ALMOST AUTOMORPHIC SOLUTIONS TO ABSTRACT VOLTERRA EQUATIONS ON THE LINE

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ABSTRACT. Given $a \in L^1(\mathbb{R})$ and A the generator of an L^1-integrable family of bounded and linear operators defined on a Banach space X, we prove the existence of almost automorphic mild solution to the semilinear integral equation $u(t) = \int_{-\infty}^{t} a(t-s)[Au(s)+f(s,u(s))]ds$ for each $f : \mathbb{R} \times X \to X$ S^p-almost automorphic in t, uniformly in $x \in X$, and satisfying diverse Lipschitz type conditions. In the scalar linear case, we prove that $a \in L^1(\mathbb{R})$ completely monotonic is already sufficient.

1. Introduction

We study almost automorphic solutions of an integral equation with infinite delay in a general Banach space X:

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s,u(s))]ds, \quad t \in \mathbb{R} \tag{1.1}$$

where the operator $A : D(A) \subset X \to X$ generates an integral resolvent and $a : \mathbb{R}_+ \to \mathbb{C}$ is an integrable function.

Such a kind of equation arises in the study of heat flow in materials of fading memory type: see for instance [3], [21] and [24]. Note that, in the finite dimensional case, the system (1.1) contains as particular cases several systems with finite or infinite delay, already considered in the literature. See e.g. [4] and [13].

The problem of existence of almost automorphic solutions to (1.1) is a very natural one. Conditions which guarantee the existence of an almost automorphic solution for any $f(t,x)$ in a given space of almost automorphic functions have been studied recently in [5].

In a recent paper [10], the authors dealt with the existence of almost automorphic solutions to certain classes of fractional differential equations, which can be represented in the form [6, Section 1]:

$$u(t) = \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[Au(s) + g(s,u(s))]ds; \quad u(0) = u_0, \quad 1 < \alpha < 2. \tag{1.2}$$

The aim of this paper is to point out that similar results hold true for the class of integral equations (1.1) containing the above equations as limiting special cases [24, Chapter II, section 11.5].

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Specifically, we consider in this paper the class of continuous data $f : \mathbb{R} \times X \to X$ of S^p- almost automorphic functions on t, and we look for solutions u belonging to the class of almost automorphic functions.

The concept of S^p-almost automorphy was introduced and applied to study the existence of solutions to some parabolic evolution equations by N'Guérékata and Pankov in [20]. We would like to point out that interesting results on S^p-almost automorphic mild solutions to evolution equations have been obtained in [14], [8] and [15].

This paper is organized as follows: in section 2 we collect all the results of [22],[8], [20], [10] and [9] we need about S^p-almost automorphic functions. In section 3, we treat equation (1.1) when $f(t, u(t)) = g(t)$ is S^p-almost automorphic, that is, the linear case. In particular, we improve some results of [5] and give new examples. Section 4 is devoted to our main results in the semilinear case. New and general existence and uniqueness theorems of almost automorphic solutions to the equation (1.1) are established. As an illustration we apply the abstract results to a concrete equation.

2. Preliminaries

Since we are going to use several concepts of almost automorphy, it is proper to provide a brief description of the spaces of almost automorphic functions to be considered.

Definition 2.1. A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t - s_n), \qquad \text{for each } t \in \mathbb{R}.$$

Almost automorphicity is a generalization of the classical concept of an almost periodic function. It was introduced in the literature by S. Bochner and recently studied by several authors, including [1, 2, 7, 11, 16, 17] among others. A complete description of their properties and further applications to evolution equations can be found in the monographs [18] and [19] by G. M. N'Guérékata.

Definition 2.2 ([22])**.** The Bochner transform $f^b(t, s), t \in \mathbb{R}, s \in [0, 1]$, of a function $f(t)$ on $\mathbb{R}$, with values in X, is defined by

$$f^b(t, s) := f(t + s).$$

Definition 2.3 ([22])**.** The space $BS^p(X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that

$$||f||_{S^p} := \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} ||f(\tau)||^p d\tau \Big)^{\frac{1}{p}} < \infty.$$

It is obvious that $L^p(\mathbb{R}; X) \subset BS^p(X) \subset L^p_{loc}(\mathbb{R}; X)$ and $BS^p(X) \subset BS^q(X)$ whenever $p \geq q \geq 1$.

