

Solutions of abstract integro-differential equations via Poisson transformation

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We study the initial value problem

$$(*) \begin{cases} \mathbf{u}(n+1) - \mathbf{u}(n) = A\mathbf{u}(n+1) + \sum_{k=0}^{n+1} \mathbf{a}(n+1-k)A\mathbf{u}(k), & n \in \mathbb{N}_0 \\ \mathbf{u}(0) = x, \end{cases}$$

where A is closed linear operator defined on a Banach space X , x belongs to the domain of A , and the kernel \mathbf{a} is a particular discretization of an integrable kernel $a \in L^1(\mathbb{R}_+)$. Assuming that A generates a resolvent family, we find an explicit representation of the solution to the initial value problem $(*)$ as well as for its inhomogeneous version, and then we study the stability of such solutions. We also prove that for a special class of kernels \mathbf{a} , it suffices to assume that A generates an immediately norm continuous C_0 -semigroup. We employ a new computational method based on the Poisson transformation.

KEYWORDS

C_0 -Semigroups, Poisson transformation, resolvent families of operators, Volterra integral equations

MSC CLASSIFICATION

39A13; 47D06; 45D05; 47A10

1 | INTRODUCTION

Consider the homogeneous initial value problem

$$\begin{cases} u'(t) = Au(t) + \int_0^t a(t-s)Au(s)ds, & t \geq 0 \\ u(0) = x, \end{cases} \quad (1)$$

where A is a closed linear operator defined in a Banach space X , x belongs to X , and the function $a : [0, \infty) \rightarrow \mathbb{R}$, known as *the relaxation function*, is an integrable kernel. This kind of initial value problems arise, for instance, in the study of the heat conduction in material with fading memory or in some population models, see the papers¹⁻⁵ for more details. In such applications, the operator A is typically the Laplacian operator or the elasticity operator and a typical choice of the relaxation function is $a(t) = \alpha \frac{t^{\mu-1}}{\Gamma(\mu)} e^{-\beta t}$ for $\alpha \in \mathbb{R}$ and $\mu, \beta > 0$.

In order to solve abstract integro-differential equations in the form of (1), Da Prato and Ianneli introduced in the paper⁶ the concept of *resolvent family* $\{S(t)\}_{t \geq 0}$ generated by the operator A (see Section 2 below for its definition). See also the monograph of Prüss.⁴ By using this concept, the *mild solution* to (1) can be written as

$$u(t) := S(t)x. \quad (2)$$

We notice here that the mild solution (2) is a classical solution to (1) if $x \in D(A)$. Now, if we consider the inhomogeneous initial problem

$$\begin{cases} u'(t) = Au(t) + \int_0^t a(t-s)Au(s)ds + f(t), & t \geq 0 \\ u(0) = x, \end{cases} \tag{3}$$

where A , a , and x are as before and the forcing term f is a continuous function, then the mild solution to problem (3) is given by

$$u(t) = S(t)x + \int_0^t S(t-r)f(r)dr. \tag{4}$$

Time discretizations of inhomogeneous integro-differential equations with memory terms of convolution type have been considered by several authors in the past decades. We mention here some few works. In Sloan and Thomé⁷ the authors considered the Equation 3 with A being an unbounded positive-definite self-adjoint operator with dense domain in a Hilbert space and the operator A in the convolution term in (3) is taken as a different operator B with domain $D(B) \supset D(A)$ and a and f are smooth functions.

More recently, in McLean et al⁸ was considered Equation 3 where A is a closed linear operator in a complex Banach space satisfying the resolvent estimate $\|(z - A)^{-1}\| \leq M_\delta/(1 + |z|)$, for $z \in \Sigma_\delta := \{z \neq 0, |\arg(z)| < \delta\} \cup \{0\}$ for some $\delta \in (\frac{1}{2}\pi, \pi)$, where M_δ is a positive constant and the kernel a satisfies some suitable assumptions. A typical example of such kernels is $a(t) = \kappa e^{-\nu t}$ with $\kappa \in \mathbb{R}$ and $\nu \geq 0$, (⁸, Section 2).

In the paper,⁹ the authors study discretizations in time for the integro-differential equation

$$\begin{cases} u'(t) = \int_0^t a(t-s)Au(s)ds + f(t), & t \geq 0 \\ u(0) = x, \end{cases} \tag{5}$$

in the context of real Hilbert spaces and self-adjoint positive-definite linear operators A . In Banach spaces, the authors in McLean and Thomé¹⁰ study time discretizations for the abstract Equation 5 where a is the weakly singular function $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $0 < \alpha < 1$ and the operator A satisfies the same conditions as in the papers^{8,11,12} In contrast, the case $1 < \alpha < 2$ is studied in another article¹³ by assuming that A is an operator of sectorial type.

Unfortunately, it is well known that the numerical discretization result in cumulate errors even in short-term domains. It becomes difficult in the real-world applications for long-term issues from the practical point of view. This paper aims to address this problem and propose a new kind of discrete time Volterra equation by use of the Poisson transformation on an isolated time scale.

