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TIME DISCRETIZATION AND CONVERGENCE TO SUPERDIFFUSION EQUATIONS VIA POISSON DISTRIBUTION

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Abstract. Let A be a closed linear operator defined on a complex Banach space X. We show a novel representation, using strongly continuous families of bounded operators defined on $\mathbb{N}_0$, for the unique solution of the following time-stepping scheme

$$(*)\begin{cases} {}_C\nabla^\alpha u^n &= Au^n + f^n, \quad n \geq 2; \\ u^0 &= u_0; \\ u^1 &= u_1; \end{cases}$$

as well as its convergence with rates to the solution of the abstract fractional Cauchy problem

$$(*)\begin{cases} \partial_t^\alpha u(t) &= Au(t) + f(t), \quad t > 0; \\ u(0) &= u_0; \\ u'(0) &= u_1; \end{cases}$$

in the superdiffusive case $1 < \alpha < 2$. Here, ${}_C\nabla^\alpha u^n$ is the Caputo-like fractional difference operator of order α.

1. Introduction.

The theory of one parameter C_0-semigroups of linear operators has many different applications in mathematical physics, probability theory, engineering, biological processes, applications in the theory of linear and nonlinear partial differential equations, problems in control theory and dynamical systems, and in some methods for numerical analysis, among others. Typically, in these applications, the phenomena can be described as an abstract evolution equation of first order

$$u'(t) = Au(t) + F(t), \quad t > 0, \tag{1.1}$$

subject to the initial condition $u(0) = u_0$. Here A is a closed linear operator (typically A corresponds to the Laplacian), F is a linear or nonlinear term and u_0 belongs to a Banach space. If A generates a C_0-semigroup of linear operators $\{T(t)\}_{t\geq 0}$, then the solution u to problem (1.1) can be written as the well known variation of

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parameters formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s)ds,$$

see for instance [2, 12]. This last formula, can be used (according to the properties of $T(t)$) to study the solution u of the problem (1.1), including its asymptotic behavior, its regularity properties or some numerical treatments to find an approximation of u. However, there are many interesting phenomena, including for example, problems in viscoelasticity, heat conduction in materials with memory, geological exploration, problems involving linear viscoelastic rods, beams or plates and many others, where the model of a first order evolution equation is not completely satisfactory. Instead, as has been widely reported in the last years, fractional differential equations (FDEs) provide a more natural framework to describe these phenomena. Unfortunately, the theory of C_0-semigroups can no longer be used to describe the evolution of FDEs. Therefore, the theory of one-parameter resolvent families of operators has become a powerful tool to describe the dynamics of the solution for this class of fractional models. See for instance, [4, 10, 13, 18, 25, 29] and references therein.

Resolvent families of operators, which can be considered as an extension of the theory of semigroups, have been marked by an increased interest, mainly due to its applications not only to the study of linear and nonlinear FDEs but also integro–differential equations. As we previously intimated, these families of operators can be used to write the solution to FDEs as a variation of parameters formula. More specifically, the solution to the superdiffusion equation

$$\left\{ \begin{array}{rcl} \partial_t^\alpha u(t) & = & Au(t) + f(t), \quad t \geq 0, \\ u(0) & = & u_0, \\ u'(0) & = & u_1, \end{array} \right. \tag{1.2}$$

where $1 < \alpha \leq 2$, f represents a loading term, A is a closed linear operator defined in a Banach space X, $u_0, u_1 \in X$, and, ∂_t^α denotes the Caputo fractional derivative of u, can be written as

$$u(t) = S_{\alpha,1}(t)u_0 + (g_1 * S_{\alpha,1})(t)u_1 + (g_{\alpha-1} * S_{\alpha,1} * f)(t), \quad t \geq 0, \tag{1.3}$$

where $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is the resolvent family generated by A (see [25, Chapter 3]) and for $\beta > 0$ the function $g_\beta(t)$ is defined by $g_\beta(t) := t^{\beta-1}/\Gamma(\beta)$ (here $\Gamma(\cdot)$ denotes the Gamma function).

The problem (1.2) has been widely studied in the last years, since it can adequately capture the dynamics of anomalous processes, as for instance in the modeling of mechanical wave propagation in viscoelastic media. See for instance [3, 5, 6, 7, 13, 22, 21, 34] and references therein. However, for various practical purposes, it is not only useful but necessary to study its discrete version.

The existence of discrete solutions to abstract fractional difference equations in the form of

$${}_C\nabla^\alpha u^n = Au^n + f^n, \quad n \in \mathbb{N},$$

where A is a closed linear operator, f^n is a given sequence and ${}_C\nabla^\alpha u^n$ is a discrete counterpart of the Caputo fractional derivative, has been marked in the last decades by a great deal of interest. See for instance [8, 9, 10, 15, 16, 19, 24, 27, 30, 33] for some developments. Note that the meaning of the fractional difference operator ${}_C\nabla^\alpha u^n$ can vary, depending on the time discretization method used [19].

Recently, in [31] the author analytically studies the time discretization scheme for the model (1.2) based on the backward Euler method in the case $0 < \alpha < 1$.

In [31] it was shown that if A is the generator of a resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$, then the analytical solution of the scheme can be represented in terms of certain resolvent families of operators defined on $\mathbb{N}_0$ by a suitable transformation of the family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ using the probability mass function, with variance t/τ defined by

$$\rho_n^\tau(t) := e^{-\frac{t}{\tau}}\left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!}, \quad t\geq 0, \quad n\in\mathbb{N}_0,$$

for a positive step size $\tau>0$. In addition, error estimates were provided in case A is a sectorial operator.

However, the extension of the results in [31] for the important case of superdiffusion, i.e. $1<\alpha<2$, was left open.

The objective of this work is to answer this problem. We provide a suitable framework to apply the theory of resolvent families of operators in order to find necessary conditions on A to have an analytical representation of the solutions of the following scheme

$$(*)\begin{cases} {}_C\nabla^\alpha u^n &= Au^n + f^n, \quad n\geq 2,\\ u^0 &= u_0,\\ u^1 &= u_1, \end{cases} \tag{1.4}$$

where A is a closed linear operator defined in a Banach space X, the initial conditions u_0, u_1 belong to X, $1<\alpha<2$, and the sequence f^n is a given vector-valued sequence. Here, ${}_C\nabla^\alpha u^n$ represents a discretization of the Caputo fractional derivative $\partial_t^\alpha u(t)$ at time $t=\tau n$, which is defined by

$${}_C\nabla^\alpha u^n := \sum_{j=2}^n k_\tau^{2-\alpha}(n-j)\frac{(u^{j+1}-2u^j+u^{j-1})}{\tau^2},$$

where, $u^n := \int_0^\infty \rho_n^\tau(t)u(t)dt$, and $k_\tau^\beta(n) := \frac{\tau^\beta\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)}$ for all $n\in\mathbb{N}_0$ and $\beta>0$. More concretely, we show that if A is the generator of a resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$, then the solution to (1.4) can be written as (see Theorem 3.1 below)

$$u^n = S_{\alpha,1}^n u_0 + \tau(g_1\star S_{\alpha,1})^n u_1 + \tau^2(g_{\alpha-1}\star S_{\alpha,1}\star f)^n,$$

where $S_{\alpha,1}^n$ is defined as

$$S_{\alpha,1}^n x := \int_0^\infty \rho_n^\tau(t)S_{\alpha,1}(t)x dt,$$

for all $x\in X$ and for $\beta>0$,

$$(g_\beta\star S_{\alpha,1})^n x := \sum_{j=0}^n k_\tau^\beta(n-j)S_{\alpha,1}^j x, \quad n\in\mathbb{N}_0.$$

We also analyze the difference $\|u(t_n)-u^n\|$, where u is the solution to (1.2) and u^n solves the difference equation (1.4) and we show that, given suitable conditions on α, there exists a constant $C=C(T)>0$ (independent of the solution, the data and the step size) such that, for $0<t_n\leq T$, there holds

$$\|u(t_n)-u^n\| \leq C\tau t_n^{\alpha\varepsilon-1}\left(\|A^\varepsilon u_0\| + \|A^\varepsilon u_1\| + \|A^\varepsilon f\|\right),$$

where $0<\varepsilon<1$ satisfies $\alpha\varepsilon<1$ and u_0, u_1 and $f(t)$ belong to the domain of A^ε. Of course, this result shows, in particular, that if $\tau\to 0$ then $\|u(t_n)-u^n\|\to 0$.

The paper is organized as follows. In Section 2 we give preliminaries on resolvent families, sectorial operators and continuous and discrete fractional calculus. Section

3 treats the existence of solutions to the Caputo fractional difference equation (1.4). Here, given a time step size $\tau > 0$, we study the connection between the continuous and the discrete resolvent families $\{S_{\alpha,1}(t)\}_{t\geq 0}$ and $\{S^n_{\alpha,1}\}_{n\in\mathbb{N}_0}$, as well as, its consequences on the representation of solutions to (1.4). In Section 4 we study error estimates for $\|u(t_n) - u^n\|$ and we give sufficient conditions on the initial data in order to obtain $\|u(t_n) - u^n\| \to 0$ as $\tau \to 0$. Finally, in Section 5 we give some numerical experiments to illustrate the theoretical results.

2. Resolvent families, continuous and discrete fractional calculus.

2.1. Resolvent families.

Given a Banach space $X \equiv (X, \|\cdot\|)$, $\mathcal{B}(X)$ denotes the Banach space of all bounded and linear operators from X into X. For a closed linear operator A defined on X, its resolvent set is denoted by $\rho(A)$, the resolvent operator is defined by $R(\lambda, A) = (\lambda - A)^{-1}$ for all $\lambda \in \rho(A)$ and $\sigma(A)$ denotes the spectrum of A. A family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called *exponentially bounded* if there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

We observe that if $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is exponentially bounded, then the Laplace transform of $S(t)$

$$\hat{S}(\lambda)x := \int_0^\infty e^{-\lambda t} S(t)x dt,$$

exists for all $\text{Re}\lambda > \omega$.

In the following we recast the main ingredients of operator theory that we will use. For an up to date review of the following concepts and their interplay with fractional differential equations in the continuous setting, we refer the reader to the recent reference [25].

Definition 2.1. Let $1 \leq \alpha \leq 2$ and $0 < \beta \leq 2$ be given. A closed linear operator A defined in a Banach space X is called the generator of an (α, β)-*resolvent family* if there exist $\omega \geq 0$ and a strongly continuous and exponentially bounded function $S_{\alpha,\beta} : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \text{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)x dt,$$

for all $\text{Re}\lambda > \omega$ and $x \in X$. The family $\{S_{\alpha,\beta}(t)\}$ is also called the (α, β)-resolvent family generated by A.

Given $\mu > 0$, we define $g_\mu(t) := \frac{t^{\mu-1}}{\Gamma(\mu)}$ for all $t > 0$, where Γ denotes the Gamma function. If we take $a(t) := g_\alpha(t)$ and $k(t) := g_\beta(t)$, where $\alpha, \beta > 0$ then the family $\{S_{\alpha,\beta}(t)\}$ corresponds to an (a, k)-regularized family according to [23]. Moreover, from [23, Lemma 2.2 and Proposition 2.5] we deduce the following properties.

