alpha-$multiplier if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y)$ such that*

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot M))(s)f(s)ds \tag{2.1}$$

for all $f \in C^\alpha(\mathbb{R}; X)$ and all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Here $(\mathcal{F}(\phi \cdot M))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)M(t)dt \in \mathcal{B}(X,Y)$. Observe that the right-hand side of (2.1) does not depend on the representative of $\dot{f}$ since

$$\int_{\mathbb{R}} (\mathcal{F}(\phi M)(s))(s)ds = 2\pi(\phi M)(0) = 0.$$

Therefore, if L exists, then it is well defined. Moreover, left-hand side of (2.1) determines the function $Lf \in C^\alpha(\mathbb{R}; X)$ uniquely up to some constant (by [3, Lemma 5.1]). Moreover, if (2.1) holds, then $L : \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y)$ is linear and continuous (see [3, Definition 5.2]) and if $f \in C^\alpha(\mathbb{R}; X)$ is bounded, then Lf is bounded as well (see [3, Remark 6.3]).

The following results are due to Arendt, Batty and Bu [3].

Lemma 2.2. *Let $f \in C^\alpha(\mathbb{R}; X)$. Then f is constant if and only if $\int_{\mathbb{R}} f(s)(\mathcal{F}\varphi)(s)ds = 0$ for all $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.*

Theorem 2.3. *Let $M \in C^2(\mathbb{R} \setminus \{0\}, \mathcal{B}(X,Y))$ be such that*

$$\sup_{t \neq 0} ||M(t)|| + \sup_{t \neq 0} ||tM'(t)|| + \sup_{t \neq 0} ||t^2 M''(t)|| < \infty. \tag{2.2}$$

Then, M is a $\dot{C}^\alpha-$multiplier.

Remark 2.4.

Recall that a Banach space X has the *Fourier type* p, with $1 \leq p \leq 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; X)$, where $1/p+1/q = 1$. As examples, the $L^p(\Omega)$, with $1 \leq p \leq 2$ has Fourier type p; the Banach space X has the Fourier type 2 if and only if X is isomorphic to a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1. A Banach space X is say to be $B-$convex if it has Fourier type p, for some $p > 1$. Every uniformly convex space is $B-$convex.

If X is $B-$convex, in particular if X is a UMD space, then the Theorem 2.3 holds if the condition (2.2) is replaced by the weaker condition

$$\sup_{t \neq 0} ||M(t)|| + \sup_{t \neq 0} ||tM'(t)|| < \infty, \tag{2.3}$$

where $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{B}(X,Y))$, see [3, Remark 5.5].

Let $0 < \alpha < 1$. We denote by $L^1(\mathbb{R}_+, t^\alpha dt)$ the set of all $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that

$$\int_0^\infty |a(t)|t^\alpha dt < \infty. \tag{2.4}$$

Observe that as consequence such an a is always in $L^1(\mathbb{R}_+)$.

Given $v \in C^\alpha(\mathbb{R}; X)$ $(0 < \alpha < 1)$ and $a \in L^1(\mathbb{R}_+, t^\alpha dt)$, we write

$$(a\dot{*}v)(t) := \int_{-\infty}^{t} a(t-s)v(s)ds = \int_0^\infty a(s)v(t-s)ds. \tag{2.5}$$

From (2.4) the above integral is well defined. Moreover, it follows from the definition that

$$\text{if } v \in C^\alpha(\mathbb{R}; X) \text{ then } a\dot{*}v \in C^\alpha(\mathbb{R}; X) \text{ and } ||a\dot{*}v||_\alpha \leq ||a||_1 ||v||_\alpha. \tag{2.6}$$

The Laplace transform of a function $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ is denoted by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \text{Re}\lambda > \omega,$$

whenever the integral is absolutely convergent for $\text{Re}\lambda > \omega$. The relation between the Laplace transform of $f \in L^1(\mathbb{R}; X)$, $f(t) = 0$ for $t < 0$, and its Fourier transform is

$$\mathcal{F}(f)(s) = \hat{f}(is), \quad s \in \mathbb{R}.$$

For $f \in L^1_{\text{loc}}(\mathbb{R}; X)$ of subexponential growth, that is

$$\int_{-\infty}^\infty e^{-\epsilon|t|}||f(t)||dt < \infty, \qquad \text{for each } \epsilon > 0,$$

we denote by $\hat{f}(\lambda)$ for the *Carleman transform of* f :

$$\hat{f}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} f(t)dt, & \text{Re}\lambda > 0, \\ \\ -\int_{-\infty}^0 e^{-\lambda t} f(t)dt, & \text{Re}\lambda < 0. \end{cases}$$

Observe that we use the same symbol for the Carleman and Laplace transform but, this will not lead to confusion.

