ON MULTIPLICATIVE PERTURBATION OF INTEGRAL RESOLVENT FAMILIES

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Abstract. In this paper we study multiplicative perturbations for the generator of a strongly continuous integral resolvent family of bounded linear operators defined on a Banach space X. Assuming that $a(t)$ is a creep function which satisfies $a(0^+) > 0$, we prove that if (A, a) generates an integral resolvent, then $(A(I + B), a)$ also generates an integral resolvent for all $B \in B(X, Z)$, where Z belongs to a class of admissible Banach spaces. In special instances of $a(t)$ the space Z is proved to be characterized by an extended class of Favard spaces.

1. Introduction

First studies on multiplicative perturbation of unbounded linear operators began with the paper of Dorroh [6] and Gustafson [10] concerning perturbation of semigroups generators. Later, multiplicative perturbations have been discussed by a number of authors, such as Clément, Diekmann, Gyllenberg, Heijmans and Thieme [3]; Gustafson and Lumer [11]; Dorroh and Holderrieth [5]; and Desch and Schappacher [4]. In [18] Piskarev and Shaw prove a general multiplicative perturbation theorem for semigroup generators which subsumes some known results on multiplicative and additive perturbations. Multiplicative perturbation theorems for cosine operator functions were proved by Piskarev and Shaw in [19], and for resolvent families by Chang and Shaw in [1] and [2]. The last results were generalized by Xiao, Liang and van Casteren in [23] to deal with some time dependent perturbed equations.

On the other hand, in dealing with linear evolution equations, we note that important linear operators can be decomposed into the form $AP+R$ or $PA+R$ where A is the generator of a simpler structure and R is a bounded operator. Several examples of this type of decomposition of linear operators as multiplicative perturbation can be founded in the paper of Zabczyk [24] and Greiner [7].

Let A be a closed linear operator, defined in a Banach space X, for which a given linear integrodifferential equation with kernel a admits solution or, equivalently, the pair (A, a) is the generator of an integral resolvent (see

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Section 2). Let $B \in \mathcal{B}(X, Z)$. Our main objective in this paper is to give sufficient conditions under which $(A(I+B), a)$ is again the generator of an integral resolvent family. We are able to prove that if $a(t)$ is a creep function which satisfies $a(0^+) > 0$ then $(A(I+B), a)$ also generates an integral resolvent for all $B \in \mathcal{B}(X, Z)$. Here Z is a Banach space continuously embedded in X that satisfies certain conditions (Definition 4.1). For instance if, in addition, $a(t)$ is positive and exponentially bounded then Z coincides with the Favard class $F_{a,A}$ with kernel (Section 5) which we prove is characterized by:

$$F_{a,A} = \{x \in X : \sup_{\lambda > \omega} || \frac{1}{\hat{a}(\lambda)} A(\frac{1}{\hat{a}(\lambda)} - A)^{-1} x|| < \infty\},$$

see Corollary 5.10. In the particular case $a(t) \equiv 1$ our results are related to and extends those of [18], [7], [21] and [24].

This paper is organized as follows: In the next section, we give some preliminaries about the concept of integral resolvent generated by (A, a), and their relationship with a linear integral equation of Volterra type with scalar kernel a.

In Section 3 we define the class $M(A; R)$ of admissible operators for multiplicative perturbation and for operators in this class we prove our main result (Theorem 3.6).

In Section 4, we investigate conditions under which an operator belong to the class $M(A; R)$. In order to do this, we introduce condition (Z) with respect to the generator (A, a) of an integral resolvent $R(t)$. The importance of this condition is that if a Banach space X verify (Z) then, under certain conditions, operators defined with range on X are always admissible of multiplicative perturbation (Theorem 4.4). We define also the subclass (Z_p) for $1 \leq p < \infty$ and prove that if a Banach space X satisfy (Z_p), then it satisfies (Z) (Theorem 4.7).

In Section 5, making use of integral resolvents, we introduce an extended notion on Favard class $F_{a,A}$, depending of the kernel a. With an appropriate norm, the class $F_{a,A}$ become a Banach space and, imposing conditions only on the kernel a we are able to prove that $F_{a,A}$ satisfy condition Z_p for $p = 1$ (Theorem 5.5). Finally, for a subclass of kernels a the Favard class is characterized solely in terms of a and the resolvent operator of A (Theorem 5.8).

In Section 6, we give some examples and a theorem on additive perturbation which can be deduced from the previous results.

2. Preliminaries

Let X be a Banach space, A a closed linear operator with dense domain $D(A)$ defined in X and $a \in L^1_{loc}(\mathbb{R}_+)$. We consider the linear Volterra equation

$$u(t) = f(t) + \int_0^t a(t-s) A u(s)\, ds\,, \qquad t \in J, \tag{2.1}$$

where $f \in C(J,X)$, with $J := [0,T]$, $T > 0$ or $J = \mathbb{R}_+$.

We denote by $[D(A)]$ the domain of A equipped with the graph-norm.

We define the convolution product of the scalar function a with the vector-valued function f by

$$(a * f)(t) := \int_0^t a(t-s)f(s)\,ds \quad , \qquad t \in J\,.$$

Definition 2.1. *A function $u \in C(J,X)$ is called*

(a) *strong solution of (2.1) on J if $u \in C(J,[D(A)])$ and (2.1) is satisfied on J.*

(b) *mild solution of (2.1) on J if $a * u \in C(J,[D(A)])$ and*

$$u(t) = f(t) + A(a * u)(t)\,, \quad \text{on } J\,. \tag{2.2}$$

Obviously, every strong solution of (2.1) is a mild solution. Conditions under which mild solutions are strong solutions are studied in [20].

Definition 2.2. *Equation (2.1) is called well-posed if, for each $x \in D(A)$, there is a unique strong solution $u(t,x)$ on $\mathbb{R}_+$ of*

$$u(t,x) = a(t)x \,+\, (a * Au)(t)\,, \quad t \geq 0\,, \tag{2.3}$$

and for a sequence $(x_n) \subset D(A)$, $x_n \to 0$ implies $u(t,x_n) \to 0$ in X, uniformly on compact intervals.