Proposition 1. *Let $1 \leq \beta \leq \alpha \leq 2$ be given. Let $\{S_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be the (α, β)-resolvent family generated by A. Then,*

1. *$S_{\alpha,\beta}(0) = I$, where I denotes the identity operator in X.*
2. *For all $x \in D(A)$ and $t \geq 0$ we have $S_{\alpha,\beta}(t)x \in D(A)$ and $AS_{\alpha,\beta}(t)x = S_{\alpha,\beta}(t)Ax$.*
3. *For $x \in X$ and $t \geq 0$ we have $\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds \in D(A)$ and*

$$S_{\alpha,\beta}(t)x = g_\beta(t)x + A\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds. \tag{2.1}$$

Moreover, the function $t \mapsto S_{\alpha,\beta}(t)$ satisfies the following functional equation (see [1, 21, 26]):

$$\begin{aligned} S_{\alpha,\beta}(s)(g_\alpha * S_{\alpha,\beta})(t) - (g_\alpha * S_{\alpha,\beta})(s)S_{\alpha,\beta}(s) \\ = g_\beta(s)(g_\alpha * S_{\alpha,\beta})(t) - g_\beta(t)(g_\alpha * S_{\alpha,\beta})(s), \end{aligned}$$

for all $t, s \geq 0$. If an operator A with domain $D(A)$ generates a resolvent family $S_{\alpha,\beta}(t)$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t\to 0^+} \frac{S_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)}.$$

For example, we notice that if $\alpha = \beta = 1$, then $S_{1,1}(t)$ corresponds to a C_0-semigroup, and if $\alpha = 2, \beta = 1$, then $S_{2,1}(t)$ is a strongly continuous cosine family of operators. Analogously, if $\alpha = \beta = 2$, then $S_{2,2}(t)$ is a strongly continuous sine family. See [2] for further details. If $\alpha > 0$ and $\beta = 1$, then $S_{\alpha,1}(t)$ is the solution operator introduced by Bazhlekova in [3, Definition 2.3].

Definition 2.2. We say that a function $v : \mathbb{R}_+ \to X$ is a *strong solution* to equation (1.2) if $v(t) \in D(A)$ for all $t \geq 0$ and satisfies (1.2).

Taking formally the Laplace transform in (1.2) we obtain

$$(\lambda^\alpha - A)\hat{u}(\lambda) = \lambda^{\alpha-1}u_0 + \lambda^{\alpha-2}u_1 + \hat{f}(\lambda),$$

for all $\mathrm{Re}(\lambda) > 0$. If $\lambda^\alpha \in \rho(A)$, then

$$\hat{u}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}u_0 + \lambda^{\alpha-2}(\lambda^\alpha - A)^{-1}u_1 + (\lambda^\alpha - A)^{-1}\hat{f}(\lambda), \tag{2.2}$$

where $u_0, u_1 \in X$. By the uniqueness of the Laplace transform and Definition 2.1, we obtain that if A generates a resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$, then for all $\lambda^\alpha \in \rho(A)$ we have

1. $\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} = \hat{S}_{\alpha,1}(\lambda)$,
2. $\lambda^{\alpha-2}(\lambda^\alpha - A)^{-1} = \hat{S}_{\alpha,2}(\lambda) \Longleftrightarrow \lambda^{\alpha-2}(\lambda^\alpha - A)^{-1} = \frac{1}{\lambda}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} = \hat{g_1}(\lambda)\hat{S}_{\alpha,1}(\lambda)$, and
3. $(\lambda^\alpha - A)^{-1} = \hat{S}_{\alpha,\alpha}(\lambda) \Longleftrightarrow (\lambda^\alpha - A)^{-1} = \frac{1}{\lambda^{\alpha-1}}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} = \hat{g}_{\alpha-1}(\lambda)\hat{S}_{\alpha,1}(\lambda)$.

The identity (2.2), the relations (1)-(2) and the uniqueness of the Laplace transform imply that the unique solution to (1.2) is given by

$$u(t) = S_{\alpha,1}(t)u_0 + (g_1 * S_{\alpha,1})(t)u_1 + (g_{\alpha-1} * S_{\alpha,1} * f)(t), \quad t \geq 0. \tag{2.3}$$

We notice that, since u_0, u_1 merely belong to X, we can not prove (by Proposition 1) that the function $u(t)$ defined by (2.3) belongs to $D(A)$ in order to obtain a strong solution. Thus, we need to introduce the following definition of solution.

Definition 2.3. Let A be the generator of a resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$. We say that a function $u : \mathbb{R}_+ \to X$ is a *mild solution* to equation (1.2) if u satisfies (2.3) for all $t \geq 0$.

We recall that a closed linear operator $A : D(A) \subset X \to X$ is said to be *sectorial of angle* θ if there are constants $\omega \in \mathbb{R}$, $M > 0$ and $\theta \in (\pi/2, \pi)$ such that $\rho(A) \supset \Sigma_{\theta,\omega} := \{z \in \mathbb{C} : z \neq \omega : |\arg(z - \omega)| < \theta\}$ and

$$\|(z - A)^{-1}\| \leq \frac{M}{|z - \omega|} \quad \text{for all} \quad z \in \Sigma_{\theta,\omega}.$$

In this case, we write $A \in \text{Sect}(\theta, \omega, M)$. We notice that we may assume, without lost of generality, that $\omega = 0$. In fact, otherwise we can take the operator $A - \omega I$, which is also sectorial. In that case, we write $A \in \text{Sect}(\theta, M)$ and we denote the sector $\Sigma_{\theta,0}$ as Σ_θ. More details and further information on sectorial operators can be found in [12, 17].

For a given linear and closed operator A whose resolvent set contains the semi real axis $(-\infty, 0]$ and $0 \le \varepsilon \le 1$, X^ε will denote the domain of the fractional power A^ε, that is $X^\varepsilon := D(A^\varepsilon)$ endowed with the norm $\|x\|_\varepsilon := \|A^\varepsilon x\|$ (see for example the monograph [28]). Examples of such operators are sectorial operators with $\omega \ge 0$. It is a well known fact that if $0 < \varepsilon < 1$, and $x \in D(A)$, then there exists a constant $\kappa \equiv \kappa_\varepsilon > 0$ such that (see [28])

$$\|A^\varepsilon x\| \le \kappa \|Ax\|^\varepsilon \|x\|^{1-\varepsilon}. \tag{2.4}$$

2.2. **Continuous and discrete fractional calculus.** For $\alpha > 0$, let $m = \lceil \alpha \rceil$ be the smallest integer m greater than or equal to α. Let $f : \mathbb{R}_+ \to X$ be a C^m-differentiable function. The *Caputo fractional derivative of order* α is defined by

$$\partial_t^\alpha f(t) := \int_0^t g_{m-\alpha}(t-s) f^{(m)}(s) ds.$$

An easy computation shows that if $\alpha = m \in \mathbb{N}$, then $\partial_t^m f = \frac{d^m f}{dt^m}$. Moreover, if $1 < \alpha < 2$, then the Laplace transform of $\partial_t^\alpha f$ verifies $\widehat{\partial_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1} f(0) - \lambda^{\alpha-2} f'(0)$. More details on fractional calculus can be found in [29].

The set of non-negative integer numbers is denoted by $\mathbb{N}_0$ and the non-negative real numbers by $\mathbb{R}_0^+$. Define $p_n(t) := \frac{t^n}{n!} e^{-t}$, $n \in \mathbb{N}_0$. We notice that $p_n(t) \ge 0$ for all $t \ge 0$, $n \in \mathbb{N}_0$, and

$$\int_0^\infty p_n(t) dt = 1, \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Take $\tau > 0$ fixed and $n \in \mathbb{N}_0$, and define the corresponding approximation to the identity ρ_n^τ by

$$\rho_n^\tau(t) := \frac{1}{\tau} p_n\left(\frac{t}{\tau}\right) = e^{-\frac{t}{\tau}} \left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!}.$$

Given a bounded and locally integrable function $u : \mathbb{R}_0^+ \to X$, we define the vector-valued sequence $(u^n)_n$ by

$$u^n := \int_0^\infty \rho_n^\tau(t) u(t) dt, \quad n \in \mathbb{N}_0. \tag{2.5}$$

It is well known that for each $f \in L^1(\mathbb{R})$ we have $f * \rho_n^\tau \to f$ as $\tau \to 0$ in the L^1-norm. However, pointwise convergence cannot be assured a priori. One of our main results shows that u^n is an approximation of $u(t_n)$, where t_n is defined by $t_n = n\tau$.

Remark 1. A calculation shows that

$$u^n = \mathcal{P}(u_\tau)(n),$$

where $u_\tau(t) := u(\tau t)$ and $\mathcal{P}$ denotes the Poisson transform [24], which is defined for a vector valued function $f : \mathbb{R}_+ \to X$ by

$$\mathcal{P}(f)(n) := \int_0^\infty p_n(t) f(t) dt.$$

The space of all vector-valued functions $v : \mathbb{R}_0^+ \to X$ is denoted by $\mathcal{F}(\mathbb{R}_0^+; X)$. The *backward Euler operator* $\nabla_\tau : \mathcal{F}(\mathbb{R}_0^+; X) \to \mathcal{F}(\mathbb{R}_0^+; X)$ is defined by

$$\nabla_\tau v^n := \frac{v^n - v^{n-1}}{\tau}, \quad n \in \mathbb{N}.$$

For $m \geq 2$, $\nabla_\tau^m : \mathcal{F}(\mathbb{R}_0^+; X) \to \mathcal{F}(\mathbb{R}_0^+; X)$ is defined recursively as

$$(\nabla_\tau^m v)^n := \nabla_\tau^{m-1}(\nabla_\tau v)^n, \qquad n \geq m \tag{2.6}$$

where $\nabla_\tau^1 \equiv \nabla_\tau$ and ∇_τ^0 is the identity operator. For $n < m$, $(\nabla_\tau^m v)^n$ is defined as 0. We call to ∇_τ^m the *backward difference operator of order* m. An easy computation shows that if $v \in \mathcal{F}(\mathbb{R}_0^+; X)$ then

$$(\nabla_\tau^m v)^n = \frac{1}{\tau^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j v^{n-j}, \quad n \in \mathbb{N}.$$

As in [16, Chapter 1, Section 1.5] we define (by convention)

$$\sum_{j=0}^{-k} v^j = 0$$

for all $k \in \mathbb{N}$.

Now, we introduce the following sequence

$$k_\tau^\alpha(n) := \tau \int_0^\infty \rho_n^\tau(t) g_\alpha(t)dt, \quad n \in \mathbb{N}_0, \alpha > 0. \tag{2.7}$$

An easy computation shows that

$$k_\tau^\alpha(n) = \frac{\tau^\alpha \Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} = \tau \frac{\Gamma(\alpha+n)}{\Gamma(n+1)} g_\alpha(\tau), \quad n \in \mathbb{N}_0, \alpha > 0. \tag{2.8}$$

Remark 2. It is easy to check using (2.7) and a change of variables that

$$k_\tau^\alpha(n) = \tau^\alpha k^\alpha(n),$$

where $k^\alpha(n) \equiv k_1^\alpha(n)$ in the notation introduced in [24].