Definition 2.4 ([20])**.** The space $AS^p(X)$ of S^p-almost automorphic functions (S^p-a.a. for short) consists of all $f \in BS^p(X)$ such that $f^b \in AA(L^p([0, 1]; X))$. In other words, a function $f \in L^p_{loc}(\mathbb{R}; X)$ is said to be S^p-almost automorphic if its Bochner transform

$f^b : \mathbb{R} \to L^p([0,1];X)$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R};X)$ such that

$$\lim_{n\to\infty}\Big(\int_0^1 ||f(t+s_n+s)-g(t+s)||^p ds\Big)^{\frac{1}{p}} = 0,$$
$$\lim_{n\to\infty}\Big(\int_0^1 ||g(t-s_n+s)-f(t+s)||^p ds\Big)^{\frac{1}{p}} = 0,$$

for each $t \in \mathbb{R}$.

Remark 2.5. It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{loc}(\mathbb{R};X)$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also if $f \in AA(X)$, then f is S^p-almost automorphic for any $1 \le p < \infty$.

Definition 2.6 ([8])**.** A function $f : \mathbb{R}\times X \to X$, $(t,u) \mapsto f(t,u)$ with $f(\cdot,u) \in L^p_{loc}(\mathbb{R},X)$ for each $u \in X$ is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly for $u \in X$, if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ and a function $g : \mathbb{R}\times X \to X$ with $g(\cdot,u) \in L^p_{loc}(\mathbb{R},X)$ such that

$$\lim_{n\to\infty}\Big(\int_0^1 ||f(t+s_n+s,u)-g(t+s,u)||^p ds\Big)^{\frac{1}{p}} = 0,$$
$$\lim_{n\to\infty}\Big(\int_0^1 ||g(t-s_n+s,u)-f(t+s,u)||^p ds\Big)^{\frac{1}{p}} = 0,$$

for each $t \in \mathbb{R}$ and for each $u \in X$. We denote by $AS^p(\mathbb{R}\times X, X)$ the set of all such functions.

Lemma 2.7 ([10])**.** *Let $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ be a strongly continuous family of bounded and linear operators such that*

$$||S(t)|| \le \phi(t), \quad \textit{for all } t \in \mathbb{R}_+,$$

where $\phi \in L^1(\mathbb{R}_+)$ is nonincreasing. Then, for each $f \in AS^1(X)$,

$$\int_{-\infty}^t S(t-s)f(s)ds \in AA(X).$$

Theorem 2.8 ([10, 9])**.** *Assume that*
(i) $f \in AS^p(\mathbb{R}\times X, X)$ with $p > 1$;
(ii) there exists a non negative function $L \in AS^r(\mathbb{R})$ with $r \ge max\{p, p/(p-1)\}$ such that for all $u, v \in X$ and $t \in \mathbb{R}$,

$$||f(t,u) - f(t,v)|| \le L(t)||u-v||;$$

(iii) $x \in AS^p(X)$ and $K = \overline{\{x(t) : t \in R\}}$ is compact in X.
Then there exists $q \in [1,p)$ such that $f(\cdot, x(\cdot)) \in AS^q(X)$.

The following definition is taken from [24, Definition 1.6, p.46].

Definition 2.9. Let X be a complex Banach space, A a closed linear unbounded operator in X and $a \in L^1_{loc}(\mathbb{R}_+)$ an scalar kernel $\not\equiv 0$. A family $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ is called an integral resolvent with generator A if the following conditions are satisfied.
(i) $S(\cdot)x \in L^1_{loc}(\mathbb{R}_+;X)$ for each $x \in X$ and $||S(t)|| \le \psi(t)$ a.e. on $\mathbb{R}_+$, for some $\psi \in$

$L^1_{loc}(\mathbb{R}_+)$;
(ii) $S(t)$ commutes with A for each $t \geq 0$;
(iii) the following integral resolvent equation holds

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)xds. \tag{2.1}$$

for all $x \in D(A)$ and a.a. $t \geq 0$.

