MAXIMAL REGULARITY OF DELAY EQUATIONS IN BANACH SPACES

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Abstract. We characterize existence and uniqueness of solutions for an inhomogeneous abstract delay equation in Hölder spaces. The main tool is the theory of operator valued Fourier multipliers.

1. Introduction

Partial differential equations with delay are a subject which has been extensively studied in the last years. In an abstract way they can be written as

$$u'(t) = Au(t) + Fu_t + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where $(A, D(A))$ is a (unbounded) linear operator on a Banach space X, $u_t(\cdot) = u(t + \cdot)$ on $[-r, 0], r > 0$, and the delay operator F is supposed to belong to $\mathcal{B}(C([-r, 0], X), X)$.

First studies on equation (1.1) goes back to J. Hale [8] and G. Webb [12]. A general and systematic study of linear delay equations with emphasis on the qualitative behavior and asymptotic properties can be found in the recent monograph by Bátkai and Piazzera [5]. See also [13]. The problem to find conditions for all solutions of (1.1) to be in the same space as f arises naturally from recent studies on maximal regularity and their application to nonlinear problems in the theory of evolution equations, see the recent monograph by Denk-Hieber-Prüss [7] and references therein.

Recently, a significant progress has been made in finding sufficient conditions for operator valued functions to be C^{α}- Fourier multipliers, see [3]. In particular, in [4] the theory of operator-valued Fourier multipliers is applied to obtain results on the hyperbolicity of delay equations and in [9] to obtain stability of linear control systems in Banach spaces. Also in [10] existence and uniqueness of periodic solutions for equation (1.1) via L^p-Fourier multiplier theorems has been recently obtained.

In this paper we are able to obtain necessary and sufficient conditions in order to guarantee well-posedness of the delay equation (1.1) in the Hölder

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spaces $C^\alpha(\mathbb{R}, X) \quad (0 < \alpha < 1)$, and under the condition that X is a B-convex space. However we stress that here A is not necessarily the generator of a C_0-semigroup.

We remark that the Fourier multiplier approach used allows to give a direct treatment of the equation, in contrast with the approach using the correspondence between (1.1) and the solutions of the abstract Cauchy problem

$$\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) + \mathcal{F}(t) \qquad t \geq 0,$$

where $\mathcal{A} = \begin{pmatrix} A & F \\ 0 & d/d\sigma \end{pmatrix}$. In this case the question of well-posedness of the delay equation reduces to the question whether or not the operator $(\mathcal{A}, D(\mathcal{A}))$ generates a C_0-semigroup; see [5, 6, 11] and references therein.

2. Preliminaries

Let X, Y be Banach spaces and let $0 < \alpha < 1$. We denote by $\dot{C}^\alpha(\mathbb{R}, X)$ the spaces

$$\dot{C}^\alpha(\mathbb{R}, X) = \{f : \mathbb{R} \to X : f(0) = 0, ||f||_\alpha < \infty\}$$

normed by

$$||f||_\alpha = \sup_{t \neq s} \frac{||f(t) - f(s)||}{|t - s|^\alpha}.$$

Let $\Omega \subset \mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all C^∞-functions in $\Omega \subseteq \mathbb{R}$ having compact support in Ω.

We denote by $\mathcal{F}f$ or $\tilde{f}$ the Fourier transform, i.e.

$$(\mathcal{F}f)(s) := \int_{\mathbb{R}} e^{-ist} f(t) dt$$

$(s \in \mathbb{R}, f \in L^1(\mathbb{R}; X))$.

Definition 2.1. *Let $M : \mathbb{R}\backslash\{0\} \to \mathcal{B}(X, Y)$ be continuous. We say that M is a $\dot{C}^\alpha$ - multiplier if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}, X) \to \dot{C}^\alpha(\mathbb{R}, Y)$ such that*

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s) ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot M))(s) f(s) ds \tag{2.1}$$

for all $f \in C^\alpha(\mathbb{R}, X)$ and all $\phi \in C_c^\infty(\mathbb{R}\backslash\{0\})$.

Here $(\mathcal{F}(\phi \cdot M))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)M(t)dt \in \mathcal{B}(X, Y)$. Note that L is well defined, linear and continuous (cf. [3, Definition 5.2]).