Definition 2.4. [31] Let $\alpha > 0$. The $\alpha^{\text{th}}-$fractional sum of $v \in \mathcal{F}(\mathbb{N}_0; X)$ is defined by

$$(\nabla_\tau^{-\alpha} v)^n := \sum_{j=0}^{n} k_\tau^\alpha(n-j) v^j, \quad n \in \mathbb{N}_0. \tag{2.9}$$

Remark 3. Using remark 2 and the definition of $\alpha^{\text{th}}-$fractional sum $\Delta^{-\alpha}$ introduced in [24] (which corresponds to (2.9) with $\tau = 1$), we observe that

$$(\nabla_\tau^{-\alpha} v)^n = (\Delta^{-\alpha} v)^n.$$

Definition 2.5. [31] Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. The *Caputo fractional backward difference operator of order* α, ${}_C\nabla^\alpha : \mathcal{F}(\mathbb{N}_0; X) \to \mathcal{F}(\mathbb{N}_0; X)$, is defined by

$$({}_C\nabla^\alpha v)^n := \nabla_\tau^{-(m-\alpha)} (\nabla_\tau^m v)^n, \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$.

In this definition, if $\alpha \in \mathbb{N}_0$, then the fractional backward difference operators ${}_C\nabla^\alpha$ is defined as the backward difference operator ∇_τ^α. Moreover, if $0 < \alpha < 1$, then ${}_C\nabla^{\alpha+1}v^n = {}_C\nabla^\alpha(\nabla^1 v)^n$. However, ${}_C\nabla^{\alpha+1}v^n \neq {}_C\nabla^1({}_C\nabla^\alpha v)^n$, (see [31, Proposition 2.6]).

Let $u : [0,\infty) \to X$ be a twice differentiable and bounded function. Since

$$\frac{d\rho_n^\tau(t)}{dt} = \frac{1}{\tau}\left(\rho_{n-1}^\tau(t) - \rho_n^\tau(t)\right),$$

for all $n \geq 1$, and u is a bounded function, then the integration by parts implies that

$$(u')^n = \frac{1}{\tau}(u^n - u^{n-1}) = \nabla_\tau u^n,$$

for all $n \geq 1$. On the other hand, since $\partial_t^{\alpha+1} f = \partial_t^\alpha \partial_t^1 f$ and ${}_C\nabla^{\alpha+1}v^n = {}_C\nabla^\alpha({}_C\nabla^1 v)^n$, for $0 < \alpha < 1$, we obtain the following result, which can be obtained directly from [31, Theorem 2.7] and relates the Caputo fractional derivative and the Caputo fractional backward difference operator.

Remark 4. In case $0 < \alpha < 1$ a calculation shows that for any $f : \mathbb{N}_0 \to X$ with $f(-1) = 0$ we have the identity

$$({}_C\nabla^\alpha f)^n = \frac{1}{\tau^\alpha}(\Delta^\alpha f)^{n-1},$$

and in case $1 < \alpha < 2$ we have

$$({}_C\nabla^\alpha f)^n = \frac{1}{\tau^\alpha}(\Delta^\alpha f)^{n-2},$$

under the assumption that $f(-1) = f(-2) = 0$.

Proposition 2. *Let $1 < \alpha < 2$. If $u : [0,\infty) \to X$ is a twice differentiable and bounded function, then*

$$\int_0^\infty \rho_n^\tau(t)\partial_t^\alpha u(t)dt = {}_C\nabla^\alpha u^n,$$

for all $n \in \mathbb{N}$.

Given a family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$, we define the sequence

$$S^n x := \int_0^\infty \rho_n^\tau(t)S(t)x dt, \quad n \in \mathbb{N}_0, x \in X. \tag{2.10}$$

On the other hand, if $c : \mathbb{R}_+ \to \mathbb{C}$ is a continuous and bounded function we define

$$c^n := \int_0^\infty \rho_n^\tau(t)c(t)dt, \quad n \in \mathbb{N}_0,$$

and the discrete convolution between c and S by

$$(c \star S)^n := \sum_{k=0}^n c^{n-k}S^k, \quad n \in \mathbb{N}_0.$$

Similarly, for a vector-valued function $f : \mathbb{R}_+ \to X$, we define the sequence f^n as

$$f^n := \int_0^\infty \rho_n^\tau(t)f(t)dt, \quad n \in \mathbb{N}_0. \tag{2.11}$$

We recall the following result, which corresponds to an extension of [24].

Theorem 2.6. [31] *Let $c : \mathbb{R}_+ \to \mathbb{C}$ be Laplace transformable such that $\hat{c}(1/\tau)$ exists, and let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists. Then, for all $x \in X$,*

$$\int_0^\infty \rho_n^\tau(t)(c * S)(t)x dt = \tau(c \star S)^n x, \quad n \in \mathbb{N}_0.$$

Similarly, we have the following results.

Proposition 3. [31] *Let $\alpha > 0$. Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists. Then, for all $x \in X$,*

$$\int_0^\infty \rho_n^\tau(t)(g_\alpha * S)(t)x dt = \sum_{j=0}^{n} k_\tau^\alpha(n-j)S^j x, \quad n \in \mathbb{N}_0.$$

Proposition 4. [31] *Let $f : \mathbb{R}_+ \to X$ be Laplace transformable such that $\hat{f}(1/\tau)$ exists, and let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists. Then,*

$$(S * f)^n x = \int_0^\infty \rho_n^\tau(t)(S * f)(t)x dt = \tau(S \star f)^n x = \tau \sum_{j=0}^{n} S^{n-j} f^j, \quad n \in \mathbb{N}_0.$$

With the above ingredients we can easily prove the following result that we will be useful later.

Lemma 2.7. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a family of exponentially bounded linear operators such that $\hat{S}(1/\tau)$ exists. If $f : \mathbb{R}_+ \to X$, $a : \mathbb{R}_+ \to \mathbb{C}$, and $\hat{a}(1/\tau)$ and $\hat{f}(1/\tau)$ exist, then*

$$\tau^2(a \star S \star f)^n = \int_0^\infty \rho_n^\tau(t)(a * S * f)(t)dt,$$

for all $n \in \mathbb{N}_0$, where $(a\star S\star f)^n := (a\star(S\star f))^n$. Moreover, $(a\star(S\star f))^n = ((a\star S)\star f)^n$ for all $n \in \mathbb{N}_0$.

Proof. Since $(a*S*f)(t) = (a*(S*f))(t)$ for all $t \geq 0$, the Theorem 2.6, Proposition 4 and the definition of discrete convolution imply that

$$\int_0^\infty \rho_n^\tau(t)(a * S * f)(t)dt = \tau(a \star (S * f))^n = \tau^2 \sum_{k=0}^{n} a^{n-k}(S \star f)^k = \tau^2(a \star (S \star f))^n,$$

for all $n \in \mathbb{N}_0$. □

3. **A fractional difference equation.** In this section we study the following fractional difference equation of order $\alpha \in (1, 2)$:

$$_C\nabla^\alpha u^n = Au^n + f^n, \tag{3.1}$$

for all $n \geq 2$ under the initial condition $u^0 = u_0$, and $u^1 = u_1$, where $u_0, u^1 \in X$.

We first assume that A is a sectorial operator, $u_0, u_1 \in D(A) \cap \ker(A)$ and $f^0 = f^1 = 0$.

As $u^0, u^1 \in \ker(A)$, by Definition (2.6), $(\nabla_\tau^2 u)^0 = (\nabla_\tau^2 u)^1 = 0$, and thus $_C\nabla^\alpha u^0 = \nabla^{-(2-\alpha)}(\nabla^2 u)^0 = 0$ and $_C\nabla^\alpha u^1 = \nabla^{-(2-\alpha)}(\nabla^2 u)^1 = 0$. Since $f^0 = f^1 = 0$, we have

$$_C\nabla^\alpha u^n = \nabla_\tau^{-(2-\alpha)}(\nabla_\tau u)^n = \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)(\nabla_\tau^2 u)^j$$

$$= \sum_{j=2}^{n-1} k_\tau^{2-\alpha}(n-j)(\nabla_\tau^2 u)^j + \tau^{-\alpha}(u^n - 2u^{n-1} + u^{n-2}),$$

for all $n \geq 2$. Therefore, for all $n \geq 2$, the scheme (3.1) is equivalent to

$$(\tau^{-\alpha} - A)u^n = 2\tau^{-\alpha}u^{n-1} - \tau^{-\alpha}u^{n-2} \sum_{j=2}^{n-1} k_\tau^{2-\alpha}(n-j)(\nabla_\tau^2 u)^j + f^n. \tag{3.2}$$

This is an implicit scheme, which means that to obtain u^n we need to find u^{n-1}, u^{n-2}, ..., u^0. If order to solve (3.2), we need to invert the operator $(\tau^{-\alpha} - A)$, which is possible, because A is a sectorial operator and therefore, we can take τ small enough (for instance $\max\{0, \omega\}\tau^\alpha < 1$) in order to obtain that $(\tau^{-\alpha} - A)$ is an invertible operator.

Using this fact, we can provide an explicit description of the solution in terms of certain sequences of bounded and linear operators. This is the content of the following result.

Theorem 3.1. *Let $\tau > 0$. Let A be the generator of a bounded $(\alpha, 1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ with $\|S_{\alpha,1}(t)\| \leq Me^{\omega t}$, where $\omega < \frac{1}{\tau}$. If $u_0, u_1 \in X$ and f is bounded, then fractional difference equation (3.1) admits a solution given by*

$$u^n = S_{\alpha,1}^n u_0 + \tau(g_1 \star S_{\alpha,1})^n u_1 + \tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f)^n \tag{3.3}$$

for all $n \geq 2$ and $u^0 = u_0$, $u^1 = u_1$.

Proof. Since $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is exponentially bounded, we obtain $S_{\alpha,1}^n x \in D(A)$ for all $n \in \mathbb{N}_0$ and $x \in X$, as in the proof of [24, Theorem 4.4] and [31, Theorem 2.8]. On the other hand, Proposition 1 implies that

$$S_{\alpha,1}(t)x = x + A\int_0^t g_\alpha(t-s)S_{\alpha,1}(s)xds,$$

for all $t \geq 0$ and $x \in X$. Multiplying this identity by $\rho_j^\tau(t)$ and then integrating over $[0, \infty)$ we obtain by Proposition 3 that

$$\begin{aligned} S_{\alpha,1}^j x = \int_0^\infty \rho_j^\tau(t) S_{\alpha,1}(t)xdt &= \int_0^\infty \rho_j^\tau(t)xdt + A\int_0^\infty \rho_j^\tau(t)(g_\alpha * S_{\alpha,1})(t)xdt \\ &= x + A\sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l x, \end{aligned} \tag{3.4}$$

$j \geq 0$. Now, by definition we have for all $n \geq 2$ that

$${}_C\nabla^\alpha(S_{\alpha,1}x)^n = \nabla_\tau^{-(2-\alpha)}\nabla_\tau^2(S_{\alpha,1}x)^n = \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)(\nabla_\tau^2 S_{\alpha,1}x)^j. \tag{3.5}$$

By (3.4), we obtain

$$\begin{aligned} (\nabla_\tau^2 S_{\alpha,1}x)^j &= \frac{1}{\tau^2}(S_{\alpha,1}^j x - 2S_{\alpha,1}^{j-1}x + S_{\alpha,1}^{j-2}x) \\ &= \frac{A}{\tau^2}\Big[\sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l x - 2\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)S_{\alpha,1}^l x \\ &\quad + \sum_{l=0}^{j-2} k_\tau^\alpha(j-2-l)S_{\alpha,1}^l x\Big], \end{aligned}$$

for all $j \geq 2$. Next, for $t \geq 0$, we define $R_\alpha(t) := (g_\alpha * S_{\alpha,1})(t)$. By Proposition 3 we obtain that

$$R_\alpha^j = \sum_{l=0}^{j} k_\tau^\alpha(j-l) S_{\alpha,1}^l,$$

for all $j \geq 0$.