3. Almost Automorphic Solutions for the Linear equation

In this section we consider the existence and uniqueness of almost automorphic solutions to the evolution equation

$$u(t) = \int_{-\infty}^t a(t-s)[Au(s) + g(s)]ds, \quad t \in \mathbb{R}, \tag{3.1}$$

where A is the generator of an integral resolvent family and $a \in L^1(\mathbb{R})$.

Proposition 3.1. *Let $a \in L^1(\mathbb{R})$. Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ on X, which satisfies*

$$||S(t)|| \leq \phi(t), \quad \textit{for all } t \in \mathbb{R}_+,$$

where $\phi \in L^1(\mathbb{R}_+)$ is nonincreasing. If $f \in AS^1(X)$ and takes values on $D(A)$ then the unique bounded solution of the problem

$$u(t) = \int_{-\infty}^t a(t-s)[Au(s) + f(s)]ds, \quad t \in \mathbb{R}, \tag{3.2}$$

is almost automorphic and is given by

$$u(t) = \int_{-\infty}^t S(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

Proof. Since $g(t) \in D(A)$ for all $t \in \mathbb{R}$, we obtain $u(t) \in D(A)$ for all $t \in \mathbb{R}$ (see [24, Proposition 1.2]). Then applying (2.1) and Fubini's theorem we obtain

$$\begin{aligned}
\int_{-\infty}^t a(t-s)Au(s)ds &= \int_{-\infty}^t a(t-s)A\int_{-\infty}^s S(s-\tau)f(\tau)d\tau ds \\
&= \int_{-\infty}^t \int_{-\infty}^s a(t-s)AS(s-\tau)f(\tau)d\tau ds \\
&= \int_{-\infty}^t \int_{\tau}^t a(t-s)AS(s-\tau)f(\tau)dsd\tau \\
&= \int_{-\infty}^t \int_0^{t-\tau} a(t-\tau-p)AS(p)dpf(\tau)d\tau \\
&= \int_{-\infty}^t (S(t-\tau)f(\tau) - a(t-\tau)f(\tau))d\tau \\
&= u(t) - \int_{-\infty}^t a(t-\tau)f(\tau)d\tau.
\end{aligned}$$

The statement follows by Lemma 2.7. $\square$

Recall that a C^∞-function $f:(0,\infty)\to\mathbb{R}$ is called completely monotonic if

$$(-1)^n f^{(n)}(\lambda)\geq 0, \text{ for all } \lambda>0, n\in\mathbb{N}_0.$$

We remark that such functions naturally occur in areas such as probability theory, numerical analysis, and elasticity. Our main result in case $X=\mathbb{R}$ is the following theorem. It is remarkable that the hypothesis given, are completely based in the data of the problem.

Theorem 3.2. *Let $a\in L^1(\mathbb{R}_+)$ be a scalar, completely monotonic function on $\mathbb{R}_+$. Let $\rho>0$ be given. If $g\in AS^1(\mathbb{R})$ then:*

a) There is $S_\rho\in L^1(\mathbb{R}_+)$ completely monotonic such that equation (2.1) is satisfied with $A=-\rho$.

b) The equation

$$u(t)=\int_{-\infty}^t a(t-s)[-\rho u(s)+g(s)]ds, \quad t\in\mathbb{R}, \tag{3.3}$$

has an unique almost automorphic solution given by

$$u(t)=\int_{-\infty}^t S_\rho(t-s)g(s)ds, \quad t\in\mathbb{R}.$$

Proof. By the hypothesis on the scalar kernel $a(t)$ and [13, Theorem 2.8, p.147] we have that $\log(a)$ is convex on $\mathbb{R}_+$. Moreover, since $a(t)$ is positive and nonincreasing, it follows by [24, Lemma 4.1, p.98] that there exists $S_\rho\in L^1(\mathbb{R}_+)$ completely monotone, such that equation (2.1) is satisfied with $A=-\rho$, that is

$$S_\rho(t)=a(t)-\rho\int_0^t a(t-s)S_\rho(s)ds. \tag{3.4}$$

Hence (a) follows. Part (b) is an immediate consequence of Lemma 2.7, since S_ρ is nonincreasing. □

In case that $g\in AA(\mathbb{R})$ we have the following result that improves [5, Corollary 3.7]. We denote by $\hat{a}(\lambda)$ the Laplace transform of $a(t)$.