We study abstract Volterra difference equations having the form

$$\mathbf{u}(n+1) - \mathbf{u}(n) = A\mathbf{u}(n+1) + \sum_{k=0}^{n+1} \mathbf{a}(n+1-k)A\mathbf{u}(k), \quad n \in \mathbb{N}_0, \tag{6}$$

where the scalar sequence $\mathbf{a}(n)$ is given in terms of the so-called Poisson transformation¹⁴ and is defined as follows:

$$\mathbf{a}(n) := \mathcal{P}(a)(n) := \int_0^\infty p_n(t)a(t)dt, \quad p_n(t) := e^{-t} \frac{t^n}{n!}, \quad n \in \mathbb{N}_0,$$

where $a(t)$ is such that the integral in the right hand side exists. Equation 6 arises by application of the Poisson transformation to Equation 1. The Poisson transformation constitutes a new method of sampling. It was introduced in the paper¹⁴ in the context of fractional abstract difference equations and their main properties studied in the article.^[15, section 4]

One of the main novelties in this paper is that we consider A as the generator of a resolvent family $\{S(t)\}_{t \geq 0}$ defined on a Banach space X . In this way, we allow unbounded operators A and, consequently, our results are applicable to mixed Volterra equations, ie, where the time variable is discrete but the space variable is continuous.

It is remarkable that Volterra difference equations in the form (6) appears very recently in discrete chaos^{16,17} and in the analysis of discrete time neural networks.^[18, Formula (10)] In such cases, $\mathbf{a}(n) = k^\mu(n) := \frac{\Gamma(n+\mu)}{\Gamma(\mu)n!}$ are the Cesáro numbers¹⁴. We notice that this sequence has fundamental importance in the recent theory of fractional sum and fractional difference operators of order μ . An account of the main properties of k^μ can be found in the recent paper.¹⁹ We recall that in the finite dimensional case, i.e., $\dim(X) < \infty$, the study of Volterra difference equations in the form (6), with A being a matrix, dates back to many years ago. See, e.g.,²⁰ and references therein.

The first and main contributions in this paper shows that if the initial value $\mathbf{u}(0) = x$ belongs to $D(A)$, then an explicit classical solution of the problem (6) is given by

$$\mathbf{u}(n) = \int_0^\infty p_n(t)S(t)(I - (1 + \mathbf{a}(0))A)xdt, \quad n \in \mathbb{N}.$$

See Theorem 2. Concerning the important issue of stability of the solutions, we consider the special case as follows:

$$\mathbf{a}(n) = \frac{\alpha}{(\beta + 1)^{\mu+n}}k^\mu(n), \quad n \in \mathbb{N}_0,$$

where $\alpha \in \mathbb{R}$, $\mu, \beta > 0$. A second contribution is that assuming the following conditions in the parameters of the kernel \mathbf{a} and on the operator A :

- (S1) $\alpha \neq 0$, $\beta > 0$, and $\mu \geq 1$ such that $\alpha + \beta^\mu > 0$;
- (S2) A generates an immediately norm continuous C_0 -semigroup on a Banach space X ; and
- (S3) $\sup\{\operatorname{Re}\lambda, \lambda \in \mathbb{C} : \lambda(\lambda + \beta)^\mu((\lambda + \beta)^\mu + \alpha)^{-1} \in \sigma(A)\} < 0$;

we prove that the solution $\mathbf{u}(n)$ of (6) exists and satisfies $\|\mathbf{u}(n)\| \rightarrow 0$ as $n \rightarrow \infty$. See Theorem 4 below.

The plan of the paper is the following: In Section 2, we give some preliminaries on resolvent families and their main properties. In Section 3, we study the Poisson discretization for the Equations 1 and (3). In Section 4, we show our main findings on the stability of the solutions, giving explicit representations in the special case $A = \rho I$ where $\rho \in \mathbb{R}$ and I is the identity operator.

2 | PRELIMINARIES

Let $(X, \|\cdot\|)$ be a Banach space. We denote by $\mathcal{B}(X)$ the space of all bounded and linear operators from X into X . If A is a closed linear operator on X , we denote by $\rho(A)$ the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent operator of A defined for all $\lambda \in \rho(A)$. The spectrum of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. A family of operators $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is said to be *exponentially bounded* if there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0. \quad (7)$$

We notice that if $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is exponentially bounded, then the Laplace transform of $S(t)$, namely

$$\hat{S}(\lambda)_X := \int_0^\infty e^{-\lambda t} S(t)xdt$$

exists for all $\operatorname{Re}\lambda > \omega$. Observe that if $\omega = 0$, then $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is bounded. On the other hand, if the constant ω in (7) is strictly negative, then $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is called *uniformly exponentially stable*.

Let $a \in L^1_{loc}(\mathbb{R}_+)$ be given. In this paper, we will assume that the Laplace transform of a exists and satisfies $1 + \hat{a}(\lambda) \neq 0$ for all $\operatorname{Re}\lambda \geq 0$.