When $f \in L^1(\mathbb{R}; X)$ is of subexponential growth, we have by [6, Chapter 4],

$$\lim_{\sigma \to 0^+} (\hat{f}(\sigma + i\rho) - \hat{f}(-\sigma + i\rho)) = \tilde{f}(\rho), \quad \rho \in \mathbb{R}. \tag{2.7}$$

If $a \in L^1(\mathbb{R}_+)$, we will always identify a with its extension on $\mathbb{R}$ by letting $a(t) = 0$ for $t < 0$. In such way, when $a \in L^1(\mathbb{R}_+)$, the Fourier transform $\tilde{a}(\rho)$ makes sense for all $\rho \in \mathbb{R}$. Moreover, by (2.7) we have

$$\lim_{\sigma \to 0^+} \hat{a}(\sigma + i\rho) = \tilde{a}(\rho)$$

and $\hat{a}(-\sigma + i\rho) = 0$ for all $\sigma > 0$ and $\rho \in \mathbb{R}$ by definition.

In what follows, we always assume that $\tilde{a}(\eta)$ and $\tilde{b}(\eta)$ exists for all $\eta \in \mathbb{R}$ and $\tilde{a}(i\eta) \neq 0$, $\tilde{b}(i\eta) \neq 0$ for all $\eta \in \mathbb{R}$ and we use the following notation:

$$a_\eta := \tilde{a}(\eta),$$
$$b_\eta := \tilde{b}(\eta).$$

We recall the notion of regular kernels (see [19, p. 69]).

Definition 2.5. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. $a(t)$ is called* k-regular *if there is a constant $c > 0$ such that*

$$|\lambda^n [\hat{a}(\lambda)]^{(n)}| \le c|\hat{a}(\lambda)|, \quad \text{for all } Re(\lambda) > 0, 0 \le n \le k.$$

For the reader's convenience, we summarize here from [19, Lemma 8.1] some properties of 1-regular kernels.

Lemma 2.6. *Suppose that $b \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is of sub-exponential growth and 1-regular. Then*
(i) $\hat{b}(i\rho) := \lim_{\lambda \to i\rho} \hat{b}(\lambda)$ exists for each $\rho \neq 0$;
(ii) $\hat{b}(\lambda) \neq 0$ for each $Re(\lambda) \ge 0$, $\lambda \neq 0$;
(iii) $\hat{b}(i\cdot) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$;
(iv) $|\rho[\hat{b}(i\rho)]'| \le c|\hat{b}(i\rho)|$ for a.a. $\rho \in \mathbb{R}$.

3. A characterization

In this section, we consider the problem of existence and uniqueness of solutions in $C^\alpha(\mathbb{R}; X)$, to equation

$$u(t) = A\int_{-\infty}^t a(t-s)u(s)ds + B\int_{-\infty}^t b(t-s)u(s)ds + f(t), \quad t \in \mathbb{R}, \tag{3.8}$$

where $(A, D(A))$, $(B, D(B))$ are closed linear operators on X, $D(A) \cap D(B) \neq \{0\}$, $a, b \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ are scalar-valued kernel and $f \in C^\alpha(\mathbb{R}; X)$, $0 < \alpha < 1$.

We define the *real resolvent* of equation (3.8) by

$$\rho(\Delta) := \{s \in \mathbb{R} : (I - \tilde{a}(s)A - \tilde{b}(s)B) : D(A) \cap D(B) \to X$$
$$\text{is invertible and } (I - \tilde{a}(s)A - \tilde{b}(s)B)^{-1} \in \mathcal{B}(X, [D(A) \cap D(B)])\}.$$

Definition 3.7. *We say that the equation (3.8) is* C^α-well posed *if, for each $f \in C^\alpha(\mathbb{R}; X)$, there exists a unique function $u \in C^\alpha(\mathbb{R}; X)$ such that $a*u \in C^\alpha(\mathbb{R}; [D(A)])$, $b*u \in C^\alpha(\mathbb{R}; [D(B)])$ and the equation (3.8) holds for all $t \in \mathbb{R}$.*

Proposition 3.8. *Let $A : D(A) \subseteq X \to X$, $B : D(B) \subseteq X \to X$ linear closed operators defined on a Banach space X satisfying $D(A) \cap D(B) \neq \{0\}$ and $a, b \in L^1(\mathbb{R}_+, t^\alpha dt)$. Suppose that the problem (3.8) is C^α-well posed. Then,*
(i) $\rho(\Delta) = \mathbb{R}$,
(ii) $\sup_{\eta \in \mathbb{R}} ||(I - a_\eta A - b_\eta B)^{-1}|| < \infty$ and $\sup_{\eta \in \mathbb{R}} ||b_\eta B(I - a_\eta A - b_\eta B)^{-1}|| < \infty$.