Definition 2.3. *Let $a \in C(\mathbb{R}_+)$ be a scalar kernel. A family $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called an integral resolvent for (2.1) if the following three conditions are satisfied*

(R1) *$R(\cdot)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = a(0)I$*

(R2) *$R(t)x \in D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t \geq 0$*

(R3) *For each $x \in D(A)$ and $t \geq 0$:*

$$R(t)x = a(t)x + \int_0^t a(t-s)AR(s)x\,ds\,.$$

If $a(t) = 1$ for all $t \geq 0$, then $R(t)$ is a C_0−semigroup with generator A.

The concept of integral resolvent, as defined above, is closely related with the concept of resolvent family (see Prüss [20, Chapter I]). The study of some of their properties is included in some recent works of Lizama [13], [14], Lizama-Prado [15] and Lizama-Sanchez [16]. A closed but weaker definition was formulated by Prüss [20, definition 1.6]. For the scalar case, where there is a large bibliography, we refer to the monograph by Gripenberg, Londen and Staffans [9], and references therein.

Suppose that $R(t)$ is an integral resolvent for (2.1), let $f \in C(J,X)$ and $u \in C(J,X)$ be a mild solution for (2.1). Then $R * u$ is well defined and continuous. From equation (2.1) and using condition (R3), we obtain

$$a * u = (R - Aa * R) * u = R * u - R * Aa * u = R * f\,,$$

that is, $R*f \in C(J,[D(A)])$ and from (2.2) we obtain

$$u(t) = f(t) + A\int_0^t R(t-s)f(s)\,ds \quad , \qquad t \in J\,. \tag{2.4}$$

Hence, if there exists an integral resolvent for (2.1) then a mild solution for (2.1) may be obtained by formula (2.4).

The following result establishes the relation between well-posedness and existence of an integral resolvent. In what follows, $\mathcal{R}$ denotes the range of a given operator.

Theorem 2.4. *Equation (2.1) is well-posed if and only if (2.1) admits an integral resolvent* $R(t)$ *. If this is the case we have in addition* $\mathcal{R}(a*R(\cdot)) \subset D(A)$ *for all* $t \geq 0$ *, and*

$$R(t)x = a(t)x + A\int_0^t a(t-s)R(s)x\,ds \quad \textit{for each} \quad x \in X\,,\, t \geq 0\,. \tag{2.5}$$

The proof is closely related to [20, proposition 1.1] and therefore omitted.

Corollary 2.5. *Equation (2.1) admits at most one integral resolvent.*

The proof, making use of Tichmarsch's theorem is omitted.

Remark 2.6.

Recall from [20, Chapter I] that given $a \in L^1_{loc}(\mathbb{R}_+)$, a strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called *resolvent family with generator* A for equation (2.1), if the following three conditions hold.

(S1) $S(0) = I$,

(S2) $S(t)$ commutes with A, that is, $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$,

(S3) For each $x \in D(A)$ and $t \geq 0$, the following equation is satisfied

$$S(t)x = x + \int_0^t a(t-s)AS(s)x\,ds\ .$$

The importance of the resolvent family $S(t)$ is that, if exists, then the formula

$$u(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)\,ds\,, \quad t \in J$$

where $f \in W^{1,1}(J,X)$, give us, analogously to formula (2.4), a mild solution for equation (2.1).

If both, $S(t)$ and $R(t)$ exist for (2.1), and additionally $t \to S(t)x$ is differentiable for all $x \in X$, then the relations between S and R are given by $R(t)Ax = S'(t)x$ for $x \in D(A)$, $t \geq 0$, and $R(t)x = (a*S)'(t)x$ for $x \in X$, $t \geq 0$.

However, in general it is possible that $R(t)$ exists but not $S(t)$ and vice versa. The following criteria can be directly deduced from [13, Proposition 2.5] and will be used in a forthcoming section.

Proposition 2.7. *Assume (2.1) admits an integral resolvent with* $a \in AC(\mathbb{R}_+)$ *and* $a(0) \neq 0$. *Then (2.1) admits a resolvent family.*

Assuming the existence of an integral resolvent family $\{R(t)\}_{t\geq 0}$, it is natural to ask how to characterize the domain $D(A)$ of the operator A in terms of the integral resolvent family. The following result was proved in [16].

Theorem 2.8. *Let* A *be a closed linear operator densely defined in a Banach space* X. *Let* $a(t)$ *be a continuous, positive and nondecreasing function. Suppose that (2.1) admits an integral resolvent* $\{R(t)\}_{t\geq 0}$, *which satisfies* $\overline{\lim_{t\to 0^+}} \dfrac{||R(t)||}{a(t)} < \infty$. *Then*

(a) $D(A) = \{\, x \in X \ : \ \lim_{t\to 0^+} \dfrac{R(t)x - a(t)x}{(a*a)(t)}$ *exists* $\}$,

(b) $\lim_{t\to 0^+} \dfrac{R(t)x - a(t)x}{(a*a)(t)} = Ax$ *for all* $x \in D(A)$.

In view of this result, in what follows instead to say that (2.1) admits an integral resolvent we will say that *the pair* (A, a) *is a generator of an integral resolvent* $R(t)$.

3. Multiplicative Perturbation

In this section we assume that $a \in C(\mathbb{R}_+)$ is Laplace transformable and there exists a constant $\omega \in \mathbb{R}$, such that $\hat{a}(\lambda) \neq 0$ for all $\lambda > \omega$.

Let X be a Banach space and let A be a closed linear operator defined in X with dense domain $D(A)$.

Henceforth, we suppose that $R(t)$ is an exponentially bounded integral resolvent, that is there exists $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|R(t)\| \leq Me^{\omega t}, \quad t \geq 0,$$

and we call the pair (M, ω) the *type* of the integral resolvent family.

The following proposition stated in [13], establishes the relation between integral resolvents and Laplace transforms.

Proposition 3.1. *Let* $R(t)$ *be a strongly continuous and exponentially bounded family of linear operators in* $\mathcal{B}(X)$ *such that the Laplace transform* $\hat{R}(\lambda)$ *exists for* $\lambda > \omega$. *Then* $R(t)$ *is an integral resolvent with generator* (A, a) *if and only if*

(i) $\dfrac{1}{\hat{a}(\lambda)} \in \rho(A)$ *for all* $\lambda > \omega$.