Definition 1. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X , $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable, and $b(t) := 1 + \int_0^t a(s)ds$. We say that the pair (b, A) is the generator of a resolvent family if there exist $\omega \geq 0$ and a strongly continuous function $S : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ such that $\{\frac{1}{\hat{b}(\lambda)}\}_{\operatorname{Re}\lambda > \omega} \subseteq \rho(A)$ and

$$\frac{1}{\lambda \hat{b}(\lambda)} \left(\frac{1}{\hat{b}(\lambda)} - A \right)^{-1} x = \int_0^\infty e^{-\lambda t} S(t)xdt, \quad \operatorname{Re}\lambda > \omega, \quad x \in X.$$

In this case, the family $\{S(t)\}_{t \geq 0}$ is called a *resolvent family generated by A* .

From the general theory,⁴ it is easy to see that if a resolvent family exists then it is unique. We notice that by the uniqueness of the Laplace transform, if $a(t) = 0$ for all $t \geq 0$, then the resolvent family $\{S(t)\}_{t \geq 0}$ is the same as a C_0 -semigroup.

On the other hand, if $k(t) = 1$ and $b(t) := 1 + \int_0^t a(s)ds$, then the resolvent family $\{S(t)\}_{t \geq 0}$ is a particular case of the theory of (b, k) -regularized families defined in the article.²¹

It is easy to show (see the papers⁴ or [21, Lemma 2.2 and Proposition 3.1]) that if A generates a resolvent family $\{S(t)\}_{t \geq 0}$ and $b(t) := 1 + \int_0^t a(s)ds$, then it satisfies the following properties:

1. $S(0) = I$.
2. If $x \in D(A)$, then

$$S(t)x = x + \int_0^t b(t-r)S(r)Axdr.$$

3. For all $x \in X$, $\int_0^t S(r)xdr \in D(A)$, and

$$S(t)x = x + A \int_0^t b(t-r)S(r)xdr. \tag{8}$$

4. For each $t \geq 0$, $S(t)x \in D(A)$ and $S(t)Ax = AS(t)x$, for all $x \in D(A)$ and $t \geq 0$.

We denote by $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, the set of non-negative integer numbers. For a given Banach space X , we denote by $s(\mathbb{N}_0, X)$ the vectorial space consisting of all vector-valued sequences $s : \mathbb{N}_0 \rightarrow X$. Let $f : \mathbb{R}_+ \rightarrow X$ be a bounded and locally integrable function. We recall from the paper¹⁴ (see also¹⁵) that the *Poisson transformation* of f is the vector-valued sequence defined by

$$\mathbf{f}(n) := \mathcal{P}(f)(n) := \int_0^\infty p_n(t)f(t)dt, \quad n \in \mathbb{N}_0,$$

where $p_n(t) := \frac{t^n e^{-t}}{n!}$ is the Poisson distribution. For a resolvent family $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)$, we denote

$$S(n)x := \mathcal{P}(S)(n)x := \int_0^\infty p_n(t)S(t)xdt, \quad n \in \mathbb{N}_0, x \in X.$$

If $c : \mathbb{R}_+ \rightarrow \mathbb{C}$ is a continuous and bounded function, we write

$$\mathbf{c}(n) := \mathcal{P}(c)(n) := \int_0^\infty p_n(t)c(t)dt, \quad n \in \mathbb{N}_0,$$

and the finite convolution of a scalar-valued sequence s and an operator-valued sequence Q is then defined as

$$(s \star Q)(n)x := \sum_{k=0}^n s(n-k)Q(k)x, \quad n \in \mathbb{N}_0, \quad x \in X.$$

For further use, we recall one of the following main results in the article.¹⁴

Theorem 1 (¹⁴, Theorem 3.4). *Let $c : \mathbb{R}_+ \rightarrow \mathbb{C}$ be Laplace transformable such that $\hat{c}(1)$ exists, and let $\{S(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1)$ exists. Then for all $x \in X$,*

$$\mathcal{P}(c * S)(n)x = (\mathcal{P}(c) \star \mathcal{P}(S))(n)x, \quad n \in \mathbb{N}_0.$$

We recall the sequence

$$k^\alpha(n) := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)\Gamma(n + 1)} = \int_0^\infty p_n(t)g_\alpha(t)dt, \quad n \in \mathbb{N}_0, \quad \alpha > 0, \tag{9}$$

where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$. This sequence has a strong connection with fractional difference equations, see for instance the papers^{14,15,19} and references therein for more details.

3 | SOLUTIONS OF VOLTERRA DIFFERENCE EQUATIONS IN BANACH SPACES

Let A be a closed linear operator defined on a Banach space X . Consider the following initial value problem:

$$\begin{cases} \mathbf{u}(n+1) - \mathbf{u}(n) = A\mathbf{u}(n+1) + \sum_{k=0}^{n+1} \mathbf{a}(n+1-k)A\mathbf{u}(k), & n \in \mathbb{N}_0 \\ \mathbf{u}(0) = x, \end{cases} \quad (10)$$

We will assume the following hypothesis:

(H1) There exists $a(t)$ such that $\mathbf{a} = \mathcal{P}(a)$.

(H2) The pair (b, A) , where $b(t) := 1 + \int_0^t a(s)ds$, is the generator of a resolvent family $\{S(t)\}_{t \geq 0}$ in X .