Analogously, since $(g_{2-\alpha} * g_\alpha)(t) = g_2(t) = (g_1 * g_1)(t)$, we obtain by Proposition 3 that

$$\begin{aligned}\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j) \sum_{l=0}^{j} k_\tau^\alpha(j-l) S_{\alpha,1}^l x &= \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j) R_\alpha^j \\ &= \int_0^\infty \rho_n^\tau(t)(g_{2-\alpha} * R_\alpha)(t) x dt \\ &= \sum_{j=0}^{n} k_\tau^1(n-j)(g_1 * S_{\alpha,1})^j x,\end{aligned}$$

for all $n \geq 2$. By definition, we have $k_\tau^1(n) = \tau$ for all n, and, once again, by Proposition 3 we obtain

$$(g_1 * S_{\alpha,1})^j x = \int_0^\infty \rho_j^\tau(t)(g_1 * S_{\alpha,1})(t) x dt = \sum_{l=0}^{j} k_\tau^1(j-l) S_{\alpha,1}^l x = \tau \sum_{l=0}^{j} S_{\alpha,1}^l x,$$

which implies that

$$\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j) \sum_{l=0}^{j} k_\tau^\alpha(j-l) S_{\alpha,1}^l x = \tau^2 \sum_{j=0}^{n} \sum_{l=0}^{j} S_{\alpha,1}^l x. \tag{3.6}$$

Since $\sum_{j=0}^{-k} v^j = 0$ for all $k \in \mathbb{N}$, by using the function R_α, we have that

$$\begin{aligned}\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j) \sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l) S_{\alpha,1}^l x &= \sum_{j=1}^{n} k_\tau^{2-\alpha}(n-j) \sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l) S_{\alpha,1}^l x \\ &= \sum_{j=1}^{n} k_\tau^{2-\alpha}(n-j) R_\alpha^{j-1} x.\end{aligned}$$

And, as above we obtain

$$\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j) \sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l) S_{\alpha,1}^l x = \tau^2 \sum_{j=0}^{n-1} \sum_{l=0}^{j} S_{\alpha,1}^l x. \tag{3.7}$$

for all $n \geq 2$. Similarly, for all $n \geq 2$ we have

$$\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j) \sum_{l=0}^{j-2} k_\tau^\alpha(j-2-l) S_{\alpha,1}^l x = \tau^2 \sum_{j=0}^{n-2} \sum_{l=0}^{j} S_{\alpha,1}^l x. \tag{3.8}$$

By (3.5)–(3.8) we obtain

$$\begin{aligned}{}_C\nabla^\alpha (S_{\alpha,1} x)^n &= \frac{A}{\tau^2} \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j) \Bigg[\sum_{l=0}^{j} k_\tau^\alpha(j-l) S_{\alpha,1}^l x \\ &\quad -2 \sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l) S_{\alpha,1}^l x + \sum_{l=0}^{j-2} k_\tau^\alpha(j-2-l) S_{\alpha,1}^l x \Bigg]\end{aligned}$$

$$= A\left[\sum_{j=0}^{n}\sum_{l=0}^{j} S_\alpha^l x - 2\sum_{j=0}^{n-1}\sum_{l=0}^{j} S_{\alpha,1}^l x + \sum_{j=0}^{n-2}\sum_{l=0}^{j} S_{\alpha,1}^l x\right]$$
$$= AS_\alpha^n x,$$

for all $n \geq 2$ and $x \in X$. Therefore

$$_C\nabla^\alpha S_{\alpha,1}^n u_0 = AS_{\alpha,1}^n u_0. \tag{3.9}$$

On the other hand, by definition we have

$$_C\nabla^\alpha(\tau(g_1\star S_{\alpha,1})^n)x = \nabla_\tau^{-(2-\alpha)}\nabla_\tau^2(\tau(g_1\star S_{\alpha,1}))^n x = \tau\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\nabla_\tau^2(g_1\star S_{\alpha,1})^j x.$$

Since $\nabla_\tau^2(g_1\star S_{\alpha,1})^j = \frac{1}{\tau^2}\left[(g_1\star S_{\alpha,1})^j - 2(g_1\star S_{\alpha,1})^{j-1} + (g_1\star S_{\alpha,1})^{j-2}\right]$ for all $j \geq 2$, and $(g_1\star S_{\alpha,1})^j x = \frac{1}{\tau}(g_1 * S_{\alpha,1})^j x$, (by Theorem 2.6), we have

$$_C\nabla^\alpha(\tau(g_1\star S_{\alpha,1})^n)x = \frac{1}{\tau^2}\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\big[(g_1 * S_{\alpha,1})^j x - 2(g_1 * S_{\alpha,1})^{j-1}x + (g_1 * S_{\alpha,1})^{j-2}x\big]. \tag{3.10}$$

Moreover, by Proposition 1 we have $(g_1 * S_{\alpha,1})(t)x = (g_1 * g_1)(t)x + A(g_1 * g_\alpha * S_{\alpha,1})(t)x$, for all $t \geq 0$ and $x \in X$. Multiplying this equation by $\rho_j^\tau(t)$ and integrating over $[0,\infty)$ we obtain by Proposition 3 that

$$(g_1 * S_{\alpha,1})^j x = \int_0^\infty \rho_j^\tau(t)(g_1 * S_{\alpha,1})(t)x\,dt = \tau j x + A\sum_{l=0}^{j} k_\tau^{\alpha+1}(j-l)S_{\alpha,1}^l x,$$

for all $j \geq 0$. Hence,

$$(g_1 * S_{\alpha,1})^j x - 2(g_1 * S_{\alpha,1})^{j-1}x + (g_1 * S_{\alpha,1})^{j-2}x$$
$$= A\left[\sum_{l=0}^{j} k_\tau^{\alpha+1}(j-l)S_{\alpha,1}^l x - 2\sum_{l=0}^{j-1} k_\tau^{\alpha+1}(j-1-l)S_{\alpha,1}^l x + \sum_{l=0}^{j-2} k_\tau^{\alpha+1}(j-2-l)S_{\alpha,1}^l x\right],$$

for all $j \geq 2$. If $Q_\alpha(t) := (g_{\alpha+1} * S_{\alpha,1})(t)$ we obtain by Proposition 3 that if $j \geq 0$, then $Q_\alpha^j x = \sum_{l=0}^{j} k_\tau^{\alpha+1}(j-l)S_{\alpha,1}^l x$. Since $(g_{2-\alpha} * Q_\alpha)(t) = (g_1 * g_1 * g_1 * S_{\alpha,1})(t)$, we obtain

$$\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^{\alpha+1}(j-l)S_{\alpha,1}^l x = \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)Q_\alpha^j$$
$$= \int_0^\infty \rho_n^\tau(t)(g_{2-\alpha} * Q_\alpha)(t)x\,dt$$
$$= \tau^2\sum_{j=0}^{n}\sum_{l=0}^{j}(g_1 * S_{\alpha,1})^l x.$$

Since $\sum_{j=0}^{-k} v^j = 0$ for all $k \in \mathbb{N}$, we obtain as above that

$$\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^{\alpha+1}(j-1-l)S_{\alpha,1}^l x = \tau^2\sum_{j=0}^{n-1}\sum_{l=0}^{j}(g_1 * S_{\alpha,1})^l x$$

and

$$\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j-2} k_\tau^{\alpha+1}(j-2-l)S_{\alpha,1}^l x = \tau^2\sum_{j=0}^{n-2}\sum_{l=0}^{j}(g_1 * S_{\alpha,1})^l x.$$

Therefore,

$$\begin{aligned}
{}_C\nabla^\alpha(\tau(g_1 \star S_{\alpha,1})^n)x =& \frac{A}{\tau^2}\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\Bigg[\sum_{l=0}^{j} k_\tau^{\alpha+1}(j-l)S_{\alpha,1}^l x \\
&- 2\sum_{l=0}^{j-1} k_\tau^{\alpha+1}(j-1-l)S_{\alpha,1}^l x + \sum_{l=0}^{j-2} k_\tau^{\alpha+1}(j-2-l)S_{\alpha,1}^l x\Bigg] \\
=& A\Bigg[\sum_{j=0}^{n}\sum_{l=0}^{j}(g_1 * S_{\alpha,1})^l x - 2\sum_{j=0}^{n-1}\sum_{l=0}^{j}(g_1 * S_{\alpha,1})^l x \\
&+ \sum_{j=0}^{n-2}\sum_{l=0}^{j}(g_1 * S_{\alpha,1})^l x\Bigg] \\
=& A(g_1 * S_{\alpha,1})^n x,
\end{aligned}$$

for all $n \geq 2$ and $x \in X$. Since $(g_1 * S_{\alpha,1})^n x = \tau(g_1 \star S_{\alpha,1})^n x$ we obtain

$$ {}_C\nabla^\alpha(\tau(g_1 \star S_{\alpha,1})^n)u_1 = A(\tau(g_1 \star S_{\alpha,1})^n u_1), \quad n \geq 2. \tag{3.11}$$

Finally, by definition we have

$$\begin{aligned}
{}_C\nabla^\alpha(\tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f)^n) &= \nabla_\tau^{-(2-\alpha)}\nabla_\tau^2(\tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f))^n \\
&= \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\nabla_\tau^2(\tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f)^j),
\end{aligned}$$

for all $n \geq 2$. Since

$$\begin{aligned}
\nabla_\tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f)^j =& \frac{1}{\tau^2}\big[(g_{\alpha-1} \star S_{\alpha,1} \star f)^j - 2(g_{\alpha-1} \star S_{\alpha,1} \star f)^{j-1} \\
&+ (g_{\alpha-1} \star S_{\alpha,1} \star f)^{j-2}\big],
\end{aligned}$$

for all $j \geq 2$ and by Lemma 2.7 we have that for each $j \geq 0$, $(g_{\alpha-1} \star S_{\alpha,1} \star f)^j = \frac{1}{\tau^2}(g_{\alpha-1} * S_{\alpha,1} * f)^j$, we have

$$\begin{aligned}
\nabla_\tau^2(\tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f)^j) =& \frac{1}{\tau^2}\big[(g_{\alpha-1} * S_{\alpha,1} * f)^j - 2(g_{\alpha-1} * S_{\alpha,1} * f)^{j-1} \\
&+ (g_{\alpha-1} * S_{\alpha,1} * f)^{j-2}\big],
\end{aligned}$$

for all $j \geq 2$. By Proposition 1 we get

$$(g_{\alpha-1} * S_{\alpha,1} * f)(t) = (g_\alpha * f)(t) + A(g_{\alpha-1} * g_\alpha * S_{\alpha,1} * f)(t),$$

and multiplying this equation by $\rho_j^\tau(t)$ and integrating over $[0,\infty)$ we obtain by 3 that

$$(g_{\alpha-1} * S_{\alpha,1} * f)^j = \sum_{l=0}^{j} k_\tau^\alpha(j-l)f^l + A\sum_{l=0}^{j} k_\tau^\alpha(j-l)(g_{\alpha-1} * S_{\alpha,1} * f)^l, \quad j \geq 0.$$

Hence

$$ {}_C\nabla^\alpha(\tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f)^n) = \sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\nabla_\tau^2(\tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f)^j)$$

$$
\begin{aligned}
=&\frac{1}{\tau^2}\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\Bigg[\sum_{l=0}^{j} k_\tau^{\alpha}(j-l)f^l \\
&-2\sum_{l=0}^{j-1} k_\tau^{\alpha}(j-1-l)f^l+\sum_{l=0}^{j-2} k_\tau^{\alpha}(j-2-l)f^l\Bigg] \\
&+\frac{A}{\tau^2}\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\Bigg[\sum_{l=0}^{j} k_\tau^{\alpha}(j-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l \\
&-2\sum_{l=0}^{j-1} k_\tau^{\alpha}(j-1-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l \\
&+\sum_{l=0}^{j-2} k_\tau^{\alpha}(j-2-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l\Bigg],
\end{aligned}
$$

for all $n \geq 2$.