Define the space $C^\alpha(\mathbb{R}, X)$ as the set

$$C^\alpha(\mathbb{R}, X) = \{f : \mathbb{R} \to X : ||f||_{C^\alpha} < \infty\}$$

with the norm

$$||f||_{C^\alpha} = ||f||_\alpha + ||f(0)||.$$

Let $C^{\alpha+1}(\mathbb{R}, X)$ be the Banach space of all $u \in C^1(\mathbb{R}, X)$ such that $u' \in C^\alpha(\mathbb{R}, X)$, equipped with the norm

$$||u||_{C^{\alpha+1}} = ||u'||_{C^\alpha} + ||u(0)||.$$

Observe from Definition (2.1) and since

$$\int_{\mathbb{R}} (\mathcal{F}(\phi M)(s))(s)ds = 2\pi(\phi M)(0) = 0,$$

that for $f \in C^\alpha(\mathbb{R}, X)$ we have $Lf \in C^\alpha(\mathbb{R}, X)$. Moreover, if $f \in C^\alpha(\mathbb{R}, X)$ is bounded then Lf is bounded as well (see [3, Remark 6.3]).

The following multiplier theorem is due to Arendt-Batty and Bu [3, Theorem 5.3].

Theorem 2.2. *Let $M \in C^2(\mathbb{R}\backslash\{0\}, \mathcal{B}(X, Y))$ be such that*

$$\sup_{t\neq 0} ||M(t)|| + \sup_{t\neq 0} ||tM'(t)|| + \sup_{t\neq 0} ||t^2 M''(t)|| < \infty. \tag{2.2}$$

Then M is a $\dot{C}^\alpha$-multiplier.

Remark 2.3.

If X is B-convex, in particular if X is a UMD space, Theorem 2.2 remains valid if condition 2.2 is replaced by the following weaker condition

$$\sup_{t\neq 0} ||M(t)|| + \sup_{t\neq 0} ||tM'(t)|| < \infty, \tag{2.3}$$

where $M \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{B}(X, Y))$ (cf. [3, Remark 5.5]).

We use the symbol $\hat{f}(\lambda)$ for the Carleman transform:

$$\hat{f}(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} f(t)dt & Re\lambda > 0 \\ \displaystyle -\int_{-\infty}^0 e^{-\lambda t} f(t)dt & Re\lambda < 0, \end{cases}$$

where $f \in L^1_{loc}(\mathbb{R}, X)$ is of subexponential growth; by this we mean

$$\int_{-\infty}^\infty e^{-\epsilon|t|}\|f(t)\|dt < \infty, \qquad \text{for each } \epsilon > 0.$$

We remark that if $u' \in L^1_{loc}(\mathbb{R}, X)$ is of subexponential growth, then

$$\hat{u'}(\lambda) = \lambda\hat{u}(\lambda) - u(0), \qquad Re\lambda \neq 0.$$

3. A Characterization

We consider in this section the equation

$$u'(t) = Au(t) + Fu_t + f(t), \qquad t \in \mathbb{T}, \tag{3.1}$$

where $A : D(A) \subseteq X \to X$ is a linear, closed operator; $f \in C^\alpha(\mathbb{R}, X)$ and, for $r > 0$, $F : C([-r, 0], X) \to X$ is a linear, bounded operator. Moreover u_t is an element of $C([-r, 0], X)$ which is defined as $u_t(\theta) = u(t + \theta)$ for $-r \leq \theta \leq 0$.

Example 3.1. Let $\mu : [-r, 0] \to \mathcal{B}(X)$ be of bounded variation. Let $F : C([-r, 0], X) \to X$ be the bounded operator given by the Riemann-Stieltjes integral

$$F(\phi) = \int_{-r}^{0} \phi d\mu \text{ for all } \phi \in C([-r, 0], X).$$

An important special case consists of operators F defined by

$$F(\phi) = \sum_{k=0}^{n} C_k \phi(\tau_k), \qquad \phi \in C([-r, 0], X),$$

where $C_k \in \mathcal{B}(X)$ and $\tau_k \in [-r, 0]$ for $k = 0, 1, ..., n$. For concrete equations dealing with the above classes of delays operators see the monograph of Bátkai and Piazzera [5, Chapter 3].

Definition 3.2. *We say that (1.1) is C^α-well posed if for each $f \in C^\alpha(\mathbb{R}, X)$ there is a unique function $u \in C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ such that (1.1) is satisfied.*

Denote by $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{F_\lambda\}_{\lambda \in \mathbb{R}} \subseteq \mathcal{B}(X)$ by

$$F_\lambda x = F(e_\lambda x), \text{ for all } \lambda \in \mathbb{R} \text{ and } x \in X. \tag{3.2}$$

We define the *real spectrum* of (3.1) by

$$\sigma(\Delta) = \{s \in \mathbb{R} : isI - F_s - A \in \mathcal{B}([D(A)], X) \text{ is not invertible } \}.$$