(H3) The initial value x belongs to $D(A)$.

We begin with the following representation of the solutions for the linear problem (10):

Theorem 2. *Suppose that (H1), (H2), and (H3) hold. Then, the Volterra difference Equation 10 admits the unique classical solution*

$$\mathbf{u}(n) = \int_0^\infty p_n(t)S(t)(I - (1 + \mathbf{a}(0))A)xdt, \quad n \in \mathbb{N}.$$

Proof. Following [14, theorem 4.4], we first prove that if $x \in X$, then $S(n)x := \mathcal{P}(S)(n)x \in D(A)$ for all $n \in \mathbb{N}_0$. In fact, if $x \in X$ and $n \in \mathbb{N}_0$, then by [22, Theorem 1.5.1]

$$S(n)x = \int_0^\infty p_n(t)S(t)xdt = \frac{(-1)^n}{n!} [\hat{S}(\lambda)]^{(n)} x|_{\lambda=1},$$

where

$$\hat{S}(\lambda)x = \frac{1}{1 + \hat{a}(\lambda)} \left(\frac{\lambda}{1 + \hat{a}(\lambda)} - A \right)^{-1} x := c(\lambda)(d(\lambda) - A)^{-1}x,$$

and the functions c and d are defined by $c(\lambda) := 1/(1 + \hat{a}(\lambda))$ and $d(\lambda) := \lambda/(1 + \hat{a}(\lambda))$. Now, if $R(\lambda) := (\lambda - A)^{-1}$, then $\hat{S}(\lambda) = c(\lambda)R(d(\lambda))$ and by the Leibniz's rule for the n^{th} -derivative of a product, we have

$$[\hat{S}(\lambda)]^{(n)} x = \sum_{k=0}^n \binom{n}{k} c(\lambda)^{(n-k)} [(R \circ d)(\lambda)]^{(k)}.$$

Moreover, by the rule for the n^{th} -derivative for the composition of functions, we obtain

$$[(R \circ d)(\lambda)]^{(k)} = \sum_{j=1}^k \frac{U_j(\lambda)}{j!} [R(\lambda)]^{(j)},$$

where

$$U_j(\lambda) := [d(\lambda)^j]^{(n)} - \frac{j}{1!} d(\lambda) [d(\lambda)^{j-1}]^{(n)} + \frac{j(j-1)}{2!} d(\lambda)^2 [d(\lambda)^{j-2}]^{(n)} - \dots + (-1)^{j-1} j d(\lambda)^{j-1} [d(\lambda)]^{(n)}.$$

On the other hand, since $[R(\lambda)]^{(m)} x = (\lambda - A)^{-(m+1)} x$ for all $x \in X$ and $m \in \mathbb{N}_0$, we obtain $[R(\lambda)]^{(j)} x|_{\lambda=1} = (I - A)^{-(j+1)} x \in D(A)$ and therefore $[(R \circ d)(\lambda)]^{(k)} x|_{\lambda=1} \in D(A)$. We conclude that $[\hat{S}(\lambda)]^{(n)} x|_{\lambda=1} \in D(A)$ and thus $S(n)x \in D(A)$ proving the claim.

Now, we take $x \in X$. Since, by (H2), $b(t) = 1 + \int_0^t a(r)dr$ we obtain from (H1) that $\mathbf{b}(n) := \mathcal{P}(b)(n) = 1 + \sum_{k=0}^n \mathbf{a}(k) =: 1 + (1 \star \mathbf{a})(n)$, for each $n \in \mathbb{N}_0$. From the resolvent identity

$$S(t)x = x + A \int_0^t b(t-r)S(r)xdr, \quad t \geq 0,$$

and taking Poisson transformation, it follows by Theorem 1, that

$$S(n)x = x + A(\mathbf{b} \star S)(n)x, \quad n \in \mathbb{N}_0. \tag{11}$$

Therefore, for $x \in X$, we obtain

$$\begin{aligned} S(n)x &= x + A \sum_{k=0}^n \mathbf{b}(n-k)S(k)x = x + A \sum_{k=0}^n (1 + (1 \star \mathbf{a}))(n-k)S(k)x \\ &= x + A \sum_{k=0}^n S(k)x + A \sum_{k=0}^n (1 \star \mathbf{a})(n-k)S(k)x = x + A \sum_{k=0}^n S(k)x + A \sum_{k=0}^n (1 \star \mathbf{a})(k)S(n-k)x. \end{aligned}$$

Since $(1 \star \mathbf{a})(n) = \sum_{k=0}^n \mathbf{a}(k)$, we obtain

$$S(n)x = x + A \sum_{k=0}^n S(k)x + A \sum_{k=0}^n \left(\sum_{j=0}^k \mathbf{a}(j) \right) S(n-k)x. \tag{12}$$

Now, by (12), we have

$$S(n+1)x - S(n)x = AS(n+1)x + A \left[\sum_{k=0}^{n+1} \left(\sum_{j=0}^k \mathbf{a}(j) \right) S(n+1-k)x - \sum_{k=0}^n \left(\sum_{j=0}^k \mathbf{a}(j) \right) S(n-k)x \right].$$

And a simple computation shows that

$$\sum_{k=0}^{n+1} \left(\sum_{j=0}^k \mathbf{a}(j) \right) S(n+1-k)x - \sum_{k=0}^n \left(\sum_{j=0}^k \mathbf{a}(j) \right) S(n-k)x = \sum_{k=0}^{n+1} \mathbf{a}(k)S(n+1-k) = (\mathbf{a} \star S)(n+1)x,$$

which implies that for $x \in X$, we have $S(n+1)x - S(n)x = AS(n+1)x + (\mathbf{a} \star AS)(n+1)x$, for all $n \in \mathbb{N}_0$. We conclude that $S(n)x$ verifies the difference equation in (10).