Theorem 3.3. *Let $f:\mathbb{R}\to\mathbb{R}$ be an almost automorphic function and let $\rho>0$ be a real number. Suppose $a\in L^1(\mathbb{R}_+)$, and $\hat{a}(\lambda)\neq-\frac{1}{\rho}$ for all $Re(\lambda)\geq 0$. Then*

a) There is $S_\rho\in L^1(\mathbb{R}_+)$ such that equation (2.1) is satisfied with $A=-\rho$;

b) The equation

$$u(t)=\int_{-\infty}^t a(t-s)[-\rho u(s)+f(s)]ds, \quad t\in\mathbb{R}, \tag{3.5}$$

has an unique almost automorphic solution given by

$$u(t)=\int_{-\infty}^t S_\rho(t-s)f(s)ds, \quad t\in\mathbb{R}.$$

Proof. The proof is a direct consequence of the half Paley-Wiener theorem [13, Theorem 4.1 p.45] and [5, Lemma 3.1] (see also the references therein).

□

Example 3.4. *Let $a(t) = e^{-bt}, b > 0$ and $\rho > 0$. Then $a(t)$ is completely monotonic and $\hat{a}(\lambda) = \frac{1}{\lambda+b} \neq -\frac{1}{\rho}$ for all $Re(\lambda) \geq 0$. Moreover a direct calculation using Laplace transform gives $S_\rho(t) = e^{-(b+\rho)t}$. Hence for any $g \in AS^1(\mathbb{R})$ (resp. $g \in AA(\mathbb{R})$) there exist a unique almost automorphic solution of the equation*

$$u(t) = \int_{-\infty}^{t} e^{-b(t-s)}[-\rho u(s) + g(s)]ds, \quad t \in \mathbb{R}, \tag{3.6}$$

given by

$$u(t) = \int_{-\infty}^{t} e^{(t-s)(b+\rho)} g(s)ds, \quad t \in \mathbb{R}.$$

The remarkable fact is that we only need $g \in AS^1(\mathbb{R})$ instead of $g \in AA(\mathbb{R})$ to have existence of almost automorphic solutions.

Example 3.5. *Let $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-bt}, b > 0, \alpha > 0$ and $\rho > 0$. We note that $a(t)$ is not completely monotonic but, under the condition $\cos(\pi/\alpha) \leq \frac{b}{\rho^{1/\alpha}}$ and since $(-\rho)^{\frac{1}{\alpha}} = [\cos(\frac{\pi}{\alpha}) + i\sin(\frac{\pi}{\alpha})]\rho^{\frac{1}{\alpha}}$, we have $\lambda \neq (-\rho)^{\frac{1}{\alpha}} - b$, for all $Re(\lambda) \geq 0$, that is $\hat{a}(\lambda) = \frac{1}{(\lambda+b)^\alpha} \neq -\frac{1}{\rho}$, for all $Re(\lambda) \geq 0$. A calculation using Laplace transform shows $S_\rho(t) = t^{\alpha-1}e^{-bt}E_{\alpha,\alpha}(\rho t^\alpha)$, where $E_{\alpha,\alpha}$ denotes the generalized Mittag-Leffler function (see e.g. [12]).*

Hence for any $g \in AA(\mathbb{R})$ there exist a unique almost automorphic solution of the equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} e^{-b(t-s)}[-\rho u(s) + g(s)]ds, \quad t \in \mathbb{R}, \tag{3.7}$$

given by

$$u(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} e^{-b(t-s)} E_{\alpha,\alpha}(\rho(t-s)^\alpha) g(s)ds, \quad t \in \mathbb{R}.$$

Note that this example improves [5, Example 3.6] *where only $1 < \alpha < 2$ was considered. In fact, $1 < \alpha < 2$ implies immediately that the more general condition $\cos(\pi/\alpha) \leq \frac{b}{\rho^{1/\alpha}}$ holds.*

4. Almost Automorphic mild Solutions for nonlinear equations

In this section we consider the existence and uniqueness of almost automorphic mild solutions to the nonlinear evolution equation

$$u(t) = \int_{-\infty}^{t} a(t-s)[Au(s) + f(s, u(s))]ds, \quad t \in \mathbb{R}, \tag{4.1}$$

where A is the generator of an integral resolvent family and $a \in L^1(\mathbb{R})$.