Proof. Let $\eta \in \mathbb{R}$ and suppose that $(I - a_\eta A - b_\eta B)x = 0$ where $x \in D(A) \cap D(B)$. Let $u(t) = e^{i\eta t}x$. Then, u is a solution to (3.8) with $f \equiv 0$. Hence, by uniqueness follows that $u \equiv 0$, that is, $x = 0$. We obtain that $(I - a_\eta A - b_\eta B)$ is injective. In order to show the surjectivity, let $y \in X$. Let $L : C^\alpha(\mathbb{R}; X) \to C^\alpha(\mathbb{R}; X)$ be the bounded operator which takes each $f \in C^\alpha(\mathbb{R}; X)$ to the unique solution u of equation (3.8). Let $\eta \in \mathbb{R}$, $f(t) = e^{i\eta t}y$ and $u = Lf$. Then, for fixed $s \in \mathbb{R}$ we have that $v_1(t) := u(t+s)$ and $v_2(t) := e^{i\eta s}u(t)$ are both solutions of (3.8) with $g(t) = e^{is\eta}f(t)$. Hence, $v_1 = v_2$, that is,

$u(t+s)=e^{is\eta}u(t)$ for all $s,t\in\mathbb{R}$. Let $x=u(0)\in D(A)\cap D(B)$. Then, $u(t)=e^{i\eta t}x$ (i.e. $u(t)=u(t+0)=v_1(t)$) satisfy the equation (3.8). Now, observe that

$$(a{*}Au)(t)=e^{i\eta t}a_\eta Ax,\quad t\in\mathbb{R}.$$

In particular, $(a{*}Au)(0)=a_\eta A$. Similarly, we obtain $(b{*}Bu)(0)=b_\eta B$. Therefore,

$$(I-a_\eta A-b_\eta B)x=x-a_\eta Ax-b_\eta Bx=u(0)-(a{*}Au)(0)-(b{*}Bu)(0).$$

Since $u(t)$ satisfy the equation (3.8) for all $t\in\mathbb{R}$, we obtain,

$$(I-a_\eta A-b_\eta B)x=u(0)-(a{*}Au)(0)-(b{*}Bu)(0)=f(0)=y. \tag{3.9}$$

Therefore $(I-a_\eta A-b_\eta B)$ is surjective. By (3.9) we have $u(t)=e^{i\eta t}(I-a_\eta A-b_\eta B)^{-1}y$. Denote $e_\eta\otimes x$ the function $t\to(e_\eta\otimes x)(t):=e^{i\eta t}x$. Since $||e_\eta\otimes x||_\alpha=\gamma_\alpha|\eta|^\alpha||x||$, where $\gamma_\alpha=2\sup_{t>0}t^{-\alpha}\sin(t/2)$ (see [3, Section 3]) we have

$$\begin{aligned}\gamma_\alpha|\eta|^\alpha||(I-a_\eta A-b_\eta B)^{-1}y|| &= ||e_\eta\otimes(I-a_\eta A-b_\eta B)^{-1}y||_\alpha=||u||_\alpha\le||u||_{C^\alpha}\\ &= ||Lf||_{C^\alpha}\le||L||\,||f||_{C^\alpha}\le||L||(||f||_\alpha+||f(0)||)\\ &= ||L||(\gamma_\alpha|\eta|^\alpha+1)||y||.\end{aligned}\tag{3.10}$$

Therefore, $(I-a_\eta A-b_\eta B)^{-1}$ is a bounded operator for every $\eta\in\mathbb{R}\setminus\{0\}$. For $\eta=0$, observe that by the closed graph theorem $(I-a_0A-b_0B)^{-1}$ is an isomorphism of X onto $D(A)\cap D(B)$ (seen as a Banach space with the graph norm). We conclude that $\rho(\Delta)=\mathbb{R}$.