Now, if $x \in D(A)$, we define $\mathbf{u}(n) := S(n)(I - \mathbf{b}(0)A)x$, where $\mathbf{b}(0) = 1 + \mathbf{a}(0)$. It then follows that $\mathbf{u}(n) \in D(A)$ for all $n \in \mathbb{N}$ and $\mathbf{u}(n)$ solves the difference equation in (10). On the other hand, from the identity $S(0)x = x + A(\mathbf{b} \star S)(0)x = x + \mathbf{b}(0)AS(0)x$, it follows that $\mathbf{u}(0) = S(0)(I - \mathbf{b}(0)A)x = x$, which means that $\mathbf{u}(n)$ is solution to the Volterra difference Eq. 10. Finally, the uniqueness of \mathbf{u} follows from the uniqueness of the resolvent family $\{S(t)\}_{t \geq 0}$, see [4, Chapter I, Section 1]. □

As a consequence, we prove the following result for the nonhomogeneous problem.

Theorem 3. *Suppose that (H1), (H2), and (H3) hold. Then, the following Volterra difference equation:*

$$\begin{cases} \mathbf{u}(n+1) - \mathbf{u}(n) = A\mathbf{u}(n+1) + \sum_{k=0}^{n+1} \mathbf{a}(n+1-k)A\mathbf{u}(k) + \mathbf{f}(n+1), & n \in \mathbb{N}_0 \\ \mathbf{u}(0) = x, \end{cases} \tag{13}$$

where $\mathbf{f} : \mathbb{N}_0 \rightarrow X$, admits the unique classical solution

$$\mathbf{u}(n) = S(n)(I - (1 + \mathbf{a}(0))A)x + (S \star \mathbf{f})(n), \quad n \in \mathbb{N}.$$

Proof. We write $y := (I - \mathbf{b}(0)A)x$, where $\mathbf{b}(0) = 1 + \mathbf{a}(0)$, and define $\mathbf{u}(n) := S(n)y + (S \star \mathbf{f})(n)$ for $n \geq 1$ and $\mathbf{u}(0) := S(0)(I - \mathbf{b}(0)A)x$. By (11), we have $\mathbf{u}(0) = x$. Moreover, as in the proof of Theorem 2, we obtain

$$\begin{aligned} \mathbf{u}(n+1) - \mathbf{u}(n) &= (S(n+1) - S(n))y + (S \star \mathbf{f})(n+1) - (S \star \mathbf{f})(n) \\ &= AS(n+1)y + (\mathbf{a} \star AS)(n+1)y + (S \star \mathbf{f})(n+1) - (S \star \mathbf{f})(n). \end{aligned}$$

We claim that

$$(S \star \mathbf{f})(n+1) - (S \star \mathbf{f})(n) = A(S \star \mathbf{f})(n+1) + A(\mathbf{a} \star S \star \mathbf{f})(n+1) + \mathbf{f}(n+1).$$

In fact, by (11), we obtain

$$(S \star \mathbf{f})(m) = \sum_{k=0}^m S(m-k)\mathbf{f}(k) = \sum_{k=0}^m \mathbf{f}(k) + A \sum_{k=0}^m (\mathbf{b} \star S)(m-k)\mathbf{f}(k) = \sum_{k=0}^m \mathbf{f}(k) + A(\mathbf{b} \star S \star \mathbf{f})(m),$$

for all $m \in \mathbb{N}_0$. Hence, $(S \star \mathbf{f})(n+1) - (S \star \mathbf{f})(n) = \mathbf{f}(n+1) + A[(\mathbf{b} \star S \star \mathbf{f})(n+1) - (\mathbf{b} \star S \star \mathbf{f})(n)]$. On the other hand, since $\mathbf{b}(m) = 1 + (1 \star \mathbf{a})(m)$ with $(1 \star \mathbf{a})(m) = \sum_{j=0}^m \mathbf{a}(j)$ for all $m \in \mathbb{N}_0$, we obtain