Proposition 3.3. *Let X be a Banach space and let $A : D(A) \subset X \to X$ be a closed linear operator. Suppose that (1.1) is C^α-well posed. Then*

(i) $\sigma(\Delta) = \emptyset$,

(ii) $\{i\eta(i\eta I - A - F_\eta)^{-1}\}_{\eta \in \mathbb{R}}$ *is bounded.*

Proof. Let $x \in D(A)$ and let $u(t) = e^{i\eta t}x$ for $\eta \in \mathbb{R}$. Then $u_t(s) = e^{it\eta}e^{is\eta}x$. Thus

$$F(u_t) = e^{it\eta}F(e_\eta x) = e^{it\eta}F_\eta x. \tag{3.3}$$

Now if $(i\eta - A - F_\eta)x = 0$, then $u(t)$ is a solution of equation (1.1) when $f \equiv 0$. Hence by uniqueness follows that $x = 0$. Now let $L : C^\alpha(\mathbb{R}, X) \to C^{\alpha+1}(\mathbb{R}, X)$ be the bounded operator which takes each $f \in C^\alpha(\mathbb{R}, X)$ to the unique solution $u \in C^{\alpha+1}(\mathbb{R}, X)$. Let $y \in X$ and let $s_0 \in \mathbb{R}$ be fixed. Then define $f(t) = e^{is_0\eta}y, t \in \mathbb{R}$. Let $u(t)$ be the unique solution of (1.1) such that $L(u) = y$. Next we claim that $v(t) := u(t + s_0)$ and $w(t) := e^{i\eta s_0}u(t)$ both satisfy equation (1.1), when $f(t) = e^{is_o\eta}y$. First we notice that

$$v_t(s) = u(t + s_0 + s) = u_{t+s_0}(s).$$

Hence $F(v_t) = F(u_{t+s_0})$. Then an easy computation shows that $v(t)$ satisfy equation (1.1). On the other hand,

$$w_t(s) = w(t + s) = e^{i\eta s_0}u(t + s) = e^{i\eta s_0}u_t(s).$$

Hence $F(w_t) = e^{is_0\eta}F(u_t)$. Thus

$$e^{i\eta s_0}u'(t) = e^{i\eta s_0}(Au(t) + F(u_t) + y) = Aw(t) + F(w_t) + f(t).$$

Thus $w(t)$ satisfy equation (1.1). By uniqueness again we have that

$$u(t + s) = e^{i\eta s}u(t)$$

for all $t, s \in \mathbb{R}$. In particular when $t = 0$ we obtain that

$$u(s) = e^{i\eta s}u(0), s \in \mathbb{R}.$$

Now let $x = u(0) \in D(A)$. Then $u(t) = e^{i\eta t}x$ satisfy (1.1), that is by (3.3)

$$i\eta u(t) = Au(t) + F(u_t) + e^{i\eta t}y = Au(t) + e^{i\eta t}F_\eta x + e^{i\eta t}y.$$

In particular if $t = 0$ we obtain that

$$i\eta x = Ax + F_\eta x + y,$$

since $x = u(0)$. Thus

$$(i\eta I - A - F_\eta)x = y \tag{3.4}$$

and hence $i\eta I - A - F_\eta$ is bijective. This shows assertion (i) of the Proposition.

Next we notice that $u(t) = (i\eta - A - F_\eta)^{-1}y$ by (3.4). Since $||e_\eta \otimes x||_\alpha = K_\alpha|\eta|^\alpha||x||$. Thus

$$\begin{aligned}
K_\alpha|\eta|^\alpha||i\eta(i\eta - A - F_\eta)^{-1}y|| &= ||e_\eta \otimes i\eta(i\eta - A - F_\eta)^{-1}y||_\alpha = ||u'||_\alpha \\
&\leq ||u||_{1+\alpha} = ||Lf||_{1+\alpha} \leq ||L||||f||_\alpha \\
&\leq ||L||(||f||_\alpha + ||f(0)||) \\
&= ||L||(||e_\eta \otimes y||_\alpha + ||y|| \\
&\leq ||L||(K_\alpha|\eta|^\alpha + 1)||y||.
\end{aligned}$$

Hence for $\epsilon > 0$ follows that

$$\sup_{|\eta|>\epsilon} ||i\eta(i\eta - A - F_\eta)^{-1}y|| \leq ||L|| \sup_{|\eta|>\epsilon}(1 + \frac{1}{K_\alpha|\eta|^\alpha}) < \infty.$$

■

Recall that a Banach space X has *Fourier type* p, where $1 \leq p \leq 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; X)$, where q is the conjugate index of p. For example, the space $L^p(\Omega)$, where $1 \leq p \leq 2$ has Fourier type p; X has Fourier type 2 if and only if X is a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1; X is B-convex if it has Fourier type p for some $p > 1$. Every uniformly convex space is B-convex.