On the other hand, using the closed graph theorem, we have that there exist a constant $C>0$ independent of $f\in C^\alpha(\mathbb{R};X)$ such that

$$||u||_{C^\alpha}+||Aa{*}u||_{C^\alpha}+||Bb{*}u||_{C^\alpha}\le C||f||_{C^\alpha}. \tag{3.11}$$

Note that for $f(t)=e^{it\eta}y$ where $y\in X$ and $\eta\in\mathbb{R}$, the solution u of (3.8) is given by $u(t)=e^{i\eta t}(I-a_\eta A-b_\eta B)^{-1}y$. An easy computation shows that $(b{*}Bu)(t)=e_\eta\otimes b_\eta B(I-a_\eta A-b_\eta B)^{-1}y$. Observe that

$$\begin{aligned}||(b{*}Bu)||_\alpha &= ||e_\eta\otimes b_\eta B(I-a_\eta A-b_\eta B)^{-1}y||_\alpha\\ &= \gamma_\alpha|\eta|^\alpha||b_\eta B(I-a_\eta A-b_\eta B)^{-1}y||.\end{aligned}$$

Since $||(b{*}Bu)||_\alpha\le C||f||_{C^\alpha}=C(\gamma_\alpha|\eta|^\alpha+1)||y||$ we have

$$\gamma_\alpha|\eta|^\alpha||b_\eta B(I-a_\eta A-b_\eta B)^{-1}y||\le C(\gamma_\alpha|\eta|^\alpha+1)||y||.$$

Therefore, we have that for $\epsilon>0$, $\sup_{|\eta|>\epsilon}||b_\eta B(I-a_\eta A-b_\eta B)^{-1}||<\infty$ and by continuity it follows that $\sup_{\eta\in\mathbb{R}}||b_\eta B(I-a_\eta A-b_\eta B)^{-1}||<\infty$. ∎

The main result of this paper is the following, which shows that under an additional hypothesis (the 2-regularity in the kernels a and b) we can prove the converse of Proposition 3.8.

Theorem 3.9. *Let $A:D(A)\subseteq X\to X$, $B:D(B)\subseteq X\to X$ linear closed operators defined on Banach space X satisfying $D(A)\cap D(B)\neq\{0\}$ and $a,b\in L^1(\mathbb{R}_+,t^\alpha dt)$. Suppose that the kernels a and b are 2-regular. Then, the following assertions are equivalent*

(*i*) *The equation (3.8) is C^α-well posed;*

(ii) $\rho(\Delta)=\mathbb{R}$, $\sup_{\eta\in\mathbb{R}}||(I-a_\eta A-b_\eta B)^{-1}||<\infty$ *and* $\sup_{\eta\in\mathbb{R}}||b_\eta B(I-a_\eta A-b_\eta B)^{-1}||<\infty$.

Proof. $(ii)\Rightarrow(i)$. For $\eta\in\mathbb{R}$, define the operator $N(\eta):=(I-a_\eta A-b_\eta B)^{-1}$. Observe that by hypothesis $N\in C^2(\mathbb{R};\mathcal{B}(X,[D(A)\cap D(B)]))$. We claim that N is a $\dot{C}^\alpha$-multiplier. In fact, the hypothesis imply that $\sup_{\eta\in\mathbb{R}}||N(\eta)||<\infty$. On the other hand,

$$\eta N'(\eta)=N(\eta)(\eta a'_\eta A+\eta b'_\eta B)N(\eta),$$

and

$$\eta^2N''(\eta)\;=\;2N'(\eta)[\eta a'_\eta A+\eta b'_\eta B]N(\eta)+N(\eta)[\eta^2a''_\eta A+\eta^2b''_\eta B]N(\eta).$$

Since a and b are 2-regular kernels, the Lemma 2.6 imply that there exist constants c_1,c_2,c_3,c_4 such that

$$||\eta N'(\eta)||\;\le\;c_1||N(\eta)a_\eta AN(\eta)||+c_2||N(\eta)b_\eta BN(\eta)||$$

and

$$\begin{aligned}||\eta^2N''(\eta)||\;\le\;&2||\eta N'(\eta)||\big[c_1||N(\eta)a_\eta AN(\eta)||+c_2||N(\eta)b_\eta BN(\eta)||\big]+\\&||N(\eta)||\big[c_3||N(\eta)a_\eta AN(\eta)||+c_4||N(\eta)b_\eta BN(\eta)||\big].\end{aligned}$$

From the hypothesis we have $\sup_{\eta\in\mathbb{R}}||\eta N'(\eta)||<\infty$ and $\sup_{\eta\in\mathbb{R}}||\eta^2N''(\eta)||<\infty$. We conclude by Theorem 2.3 that N is a $\dot{C}^\alpha$-multiplier, with $N\in C^2(\mathbb{R};\mathcal{B}(X,[D(A)\cap D(B)]))$.