Definition 4.1. Let A be the generator of an integral resolvent family $\{S(t)\}_{t\geq 0}$. A continuous function $u : \mathbb{R} \to X$ satisfying the integral equation

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s, u(s))ds, \text{ for all } t \in \mathbb{R}, \tag{4.2}$$

is called a *mild* solution on $\mathbb{R}$ to the equation (4.1).

Theorem 4.2. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ such that*

$$\|S(t)\| \leq \phi(t), \text{ for all } t \geq 0,$$

where $\phi \in L^1(\mathbb{R}_+)$ is nonincreasing. Suppose that
(i) $f \in AS^p(\mathbb{R} \times X, X)$ with $p > 1$;
(ii) there exists a non negative function $L \in AS^r(\mathbb{R})$ with $r \geq max\{p, p/(p-1)\}$ such that for all $u, v \in X$ and $t \in \mathbb{R}$,

$$||f(t,u) - f(t,v)|| \leq L(t)||u - v||;$$

If $||L||_{S^1} < ||\phi||_1^{-1}$, then the equation (4.1) has a unique almost automorphic mild solution.

Proof. We define the operator $F : AA(X) \mapsto AA(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^{t} S(t-s)f(s,\varphi(s))\,ds, \quad t \in \mathbb{R}. \tag{4.3}$$

Since $\varphi \in AA(X)$, we have $\overline{\{\varphi(t) : t \in \mathbb{R}\}}$ is compact in X, by Theorem 2.8, there exists $q \in [1,p)$ such that $f(\cdot,\varphi(\cdot)) \in AS^q(X) \subset AS^1(X)$. Then, by Lemma 2.7, we conclude that F is well defined. Then for $\varphi_1, \varphi_2 \in AA(X)$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned}
\|F\varphi_1(t) - F\varphi_2(t)\| &\leq \int_{-\infty}^{t} ||S(t-s)|| \cdot ||f(s,\varphi_1(s)) - f(s,\varphi_2(s))||ds \\
&\leq \int_{0}^{\infty} L(t-\tau)\|S(\tau)\| \cdot \|\varphi_1(t-\tau) - \varphi_2(t-\tau)\|d\tau \\
&\leq ||\varphi_1 - \varphi_2||_\infty \int_0^\infty L(t-\tau)\phi(\tau)d\tau \\
&= ||\varphi_1 - \varphi_2||_\infty \sum_{k=0}^{\infty} \int_k^{k+1} L(t-\tau)\phi(\tau)d\tau \\
&\leq ||\varphi_1 - \varphi_2||_\infty ||L||_{S^1} \sum_{k=0}^{\infty} \int_k^{k+1} \phi(\tau)d\tau \\
&= ||\varphi_1 - \varphi_2||_\infty ||L||_{S^1} ||\phi||_1 \\
&< ||\varphi_1 - \varphi_2||_\infty.
\end{aligned}$$

This proves that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in AA(X)$, such that $Fu = u$, that is $u(t) = \displaystyle\int_{-\infty}^{t} S(t-s)f(s,u(s))ds.$ ∎

We remark that in the case of $L(t) \equiv L$, by following the proof of previous theorem, one can get the same conclusion. Our next result reads as follows.

Theorem 4.3. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ such that*

$$\|S(t)\| \leq \phi(t), \text{ for all } t \geq 0,$$

where $\phi \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is nonincreasing. Suppose that
(i) $f \in AS^p(\mathbb{R} \times X, X)$ with $p > 1$;

(ii) there exists a non negative function $L \in AS^r(\mathbb{R}) \cap L^1(\mathbb{R})$ with $r \geq max\{p, p/(p-1)\}$ such that for all $u, v \in X$ and $t \in \mathbb{R}$,

$$||f(t,u) - f(t,v)|| \leq L(t)||u - v||;$$

Then equation (4.1) has a unique almost automorphic mild solution.