Our main result in this paper, establish that the converse of Proposition 3.3 is true.

Theorem 3.4. *Let A be a closed linear operator defined on a B-convex space X. Then the following assertions are equivalent*

(i) Equation (1.1) is C^α-well posed.

(ii) $\sigma(\Delta) = \emptyset$ and $\sup_{\eta\in\mathbb{R}} ||i\eta(i\eta I - A - F_\eta)^{-1}|| < \infty$.

Proof.

$(ii) \Rightarrow (i)$. Define the operator $M(t) = (B_t - A)^{-1}$, with $B_t = itI - F_t$. Note that by hypothesis $M \in C^2(\mathbb{R}, \mathcal{B}(X, [D(A)]))$.

We claim that M is a $C^\alpha-$multiplier. In fact, by hypothesis it is clear that $\sup_{t\in\mathbb{R}} ||M(t)|| < \infty$. On the other hand, we have

$$M'(t) = -M(t)\, B_t'\, M(t)$$

with $B_t' = iI - F_t'$ and $F_t'(x) = F(e_t'x)$ where $e_t'(s) = ise^{ist}$. Note that for each $x \in X$

$$||F_t x||_X \leq ||F(e_t x)||_X \leq ||F||\, ||e_t x||_\infty \leq ||F||\, ||x||_X\,, \tag{3.5}$$

and

$$||F_t' x||_X \leq ||F(e_t' x)||_X \leq ||F||\, ||e_t' x||_\infty \leq r||F||\ \ ||x||_X. \tag{3.6}$$

Hence B_t' is uniformly bounded with respect to $t \in \mathbb{R}$ and we conclude from the hypothesis that

$$\sup_{t\in\mathbb{R}} ||tM'(t)|| = \sup_{t\in\mathbb{R}} ||\, [tM(t)]\, B_t'\, M(t)|| < \infty\,, \tag{3.7}$$

and hence the claim follows from Theorem 2.2 and Remark 2.3.

Now, define $N \in C^1(\mathbb{R}, \mathcal{B}(X))$ by $N(t) = (id \cdot M)(t)$, where $id(t) := it$ for all $t \in \mathbb{R}$. We will prove that N is a $C^\alpha-$multiplier. In fact, with a direct calculation, we have

$$\begin{aligned} t\,N'(t) &= itM(t) + it^2 M'(t) = itM(t) + i[itM(t)]\,B'_t\,[itM(t)] \\ &= N(t) + iN(t)\,B'_t\,N(t). \end{aligned}$$

By hypothesis and (3.6) it follows that $\sup\limits_{t\in\mathbb{R}} ||tN'(t)|| \leq \sup\limits_{t\in\mathbb{R}} ||N(t)|| + \sup\limits_{t\in\mathbb{R}} ||N(t)\,B'_t\,N(t)|| < \infty$, hence from Theorem 2.2 and Remark 2.3 the claim is proved.

A similar calculation prove that $P \in C^1(\mathbb{R}\backslash\{0\}, \mathcal{B}(X))$ defined by $P(t) = F_t\,M(t)$ is a $C^\alpha-$multiplier.
In fact, we have $t\,P'(t) = F'_t N(t) + F_t\,t\,M'(t)$, and hence from (3.5), (3.6) and (3.7) we obtain that $\sup\limits_{t\in\mathbb{R}} ||P(t)|| + \sup\limits_{t\in\mathbb{R}} ||tP'(t)|| < \infty$.

Let $f \in C^\alpha(\mathbb{R}, X)$, since M, N and P are $C^\alpha-$multiplier, there exist $\bar{u} \in C^\alpha(\mathbb{R}, [D(A)])$, $v \in C^\alpha(\mathbb{R}, X)$ and $w \in C^\alpha(\mathbb{R}, X)$, respectively, such that

$$\int_{\mathbb{R}} \bar{u}(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi \cdot M)(s) f(s)\,ds\,, \tag{3.8}$$

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\psi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\psi \cdot id \cdot M)(s) f(s)\,ds\,, \tag{3.9}$$

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot F_. M)(s) f(s)\,ds\,, \tag{3.10}$$

for all ϕ, ψ, $\varphi \in C_c^\infty(\mathbb{R})$.