Now, define $M\in C^2(\mathbb{R};\mathcal{B}(X))$ by $M(\eta):=b_\eta BN(\eta)$. Observe that by hypothesis $\sup_{\eta\in\mathbb{R}}||M(\eta)||<\infty$. Moreover,

$$\eta M'(\eta)\;=\;\eta b'_\eta BN(\eta)+\eta b_\eta BN'(\eta)=\eta b'_\eta BN(\eta)+\eta b_\eta BN(\eta)[a'_\eta A+b'_\eta B]N(\eta),$$

and

$$\eta^2M''(\eta)\;=\;2\eta b'_\eta B\eta N'(\eta)+\eta^2b''_\eta BN(\eta)+b_\eta B\eta^2N''(\eta).$$

From the 2-regularity of the kernels a, b and Lemma 2.6 we have $\sup_{\eta\in\mathbb{R}}||\eta M'(\eta)||<\infty$, and $\sup_{\eta\in\mathbb{R}}||\eta^2M''(\eta)||<\infty$, and therefore M is a $\dot{C}^\alpha$-multiplier by Theorem 2.3. A similar computation shows that $T\in C^2(\mathbb{R};\mathcal{B}(X))$ defined by $T(\eta):=a_\eta AN(\eta)$ is a $\dot{C}^\alpha$-multiplier.

Let $f\in C^\alpha(\mathbb{R};X)$. Since N,M and T are $\dot{C}^\alpha$-multipliers, there exists $u\in C^\alpha(\mathbb{R};[D(A)\cap D(B)])$, $v\in C^\alpha(\mathbb{R};X)$ and $w\in C^\alpha(\mathbb{R};X)$ such that

$$\int_{\mathbb{R}}u(s)(\mathcal{F}\phi)(s)ds=\int_{\mathbb{R}}\mathcal{F}(\phi\cdot N)(s)f(s)ds, \tag{3.12}$$

$$\int_{\mathbb{R}}v(s)(\mathcal{F}\varphi)(s)ds=\int_{\mathbb{R}}\mathcal{F}(\varphi\cdot M)(s)f(s)ds, \tag{3.13}$$

$$\int_{\mathbb{R}}w(s)(\mathcal{F}\psi)(s)ds=\int_{\mathbb{R}}\mathcal{F}(\psi\cdot T)(s)f(s)ds, \tag{3.14}$$

for all $\phi,\varphi,\psi\in C_c^\infty(\mathbb{R}\setminus\{0\})$. Since $u\in C^\alpha(\mathbb{R};[D(A)\cap D(B)])$ and $a,b\in L^1(\mathbb{R}_+,t^\alpha dt)$ we have by (2.6) that $a*u\in C^\alpha(\mathbb{R};[D(A)])$, and $b*u\in C^\alpha(\mathbb{R};[D(B)])$.

Choosing $\phi = b_\eta \varphi$, we have that $\phi \in C_c^1(\mathbb{R} \setminus \{0\})$ (see Lemma 2.6). Thus, replacing ϕ in (3.12) we obtain

$$\int_{\mathbb{R}} u(s)(\mathcal{F}b_s\varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(b_s\varphi \cdot N)(s)f(s)ds. \tag{3.15}$$

Since $u(t) \in D(A) \cap D(B)$ and $\mathcal{F}(\phi N)(s)x \in D(A) \cap D(B)$ for all $x \in X$ and using the fact that B is a closed operator, we have from (3.15) and (3.13)

$$B \int_{\mathbb{R}} u(s)\mathcal{F}(b_s\varphi)(s)ds \;\; = \;\; \int_{\mathbb{R}} v(s)\mathcal{F}(\varphi)(s)ds.$$

By Lemma [15, Lemma 3.2] we obtain $B(b{*}u) = v + y_1$ where $y_1 \in X$. Similarly, choosing $\phi = a_s\psi$ in (3.12) and replacing in (3.14) we have $A(a{*}u) = w + y_2$, where $y_2 \in X$. Since $N(\eta) = I + a_\eta AN(\eta) + b_\eta BN(\eta)$ we have from (3.12), (3.13) and (3.14)

$$\begin{aligned} \int_{\mathbb{R}} u(s)(\mathcal{F}\phi)(s)ds &= \int_{\mathbb{R}} \mathcal{F}(\phi \cdot N)(s)f(s)ds \\ &= \int_{\mathbb{R}} \mathcal{F}(\phi \cdot I)(s)f(s)ds + \int_{\mathbb{R}} \mathcal{F}(\phi \cdot a_sAN)(s)f(s)ds + \\ &\qquad \int_{\mathbb{R}} \mathcal{F}(\phi \cdot b_sBN)(s)f(s)ds \\ &= \int_{\mathbb{R}} \mathcal{F}(\phi \cdot I)(s)f(s)ds + \int_{\mathbb{R}} w(s)(\mathcal{F}\phi)(s)ds + \int_{\mathbb{R}} v(s)(\mathcal{F}\phi)(s)ds. \end{aligned}$$