On the other hand, we notice that defining $h(t) := (g_\alpha * f)(t)$, we have $h^j = \int_0^\infty \rho_j^\tau(t)(g_\alpha * f)(t)dt = \sum_{l=0}^{j} k_\tau^\alpha(j-l)f^l$, which implies (by similar computations as above) that

$$
\begin{aligned}
\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^{\alpha}(j-l)f^l &= \tau^2\sum_{j=0}^{n}\sum_{l=0}^{j} f^l, \\
\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^{\alpha}(j-1-l)f^l &= \tau^2\sum_{j=0}^{n-1}\sum_{l=0}^{j} f^l,
\end{aligned}
$$

and

$$
\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j-2} k_\tau^{\alpha}(j-2-l)f^l = \tau^2\sum_{j=0}^{n-2}\sum_{l=0}^{j} f^l.
$$

Thus,

$$
\frac{1}{\tau^2}\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\Bigg[\sum_{l=0}^{j} k_\tau^{\alpha}(j-l)f^l-2\sum_{l=0}^{j-1} k_\tau^{\alpha}(j-1-l)f^l+\sum_{l=0}^{j-2} k_\tau^{\alpha}(j-2-l)f^l\Bigg] = f^n,
$$

for all $n \geq 2$. Similarly, if $T_\alpha(t) := (g_\alpha * (g_{\alpha-1} * S_{\alpha,1} * f))(t)$ then $T_\alpha^j = \sum_{l=0}^{j} k_\tau^\alpha(j-l)(g_{\alpha-1} * S_{\alpha,1} * f)^l$, and therefore

$$
\begin{aligned}
\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^{\alpha}(j-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l &= \tau^2\sum_{j=0}^{n}\sum_{l=0}^{j}(g_{\alpha-1}*S_{\alpha,1}*f)^l, \\
\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^{\alpha}(j-1-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l &= \tau^2\sum_{j=0}^{n-1}\sum_{l=0}^{j}(g_{\alpha-1}*S_{\alpha,1}*f)^l,
\end{aligned}
$$

and

$$
\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\sum_{l=0}^{j-2} k_\tau^{\alpha}(j-2-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l = \tau^2\sum_{j=0}^{n-2}\sum_{l=0}^{j}(g_{\alpha-1}*S_{\alpha,1}*f)^l.
$$

Thus,

$$\frac{A}{\tau^2}\sum_{j=0}^{n} k_\tau^{2-\alpha}(n-j)\Bigg[\sum_{l=0}^{j} k_\tau^{\alpha}(j-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l - 2\sum_{l=0}^{j-1} k_\tau^{\alpha}(j-1-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l + \sum_{l=0}^{j-2} k_\tau^{\alpha}(j-2-l)(g_{\alpha-1}*S_{\alpha,1}*f)^l\Bigg]$$
$$= A(g_{\alpha-1}*S_{\alpha,1}*f)^n.$$

Since $(g_{\alpha-1}*S_{\alpha,1}*f)^n = \tau^2(g_{\alpha-1}\star S_{\alpha,1}\star f)^n$ we conclude that

$$_C\nabla^\alpha(\tau^2(g_{\alpha-1}\star S_{\alpha,1}\star f)^n) = f^n + A(\tau^2(g_{\alpha-1}\star S_{\alpha,1}\star f)^n), \tag{3.12}$$

for all $n \geq 2$. We conclude that if we define the sequence $(u^n)_{n\in\mathbb{N}_0}$ by $u^n := S^n_{\alpha,1}u_0 + \tau(g_1\star S_{\alpha,1})^n u_1 + \tau^2(g_{\alpha-1}\star S_{\alpha,1}\star f)^n)$, for $n\geq 2$ and $u^0 := u_0$, $u^1 := u_1$, then by (3.9), (3.11) and (3.12) we have that

$$\begin{aligned} _C\nabla^\alpha(u^n) &= {}_C\nabla^\alpha\left(S^n_{\alpha,1}u_0 + \tau(g_1\star S_{\alpha,1})^n u_1 + \tau^2(g_{\alpha-1}\star S_{\alpha,1}\star f)^n)\right) \\ &= Au^n + f^n, \end{aligned}$$

for all $n \geq 2$, that is, $(u^n)_{n\in\mathbb{N}_0}$ solves the equation

$$_C\nabla^\alpha u^n = Au^n + f^n, \quad n\geq 2,$$

under the initial conditions $u^0 = u_0$, and $u^1 = u_1$. □

4. **Error estimates.** In this section we compare the mild solution u to the Caputo fractional Cauchy problem (1.2) at t_n and the solution u^n of one solution of the fractional difference equation (3.1). More concretely, we study the norm difference $\|u(t_n)-u^n\|$, where u is the mild solution to Problem (1.2) and u^n solves the discrete difference equation (3.1).

For a closed operator $A\in \mathrm{Sec}(\theta, M)$, we will consider the following path Γ_t : For $\frac{\pi}{2} < \theta < \pi$, we take ϕ such that $\frac{1}{2}\phi < \frac{\pi}{2}\alpha < \phi < \theta$. Next, we define Γ_t (see Figure 1) as the union $\Gamma_t^1 \cup \Gamma_t^2$, where

$$\Gamma_t^1 := \left\{\frac{1}{t}e^{i\psi/\alpha} : -\phi < \psi < \phi\right\} \quad \text{and} \quad \Gamma_t^2 := \left\{re^{\pm i\phi/\alpha} : \frac{1}{t} \leq r\right\}.$$

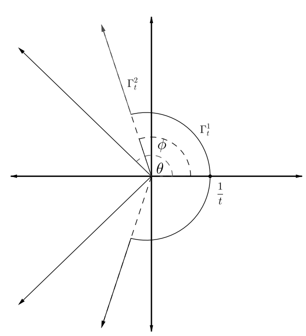

FIGURE 1. Plot of path Γ_t.

The next result will be useful to prove the main theorem in this section. A similar result can be found in [11, 31]. We give its proof for the sake of completeness.

Lemma 4.1. *Let $A \in \mathrm{Sec}(\theta, M)$ and Γ be the complex path defined above. If $\mu \geq 0$, then there exist positive constants C_α, depending only on α such that*

$$\int_{\Gamma_t} \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq C_\alpha t^{\mu-1},$$

for all $t > 0$, where

$$C_\alpha := \left(2\phi \int_{-\phi}^{\phi} e^{\cos(\psi/\alpha)} d\psi + \frac{2}{-\cos(\phi/\alpha)} \right).$$

Proof. On Γ_t^1 we have

$$\int_{\Gamma_t^1} \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq 2\phi \int_{-\phi}^{\phi} \frac{e^{(t\frac{\cos(\psi/\alpha)}{t})}}{\left| \frac{e^{(i\mu\psi/\alpha)}}{t^\mu} \right|} \frac{1}{t} d\psi = 2\phi \int_{-\phi}^{\phi} e^{\cos(\psi/\alpha)} d\psi t^{\mu-1}.$$

On the other hand, since $\frac{1}{2}\phi < \frac{\pi}{2}\alpha < \phi$ we obtain $\frac{\pi}{2} < \frac{\phi}{\alpha} < \pi$, and thus $\cos(\phi/\alpha) < 0$, which implies that on Γ_t^2 we have

$$\int_{\Gamma_t^2} \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq 2 \int_{\frac{1}{t}}^{\infty} \frac{e^{rt\cos(\phi/\alpha)}}{r^\mu} dr \leq 2t^\mu \int_0^\infty e^{rt\cos(\phi/\alpha)} dr = 2\frac{t^{\mu-1}}{-\cos(\phi/\alpha)}.$$

We conclude that

$$\int_{\Gamma_t} \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq \left(2\phi \int_{-\phi}^{\phi} e^{\cos(\psi/\alpha)} d\psi + \frac{2}{-\cos(\phi/\alpha)} \right) t^{\mu-1}. \qquad \square$$

Take $A \in \mathrm{Sec}(\theta, M)$. If $z = \frac{1}{t} e^{i\phi/\alpha}$, then $z^\alpha = \frac{1}{t^\alpha} e^{i\phi}$ and $\arg(z^\alpha) = \phi < \theta$. This implies that $z^\alpha \in \rho(A)$. If we take the complex path $\Gamma \equiv \Gamma_t$ defined in Lemma 4.1, then, by the inversion formula of the Laplace transform, we can write

$$S_{\alpha,1}(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} z^{\alpha-1} (z^\alpha - A)^{-1} dz, \quad t > 0. \tag{4.1}$$

Let $0 < \varepsilon < 1$ be given. The space of all continuous function $f : [0, \infty) \to D(A^\varepsilon)$ endowed with the norm $\|f\|_\varepsilon := \sup_{t\geq 0} \|f(t)\|_\varepsilon = \sup_{t\geq 0} \|A^\varepsilon f(t)\|$ will be denoted by $C([0,\infty), D(A^\varepsilon))$.

Our main result is the following theorem.