(ii) $\hat{a}(\lambda)(I - \hat{a}(\lambda)A)^{-1}x = \int_0^\infty e^{-\lambda s} R(s)x\, ds$ *for all* $x \in X$ *and all* $\lambda > \omega$.

Let $C \in \mathcal{B}(X)$ be a bounded operator and suppose that (A,a) generates an integral resolvent. In this section we want to answer the following question. Under which conditions (AC,a) with $D(AC)=\{x \in X : Cx \in D(A)\}$ generates an integral resolvent?

Our next definition states a class of admissible operators to give a positive answer.

Definition 3.2. *Let (A,a) be the generator of an integral resolvent $\{R(t)\}_{t\geq 0}$. We say that an operator $C \in \mathcal{B}(X)$ belongs to the class $M(A,R)$ if the operator $B = C - I$ satisfy the following condition: For all continuous function $f \in C([0,\infty),X)$ and all $h \geq 0$ we have*

(Ma) $\displaystyle\int_0^h R(h-s)Bf(s)\,ds \in D(A)$,

(Mb) there exists $\omega \in \mathbb{R}_+$ such that

$$||A\int_0^h R(t+h-s)Bf(s)\,ds\,|| \leq e^{\omega t}\gamma_B(h)||f||_{[0,h]}\,,$$

for all $t \geq 0$, where $\gamma_B : [0,+\infty) \to [0,+\infty)$ is a continuous, non-decreasing function with $\gamma_B(0)=0$.

Remark 3.3.

(1) We denote by $||f||_{[0,h]} = \sup\limits_{0\leq s\leq h}||f(s)||$ the norm of $f \in C([0,h],X)$.

(2) From the identity

$$\int_0^{t+h} R(t+h-s)Bf(s)\,ds = \int_0^h R(t+h-s)Bf(s)\,ds + \int_0^t R(t-s)Bf(s+h)\,ds\,,$$

and condition (Ma), we obtain that $\displaystyle\int_0^h R(t+h-s)Bf(s)\,ds \in D(A)$ for all $h \geq 0$ and all $t \geq 0$.

(3) We note that condition (Mb) implies the following condition (Mb'):

$$||A\int_0^h R(h-s)Bf(s)\,ds\,|| \leq \gamma_B(h)||f||_{[0,h]}\,,\ h \geq 0\,,\ f \in C([0,h],X)\,,$$

where $\gamma_B : [0,+\infty) \to [0,+\infty)$ is a continuous non-decreasing function with $\gamma_B(0)=0$, see [18]. For C_0-semigroups we have that (Mb) is equivalent to (Mb').

(4) A natural question is when (CA,a) is also the generator of an integral resolvent. In [21] Rhandi has proved that problem (2.1) with (AC,a) and (CA,a) are essentially equivalent, requiring only $\rho(CA) \neq \emptyset$.

In what follows we denote by $M(A,R)-I = \{\,C-I \,:\, C \in M(A,R)\,\}$, where I is the identity operator.

Lemma 3.4. *Let (A,a) be the generator of an integral resolvent $\{R(t)\}_{t\geq 0}$ of type (M,ω). Let $B \in M(A,R)-I$ and $f \in C([0,\infty),X)$ such that*

$e^{-\mu t}||f(t)||$ is bounded on $[0,\infty)$ for some $\mu>\omega$. Let $h>0$ be fixed. Then for all $t>0$ we have

$$e^{-\mu t}||A\int_0^t R(t-s)Bf(s)\,ds|| \le \gamma_B(h)\frac{e^{(\mu-w)h}}{1-e^{-(\mu-\omega)h}}\sup_{s\in[0,t]}||e^{-\mu s}f(s)||\,.$$

Proof. Let $t=kh+\tau$ where k is an integer positive and $\tau\in[0,h)$. Since by assumption $B\in M(A,R)-I$, we have that

$$\begin{aligned}
&||A\int_0^t R(t-s)Bf(s)\,ds||\\
&=||A\int_0^{kh} R(t-s)Bf(s)\,ds \;+\; A\int_{kh}^{kh+\tau} R(t-s)Bf(s)\,ds||\\
&=||A\int_0^{kh} R(t-s)Bf(s)\,ds \;+\; A\int_0^{\tau} R(t-s-kh)Bf(s+kh)\,ds||\\
&=||\sum_{j=0}^{k-1} A\int_{jh}^{jh+h} R(t-s)Bf(s)\,ds \;+\; A\int_0^{\tau} R(t-s-kh)Bf(s+kh)\,ds||\\
&\le \sum_{j=0}^{k-1}||A\int_0^h R(t-s-jh)Bf(s+jh)\,ds||\\
&\quad +||A\int_0^{\tau} R(t-s-kh)Bf(s+kh)\,ds||\\
&=\sum_{j=0}^{k-1}||A\int_0^h R(t-jh-h+h-s)Bf(s+jh)\,ds||\\
&\quad +||A\int_0^{\tau} R(t-kh-\tau+\tau-s)Bf(s+kh)\,ds||\\
&\le \sum_{j=0}^{k-1} e^{\omega(t-jh-h)}\,\gamma_B(h)\,||f||_{[0,h+jh]} \;+\; e^{\omega(t-kh-\tau)}\gamma_B(\tau)||f||_{[0,t]}\\
&=e^{\omega t}\,\gamma_B(h)\sum_{j=0}^{k-1} e^{-\omega(h+jh)}\,||f||_{[0,h+jh]} \;+\; \gamma_B(\tau)||f||_{[0,t]}\\
&=e^{\omega t}\,\gamma_B(h)\sum_{j=0}^{k-1} e^{(\mu-\omega)(h+jh)}e^{-\mu(h+jh)}||f||_{[0,h+jh]} \;+\; \gamma_B(\tau)e^{\mu t}e^{-\mu t}||f||_{[0,t]}\\
&\le \sup_{s\in[0,t]}\|e^{-\mu s}f(s)\|e^{\omega t}\,\gamma_B(h)\sum_{j=0}^{k-1} e^{(\mu-\omega)(h+jh)} \;+\; \sup_{s\in[0,t]}\|e^{-\mu s}f(s)\|\gamma_B(\tau)e^{\mu t}\\
&=\sup_{s\in[0,t]}\|e^{-\mu s}f(s)\|\left[e^{\omega t}\,\gamma_B(h)\sum_{j=0}^{k-1} e^{(\mu-\omega)(h+jh)} \;+\; e^{\omega t}\gamma_B(\tau)e^{(\mu-\omega)(kh+\tau)}\right]\\
&\le \sup_{s\in[0,t]}\|e^{-\mu s}f(s)\|\left[e^{\omega t}\,\gamma_B(h)\sum_{j=0}^{k-1} e^{(\mu-\omega)(h+jh)} \;+\; e^{\omega t}\gamma_B(h)e^{(\mu-\omega)(kh+h)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{s\in[0,t]} \|e^{-\mu s} f(s)\| e^{\omega t}\, \gamma_B(h) \sum_{j=1}^{k+1} e^{(\mu-\omega)jh} \\
&= \sup_{s\in[0,t]} \|e^{-\mu s} f(s)\| e^{\omega t}\, \gamma_B(h) \left[\frac{e^{(\mu-\omega)h}}{1-e^{(\mu-\omega)h}} - \frac{e^{(\mu-\omega)h(k+2)}}{1-e^{(\mu-\omega)h}} \right] \\
&\le \sup_{s\in[0,t]} \|e^{-\mu s} f(s)\| e^{\omega t}\, \gamma_B(h) \frac{-e^{(\mu-\omega)h(k+2)}}{1-e^{(\mu-\omega)h}} \\
&\le \sup_{s\in[0,t]} \|e^{-\mu s} f(s)\| e^{\mu t}\, \gamma_B(h) \frac{e^{2(\mu-\omega)h}}{e^{(\mu-\omega)h}-1} \\
&= \sup_{s\in[0,t]} \|e^{-\mu s} f(s)\| e^{\mu t}\, \gamma_B(h) \frac{e^{(\mu-w)h}}{1-e^{-(\mu-\omega)h}} \,,
\end{aligned}
$$