Therefore, by Lemma 2.2, $u(t) = f(t) + w(t) + v(t) + y_3$ where $y_3 \in X$. Thus, $u(t) = f(t) + A(a{*}u)(t) + B(b{*}u)(t) + y_4$, where $y_4 = y_1 + y_2 + y_3$. Let $\overline{u}(t) = u(t) + x$ where $x = (I - a_0A - b_0B)^{-1}y_4$. Note that x is well defined since $\rho(\Delta) = \mathbb{R}$. Clearly $a{*}\overline{u} \in C^\alpha(\mathbb{R}; [D(A)])$ and $b{*}\overline{u} \in C^\alpha(\mathbb{R}; [D(B)])$. An easy computation shows that $\overline{u}$ is a solution to equation (3.8).

In order to prove the uniqueness, suppose that

$$u(t) = A(a{*}u)(t) + B(b{*}u)(t), \quad t \in \mathbb{R}, \tag{3.16}$$

where $u \in C^\alpha(\mathbb{R}; X)$ is such that $a{*}u \in C^\alpha(\mathbb{R}; [D(A)])$, $b{*}u \in C^\alpha(\mathbb{R}; [D(B)])$. Take Carleman transform in (3.16). As in [15, Appendix A], for $\sigma > 0$, we denote $L_\sigma(u)(\rho)$ by $L_\sigma(u)(\rho) := \hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho)$, where $\rho \in \mathbb{R}$. From [15, Proposition A.2.(iv)], we have

$$L_\sigma(u)(\rho) = A\hat{a}(\sigma + i\rho)L_\sigma(u)(\rho) + G_a^{Au}(\sigma, \rho) + B\hat{b}(\sigma + i\rho)L_\sigma(u)(\rho) + G_b^{Bu}(\sigma, \rho),$$

with

$$\lim_{\sigma \to 0^+} \int_{\mathbb{R}} G_a^{Au}(\sigma, \rho)\phi(\rho)d\rho = \lim_{\sigma \to 0^+} \int_{\mathbb{R}} G_b^{Bu}(\sigma, \rho)\phi(\rho)d\rho = 0,$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Since,

$$\left(I - A\hat{a}(\sigma + i\rho) - B\hat{b}(\sigma + i\rho)\right) L_\sigma(u)(\rho) = [G_a^{Au}(\sigma, \rho) + G_b^{Bu}(\sigma, \rho)],$$

and $\rho(\Delta)=\mathbb{R}$, we have

$$\begin{aligned} L_\sigma(u)(\rho) &= [G_a^{Au}(\sigma,\rho)+G_b^{Bu}(\sigma,\rho)]R_\rho - \\ &\quad (\hat{a}(i\rho)-\hat{a}(\sigma+i\rho))AR_\rho L_\sigma(u)(\rho)-(\hat{b}(i\rho)-\hat{b}(\sigma+i\rho))BR_\rho L_\sigma(u)(\rho), \end{aligned}$$

where R_ρ denotes $R_\rho=(I-\hat{a}(i\rho)A-\hat{b}(i\rho)B)^{-1}$.

A similar argument to used in [15, Lemma A.4] shows that

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}(\hat{a}(i\rho)-\hat{a}(\sigma+i\rho))AR_\rho L_\sigma(u)(\rho)\phi(\rho)d\rho=0,$$

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}(\hat{b}(i\rho)-\hat{b}(\sigma+i\rho))BR_\rho L_\sigma(u)(\rho)\phi(\rho)d\rho=0,$$

for all $\phi\in C_c^\infty(\mathbb{R})$. Moreover, applying the dominated convergence theorem, we have

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}[G_a^{Au}(\sigma,\rho)+G_b^{Bu}(\sigma,\rho)]R_\rho\phi(\rho)d\rho=0.$$

Therefore, we have by [15, Proposition A.2.(i)]

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}L_\sigma(u)(\rho)\phi(\rho)d\rho=\int_{\mathbb{R}}u(\rho)\mathcal{F}(\phi)(\rho)d\rho=0,$$

for all $\phi\in\mathcal{S}(\mathbb{R})$. We conclude from Lemma 2.2 that u is constant. Since $u\in C^\alpha(\mathbb{R};X)$ is solution of equation (3.8) and $\rho(\Delta)=\mathbb{R}$, we have $u\equiv 0$.