Theorem 4.2. *Let $1 < \alpha < 2$ and $A \in \mathrm{Sect}(\theta, M)$ which generates an $(\alpha, 1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$. Let $0 < \varepsilon < 1$ such that $0 < \alpha\varepsilon < 1$ and $\alpha(\varepsilon + 1) < 2$. Suppose that $f \in C([0,\infty), D(A^\varepsilon))$. Let Γ be the complex path defined above. If $u_0, u_1 \in D(A^\varepsilon)$, then for each $T > 0$ there exists a constant $C = C(T) > 0$ (independent of the solution, the data and the step size) such that, for $0 < t_n \leq T$, there holds*

$$\|u^n - u(t_n)\| \leq C\tau t_n^{\alpha\varepsilon-1} \left(\|u_0\|_\varepsilon + \|u_1\|_\varepsilon + \|f\|_\varepsilon \right). \tag{4.2}$$

Proof. Since A generates a resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$, the solution to (1.2) and (3.1) are given, respectively, by

$$u(t) = S_{\alpha,1}(t)u_0 + (g_1 * S_{\alpha,1})(t)u_1 + (g_{\alpha-1} * S_{\alpha,1} * f)(t),$$

and

$$u^n = S_{\alpha,1}^n u_0 + \tau(g_1 \star S_{\alpha,1})^n u_1 + \tau^2 (g_{\alpha-1} \star S_{\alpha,1} \star f)^n.$$

Now, we fix $n \in \mathbb{N}$ such that $0 < t_n \leq T$, where $t_n := \tau n$. Then, we have

$$\|u^n - u(t_n)\| \leq \|(S_{\alpha,1}(t_n) - S_{\alpha,1}^n)u_0\| + \|((g_1 * S_{\alpha,1})(t_n) - \tau(g_1 \star S_{\alpha,1}^n))u_1\|$$

$$+\|(g_{\alpha-1} * S_{\alpha,1} * f)(t_n) - \tau^2(g_{\alpha-1} \star S_{\alpha,1} \star f)^n\| := I_1 + I_2 + I_3.$$

Now, we estimate I_1, I_2 and I_3. Since $\int_0^\infty \rho_n^\tau(t)dt = 1$, we can write

$$(S_{\alpha,1}(t_n) - S_{\alpha,1}^n)u_0 = \int_0^\infty \rho_n^\tau(t)((S_{\alpha,1}(t_n) - S_{\alpha,1}(t))u_0 dt,$$

and therefore

$$I_1 \leq \int_0^\infty \rho_n^\tau(t)\|(S_{\alpha,1}(t) - S_{\alpha,1}(t_n))u_0\|dt.$$

Now, if $\Gamma = \Gamma_{t_n}$, by (4.1) we can write

$$(S_{\alpha,1}(t) - S_{\alpha,1}(t_n))u_0 = \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt} - e^{zt_n})}{z} z^\alpha (z^\alpha - A)^{-1} u_0 dz.$$

On the other hand, we have $A(z^\alpha - A)^{-1} = A^{1-\varepsilon}(z^\alpha - A)^{-1}A^\varepsilon$ and

$$z^\alpha(z^\alpha - A)^{-1} = A(z^\alpha - A)^{-1} + I = A^{1-\varepsilon}(z^\alpha - A)^{-1}A^\varepsilon + I. \tag{4.3}$$

Thus,

$$\begin{aligned}(S_{\alpha,1}(t) - S_{\alpha,1}(t_n))u_0 =& \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt} - e^{zt_n})}{z} u_0 dz \\ &+ \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt} - e^{zt_n})}{z} A^{1-\varepsilon}(z^\alpha - A)^{-1}A^\varepsilon u_0 dz.\end{aligned}$$

The function $h(z) := \frac{(e^{zt} - e^{zt_n})}{z}$ has a unique removable singularity at $z = 0$. Since $t \geq t_n$, $h(z)$ can be analytically extended to the region enclosed by the path $\Gamma^R := \Gamma_{t_n}^R$ where Γ^R is the path given in Figure 2, and therefore

$$\frac{1}{2\pi i}\int_{\Gamma^R} \frac{(e^{zt} - e^{zt_n})}{z} u^0 dz = 0.$$

Since

$$\frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt} - e^{zt_n})}{z} u^0 dz = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{\Gamma^R} \frac{(e^{zt} - e^{zt_n})}{z} u^0 dz,$$

we get

$$\frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt} - e^{zt_n})}{z} u^0 dz = 0.$$

On the other hand, since A is a sectorial operator, we get by (2.4)

$$\|A^{1-\varepsilon}(z^\alpha - A)^{-1}A^\varepsilon x\| \leq \kappa(M+1)\frac{\|A^\varepsilon x\|}{|z^\alpha|^\varepsilon}, \tag{4.4}$$

for all $x \in D(A^\varepsilon)$, which implies that

$$\|(S_{\alpha,1}(t) - S_{\alpha,1}(t_n))u_0\| \leq \frac{\kappa(M+1)}{2\pi}\int_\Gamma \frac{|e^{zt} - e^{zt_n}|}{|z|}\frac{1}{|z|^{\alpha\varepsilon}}|dz|\|A^\varepsilon u_0\|.$$

The mean value theorem for complex-valued functions ensures the existence of t_0, t_1 with $0 < t_n < t_0 < t_1 < t$ satisfying

$$\frac{|e^{zt} - e^{zt_n}|}{|z|} \leq (t - t_n)\left(|e^{t_0 z}| + |e^{t_1 z}|\right). \tag{4.5}$$

By Lemma 4.1 and (4.5) we conclude that

$$\|(S_{\alpha,1}(t) - S_{\alpha,1}(t_n))u_0\| \leq \frac{\kappa(M+1)}{2\pi}(t - t_n)C_\alpha(t_0^{\alpha\varepsilon-1} + t_1^{\alpha\varepsilon-1})\|A^\varepsilon u_0\|.$$

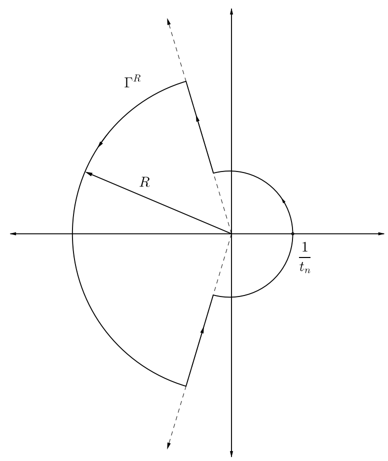

FIGURE 2. Plot of path Γ^R.

Since $0 < t_n < t_0 < t_1$ and $0 < \alpha\varepsilon < 1$ we have $t_1^{\alpha\varepsilon-1} < t_0^{\alpha\varepsilon-1} < t_n^{\alpha\varepsilon-1}$, which implies that

$$\|(S_{\alpha,1}(t) - S_{\alpha,1}(t_n))u_0\| \leq \frac{\kappa(M+1)}{\pi} C_\alpha (t-t_n) t_n^{\alpha\varepsilon-1} \|A^\varepsilon u_0\|.$$

Now, an easy computation shows that

$$\int_0^\infty \rho_n^\tau(t)(t-t_n)dt = \tau, \tag{4.6}$$

for all $n \in \mathbb{N}$, and thus

$$I_1 \leq \int_0^\infty \rho_n^\tau(t) \|(S_{\alpha,1}(t) - S_{\alpha,1}(t_n))u_0\| dt \leq D_1 \tau t_n^{\alpha\varepsilon-1} \|A^\varepsilon u_0\|,$$

for all $n \in \mathbb{N}$, where $D_1 := \frac{\kappa(M+1)}{\pi} C_\alpha$. We conclude that

$$I_1 \leq D_1 \tau t_n^{\alpha\varepsilon-1} \|A^\varepsilon u_0\|. \tag{4.7}$$

Now, to estimate I_2 we notice that by Theorem 2.6 we can write

$$\|(g_1 * S_{\alpha,1})(t_n) - \tau(g_1 \star S_{\alpha,1})^n\| = \left\| \int_0^\infty \rho_n^\tau(t)[(g_1 * S_{\alpha,1})(t_n) - (g_1 * S_{\alpha,1})(t)]dt \right\|.$$

Since $(\widehat{g_1 * S_{\alpha,1}})(z) = \frac{1}{z^2} z^\alpha (z^\alpha - A)^{-1}$, the inversion theorem for the Laplace transform and (4.3) allow us to write

$$\begin{aligned}(g_1 * S_{\alpha,1})(t)x - (g_1 * S_{\alpha,1})(t_n)u_1 =& \frac{1}{2\pi i}\int_\Gamma (e^{zt} - e^{zt_n})(\widehat{g_1 * S_{\alpha,1}})(z)u_1 dz \\ =& \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt} - e^{zt_n}}{z^2} u_1 dz \\ &+ \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt} - e^{zt_n}}{z^2} A^{1-\varepsilon}(z^\alpha - A)^{-1} A^\varepsilon u_1 dz.\end{aligned}$$

Since $\|u_1\| \leq \|A^\varepsilon u_1\|$, by (4.4)-(4.5) and Lemma 4.1, we have

$$\|(g_1 * S_{\alpha,1})(t_n)u_1 - (g_1 * S_{\alpha,1})(t)u_1\|$$

$$\leq \frac{1}{2\pi}\int_\Gamma \frac{|e^{zt}-e^{zt_n}|}{|z|^2}\|u_1\||dz| + \frac{1}{2\pi}\int_\Gamma \frac{|e^{zt_n}-e^{zt}|}{|z|}\frac{1}{|z|}\|A^{1-\varepsilon}(z^\alpha - A)^{-1}A^\varepsilon u_1\||dz|$$
$$\leq \frac{(t-t_n)}{\pi}C_\alpha\|A^\varepsilon u_1\| + \frac{\kappa(M+1)}{2\pi}(t-t_n)C_\alpha(t_0^{\alpha\varepsilon}+t_1^{\alpha\varepsilon})\|A^\varepsilon u_1\|.$$

Since $\alpha\varepsilon > 0$ and $t_0 < t_1 < t$ we have

$$\|(g_1 * S_{\alpha,1})(t_n)u_1 - (g_1 * S_{\alpha,1})(t)u_1\| \leq \frac{(t-t_n)}{\pi}C_\alpha\|A^\varepsilon u_1\| + \frac{\kappa(M+1)C_\alpha}{\pi}(t-t_n)t^{\alpha\varepsilon}\|A^\varepsilon u_1\|.$$

On the other hand, an easy computation shows that if $\gamma > 0$, then

$$\int_0^\infty \rho_n^\tau(t)t^\gamma dt = \frac{\tau^\gamma}{n!}\Gamma(n+\gamma+1), \tag{4.8}$$

for all $n \in \mathbb{N}$, and therefore

$$\int_0^\infty \rho_n^\tau(t)(t-t_n)t^\gamma dt = \frac{\tau^{\gamma+1}}{n!}\Gamma(n+\gamma+2) - \frac{\tau^\gamma}{n!}\Gamma(n+\gamma+1)t_n =: c_n^\gamma. \tag{4.9}$$

We notice that c_n^γ can be written as

$$\begin{aligned} c_n^\gamma &= \frac{t_n^{\gamma+1}\Gamma(n+\gamma+2)}{n^{\gamma+1}\cdot n!} - \frac{t_n^{\gamma+1}\Gamma(n+\gamma+1)}{n^\gamma\cdot n!} \\ &= \frac{t_n^{\gamma+1}}{n^\gamma\cdot n!}\left(\frac{\Gamma(n+\gamma+2)}{n} - \Gamma(n+\gamma+1)\right) \\ &= \tau(\gamma+1)(n+\gamma)t_n^\gamma\frac{\Gamma(n+\gamma)}{\Gamma(n+1)}\frac{1}{n^\gamma}. \end{aligned}$$