proving the Lemma. ■

Proposition 3.5. *Let (A,a) be the generator of an integral resolvent $\{R(t)\}_{t\ge 0}$ of type (M,ω). Let $B \in M(A,R) - I$. Then there exists a strongly continuous and exponentially bounded family $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$ which satisfies the equation*

$$S(t)x = R(t)x + A\int_0^t R(t-s)BS(s)x\,ds\,,$$

for all $t \ge 0$ and $x \in X$.

Proof. Let E be the Banach space

$$E = \{\, f \in C([0,\infty), X) \;:\; e^{-\mu t}||f(t)|| \text{ is bounded} \,\}$$

with norm

$$||f||_E := \sup_{t\in[0,\infty)} e^{-\mu t}||f(t)||\,.$$

For each $x \in X$, we define the operator $T_x : E \to E$ by

$$T_x(f)(t) = R(t)x + A\int_0^t R(t-s)Bf(s)\,ds\,, \quad t \ge 0\,.$$

Note that $T_x(f) \in E$. This follows from (Mb) and the following identity (cf. Remark 3.3(2))

$$
\begin{aligned}
&A\int_0^{t+h} R(t+h-s)Bf(s)ds - A\int_0^t R(t-s)Bf(s)ds \\
&= A\int_0^h R(t+h-s)Bf(s) + A\int_0^t R(t-s)B[f(s+h)-f(s)]ds.
\end{aligned}
$$

We claim that T_x is a contraction on E. In fact, by Lemma 3.4, we have

$$\begin{aligned} e^{-\mu t}||T_x(f)(t)-T_x(g)(t)|| &= e^{-\mu t}||A\int_0^t R(t-s)B(f-g)(s)\,ds|| \\ &\le \gamma_B(h)\,\frac{e^{(\mu-w)h}}{1-e^{-(\mu-\omega)h}}\sup_{s\in[0,t]} e^{-\mu s}||(f-g)(s)|| \\ &\le \frac{1}{8}\frac{2^2}{2-1}\sup_{s\in[0,\infty)} e^{-\mu s}||(f-g)(s)||\,, \end{aligned}$$

for h sufficiently small and μ sufficiently large such that $\gamma_B(h)<\frac{1}{8}$ and $e^{(\mu-w)h}=2$. From this inequality follows that

$$||T_x(f)-T_x(g)||_E \le \frac{1}{2}||f-g||_E\,.$$

Let $f_x\in E$ be the unique fixed point of T_x and define $S(t)x=T_x(f_x)(t)$. That $S(t)$ thus defined is a linear operator follows easily from the uniqueness of fixed point of T_x for every $x\in X$. The claim follows from the strong continuity of $R(t)$. ∎

The previous result, enable us to work with the method of Laplace transforms to prove, in the next theorem, the main result of this paper.

Theorem 3.6. *Let $a\in C(\mathbb{R}_+)$ be Laplace transformable and satisfying $\hat{a}(\lambda)\neq 0$ for $\lambda>\omega$. Let (A,a) be the generator of an integral resolvent $\{R(t)\}_{t\ge 0}$ of type (M,ω) and $B\in M(A,R)-I$. Then $(A(I+B),a)$ is the generator of an exponentially bounded integral resolvent $\{S(t)\}_{t\ge 0}$.*