$(i)\Rightarrow(ii)$. Follows from Proposition 3.8. ■

Remark 3.10. *When the underlying Banach space X is B-convex, we may replace the assumption that a and b are 2-regular kernels in Theorem 3.9, by the weaker condition that a and b are 1-regular kernels. This follows from Remark 2.4 and the proof of Theorem 3.9.*

Corollary 3.11. *In the context of Theorem 3.9, if condition (ii) is fulfilled, we have that $u, Aa{*}u, Bb{*}u\in C^\alpha(\mathbb{R};X)$. Moreover, there exists a constant $C>0$ independent of $f\in C^\alpha(\mathbb{R};X)$ such that*

$$||u||_{C^\alpha}+||Aa{*}u||_{C^\alpha}+||Bb{*}u||_{C^\alpha}\le C||f||_{C^\alpha}. \tag{3.17}$$

4. Examples

Example 4.12.

Consider the problem

$$u(t)=-A\int_{-\infty}^t e^{-\gamma(t-s)}(t-s)u(s)ds-B\int_{-\infty}^t e^{-\gamma(t-s)}u(s)ds+f(t),\ t\in\mathbb{R}, \tag{4.18}$$

where A,B are closed linear operators defined on a Banach space X, $\gamma>0$ and $f\in C^\alpha(\mathbb{R};X)$. We observe that, in the limit case $\gamma=0$, the equation (4.18) can be considered as a integral version of the problem

$$u''(t)+Bu'(t)+Au(t)=F(t). \tag{4.19}$$

L^p-Maximal regularity for equation (4.19) with $B = A^{\varepsilon}$, $(\varepsilon > 0)$ and initial conditions $u(0) = u'(0) = 0$ have been studied in [10] in the case when A is sectorial and admits a bounded RH^{∞} calculus of angle less that $\pi/2\varepsilon$. On Hölder space, the problem (4.19) have been recently studied in [18] whereas on periodic Lebesgue space $L^p_{2\pi}(\mathbb{R};X)$, with $1 < p < \infty$ and X being a UMD space in [16]. On periodic Besov space, see [7].

We recall that a closed, densely defined operator A is *sectorial of angle* $\beta \in (0,\pi)$ if $\sigma(A) \subset \overline{\Sigma}_{\beta}$, and for every $\beta' \in (\beta,\pi)$

$$\sup_{z\in\mathbb{C}\setminus\Sigma_{\beta'}} ||z(z-A)^{-1}|| < \infty,$$

where $\Sigma_{\gamma} := \{z \in \mathbb{C} : |\arg z| < \gamma\}$. For a sectorial operator, define the *sectorial angle* $\omega(A)$ by

$$\omega(A) := \inf\{\beta \in (0,\pi) : A \text{ is sectorial of angle } \beta\}.$$

For every $\beta \in (0,\pi)$ we put

$$H^{\infty}(\Sigma_{\beta}) := \{f : \Sigma_{\beta} \to \mathbb{C} : ||f||_{\infty} < \infty\},$$

$$H^{\infty}_0(\Sigma_{\beta}) := \left\{ f \in H^{\infty}(\Sigma_{\beta}) : \exists \varepsilon > 0 \text{ such that } \sup_{z\in\Sigma_{\beta}} |f(z)| \left| \frac{1+z^2}{z} \right|^{\varepsilon} < \infty \right\}.$$

If A is a sectorial operator of angle $\beta \in (0,\pi)$, then

$$\Phi_A(f) := f(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_{\beta'}} f(z)(z-A)^{-1}dz$$

defines a functional calculus from $H^{\infty}_0(\Sigma_{\beta'})$ into $\mathcal{B}(X)$ for every $\beta' > \beta$. A sectorial operator A is say to admits a *bounded* H^{∞} *functional calculus of angle* $\beta \in [\omega(A),\pi)$ if the functional calculus on $H^{\infty}_0(\Sigma_{\beta'})$ extends to a bounded linear operator on $H^{\infty}(\Sigma_{\beta'})$ for every $\beta' \in (\beta,\pi)$.

Now, in the equation (4.18) suppose that A is a sectorial operator on a Banach space X which admits a bounded H^{∞} functional calculus of angle $\omega \in (0,\pi/3)$ on X. In this conditions, the fractional powers of A for every $\varepsilon > 0$, A^{ε}, can be defined (see [14, 17]). An easy computation shows that the kernels $a(t) = -te^{-\gamma t}$ and $b(t) = -e^{-\gamma t}, \gamma > 0$ are 2-regular. Observe that

$$\begin{aligned} (I - a_{\eta}A - b_{\eta}B)^{-1} &= (i\eta)^2((i\eta)^2 - A - (i\eta)A^{\varepsilon})^{-1}, \\ b_{\eta}B(I - a_{\eta}A - b_{\eta}B)^{-1} &= A^{\varepsilon}(i\eta)((i\eta)^2 - A - (i\eta)A^{\varepsilon})^{-1}, \end{aligned}$$