Since $\frac{\Gamma(n+\gamma)}{\Gamma(n+1)} < n^{\gamma-1}$ for all $0<\gamma<1$ and $n \in \mathbb{N}$ (see for instance [14]), we have

$$c_n^\gamma < \tau(\gamma+1)(n+\gamma)t_n^\gamma n^{\gamma-1}\frac{1}{n^\gamma} = \tau(\gamma+1)t_n^\gamma\left(1+\frac{\gamma}{n}\right) \leq 2\tau(\gamma+1)t_n^\gamma, \tag{4.10}$$

for all $n\in\mathbb{N}$. Hence, if $\gamma=\alpha\varepsilon$, then the hypothesis implies that $c_n^{\alpha\varepsilon} \leq 2\tau(\alpha\varepsilon+1)t_n^{\alpha\varepsilon} = 2\tau t_n(\alpha\varepsilon+1)t_n^{\alpha\varepsilon-1} \leq 2\tau(\alpha\varepsilon+1)Tt_n^{\alpha\varepsilon-1}$. Therefore, by (4.6), (4.9) and (4.10) we obtain

$$\begin{aligned} I_2 &\leq \int_0^\infty \rho_n^\tau(t)\|(g_1 * S_{\alpha,1})(t_n)u_1 - (g_1 * S_{\alpha,1})(t)u_1\|dt \\ &\leq \frac{C_\alpha}{\pi}\int_0^\infty \rho_n^\tau(t)(t-t_n)dt\|A^\varepsilon u_1\| + \frac{\kappa(M+1)C_\alpha}{\pi}\int_0^\infty \rho_n^\tau(t)(t-t_n)t^{\beta\varepsilon}dt\|A^\varepsilon u_1\| \\ &\leq \left(\frac{C_\alpha T^{1-\alpha\varepsilon}}{\pi} + \frac{2\kappa(M+1)C_\alpha(\alpha\varepsilon+1)T}{\pi}\right)\tau t_n^{\alpha\varepsilon-1}\|A^\varepsilon u_1\|. \end{aligned}$$

We conclude that

$$I_2 \leq D_2\tau t_n^{\alpha\varepsilon-1}\|A^\varepsilon u_1\|, \tag{4.11}$$

where

$$D_2 := \frac{C_\alpha T^{1-\alpha\varepsilon}}{\pi} + \frac{2\kappa(M+1)C_\alpha(\alpha\varepsilon+1)T}{\pi}.$$

Finally, we estimate the integral I_3. By Lemma 2.7 we can write

$$I_3 = \left\|\int_0^\infty \rho_n^\tau(t)[(g_{\alpha-1} * S_{\alpha,1} * f)(t) - (g_{\alpha-1} * S_{\alpha,1} * f)(t_n)]dt\right\|.$$

Moreover, we have

$$\begin{aligned}&(g_{\alpha-1}*S_{\alpha,1}*f)(t)-(g_{\alpha-1}*S_{\alpha,1}*f)(t_n)\\&=\int_0^{t_n}[(g_{\alpha-1}*S_{\alpha,1})(t-r)-(g_{\alpha-1}*S_{\alpha,1})(t_n-r)]f(r)dr\\&\quad+\int_{t_n}^{t}(g_{\alpha-1}*S_{\alpha,1})(t-r)f(r)dr\\&:=J_1+J_2.\end{aligned}$$

To estimate J_1 we notice that $(\widehat{g_{\alpha-1}*S}_{\alpha,1})(z)=(z^\alpha-A)^{-1}$, which implies by (4.3) that

$$\begin{aligned}&(g_{\alpha-1}*S_{\alpha,1})(t)x-(g_{\alpha-1}*S_{\alpha,1})(s)x\\&=\frac{1}{2\pi i}\int_\Gamma(e^{zt}-e^{zs})(\widehat{g_{\alpha-1}*S}_{\alpha,1})(z)xdz\\&=\frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zs})}{z^\alpha}A^{1-\varepsilon}(z^\alpha-A)^{-1}A^\varepsilon xdz+\frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zs})}{z^\alpha}dz,\end{aligned}\tag{4.12}$$

for all $x\in D(A^\varepsilon)$ and $t>s>0$. Since $q(z):=\frac{(e^{zt}-e^{zs})}{z^\alpha}$ has a unique removable singularity at $z=0$, we can prove that the second integral in this last equality is equal to zero (following the same method used to prove that $\int_\Gamma h(z)u_0dz=0$). By (4.4) we obtain

$$\|(g_{\alpha-1}*S_{\alpha,1})(t)x-(g_{\alpha-1}*S_{\alpha,1})(s)x\|\le\frac{\kappa(M+1)}{2\pi}\int_\Gamma\frac{|e^{zt}-e^{zs}|}{|z|^{\alpha(\varepsilon+1)}}\|A^\varepsilon x\||dz|.$$

Once again,applying the mean value theorem for complex-valued functions, we obtain the existence of t_0',t_1' with $0<s<t_0'<t_1'<t$ such that

$$\frac{|e^{zt}-e^{zs}|}{|z|}\le(t-s)\left(|e^{t_0'z}|+|e^{t_1'z}|\right).$$

Hence, by Lemma 4.1 we get

$$\begin{aligned}&\|(g_{\alpha-1}*S_{\alpha,1})(t)x-(g_{\alpha-1}*S_{\alpha,1})(s)x\|\\&\le\frac{\kappa(M+1)}{2\pi}(t-s)\int_\Gamma\frac{|e^{zt_0'}|+|e^{zt_1'}|}{|z|^{\alpha(\varepsilon+1)-1}}\|A^\varepsilon x\||dz|\\&\le\frac{\kappa(M+1)}{2\pi}(t-s)C_\alpha(t_0'^{\beta(\varepsilon+1)-2}+t_1'^{\beta(\varepsilon+1)-2})\|A^\varepsilon x\|.\end{aligned}$$

By hypothesis, $t_0'^{\beta(\varepsilon+1)-2}<s^{\beta(\varepsilon+1)-2}$ and $t_1'^{\alpha(\varepsilon+1)-2}<s^{\alpha(\varepsilon+1)-2}$, because $0<s<t_0'<t_1'<t$. Thus,

$$\|(g_{\alpha-1}*S_{\alpha,1})(t)x-(g_{\alpha-1}*S_{\alpha,1})(s)x\|\le\frac{\kappa(M+1)}{\pi}C_\alpha(t-s)s^{\alpha(\varepsilon+1)-2}\|A^\varepsilon x\|,$$

for all $x\in D(A^\varepsilon)$ and $0<s<t$. Replacing t by $t-r$ and s by t_n-r we have

$$\begin{aligned}\|J_1\|&\le\int_0^{t_n}\|[(g_{\alpha-1}*S_{\alpha,1})(t-r)-(g_{\alpha-1}*S_{\alpha,1})(t_n-r)]f(r)\|dr\\&\le\frac{\kappa(M+1)}{\pi}C_\alpha(t-t_n)\int_0^{t_n}(t_n-r)^{\alpha(\varepsilon+1)-2}\|A^\varepsilon f(r)\|dr\\&\le\frac{\kappa(M+1)}{\pi}C_\alpha(t-t_n)\|f\|_\varepsilon(g_1*g_{\alpha(\varepsilon+1)-1})(t_n)\Gamma(\alpha(\varepsilon+1)-1)\end{aligned}$$

$$= \frac{\kappa(M+1)T^\alpha}{\pi(\alpha(\varepsilon+1)-1)}C_\alpha(t-t_n)\|f\|_\varepsilon t_n^{\alpha\varepsilon-1}.$$

By (4.6) we have

$$\|J_1\| \le \int_0^\infty \rho_n^\tau(t)\int_0^{t_n} \|[(g_{\alpha-1}*S_{\alpha,1})(t-r)-(g_{\alpha-1}*S_{\alpha,1})(t_n-r)]f(r)\|drdt$$
$$\le \frac{\kappa(M+1)T^\alpha C_\alpha}{\pi(\alpha(\varepsilon+1)-1)}\|f\|_\varepsilon \tau t_n^{\alpha\varepsilon-1}.$$

Now, we estimate J_2. For $t>0$ and $x\in D(A^\varepsilon)$ we have as in (4.12) that

$$(g_{\alpha-1}*S_{\alpha,1})(t)x = \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}}{z^\alpha}A^{1-\varepsilon}(z^\alpha-A)^{-1}A^\varepsilon x dz + \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}}{z^\alpha}xdz.$$

The inequality (4.4) and Lemma 4.1 imply that

$$\|(g_{\alpha-1}*S_{\alpha,1})(t)x\| \le \frac{\kappa(M+1)}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^{\alpha(\varepsilon+1)}}\|A^\varepsilon x\||dz| + \frac{1}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^\alpha}\|x\||dz|$$
$$\le \frac{\kappa(M+1)}{2\pi}C_\alpha t^{\alpha(\varepsilon+1)-1}\|A^\varepsilon x\| + \frac{C_\alpha}{2\pi}t^{\alpha-1}\|A^\varepsilon x\|,$$

for all $x\in D(A^\varepsilon)$ and $t>0$. Replacing t by $t-r$ we get

$$\int_{t_n}^t \|(g_{\alpha-1}*S_{\alpha,1})(t-r)f(r)\|dr$$
$$\le \frac{\kappa(M+1)}{2\pi}C_\alpha\|f\|_\varepsilon\int_{t_n}^t (t-r)^{\alpha(\varepsilon+1)-1}dr + \frac{C_\alpha}{2\pi}\|f\|_\varepsilon\int_{t_n}^t (t-r)^{\alpha-1}dr.$$

Since $\int_{t_n}^t (t-r)^{\alpha(\varepsilon+1)-1}dr = \int_0^t (t-r)^{\alpha(\varepsilon+1)-1}dr - \int_0^{t_n}(t-r)^{\alpha(\varepsilon+1)-1}dr$ and the function $x\mapsto x^{\alpha(\varepsilon+1)-1}$ is increasing, we obtain for $t_n\le t$ that

$$\int_{t_n}^t (t-r)^{\alpha(\varepsilon+1)-1}dr \le \frac{1}{\alpha(\varepsilon+1)}(t^{\alpha(\varepsilon+1)}-t_n^{\alpha(\varepsilon+1)}).$$

Similarly, we obtain $\int_{t_n}^t (t-r)^{\alpha-1}dr \le \frac{1}{\alpha}(t^\alpha - t_n^\alpha)$. Thus

$$\int_{t_n}^t \|(g_{\alpha-1}*S_{\alpha,1})(t-r)f(r)\|$$
$$\le \frac{\kappa(M+1)}{2\pi\alpha(\varepsilon+1)}C_\alpha\|f\|_\varepsilon(t^{\alpha(\varepsilon+1)}-t_n^{\alpha(\varepsilon+1)}) + \frac{C_\alpha}{2\pi\alpha}\|f\|_\varepsilon(t^\alpha-t_n^\alpha).$$

On the other hand, by (4.8)

$$\int_0^\infty \rho_n^\tau(t)(t^{\alpha(\varepsilon+1)}-t_n^{\alpha(\varepsilon+1)})dt = \frac{\tau^{\alpha(\varepsilon+1)}}{n!}\Gamma(n+\alpha(\varepsilon+1)+1)-t_n^{\alpha(\varepsilon+1)}.$$