Proof. Let $B=C-I$, with $C\in M(A,R)$. To prove the Theorem we use Proposition 3.1. We first claim that for $\lambda>\omega$, the operator $\hat{a}(\lambda)^{-1}I-A(I+B)$ is invertible. In order to prove the claim, we choose τ such that $\gamma_B(\tau)<1/2$. For all $x\in X$, $\lambda>\omega$ and using the condition (Mb), we obtain

$$\begin{aligned} ||A\ \hat{R}(\lambda)\ Bx|| &= ||A\int_0^\infty e^{-\lambda s}R(s)Bx\,ds|| \\ &= ||\sum_{j=0}^\infty A\int_{j\tau}^{(j+1)\tau} e^{-\lambda s}R(s)Bx\,ds|| \\ &= ||\sum_{j=0}^\infty A\int_0^\tau e^{-\lambda(s+j\tau)}R(s+j\tau)Bx\,ds|| \\ &= ||\sum_{j=0}^\infty e^{-\lambda j\tau}A\int_0^\tau e^{-\lambda s}R(s+j\tau)Bx\,ds|| \\ &= ||\sum_{j=0}^\infty e^{-\lambda j\tau}A\int_0^\tau e^{-\lambda(\tau-s)}R(j\tau+\tau-s)Bx\,ds|| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{\infty} e^{-\lambda j\tau}||A\int_0^{\tau} R(j\tau+\tau-s)B(e^{-\lambda(\tau-s)})x\,ds|| \\
&\leq \sum_{j=0}^{\infty} e^{-\lambda j\tau}e^{j\tau\omega}\gamma_B(\tau)||e^{-\lambda(\tau-\cdot)}x||_{[0,\tau]} \\
&\leq \sum_{j=0}^{\infty} e^{-(\lambda-\omega)j\tau}\gamma_B(\tau)||x|| \\
&= (1-e^{-(\lambda-\omega)\tau})^{-1}\gamma_B(\tau)||x||\,.
\end{aligned}
$$

Let $\omega_1 > \omega$ be such that $e^{-(\omega_1-\omega)\tau} < 1/2$. Then

$$||A\ \hat{R}(\lambda)\ B|| < (1-e^{-(\omega_1-\omega)\tau})^{-1}\gamma_B(\tau) < 1$$

for all $\lambda > \omega_1$. Hence, the series $\sum_{j=0}^{\infty}[A\hat{R}(\lambda)B]^j$ converges in $\mathcal{B}(X)$, for all $\lambda > \omega_1$.

Let $J(\lambda) := \sum_{j=0}^{\infty}\ [A\hat{R}(\lambda)B]^j\ \hat{R}(\lambda)$. We will prove that

$$(\hat{a}(\lambda)^{-1}I - A(I+B))\,J(\lambda)x = x$$

for all $x \in X$. To this end, we note that for all $x \in X$, we have

$$
\begin{aligned}
(I+B)\,J(\lambda)x\ &= \sum_{j=0}^{\infty}\ [A\hat{R}(\lambda)B]^j\ \hat{R}(\lambda)x\ +\ \sum_{j=0}^{\infty}\ B[A\hat{R}(\lambda)B]^j\ \hat{R}(\lambda)x \\
&= \hat{R}(\lambda)x\ +\ \sum_{j=0}^{\infty}\ [A\hat{R}(\lambda)+I]\,B\,[A\hat{R}(\lambda)B]^j\ \hat{R}(\lambda)x \\
&= \hat{R}(\lambda)x + \hat{a}(\lambda)^{-1}\hat{R}(\lambda)BJ(\lambda)x\ \in D(A)\,.
\end{aligned}
$$

Hence $(I+B)\,J(\lambda)$ maps X in $D(A)$. In particular $D(A(I+B)) \neq \emptyset$, and for all $x \in X$,

$$
\begin{aligned}
&(\hat{a}(\lambda)^{-1}I - A(I+B))\,J(\lambda)x \\
&= \hat{a}(\lambda)^{-1}J(\lambda)x - A(I+B)J(\lambda)x \\
&= \hat{a}(\lambda)^{-1}J(\lambda)x - A(\hat{R}(\lambda)x + \hat{a}(\lambda)^{-1}\hat{R}(\lambda)BJ(\lambda)x) \\
&= \hat{a}(\lambda)^{-1}J(\lambda)x - A\hat{R}(\lambda)x - \hat{a}(\lambda)^{-1}A\hat{R}(\lambda)BJ(\lambda)x \\
&= -A\hat{R}(\lambda)x + (I - A\hat{R}(\lambda)B)\hat{a}(\lambda)^{-1}J(\lambda)x \\
&= -A\hat{R}(\lambda)x + (I + B - \hat{a}(\lambda)^{-1}\hat{R}(\lambda)B)\hat{a}(\lambda)^{-1}J(\lambda)x \\
&= -A\hat{R}(\lambda)x + (I+B)\hat{a}(\lambda)^{-1}J(\lambda)x - \hat{a}(\lambda)^{-2}\hat{R}(\lambda)BJ(\lambda)x \\
&= -A\hat{R}(\lambda)x + \hat{a}(\lambda)^{-1}\hat{R}(\lambda)x + \hat{a}(\lambda)^{-2}\hat{R}(\lambda)BJ(\lambda)x - \hat{a}(\lambda)^{-2}\hat{R}(\lambda)BJ(\lambda)x \\
&= (\hat{a}(\lambda)^{-1}I - A)\hat{R}(\lambda)x = x\,.
\end{aligned}
$$

Next, we will verify that

$$J(\lambda) = (\hat{a}(\lambda)^{-1}I - A(I+B))^{-1}\,.$$

In fact, we have for all $x \in D(A(I+B))$

$$J(\lambda)[\hat{a}(\lambda)^{-1}I - A(I+B)]x \ = \sum_{j=0}^{\infty} [A\hat{R}(\lambda)B]^j \ \hat{R}(\lambda)[\hat{a}(\lambda)^{-1}I - A(I+B)]x$$

$$\begin{aligned} &= \sum_{j=0}^{\infty} [A\hat{R}(\lambda)B]^j \ [\hat{R}(\lambda)\hat{a}(\lambda)^{-1} - \hat{R}(\lambda)A(I+B)]x \\ &= \sum_{j=0}^{\infty} [A\hat{R}(\lambda)B]^j \ [\hat{a}(\lambda)^{-1}\hat{R}(\lambda) - A\hat{R}(\lambda)(I+B)]x \\ &= \sum_{j=0}^{\infty} [A\hat{R}(\lambda)B]^j \ [\hat{a}(\lambda)^{-1}\hat{R}(\lambda) - (\hat{a}(\lambda)^{-1}\hat{R}(\lambda) - I)(I+B)]x \\ &= \sum_{j=0}^{\infty} [A\hat{R}(\lambda)B]^j \ [I + B - \hat{a}(\lambda)^{-1}\hat{R}(\lambda)B]x \\ &= \sum_{j=0}^{\infty} [A\hat{R}(\lambda)B]^j \ [I - A\hat{R}(\lambda)B]x = x\,. \end{aligned}$$

It shows that $(\hat{a}(\lambda)^{-1}I - A(I+B))$ is invertible for $\lambda > \omega$ proving the claim.