Note that for $x \in X$ and $\phi \in C_c^\infty(\mathbb{R})$ we have

(3.11)
$$\mathcal{F}(\phi\,F_.\,M)(s)\,x = \int_{\mathbb{R}} e^{-ist}\phi(t)\,F_t\,M(t)\,x\,dt = \int_{\mathbb{R}} e^{-ist}\phi(t)\,F(e_t\,M(t)\,x)\,dt\,.$$

where $\int_{\mathbb{R}} e^{-ist}\phi(t)\,e_t\,M(t)\,x\,dt \in C([-r, 0], X)$. Now, for all $\theta \in [-r, 0]$ we have

$$\left\| \int_{\mathbb{R}} e^{-ist}\phi(t)\,e_t(\theta)\,M(t)\,x\,dt \right\|_X \leq \int_{\mathbb{R}} |\,\phi(t)\,|\;||M(t)\,x\,||_X dt.$$

Since F is bounded, we deduce that

$$\mathcal{F}(\phi \cdot F_.\,M)(s)x = F(\mathcal{F}(\phi \cdot e_.\,M)(s)\,x)\,. \tag{3.12}$$

Furthermore, observe that for $\theta \in [-r,0]$ fixed we have that $e.(\theta)\phi \in C_c^\infty(\mathbb{R})$. Using (3.8) we obtain

$$\begin{aligned}\int_{\mathbb{R}} \bar{u}(s+\theta)(\mathcal{F}\phi)(s)ds &= \int_{\mathbb{R}} \bar{u}(s+\theta)\int_{\mathbb{R}} e^{-ist}\phi(t)\,dt\ ds\\ &= \int_{\mathbb{R}} \bar{u}(s+\theta)\int_{\mathbb{R}} e^{-i(s+\theta)t}e_t(\theta)\phi(t)\,dt\ ds\\ &= \int_{\mathbb{R}} \bar{u}(s+\theta)(\mathcal{F}e.(\theta)\phi)(s+\theta)\ ds\\ &= \int_{\mathbb{R}} \bar{u}(s)(\mathcal{F}e.(\theta)\phi)(s)\ ds\\ &= \int_{\mathbb{R}} \mathcal{F}(e.(\theta)\phi\cdot M)(s)f(s)\ ds,\end{aligned}$$

hence $\displaystyle\int_{\mathbb{R}} \bar{u}_s(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(e.\phi\cdot M)(s)f(s)\ ds$.
Since the function $\theta \to \displaystyle\int_{\mathbb{R}} \bar{u}_s(\theta)(\mathcal{F}\phi)(s)ds \in C([-r,0],X)$ (see [3, p.3]), due to the boundedness of F and (3.12) it follows that

(3.13)
$$\int_{\mathbb{R}} \mathcal{F}(\phi\cdot\ F_{.}M)(s)f(s)ds = \int_{\mathbb{R}} F\mathcal{F}(\phi\cdot\ e_{.}M)(s)f(s)ds = \int_{\mathbb{R}} F\bar{u}_s\,(\mathcal{F}\phi)(s)ds,$$

for all $\phi \in C_c^\infty(\mathbb{R})$. Since $F_{.}M$ is $C^\alpha-$multiplier, we obtain from (3.10)

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} F\bar{u}_s\,(\mathcal{F}\phi)(s)ds\,.$$

for all $\phi \in C_c^\infty(\mathbb{R})$. We conclude that there exists $y_1 \in X$ satisfying $w(t) = F\bar{u}_t + y_1$, proving that $F\bar{u}. \in C^\alpha(\mathbb{R},X)$.

Choosing $\phi = id\cdot\psi$ in (3.8) we obtain from (3.9) that

$$\int_{\mathbb{R}} \bar{u}(s)\mathcal{F}(id\cdot\psi)(s)\,ds = \int_{\mathbb{R}} v(s)\,(\mathcal{F}\psi)(s)\,ds\,, \tag{3.14}$$

and it follows from Lemma 6.2 in [3] that $\bar{u} \in C^{\alpha+1}(\mathbb{R},X)$ and $\bar{u}' = v + y_2$ for some $y_2 \in X$.

Since $(id\,I - F_{.} - A)\,M = I$ we have $id\cdot M = I + F_{.}M + AM$ and replacing in (3.9) gives

(3.15)
$$\begin{aligned}\int_{\mathbb{R}} v(s)\,(\mathcal{F}\phi)(s)\,ds &= \int_{\mathbb{R}} \mathcal{F}(\phi\cdot(I+F_{\cdot}M+AM))(s)\,f(s)\,ds\\ &= \int_{\mathbb{R}} (\mathcal{F}\phi)(s)\,f(s)\,ds + \int_{\mathbb{R}} \mathcal{F}(\phi\cdot F_{\cdot}\,M)(s)\,f(s)\,ds\\ &+ \int_{\mathbb{R}} \mathcal{F}(\phi\cdot A\,M)(s)\,f(s)\,ds\,,\end{aligned}$$

for all $\phi\in C_c^\infty(\mathbb{R})$.