Proof. Define the operator F as in (4.3). Let φ_1, φ_2 be in $AA(X)$. We have:

$$\begin{aligned}
||(F\varphi_1)(t) - (F\varphi_2)(t)|| &\leq \int_{-\infty}^{t} ||S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]||ds \\
&\leq \int_{-\infty}^{t} L(s)\|S(t-s)\|\|\varphi_1(s) - \varphi_2(s)\|ds \\
&\leq \|\varphi_1 - \varphi_2\|_\infty \int_{-\infty}^{t} L(s)\phi(t-s)ds \\
&= \|\varphi_1 - \varphi_2\|_\infty \int_{0}^{\infty} L(t-\tau)\phi(\tau)d\tau \\
&\leq ||\varphi_1 - \varphi_2||_\infty ||\phi||_\infty \int_{0}^{\infty} L(t-\tau)d\tau \\
&= ||\varphi_1 - \varphi_2||_\infty ||\phi||_\infty \int_{-\infty}^{t} L(s)ds.
\end{aligned}$$

In the same way

$$\begin{aligned}
||(F^2\varphi_1)(t) - (F^2\varphi_2)(t)|| &\leq \int_{-\infty}^{t} ||S(t-s)[f(s,F\varphi_1(s)) - f(s,F\varphi_2(s))]||ds \\
&\leq \int_{-\infty}^{t} L(s)\phi(t-s)\|F\varphi_1(s) - F\varphi_2(s)\|ds \\
&\leq ||\varphi_1 - \varphi_2||_\infty ||\phi||_\infty^2 \int_{-\infty}^{t} L(s) \int_{-\infty}^{s} L(\tau)d\tau ds.
\end{aligned}$$

In general we get

$$\begin{aligned}
||(F^n\varphi_1)(t) - (F^n\varphi_2)(t)|| &\leq ||\varphi_1 - \varphi_2||_\infty \frac{||\phi||_\infty^n}{(n-1)!} \left(\int_{-\infty}^{t} L(s) \left(\int_{-\infty}^{s} L(\tau)d\tau \right)^{n-1} ds \right) \\
&\leq ||\varphi_1 - \varphi_2||_\infty \frac{||\phi||_\infty^n}{n!} \left(\int_{-\infty}^{t} L(s)ds \right)^n \\
&\leq ||\varphi_1 - \varphi_2||_\infty \frac{(||L||_1||\phi||_\infty)^n}{n!}.
\end{aligned}$$

Hence, since $\frac{(||L||_1||\phi||_\infty)^n}{n!} < 1$ for n sufficiently large, by the contraction principle F has a unique fixed point $u \in AA(X)$. □

Theorem 4.4. *Assume that A generates an integral resolvent family $\{S(t)\}_{t\geq 0}$ such that*

$$\|S(t)\| \leq \phi(t), \text{ for all } t \geq 0,$$

where $\phi \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is nonincreasing. Suppose that
(i) $f \in AS^p(\mathbb{R} \times X, X)$ with $p > 1$;
(ii) there exists a non negative function $L \in AS^r(\mathbb{R})$ with $r \geq max\{p, p/(p-1)\}$ such that for all $u, v \in X$ and $t \in \mathbb{R}$,

$$||f(t,u) - f(t,v)|| \leq L(t)||u - v||;$$

(iii) the integral $\displaystyle\int_{-\infty}^{t} L(s)ds$ exists for all $t \in \mathbb{R}$.

Then equation (4.1) has a unique almost automorphic mild solution.