From Proposition 3.5 , we have that the solution $\{S(t)\}_{t\geq 0}$ of the equation

$$S(t)x = R(t)x + A\int_0^t R(t-s)BS(s)x\,ds\,, \quad x \in X\,,\ t \geq 0\,,$$

is strongly continuous and exponentially bounded, that is, there are constants $K > 0$ and $\omega \in \mathbb{R}$ such that $||S(t)|| < Ke^{\omega t}$, for each $t \geq 0$, and hence there exists its Laplace transform. Also by hypothesis $R(t)$ is Laplace transformable, and hence by the convolution theorem we have

$$\hat{S}(\lambda)x = \hat{R}(\lambda)x + A\hat{R}(\lambda)B\hat{S}(\lambda)x\,,$$

for all $x \in X$ and $\lambda > \omega$.
Hence, for all $x \in X$ and $\lambda > \omega$;

$$\begin{aligned} \hat{S}(\lambda)x \ &= (\hat{a}(\lambda)^{-1}I - A)^{-1}x + A(\hat{a}(\lambda)^{-1}I - A)^{-1}B\hat{S}(\lambda)x \\ &= (\hat{a}(\lambda)^{-1}I - A)^{-1}x + [\hat{a}(\lambda)^{-1}(\hat{a}(\lambda)^{-1}I - A)^{-1} - I]B\hat{S}(\lambda)x \\ &= (\hat{a}(\lambda)^{-1}I - A)^{-1}x + \hat{a}(\lambda)^{-1}(\hat{a}(\lambda)^{-1}I - A)^{-1}B\hat{S}(\lambda)x - B\hat{S}(\lambda)x\,. \end{aligned}$$

Therefore

$$(I+B)\hat{S}(\lambda)x = (\hat{a}(\lambda)^{-1}I - A)^{-1}x + \hat{a}(\lambda)^{-1}(\hat{a}(\lambda)^{-1}I - A)^{-1}B\hat{S}(\lambda)x \in D(A)\,.$$

We conclude that $\mathcal{R}((I+B)\hat{S}(\lambda)) \subset D(A)$. Hence we can apply $(\hat{a}(\lambda)^{-1}I - A)$ in the above identity to obtain

$$\begin{aligned} x &= (\hat{a}(\lambda)^{-1}I - A)(I+B)\hat{S}(\lambda)x - \hat{a}(\lambda)^{-1}B\hat{S}(\lambda)x \\ &= [(\hat{a}(\lambda)^{-1}I - A)(I+B) - \hat{a}(\lambda)^{-1}B]\hat{S}(\lambda)x \\ &= [\hat{a}(\lambda)^{-1}(I+B) - A(I+B) - \hat{a}(\lambda)^{-1}B]\hat{S}(\lambda)x \\ &= [\hat{a}(\lambda)^{-1}I - A(I+B)]\hat{S}(\lambda)x\,. \end{aligned}$$

Since by the claim $(\hat{a}(\lambda)^{-1}I - A(I+B))$ is invertible, we obtain that

$$\hat{S}(\lambda)x = (\hat{a}(\lambda)^{-1} - A(I+B))^{-1}x\,.$$

Hence, applying Proposition 3.1, we conclude that $S(t)$ is an integral resolvent generated by $(A(I+B), a)$.

□

The particular case $a(t) = 1$ recover the following result due to Desch and Schappacher [4].

Corollary 3.7. *Let A be the generator of a C_0-semigroup $T(t)$ defined in a Banach space X. Let $B \in M(A,T) - I$. Then $A(I+B)$ generates a C_0-semigroup in X.*

Proof. Taking $a(t) \equiv 1$ we have $T(t)x = a(t)x + \int_0^t a(t-s)AT(s)x\,ds$ for all $x \in D(A)$ and $t \geq 0$. Hence $T(t)$ is an integral resolvent. Since $C = I + B$ satisfy the condition $M(A,T)$, then by Proposition 3.5 and Theorem 3.6, $(A(I+B), 1)$ generates an integral resolvent $\{S(t)\}_{t\geq 0}$ strongly continuous and exponentially bounded. Note also that, because $a(t) \equiv 1$, we have $S(t)x = x + \int_0^t A(I+B)S(s)x\,ds$ for $x \in D(A(I+B))$. Hence $S'(t)x = A(I+B)S(t)x$ and $S(0)x = x$, for $x \in D(A(I+B))$. Therefore by a classical result on C_0-semigroups (see e.g. [17]), $S(t)$ is a C_0-semigroup with generator $A(I+B)$.

□

In contrast with the above corollary the next result is new for the theory of sine families.

Corollary 3.8. *Let A be the generator of a sine family $\{S(t)\}_{t\geq 0}$ in X and $B \in M(A,S) - I$. Then $A(I+B)$ generates a sine family.*

Proof. Take $a(t) \equiv t$. For each $x \in D(A)$ and $t \geq 0$ we have $S(t)x = tx + \int_0^t (t-s)AS(s)x\,ds$. Therefore, $S(t)$ is an integral resolvent with generator (A,t). By Theorem 3.6, $(A(I+B), t)$ is the generator of an exponentially bounded integral resolvent $\{Si(t)\}_{t\geq 0}$. Hence for all $x \in D(A)$ and $t \geq 0$, we have

$$Si(t)x = tx + \int_0^t (t-s)A(I+B)Si(s)x\,ds\,.$$

Finally, is clear from the definition and classical results of cosine families (see e.g. [22]), that $Si(t)$ is a sine family generated by $A(I+B)$.

□

4. Sufficient conditions for $M(A,R)$

In what follows we investigate conditions under which we verify the hypothesis of the multiplicative perturbation theorem.