Since $\bar u(t)\in D(A)$ and $\mathcal{F}(\phi\cdot M)(s)x\in D(A)$ for all $x\in X$, using the fact that A is closed and setting (3.8) and (3.13) in (3.15) we obtain that

(3.16)
$$\begin{aligned}\int_{\mathbb{R}} v(s)\,(\mathcal{F}\phi)(s)\,ds &= \int_{\mathbb{R}} F\bar u_s\,(\mathcal{F}\phi)(s)ds + \int_{\mathbb{R}} A\bar u(s)(\mathcal{F}\phi)(s)\,f(s)\,ds\\ &+ \int_{\mathbb{R}} f(s)\,(\mathcal{F}\phi)(s)\,ds\,,\end{aligned}$$

for all $\phi\in C_c^\infty(\mathbb{R})$.

By Lemma 5.1 in [3] this implies that for some $y_3\in X$ one has

$$v(t)=F\bar u_t+A\bar u(t)+f(t)+y_3,\quad t\in\mathbb{R}\,.$$

Consequently, $\bar u'(t)=v(t)+y_2=F\bar u_t+A\bar u(t)+f(t)+y$ where $y=y_2+y_3$. In particular $A\bar u\in C^\alpha(\mathbb{R},X)$. Now, by hypothesis we can define $x=(A+F)^{-1}y\in D(A)$, and then is clear that $u(t):=\bar u(t)+x$ is in $C^{\alpha+1}(\mathbb{R},X)\cap C^\alpha(\mathbb{R},[D(A)])$ and satisfies (1.1). We have shown that a solution of (1.1) exists.

In order to prove uniqueness, suppose that

$$u'(t)=Au(t)+Fu_t\,,\qquad t\in\mathbb{R}, \tag{3.17}$$

where $u\in C^{\alpha+1}(\mathbb{R},X)\cap C^\alpha(\mathbb{R},[D(A)])$ and, as showed, $Au, Fu.\in C^\alpha(\mathbb{R},X)$.

We claim that $\hat u.(\lambda)\in C([-r,0],X)$ for $Re\lambda\neq 0$. In fact, let $Re\lambda>0$ then

$$\begin{aligned} ||e^{-\lambda t}u_t||_\infty &= \sup_{\theta\in[-r,0]}||e^{-\lambda t}u(t+\theta)||_X \le \sup_{\theta\in[-r,0]} e^{-Re\lambda t}(1+|t+\theta|^\alpha) \\ &\le e^{-Re\lambda t}(1+(|t|+r)^\alpha). \end{aligned}$$

Since $e^{-Re\lambda t}(1+(|t|+r)^\alpha)\in L^1(\mathbb{R}_+)$ applying the dominated convergence theorem, we obtain the claim. Analogously we obtain the claim for $Re\lambda<0$.

Now, note that for $Re\lambda>0$ and $\theta\in[-r,0]$

$$\begin{aligned} \int_0^\infty e^{-\lambda t}u_t(\theta)dt &= \int_0^\infty e^{-\lambda t}u(t+\theta)dt \\ &= \int_\theta^\infty e^{-\lambda(t-\theta)}u(t)dt \\ &= e^{\lambda\theta}\int_\theta^\infty e^{-\lambda t}u(t)dt \\ &= e^{\lambda\theta}\left(\int_0^\infty e^{-\lambda t}u(t)dt+\int_\theta^0 e^{-\lambda t}u(t)dt\right) \\ &= e^{\lambda\theta}\hat{u}(\lambda)+e^{\lambda\theta}\int_\theta^0 e^{-\lambda t}u(t)dt. \end{aligned}$$

Analogously if $Re\lambda<0$ and $\theta\in[-r,0]$, then

$$\begin{aligned} -\int_{-\infty}^0 e^{-\lambda t}u_t(\theta)dt &= -\int_{-\infty}^0 e^{-\lambda t}u(t+\theta)dt \\ &= -\int_{-\infty}^\theta e^{-\lambda(t-\theta)}u(t)dt \\ &= -e^{\lambda\theta}\left(\int_{-\infty}^0 e^{-\lambda t}u(t)dt-\int_\theta^0 e^{-\lambda t}u(t)dt\right) \\ &= e^{\lambda\theta}\hat{u}(\lambda)+e^{\lambda\theta}\int_\theta^0 e^{-\lambda t}u(t)dt. \end{aligned}$$

Since F is bounded, we obtain that

$$\widehat{Fu.}(\lambda)=F\hat{u}.(\lambda)=Fg\hat{u}(\lambda)+Fgh,\quad \text{for } Re(\lambda)\neq 0 \tag{3.18}$$

where $g(\theta)=e^{\lambda\theta}$ and $h(\theta)=\displaystyle\int_\theta^0 e^{-\lambda t}u(t)dt$. Note that $gh\in C([-r,0],X)$.