$$\begin{aligned} (\mathbf{b} \star S \star \mathbf{f})(n+1) - (\mathbf{b} \star S \star \mathbf{f})(n) &= \sum_{k=0}^{n+1} (\mathbf{b} \star S)(n+1-k)\mathbf{f}(k) - \sum_{k=0}^n (\mathbf{b} \star S)(n-k)\mathbf{f}(k) \\ &= (S \star \mathbf{f})(n+1) + \sum_{k=0}^{n+1} (1 \star \mathbf{a})(n+1-k)(S \star \mathbf{f})(k) - \sum_{k=0}^n (1 \star \mathbf{a})(n-k)(S \star \mathbf{f})(k) \\ &= (S \star \mathbf{f})(n+1) + \sum_{k=0}^{n+1} \left(\sum_{j=0}^{n+1-k} \mathbf{a}(j) \right) (S \star \mathbf{f})(k) - \sum_{k=0}^n \left(\sum_{j=0}^{n-k} \mathbf{a}(j) \right) (S \star \mathbf{f})(k) \\ &= (S \star \mathbf{f})(n+1) + \sum_{k=0}^{n+1} \mathbf{a}(n+1-k)(S \star \mathbf{f})(k) = (S \star \mathbf{f})(n+1) + (\mathbf{a} \star S \star \mathbf{f})(n+1), \end{aligned}$$

which proves the claim. Therefore,

$$\begin{aligned} \mathbf{u}(n+1) - \mathbf{u}(n) &= AS(n+1)y + (\mathbf{a} \star AS)(n+1)y + A(S \star \mathbf{f})(n+1) + A(\mathbf{a} \star S \star \mathbf{f})(n+1) + \mathbf{f}(n+1) \\ &= A[S(n+1)y + (S \star \mathbf{f})(n+1)] + (\mathbf{a} \star AS)(n+1)y + (\mathbf{a} \star AS \star \mathbf{f})(n+1) + \mathbf{f}(n+1) \\ &= A\mathbf{u}(n+1) + (\mathbf{a} \star A\mathbf{u})(n+1) + \mathbf{f}(n+1), \end{aligned}$$

for all $n \in \mathbb{N}$. The uniqueness follows as in Theorem 2, proving the Theorem. \square

4 | STABILITY PROPERTIES

In this section, we study the stability of the solution to the integro-difference Equation 10. We recall that a vector-valued sequence $\mathbf{u} \in s(\mathbb{N}_0, X)$ is said to be *stable* if $\|\mathbf{u}(n)\| \rightarrow 0$ as $n \rightarrow \infty$. From [23, Section 2.3], we recall the following definition:

Definition 2. Let X be a Banach space. A strongly continuous function $T : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is said to be *immediately norm continuous* if $T : (0, \infty) \rightarrow \mathcal{B}(X)$ is continuous.

Now, we consider Equation 10 where the kernel $\mathbf{a}(t)$ is given by

$$\mathbf{a}(n) = \frac{\alpha}{(\beta + 1)^{\mu+n}} k^\mu(n), \quad n \in \mathbb{N}_0.$$

with $\alpha \in \mathbb{R}$ and $\mu, \beta > 0$. We assume the following hypothesis in the parameters of the kernel \mathbf{a} and on the operator A :

- (S1) $\alpha \neq 0, \beta > 0$ and $\mu \geq 1$ such that $\alpha + \beta^\mu > 0$.
- (S2) A generates an immediately norm continuous C_0 -semigroup.
- (S3) $\sup\{\operatorname{Re}\lambda, \lambda \in \mathbb{C} : \lambda(\lambda + \beta)^\mu((\lambda + \beta)^\mu + \alpha)^{-1} \in \sigma(A)\} < 0$.

Theorem 4. *Let $x \in D(A)$. Suppose that hypotheses (S1), (S2), and (S3) hold. Then, there exists a unique stable solution to the problem (10).*

Proof. We verify (H1) because $\mathbf{a} = \mathcal{P}(a)$ where $a(t) = \alpha \frac{t^{\mu-1}}{\Gamma(\mu)} e^{-\beta t}$. Since (S1)-(S3) hold, we have by [24, Proposition 3.1], (see also the paper²⁵), that the pair (b, A) generates a resolvent family $\{S_{\alpha,\beta}^\mu(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ where $b(t) = 1 + \alpha \int_0^t \frac{r^{\mu-1}}{\Gamma(\mu)} e^{-\beta r} dr$, and therefore (H3) holds. Moreover, Proposition 3.1 in the paper²⁴ asserts that $S_{\alpha,\beta}^\mu(t)$ is uniformly exponentially stable, that is, there exist $M, \omega > 0$ such that $\|S_{\alpha,\beta}^\mu(t)\| \leq M e^{-\omega t}$ for all $t \geq 0$. Consequently, by Theorem 2, we have

$$\begin{aligned} \|\mathbf{u}(n)\| &\leq \int_0^\infty p_n(t) \|S_{\alpha,\beta}^\mu(t)(I - (1 - \mathbf{a}(0)A)x)\| dt \\ &\leq M \int_0^\infty p_n(t) e^{-\omega t} dt \|I - (1 - \mathbf{a}(0)A)x\| = \frac{M}{(\omega + 1)^{n+1}} \|I - (1 - \mathbf{a}(0)A)x\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. This proves the theorem. □