Definition 4.1. *We say that a Banach space $(Z, |\cdot|)$ satisfy condition (Z) with respect to the generator (A,a) of an integral resolvent $\{R(t)\}_{t\geq 0}$ if the following three conditions are satisfied.*

(Za) *Z is continuously embedded in X.*

(Zb) *For all $\phi \in C([0,h],Z)$ we have*

$$\int_0^h R(h-s)\phi(s)\,ds \in D(A)\,,\ h \geq 0\,.$$

(Zc) *There exists $\omega \in \mathbb{R}_+$ such that*

$$||A\int_0^h R(t+h-s)\phi(s)\,ds\,|| \leq e^{\omega t}\gamma(h) \sup_{0\leq s\leq h} |\phi(s)|_Z\,,$$

for all $h \geq 0$, $t \geq 0$, where $\gamma : [0,+\infty) \to [0,+\infty)$ is a continuous nondecreasing function with $\gamma(0) = 0$.

If X is a Banach space, then $\mathcal{B}(X,Z)$ will denote the set of all linear and bounded operators from X to Z.

Theorem 4.2. *Let Z be a Banach space that satisfies condition (Z) with respect to the generator (A,a) of an exponentially bounded integral resolvent $R(t)$. Then $I + \mathcal{B}(X,Z) \subset M(A,R)$.*

Proof. Let $C \in I + \mathcal{B}(X,Z)$, that is, $C = I + B$ with $B \in \mathcal{B}(X,Z)$. Let $h \geq 0$ and $f \in C([0,h],X)$. We define $\phi(s) := Bf(s)$. Clearly $\phi \in C([0,h],Z)$, and hence condition (Zb) implies that (Ma) is satisfied, that is, $\int_0^h R(t+h-s)Bf(s)\,ds \in D(A)$. (See Remark 3.3 (2)). On the other hand, by (Zc) there exists $\omega \in \mathbb{R}_+$ such that

$$\begin{aligned} ||A\int_0^h R(t+h-s)\phi(s)\,ds\,|| &\leq e^{\omega t}\gamma(h) \sup_{s\in[0,h]} ||Bf(s)||_Z \\ &\leq e^{\omega t}\,\gamma(h)\,||B||\,\sup_{s\in[0,h]} ||f(s)||_X \\ &\leq e^{\omega t}\,\gamma(h)\,||B||\,||f||_{[0,h]} \end{aligned}$$

Therefore defining $\gamma_B(h) = ||B||\gamma(h)$, we obtain that γ_B is a continuous function, non decreasing and $\gamma_B(0) = 0$. The proof is complete.

□

Remark 4.3.

Usually, the space Z corresponds to the domain of A equipped with the graph norm. The following result is a direct application of Theorem 3.6.

Theorem 4.4. *Let Z be a Banach space with satisfy condition (Z) with respect to the generator (A,a) of an exponentially bounded integral resolvent $R(t)$, then for all $B \in \mathcal{B}(X,Z)$ the pair $(A(I+B),a)$ is the generator of an integral resolvent.*

Proof. Let $B \in \mathcal{B}(X,Z)$, by Theorem 4.2 we have $C = I+B \in M(A,R)$. Applying Theorem 3.6 we conclude that $(A(I+B),a)$ is the generator of an integral resolvent.

□

Taking $a(t) \equiv 1$, we obtain the following result, which corresponds to Corollary 2.3 in [18].

Corollary 4.5. *Let Z be a Banach space which satisfies condition (Z) with respect to the generator A of a C_0-semigroup, then for all $B \in \mathcal{B}(X,Z)$, the operator $A(I+B)$ is the generator of a C_0-semigroup.*

Definition 4.6. *Let $1 \le p < \infty$ and (A,a) the generator of an integral resolvent $R(t)$ on X. Suppose that $(Z,|\cdot|)$ is a Banach space continuously embedded in X.*

We say that $(Z,|\cdot|)$ satisfy condition (Z_p) with respect to A if there is some $T>0$ such that for all $\xi \in L^p([0,T],Z)$ the following two conditions are satisfied

(Z_pa) *For all $t_1 \ge 0$, we have* $\displaystyle\int_0^T R(T+t_1-s)\xi(s)\,ds \in D(A)$,

(Z_pb) *There exists $\omega \in \mathbb{R}_+$ such that*

$$||A\int_0^T R(T+t_1-s)\xi(s)\,ds|| \le Ne^{\omega t_1} \left(\int_0^T |\xi(s)|_Z^p\,ds\right)^{1/p},$$

for all $t_1 \ge 0$.

Theorem 4.7. *For all $1 \le p < \infty$, condition (Z_p) implies condition (Z).*

Proof. By definition condition (Za) is satisfied. Let $\phi \in C([0,\infty),Z)$. and $t \ge 0$. Then

$$\int_0^h R(h+t-s)\phi(s)\,ds \in D(A). \tag{4.1}$$

In fact, given $T>0$ and for $h \ge 0$ fixed there exists $n \in \mathbb{N}$ such that $h = nT + r$ where $0 \le r < T$.

We extend ϕ to the negative real axis as $\phi(s) = 0$ for $s<0$. Then for $0 \le r < T$, we have

$$\int_0^r R(r+t-s)\phi(s)\,ds = \int_{T-r}^T R(T+t-s)\phi(s-T+r)\,ds \tag{4.2}$$

$$= \int_{T-r}^{T} R(T+t-s)\phi(s-T+r)\,ds + \int_{0}^{T-r} R(T+t-s)\phi(s-T+r)\,ds$$
$$= \int_0^T R(T+t-s)\phi(s-T+r)\,ds\,,$$

where by $(Z_p\mathrm{a})$, the last integral belongs to the domain of the operator A because $\xi(s) := \phi(s-T+r)$, $s \in [0,T]$ is such that $\xi \in L^p([0,T],Z)$. Hence

$$\int_0^h R(h+t-s)\phi(s)ds \tag{4.3}$$
$$= \sum_{j=0}^{n-1} \int_{jT}^{(j+1)T} R(h+t-s)\phi(s)ds + \int_{nT}^{h} R(h+t-s)\phi(s)ds$$
$$= \sum_{j=0}^{n-1} \int_0^T R(T+h+t-(j+1)T-s)\phi(s+jT)ds$$
$$+ \int_0^r R(r+t-s)\phi(s+nT)\,ds\,.$$

Using the hypothesis $(Z_p\mathrm{a})$ with $t_1 = h+t-(j+1)T$ and $\xi(s) := \phi(s+jT)$ we conclude that each integral term in the sum belongs to $D(A)$. Analogously, according (4.2) the second integral is in $D(A)$. Hence (4.1) holds. In particular taking $t=0$ we obtain (Zb).