Since $\hat{u'}(\lambda)=\lambda\hat{u}(\lambda)-u(0)$ for $Re(\lambda)\neq 0$, one has $\hat{u}(\lambda)\in D(A)$ and

$$\hat{u'}(\lambda)=\widehat{Au}(\lambda)+\widehat{Fu.}(\lambda),\quad \text{for}\quad Re(\lambda)\neq 0. \tag{3.19}$$

Using the fact that A is closed, from (3.18) and (3.19) we get

$$(\lambda I - Fg - A)\,\hat{u}(\lambda) = u(0) + Fgh \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus i\mathbb{R}.$$

Since $i\mathbb{R} \subset \rho(A)$, it follows that the Carleman spectrum $sp_C(u)$ of u is empty. Hence $u \equiv 0$ by [2, Theorem 4.8.2]. ∎

We denote by $\mathcal{K}_F(X)$ the class of operators in X satisfying (ii) in the above theorem. If $A \in \mathcal{K}_F(X)$ we have $u', Au, Fu. \in C^\alpha(\mathbb{R}, X)$, and hence we deduce the following result.

Corollary 3.5. *Let X be B-convex and $A \in \mathcal{K}_F(X)$. Then*

(i) (1.1) has a unique solution in $Z := C^{\alpha+1}(\mathbb{R}, X) \cap C^\alpha(\mathbb{R}, [D(A)])$ if and only if $f \in C^\alpha(\mathbb{R}, X)$.

(ii) There exists a constant $M > 0$ independent of $f \in C^\alpha(\mathbb{R}, X)$ such that

$$\|u'\|_{C^\alpha(\mathbb{R},X)} + \|Au\|_{C^\alpha(\mathbb{R},X)} + \|Fu.\|_{C^\alpha(\mathbb{R},X)} \le M\|f\|_{C^\alpha(\mathbb{R},X)}. \tag{3.20}$$

Remark 3.6. The inequality (3.20) is a consequence of the closed graph theorem and known as the *maximal regularity property* for equation (1.1). From it we deduce that the operator L defined by

$$D(L) = Z$$
$$(Lu)(t) = u'(t) - Au(t) - Fu_t$$

is an isomorphism onto. In fact, since A is closed, the space Z becomes a Banach space under the norm

$$||\ u\ ||_Z \ := \|u\|_{C^\alpha(\mathbb{R},X)} + \|u'\|_{C^\alpha(\mathbb{R},X)} + \|Au\|_{C^\alpha(\mathbb{R},X)}.$$

Such isomorphisms are crucial for the treatment of nonlinear versions of (1.1).

Indeed, assume X be B-convex and $A \in \mathcal{K}_F(X)$ and consider the semilinear problem

$$u'(t) = Au(t) + Fu_t + f(t, u(t)), \quad t \ge 0. \tag{3.21}$$

Define the Nemytskii's superposition operator $N : Z \to C^\alpha(\mathbb{R}, X)$ given by $N(v)(t) = f(t, v(t))$ and the bounded linear operator

$$S := L^{-1} : C^\alpha(\mathbb{R}, X) \to Z$$

by $S(g) = u$ where u is the unique solution of the linear problem

$$u'(t) = Au(t) + Fu_t + g(t).$$

Then to solve (3.21) we have to show that the operator $H : Z \to Z$ defined by $H = SN$ has a fixed point.

For related information we refer to Amann [1] where results in quasilinear delay equations involving the method of maximal regularity are presented.

We finish this paper with the following result which give us a useful criterion to verify condition (ii) in the above theorem.

Theorem 3.7. *Let X be a B-convex space and let $A : D(A) \subset X \to X$ be a closed linear operator such that $i\mathbb{R} \subset \rho(A)$ and $\sup_{s\in\mathbb{R}} \|A(isI - A)^{-1}\| =: M < \infty$. Suppose that*

$$||F|| < \frac{1}{||A^{-1}||M}. \tag{3.22}$$

Then for each $f \in C^{\alpha}(\mathbb{R}, X)$ there is a unique function $u \in C^{\alpha+1}(\mathbb{R}, X) \cap C^{\alpha}(\mathbb{R}, [D(A)])$ such that (1.1) is satisfied.