Next, we notice that

$$d_n := \frac{\tau^{\alpha(\varepsilon+1)}}{n!}\Gamma(n+1+\alpha(\varepsilon+1))$$
$$= \tau\tau^{\alpha(\varepsilon+1)-1}\frac{\Gamma(n+1+\alpha(\varepsilon+1)-1)}{\Gamma(n+2)}\times(n+1)(n+\alpha(\varepsilon+1))$$
$$< t_n t_{n+1} t_{n+1}^{\alpha(\varepsilon+1)-2} + \alpha(\varepsilon+1)\tau t_{n+1}^{\alpha(\varepsilon+1)-1}$$

for all $n \in \mathbb{N}$, because $0 < \alpha(\varepsilon+1) - 1 < 1$ and $\frac{\Gamma(n+1+\eta)}{\Gamma(n+2)} < (n+1)^{\eta-1}$ for all $n \in \mathbb{N}$ and $0 < \eta < 1$. Moreover, the map $x \mapsto x^{\alpha(\varepsilon+1)-2}$ is a decreasing function on $[1,\infty)$, and therefore $t_{n+1}^{\alpha(\varepsilon+1)-2} \le t_n^{\alpha(\varepsilon+1)-2}$ for all $n \in \mathbb{N}$. This implies that

$$\begin{aligned} t_{n+1}^{\alpha(\varepsilon+1)-1} &= (n+1)\tau t_{n+1}^{\alpha(\varepsilon+1)-2} \le (n+1)\tau t_n^{\alpha(\varepsilon+1)-2} \\ &\le t_n^{\alpha(\varepsilon+1)-1} + \tau t_n^{\alpha(\varepsilon+1)-2} \le 2t_n^{\alpha(\varepsilon+1)-1}, \end{aligned}$$

and thus

$$d_n < t_n t_{n+1} t_{n+1}^{\alpha(\varepsilon+1)-2} + \alpha(\varepsilon+1)\tau t_{n+1}^{\alpha(\varepsilon+1)-1} \le t_{n+1} t_n^{\alpha(\varepsilon+1)-1} + 2\alpha(\varepsilon+1)\tau t_n^{\alpha(\varepsilon+1)-1},$$

for all $n \in \mathbb{N}$. Since $0 < t_n \le T$ and

$$t_{n+1} t_n^{\alpha(\varepsilon+1)-1} - t_n^{\alpha(\varepsilon+1)} = t_n^{\alpha(\varepsilon+1)} \left(\frac{t_{n+1}}{t_n} - 1 \right) = t_n^{\alpha(\varepsilon+1)} \left(\frac{t_{n+1} - t_n}{t_n} \right) = \tau t_n^{\alpha(\varepsilon+1)-1},$$

we obtain

$$\begin{aligned} \int_0^\infty \rho_n^\tau(t)(t^{\alpha(\varepsilon+1)} - t_n^{\alpha(\varepsilon+1)})dt &\le d_n - t_n^{\alpha(\varepsilon+1)} \\ &\le \tau t_n^{\alpha(\varepsilon+1)-1} + 2\alpha(\varepsilon+1)\tau t_n^{\alpha(\varepsilon+1)-1} \\ &\le (1+2\alpha(\varepsilon+1))\tau T^\alpha t_n^{\alpha\varepsilon-1}. \end{aligned}$$

Analogously, we can prove that

$$\int_0^\infty \rho_n^\tau(t)(t^\alpha - t_n^\alpha)dt \le (1+2\alpha)\tau T^{\alpha(1-\varepsilon)} t_n^{\alpha\varepsilon-1}.$$

Therefore,

$$\begin{aligned} \|J_2\| &\le \int_0^\infty \rho_n^\tau(t) \int_{t_n}^t \|(g_{\alpha-1} * S_{\alpha,1})(t-r)f(r)\| dr dt \\ &\le \frac{\kappa(M+1)}{2\pi\alpha(\varepsilon+1)} C_\alpha \|f\|_\varepsilon \int_0^\infty \rho_n^\tau(t)(t^{\alpha(\varepsilon+1)} + \frac{C_\alpha}{2\pi\alpha}\|f\|_\varepsilon \int_0^\infty \rho_n^\tau(t)(t^\alpha - t_n^\alpha)dt \\ &\le \frac{\kappa(M+1)}{2\pi\alpha(\varepsilon+1)} C_\alpha (1+2\alpha(\varepsilon+1)) T^\alpha \tau t_n^{\alpha\varepsilon-1} \|f\|_\varepsilon \\ &\quad + \frac{(1+2\alpha)C_\alpha}{2\pi\alpha} T^{\alpha(1-\varepsilon)} \tau t_n^{\alpha\varepsilon-1} \|f\|_\varepsilon. \end{aligned}$$

We conclude that

$$I_3 \le \|J_1\| + \|J_2\| \le D_3 \|f\|_\varepsilon \tau t_n^{\alpha\varepsilon-1}, \tag{4.13}$$

where

$$D_3 := \frac{\kappa(M+1)T^\alpha C_\alpha}{\pi} \left(\frac{1}{(\alpha(\varepsilon+1)-1)} + \frac{1+2\alpha(\varepsilon+1)}{2\alpha(\varepsilon+1)} \right) + \frac{(1+2\alpha)C_\alpha}{2\pi\alpha} T^{\alpha(1-\varepsilon)}.$$

Summarizing, by (4.7), (4.11) and (4.13), we obtain

$$\begin{aligned} \|u^n - u(t_n)\| &\le D_1 \tau t_n^{\alpha\varepsilon-1} \|A^\varepsilon u_0\| + D_2 \tau t_n^{\alpha\varepsilon-1} \|A^\varepsilon u_1\| + D_3 \|f\|_\varepsilon \tau t_n^{\alpha\varepsilon-1} \\ &\le C(\|A^\varepsilon u_0\| + \|A^\varepsilon u_1\| + \|f\|_\varepsilon) \tau t_n^{\alpha\varepsilon-1}, \end{aligned}$$

where the constant $C = C(T)$ is defined by

$$C := \max\{D_1, D_2, D_3\}.$$

The proof is finished. $\square$

5. **Some Experiments.** In this section, we illustrate the exact solution $u(t)$ at t_n to the fractional differential equation (1.2) and the approximated solution u^n to the Caputo difference equation (3.1) given in Theorem 3.1. Suppose that $A = \rho I$ for some $\rho > 0$. Then, the Laplace transform of the resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ satisfies

$$\hat{S}_{\alpha,1}(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^\alpha - \rho},$$

for all $\mathrm{Re}(\lambda) > \rho^{1/\alpha}$. By [18, Formula 17.6], we obtain that

$$S_{\alpha,1}(t) = E_{\alpha,1}(\rho t^\alpha), \tag{5.1}$$

where, for $p, q, r > 0$, $E_{p,q}(z)$ is the Mittag-Leffler defined by

$$E_{p,q}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(pj+q)}, \quad z \in \mathbb{C}.$$

Therefore, the solution to

$$\begin{cases} \partial_t^\alpha u(t) &= \rho u(t) + f(t), \quad t \geq 0, \\ u(0) &= u_0 \\ u'(0) &= u_1, \end{cases} \tag{5.2}$$

is given by

$$u(t) = S_{\alpha,1}(t)u_0 + (g_1 * S_{\alpha,1})(t)u_1 + +(g_{\alpha-1} * S_{\alpha,1} * f)(t), \tag{5.3}$$

where $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is given in (5.1). Now, we consider the exact and approximated solution to (5.2) on the interval $[0, 4]$. We take $\tau = 4/N$ for $N = 40$, $N = 80$ and $N = 100$. In Figure 3 we have the exact solution u to the initial value problem (5.2) given by (5.3) evaluated at $t_n = n\tau$, that is, $u(t_n)$ for $2 \leq n \leq N$, and the approximated solution $(u^n)_{n=2}^N$ given by

$$u^n = \int_0^\infty \rho_n^\tau(t)\Big[S_{\alpha,1}(t)u_0 + (g_1 * S_{\alpha,1})(t)u_1 + (g_{\alpha-1} * S_{\alpha,1} * f)(t)\Big]dt.$$

To illustrate the theoretical results, we take $u_0 = u_1 = 1$ and $f(t) = t^2 e^{-t}$ for all $t \in [0, 4]$ and in the numerical computations we consider $\alpha = 1.5$.

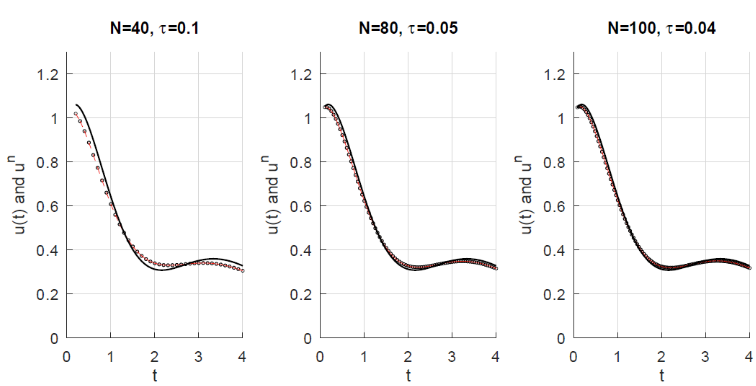

FIGURE 3. Exact (line) and approximated (circles) solution u and u^n, respectively, for $N = 40$, $N = 80$ and $N = 100$. Here we take $\alpha = 1.5$.

Next, we illustrate Theorem 4.2 where we compare $u(t_n)$ and u^n for $2 \leq n \leq N$, where $N \in \mathbb{N}$ is given integer number. In Figure 4 we have the error $e_n := |u(t_n) - u^n|$ for $2 \leq n \leq N$, where $N = 40, 80$ and $N = 100$. As before, we take $\alpha = 1.5$, $u_0 = u_1 = 1$ and $f(t) = t^2 e^{-t}$ for all $t \in [0, 4]$. We notice that, as in Theorem 4.2, the error e_n behaves as $\tau = 4/N$.

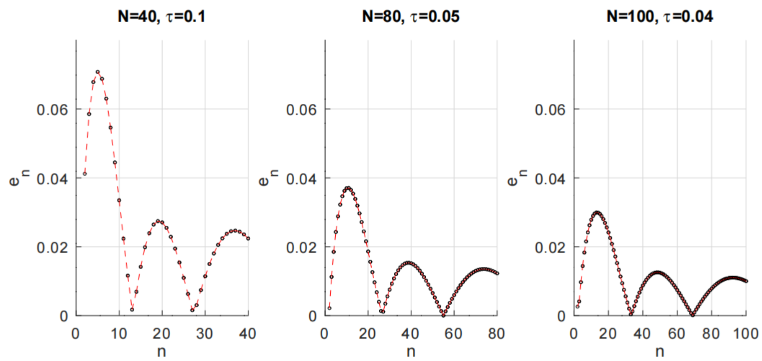

FIGURE 4. Plot of e_n for $N = 40$, $N = 80$ and $N = 100$. Here we take $\alpha = 1.5$.

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