Now, we consider the scalar case, that is, we assume that $A = \rho I$, where ρ is a real number. In such case, we will show that we can find an explicit representation of the resolvent families $\{S_{\alpha,\beta}^\mu(t)\}_{t \geq 0}$ and $S_{\alpha,\beta}^\mu(n)$ for all $n \in \mathbb{N}_0$. In fact, from (8), it is not difficult to see that the Laplace transform of the family $\{S_{\alpha,\beta}^\mu(t)\}_{t \geq 0}$ is given by

$$\hat{S}_{\alpha,\beta}^\mu(\lambda) = \frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^{\mu+1} - (\rho + \beta)(\lambda + \beta)^\mu - \alpha\rho}.$$

Using the properties of the Laplace transform, we obtain $S_{\alpha,\beta}^\mu(t) = e^{-\beta t} R_{\alpha,\beta}^\mu(t)$, where $R_{\alpha,\beta}^\mu(t)$ is a function whose Laplace transform is given by

$$\hat{R}_{\alpha,\beta}^\mu(\lambda) = \frac{\lambda^\mu}{\lambda^{\mu+1} - (\rho + \beta)\lambda^\mu - \alpha\rho}.$$

Since

$$\left| \frac{(\rho + \beta)\lambda^\mu}{\lambda^{\mu+1} - \alpha\rho} \right| < 1,$$

for λ is large enough, we obtain by [26, Formula 17.6],

$$R_{\alpha,\beta}^\mu(t) = \sum_{j=0}^\infty (\rho + \beta)^j t^j E_{\mu+1, j+1}^{j+1}(\alpha\rho t^{\mu+1}), \quad t \geq 0,$$

where $E_{p,q}^\gamma(z) := \sum_{j=0}^\infty \frac{k^\gamma(j)z^j}{\Gamma(q+pj)}$ is the generalized Mittag-Leffer type function defined for $p, q, \gamma > 0$, see for instance [26, Section 11]. Therefore, we conclude that

$$S_{\alpha,\beta}^\mu(t) = e^{-\beta t} \sum_{j=0}^\infty (\rho + \beta)^j t^j E_{\mu+1, j+1}^{j+1}(\alpha\rho t^{\mu+1}), \quad t \geq 0. \tag{14}$$

Now, we compute $S_{\alpha,\beta}^\mu(n)$ when $A = \rho I$. By (14), we have

$$\begin{aligned} S_{\alpha,\beta}^\mu(n) &= \int_0^\infty p_n(t) S_{\alpha,\beta}^\mu(t) dt = \int_0^\infty e^{-t} \frac{t^n}{n!} e^{-\beta t} \sum_{j=0}^\infty (\rho + \beta)^j t^j E_{\mu+1, j+1}^{j+1}(\alpha\rho t^{\mu+1}) dt \\ &= \frac{1}{\Gamma(n+1)} \sum_{j=0}^\infty (\rho + \beta)^j \int_0^\infty e^{-(\beta+1)t} t^{(n+j+1)-1} E_{\mu+1, j+1}^{j+1}(\alpha\rho t^{\mu+1}) dt. \end{aligned}$$

Observe that the last integral corresponds to the Laplace transform of the function

$$h(t) := t^{(n+j+1)-1} E_{\mu+1, j+1}^{j+1}(\alpha \rho t^{\mu+1})$$

evaluated at $\lambda = (\beta + 1)$. Using [26, Formula 11.15], we obtain

$$S_{\alpha, \beta}^{\mu}(n) = \frac{1}{\Gamma(n+1)} \sum_{j=0}^{\infty} (\rho + \beta)^j \frac{(\beta + 1)^{-(n+j+1)}}{\Gamma(j+1)} {}_2\psi_1 \left[\begin{matrix} (j+1, 1); (n+j+1, \mu+1) \\ (j+1, \mu+1) \end{matrix} \middle| \frac{\alpha \rho}{(\beta + 1)^{\mu+1}} \right],$$

where ${}_2\psi_1[\cdot | z]$ is the generalized Wright function. From [26, Formula 8.5], it follows that

$$\begin{aligned} & {}_2\psi_1 \left[\begin{matrix} (j+1, 1); (n+j+1, \mu+1) \\ (j+1, \mu+1) \end{matrix} \middle| \frac{\alpha \rho}{(\beta + 1)^{\mu+1}} \right] \\ &= \sum_{r=0}^{\infty} \frac{\Gamma(j+1+r)\Gamma(j+1+(\mu+1)r+n)}{\Gamma(j+1+(\mu+1)r)} \left(\frac{\alpha \rho}{(\beta + 1)^{\mu+1}} \right)^r \frac{1}{r!} \end{aligned}$$

Using the sequence k^{α} defined in (9), we can write

$$\begin{aligned} S_{\alpha, \beta}^{\mu}(n) &= \sum_{j=0}^{\infty} (\rho + \beta)^j (\beta + 1)^{-(n+j+1)} \\ &\quad \times \sum_{r=0}^{\infty} \frac{\Gamma(j+1+r)}{\Gamma(j+1)\Gamma(r+1)} \frac{\Gamma(j+1+(\mu+1)r+n)}{\Gamma(j+1+(\mu+1)r)\Gamma(n+1)} \left(\frac{\alpha \rho}{(\beta + 1)^{\mu+1}} \right)^r \\ &= \sum_{j=0}^{\infty} \frac{(\rho + \beta)^j}{(\beta + 1)^{(n+j+1)}} \sum_{r=0}^{\infty} k^{j+1}(r) k^{(j+1)+(\mu+1)r}(n) \left(\frac{\alpha \rho}{(\beta + 1)^{\mu+1}} \right)^r. \end{aligned}$$

This gives the claimed representation.