Proof. From the identity

$$isI - A - F_s = (isI - A)(I - F_s(isI - A)^{-1}) \quad s \in \mathbb{R},$$

it follows that $isI - A - F_s$ is invertible whenever $||F_s(isI - A)^{-1}|| < 1$. Next observe that

$$||F_s|| \le ||F||, \tag{3.23}$$

and hence

$$||F_s(isI - A)^{-1}|| = ||F_sA^{-1}A(isI - A)^{-1}|| \le ||F||||A^{-1}||M =: \alpha.$$

Therefore, under the condition (3.22) we obtain that $\sigma(\Delta) = \emptyset$, and the identity

(3.24)

$$(isI-A-F_s)^{-1} = (isI-A)^{-1}(I-F_s(isI-A)^{-1}) = (isI-A)^{-1}\sum_{n=0}^{\infty}[F_s(isI-A)^{-1}]^n.$$

For all $n \in \mathbb{N}$ we have

$$\begin{aligned} &\|is(isI - A)^{-1}[F_s(isI - A)^{-1}]^n\| \\ &\qquad \le \|is(isI - A)^{-1}\|[\|F_sA^{-1}A(isI - A)^{-1}\|]^n \\ &\qquad \le \|is(isI - A)^{-1})[\|F_sA^{-1}\|]^n[\|(A(is - A)^{-1}\|]^n \\ &\qquad \le \|is(isI - A)^{-1}\|||A^{-1}||^n[\|F_s\|]^n[\|A(isI - A)^{-1}\|]^n. \end{aligned}$$

By (3.23) we obtain

$$\begin{aligned} \|is(isI - A)^{-1}[F_s(isI - A)^{-1}]^n\| &\le \|is(isI - A)^{-1}\|||A^{-1}||^n||F||^nM^n \\ &= \|is(isI - A)^{-1}\|\alpha^n. \end{aligned}$$

Finally by (3.24), one has

$$\|is(isI - A - F_s)^{-1}\| \leq \|is(isI - A)^{-1}\| \frac{1}{1-\alpha} \leq \frac{M+1}{1-\alpha}.$$

This proves that $\{is(isI - A - F_s)^{-1}\}$ is bounded and the conclusion follows from Theorem 3.4.

■

References

[1] H. Amann, *Quasilinear parabolic functional evolution equations.* Preprint.

[2] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems.* Monographs in Mathematics, vol 96., Birkhäuser, Basel-Boston-Berlin, 2001.

[3] W. Arendt, C. Batty, S. Bu, *Fourier multipliers for Hölder continuous functions and maximal regularity,* Studia Math. **160** (2004), 23-51.

[4] A. Bátkai, E. Fašanga, R. Shvidkoy,*Hyperbolicity of delay equations via Fourier multipliers*, Acta Sci. Math. (Szeged) **69** (2003), 131-145.

[5] A. Bátkai, S. Piazzera,*Semigroups for Delay Equations,* Research Notes in Mathematics, vol 10, A.K. Peters, Ltd., Boston, Mass., 2005.

[6] A. Bátkai, S. Piazzera, *Semigroups and linear partial differential equations with delay.* J. Math. Anal. Appl. **264** (2001), 1-20.

[7] R. Denk, M. Hieber, J. Prüss, *R-boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type,* Mem. Amer. Math. Soc. **166** (788), 2003.

[8] J.K. Hale, *Functional Differential Equations*, Appl. Math. Sci., **3**, Springer-Verlag, 1971.

[9] Y. Latushkin, F. Räbiger, *Operator valued Fourier multipliers and stability of strongly continuous semigroups.* Integral Equations Operator Theory **51** (2005), no. 3, 375–394.

[10] C. Lizama. *Fourier multipliers and periodic solutions of delay equations in Banach spaces.* J. Math. Anal. Appl. (to appear).

[11] M. Stein, H. Vogt, J. Vöigt, *The modulus semigroup for linear delay equations III.* J. Funct. Anal. **220** (2005), no. 2, 388–400.

[12] G. Webb, *Functional differential equations and nonlinear semigroups in L^p-spaces,* J. Differential Equations **29** (1976), 71-89.

[13] J. Wu, *Theory and Applications of Partial Differential Equations,* Appl. Math. Sci. **119**, Springer-Verlag, 1996.

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