Let $\varepsilon = 1/2$, that is $B = A^{1/2}$. For $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}\setminus(-\infty,0]$, define $F^1(\lambda,z) := \lambda^2(\lambda^2 I - \lambda z^{1/2} - z)^{-1}$ and $F^2(\lambda,z) := \lambda z^{1/2}(\lambda^2 I - \lambda z^{1/2} - z)^{-1}$. As in the proof of [10, Lemma 4.1], we can choose $\omega' > \omega$ and $\delta > 0$, with $\frac{\omega'}{2} < \frac{\pi}{6} - \delta$, and such that there exists a constant $M \geq 0$ independent of $\lambda \in \Sigma_{\delta+\pi/2}$ and $z \in \Sigma_{\omega'}$ such that $|F^j(\lambda,z)| \leq M$ for $j = 1,2$. Therefore

$$\sup_{\eta\in\mathbb{R}} ||(I - a_{\eta}A - b_{\eta}A^{1/2})^{-1}|| < \infty \quad \text{and} \quad \sup_{\eta\in\mathbb{R}} ||b_{\eta}A^{1/2}(I - a_{\eta}A - b_{\eta}A^{1/2})^{-1}|| < \infty.$$

Thus, we have the following proposition.

Proposition 4.13. *Let A be a sectorial operator which admits a bounded H^∞ functional calculus of angle $\beta \in (0, \pi/3)$ on a Banach space X. Then (4.18) with $B = A^{1/2}$ is C^α-well posed.*

Example 4.14.

Let $k \in L^1_{\text{loc}}(\mathbb{R}_+)$ be a 2-regular kernel. Define $a(t) := 1 + \int_0^t k(s)ds$. The remarks following [19, Definition 3.3] show that $a(t)$ is 2-regular again. Let A a closed operator and $B \equiv 0$, $\Omega \subset \mathbb{R}^n$ a bounded domain and $X = H^{-1}(\Omega)$. From Theorem 3.9, we have that if $\sup_{\eta\in\mathbb{R}} ||(I - a_\eta A)^{-1}|| < \infty$, then the equation

$$u(x,t) = A\int_{-\infty}^t a(t-s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times \mathbb{R}, \tag{4.20}$$

is C^α-well posed.

In particular, if we take $A = \Delta$, where Δ denotes the Laplacian in $H^{-1}(\Omega)$ and $k(t) = \alpha e^{-\beta t}$ with $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, we have $a(t) = 1 + \frac{\alpha}{\beta}[1 - e^{-\beta t}]$, $a_\eta = \frac{i\eta+\alpha+\beta}{i\eta(i\eta+\beta)}$ and

$$(I - a_\eta A)^{-1} = \frac{i\eta(i\eta+\beta)}{i\eta+\alpha+\beta}\left(\frac{i\eta(i\eta+\beta)}{i\eta+\alpha+\beta}I - \Delta\right)^{-1}.$$

Observe that, in this case, the equation (4.21) can be considered as the integral version of the problem

$$u'(x,t) = Au(x,t) + g(x,t), \quad (x,t) \in \Omega \times \mathbb{R}, \tag{4.21}$$

in the limit case $\beta = 0$.

Denote $w_\eta := \frac{i\eta(i\eta+\beta)}{i\eta+\alpha+\beta}$. Then, $\text{Re}(w_\eta) = \frac{-\alpha\eta^2}{(\alpha+\beta)^2+\eta^2}$ and $\text{Im}(w_\eta) = \frac{\eta^3+(\alpha+\beta)\beta\eta}{(\alpha+\beta)^2+\eta^2}$. From [13, p. 75] we have that there exist a constant $c > 0$ such that

$$||(zI - \Delta)^{-1}|| \le \frac{c}{1+|z|}, \tag{4.22}$$

whenever $\text{Re}(z) \ge -c(1 + |\text{Im}(z)|)$. If $\alpha \le 0$, then clearly $\text{Re}(w_\eta) \ge -(1 + |\text{Im}(w_\eta)|)$ for all $\eta \in \mathbb{R}$, that is, $c = 1$ in inequality (4.22) and therefore

$$\sup_{\eta\in\mathbb{R}} ||(I - a_\eta A)^{-1}|| = \sup_{\eta\in\mathbb{R}} ||w_\eta(w_\eta I - \Delta)^{-1}|| < \infty.$$

We conclude by Theorem 3.9 that the equation (4.21) is C^α-well posed.

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