Now, we consider the difference equation

$$\begin{cases} \mathbf{u}(n+1) - \mathbf{u}(n) = \rho \mathbf{u}(n+1) + \frac{\alpha \rho}{(\beta+1)^{\mu}} \sum_{k=0}^{n+1} \frac{k^{\mu(n+1-k)}}{(\beta+1)^{\mu+1-k}} \mathbf{u}(k), & n \in \mathbb{N}_0 \\ \mathbf{u}(0) = x, \end{cases} \quad (15)$$

Since $A = \rho I$ trivially generates an immediately norm continuous C_0 -semigroup, we obtain the following result.

Proposition 1. *Assume the hypothesis (S1) and $x \in X$. Then, the difference Eq. 15 has the classical solution*

$$\mathbf{u}(n) = \frac{(\beta + 1)^{\mu}(1 - \rho) - \alpha \rho}{(\beta + 1)^{\mu}} S_{\alpha, \beta}^{\mu}(n)x.$$

Moreover, $\|\mathbf{u}(n)\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. An easy computation shows that the pair $(a, \rho I)$ generates the resolvent family $\{S_{\alpha, \beta}^{\mu}(t)\}_{t \geq 0}$, where $a(t) = \alpha \frac{t^{\mu-1}}{\Gamma(\mu)} e^{-\beta t}$. Since $\mathbf{a} = \mathcal{P}(a)$ and $\mathbf{b}(0) = 1 + \mathbf{a}(0) = 1 + \frac{\alpha}{(\beta+1)^{\mu}} k^{\mu}(0) = 1 + \frac{\alpha}{(\beta+1)^{\mu}}$, we have $(1 - \mathbf{b}(0)\rho) = \frac{(\beta+1)^{\mu}(1-\rho) - \alpha \rho}{(\beta+1)^{\mu}}$ and the conclusion follows from Theorem 2. \square

For the next result, we need to recall the following definition.

Definition 3 (⁴, Section 3.2). Let $k \in \mathbb{N}$ and $a \in L_{\text{loc}}^1(\mathbb{R}_+)$. The kernel a is called k -regular if there is a constant $c > 0$ such that $|\lambda^n \hat{a}^{(n)}(\lambda)| \leq c |\hat{a}(\lambda)|$, for all $\text{Re} \lambda > 0$ and $1 \leq n \leq k$.

We obtain the following criteria for the stability of the solutions for (10) in Hilbert spaces:

Proposition 2. Let H be a Hilbert space. We assume the following hypothesis:

(P1) $\mathbf{a} = \mathcal{P}(a)$ where $a \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular.

(P2) the pair (b, A) is the generator of an exponentially bounded resolvent family $\{S(t)\}_{t \geq 0}$ in H .

(P3) $0 \in \rho(A)$.

(P4) $\frac{1}{\hat{b}(\lambda)} \in \rho(A)$ for all $\operatorname{Re}(\lambda) \geq 0, \lambda \neq 0$.

(P5) $H(\lambda) := (\lambda - (1 + \hat{a}(\lambda))A)^{-1}$ is uniformly bounded in $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$.

Then, the classical solution to the difference initial value problem (10) exists and is stable.

Proof. The 1-regularity of the kernel a implies the existence of a constant $d > 0$ such that $|\lambda \hat{a}'(\lambda)| \leq d|\hat{a}(\lambda)|$. Since $b(t) = 1 + (1 * a)(t)$, we obtain

$$|\lambda \hat{b}'(\lambda)| = \left| -\frac{1}{\lambda} - \frac{1}{\lambda} \hat{a}(\lambda) + \hat{a}'(\lambda) \right| \leq |\hat{b}(\lambda)| + d \frac{|\hat{a}(\lambda)|}{|\lambda|} \leq |\hat{b}(\lambda)| + d \frac{|1 + \hat{a}(\lambda)|}{|\lambda|} = (d + 1)|\hat{b}(\lambda)|, \quad (16)$$

for all $\operatorname{Re} \lambda > 0$, which means that b is a 1-regular kernel. On the other hand, $\lim_{\lambda \rightarrow 0} \lambda \hat{b}'(\lambda) = \lim_{\lambda \rightarrow 0} (1 + \hat{a}(\lambda)) = 1 + \hat{a}(0) \neq 0$. By [27, Theorem 1], we conclude that the resolvent family $\{S(t)\}_{t \geq 0}$ is uniformly stable. Then, the dominated convergence theorem shows that $\lim_{n \rightarrow \infty} \|S(n)\| = 0$. Finally, an application of Theorem 2 finishes the proof. \square

CONFLICT OF INTERESTS

This work does not have any conflicts of interest

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