Proof. Define a new norm $|||\varphi||| := \sup_{t\in\mathbb{R}}\{v(t)||\varphi(t)||\}$, where $v(t) := e^{-k\int_{-\infty}^{t} L(s)ds}$ and k is a fixed positive constant greater than $C := ||\phi||_\infty$. Define the operator F as in (4.3). Let φ_1, φ_2 be in $AA(X)$, then we have

$$\begin{aligned}
v(t)||(F\varphi_1)(t) - (F\varphi_2)(t)|| &= v(t)\left\|\int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds\right\| \\
&\leq C\int_{-\infty}^{t} v(t)L(s)\|\varphi_1(s) - \varphi_2(s)\|ds \\
&\leq C\int_{-\infty}^{t} v(t)v(s)^{-1}L(s)v(s)\|\varphi_1(s) - \varphi_2(s)\|ds \\
&\leq C|||\varphi_1 - \varphi_2|||\int_{-\infty}^{t} v(t)v(s)^{-1}L(s)ds \\
&= \frac{C}{k}|||\varphi_1 - \varphi_2|||\int_{-\infty}^{t} ke^{k\int_t^s L(\tau)d\tau}L(s)ds \\
&= \frac{C}{k}|||\varphi_1 - \varphi_2|||\int_{-\infty}^{t} \frac{d}{ds}\left(e^{k\int_t^s L(\tau)d\tau}\right)ds \\
&= \frac{C}{k}[1 - e^{-k\int_{-\infty}^{t} L(\tau)d\tau}]|||\varphi_1 - \varphi_2||| \\
&\leq \frac{C}{k}|||\varphi_1 - \varphi_2|||.
\end{aligned}$$

Hence, since $C/k < 1$, F has a unique fixed point $u \in AA(X)$. □

Example 4.5. *Suppose that A is the generator of integral resolvent $\{S(t)\}_{t\geq 0}$ such that $||S(t)|| \leq Me^{-\omega t}$ where $M, \omega > 0$. For concrete examples of operators satisfying this condition we refer to* [23, Section 5]*. If $\phi(t) = Me^{-\omega t}$, we observe that ϕ is nonincreasing and $\phi \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. Define*

$$l(t) := \begin{cases} \sin\left(\frac{1}{2+\cos n+\cos\pi n}\right), & t \in (n - \frac{\varepsilon}{2^{|n|}}, n + \frac{\varepsilon}{2^{|n|}}),\ n \in \mathbb{Z} \\ 0, & \text{otherwise}, \end{cases}$$

for some $0 < \varepsilon < \frac{1}{2}$. By [20, Example 2.3]*, we conclude that $l \in AS^r(\mathbb{R})$ for all $r \in [1, \infty)$. Moreover,*

$$\begin{aligned}
\int_{-\infty}^{\infty} |l(t)|dt &\leq \sum_{n=-\infty}^{\infty} \int_{n-\frac{\varepsilon}{2^{|n|}}}^{n+\frac{\varepsilon}{2^{|n|}}} \Big|\sin\Big(\frac{1}{2+\cos n+\cos \pi n}\Big)\Big|dt \\
&\leq \sum_{n=-\infty}^{\infty} \int_{n-\frac{\varepsilon}{2^{|n|}}}^{n+\frac{\varepsilon}{2^{|n|}}} dt \\
&= \sum_{n=-\infty}^{\infty} \frac{\varepsilon}{2^{|n|-1}} = 6\varepsilon.
\end{aligned}$$

Therefore, $l \in AS^r(\mathbb{R}) \cap L^1(\mathbb{R})$. *On* $X = L^p([0,\pi])$, $p > 1$, *define* $f(t,u)(s) = l(t)\cos(u(s))$ *for* $u \in X$ *and* $s \in [0,\pi]$. *Then,* $f \in AS^p(\mathbb{R}\times X, X)$ *and for* $u, v \in X$, $t \in \mathbb{R}$ *we have*

$$\begin{aligned}
||f(t,u) - f(t,v)||_p^p &= \int_0^\pi |l(t)\cos(u(s)) - l(t)\cos(v(s))|^p ds \\
&\leq |l(t)|^p \cdot ||u-v||_p^p.
\end{aligned}$$

We conclude from Theorem 4.3, with $L(t) = |l(t)|$, *that the equation (4.1) has a unique almost automorphic mild solution. Note that, we obtain the same conclusion from Theorem 4.4. Finally, note that we can apply Theorem 4.2 under the condition* $0 < \varepsilon < \frac{\omega}{2M}$.

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