In order to verify (Zc) we observe that, by $(Z_p\mathrm{b})$, there exists $\omega \in \mathbb{R}_+$ such that for all $t_1 \geq 0$,

$$||A\int_0^T R(T+t_1-s)\xi(s)\,ds\,|| \leq Ne^{\omega t_1}\,||\xi||_{L^p([0,T],Z)}\,. \tag{4.4}$$

Let $\phi \in C([0,\infty),Z)$. We extend as before ϕ to the negative real axis as $\phi(s)=0$. Then from (4.2) and (4.4) we obtain

(4.5)
$$\begin{aligned} ||A\int_0^r R(r+t-s)\phi(s)\,ds|| &= ||A\int_0^T R(T+t-s)\phi(s-T+r)\,ds|| \\ &\leq Ne^{\omega t}\left(\int_0^T |\phi(s-T+r)|_Z^p\,ds\right)^{1/p} \\ &= Ne^{\omega t}\left(\int_0^r |\phi(s)|_Z^p\,ds\right)^{1/p}, \end{aligned}$$

where $0 \leq r < T$, and $h = nT + r$.

Hence from (4.3) and making use of (4.4) and (4.5) we have

$$||A\int_0^h R(h+t-s)\phi(s)ds||$$

$$\begin{aligned}
&\leq \sum_{j=0}^{n-1} ||A\int_0^T R(T+h+t-(j+1)T-s)\phi(s+jT)\,ds|| \\
&+ ||A\int_0^r R(r+t-s)\phi(s+nT)\,ds|| \\
&\leq \sum_{j=0}^{n-1} Ne^{\omega(h+t-(j+1)T)}\left(\int_0^T |\phi(s+jT)|_Z^p\,ds\right)^{1/p} \\
&+ Ne^{\omega t}\left(\int_0^r |\phi(s+nT)|_Z^p\,ds\right)^{1/p}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&||A\int_0^h R(h+t-s)\phi(s)ds|| \\
&\leq \sum_{j=0}^{n-1} Ne^{\omega(h+t-(j+1)T)}\left(\int_0^T |\sup_{jT<v<(j+1)T}\phi(v)|_Z^p\,ds\right)^{1/p} \\
&+ Ne^{\omega t}\left(\int_0^r |\sup_{nT<v<nT+r}\phi(v)|_Z^p\,ds\right)^{1/p} \\
&\leq \sum_{j=0}^{n-1} Ne^{\omega(h+t-(j+1)T)}T^{1/p}\sup_{0<s<h}|\phi(s)|_Z + Ne^{\omega t}r^{1/p}\sup_{0<s<h}|\phi(s)|_Z \\
&= Ne^{\omega t}\,[\sum_{j=0}^{n-1} e^{\omega(h-(j+1)T)}T^{1/p} + r^{1/p}\,]\sup_{0<s<h}|\phi(s)|_Z \\
&\leq Ne^{\omega t}\,[\,ne^{\omega h}T^{1/p} + r^{1/p}\,]\sup_{0<s<h}|\phi(s)|_Z\,.
\end{aligned}$$

Let $\gamma(h) := Nne^{\omega h}T^{1/p} + Nr^{1/p}$. We observe that γ is continuous and non decreasing. Moreover, if $h = nT + r = 0$, then $n = 0$ and $r = 0$ therefore $\gamma(0) = 0$. This proves that (Zc) is satisfied and the proof is complete.

□

In a forthcoming section we will apply the above result. To this end we need to introduce a new class of spaces. This is the objective of the next section.

5. A Favard class with kernel

The following definition corresponds to a natural extension, in our context, of the Favard class frequently used in approximation theory for semigroups.

Definition 5.1. *Let $a(t)$ be a continuous and positive scalar function. Let (A,a) be the generator of a bounded integral resolvent $\{R(t)\}_{t\geq 0}$ on X.*

We define the Favard class of A with kernel $a(t)$, by

$$F_{a,A} = \{ \; x \in X \quad : \quad \sup_{t>0} \frac{||R(t)x - a(t)x||}{(a*a)(t)} < \infty \; \} \tag{5.1}$$

Remark 5.2.

(1) From the definition it is clear that $D(A) \subset F_{a,A}$. In this way, for different functions $a(t)$ we obtain different Favard classes which may be considered as extrapolation spaces between $D(A)$ and X.

(2) For $a(t) \equiv 1$, and $T(t)$ a bounded C_0-semigroup generated by A, the Favard class is

$$F_{1,A} = \{ \; x \in X \quad : \quad \sup_{t>0} \frac{||T(t)x - x||}{t} < \infty \; \} \tag{5.2}$$

This case is well-known. See for example [12].

(3) The Favard class of A with kernel $a(t)$ can be alternatively defined as the subspace of X given by

$$\{x \in X : \lim \sup_{h \to 0^+} \frac{||R(t)x - a(t)x||}{(a*a)(t)} < \infty\}.$$

As a consequence of $R(t)$ being bounded, the above space coincides with $F_{a,A}$ in Definition 5.1.

Proposition 5.3. *The Favard class A with kernel $a(t)$, $F_{a,A}$, is a Banach space with respect to the norm* $|x|_{F_{a,A}} = ||x|| + \sup_{t>0} \frac{||R(t)x - a(t)x||}{(a*a)(t)}$.

The easy proof is omitted.

Definition 5.4. *A scalar function $a : [0, \infty) \to \mathbb{R}$ is creep if it is continuous, non negative, non decreasing and concave.*

According to [20, Definition 4.4] a creep function have the following standard form

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(\tau)\, d\tau \,,$$

where $a_0 = a(0^+) \geq 0$, $a_\infty = \lim_{t\to\infty} \frac{a(t)}{t} \geq 0$ and $a_1(t) = \dot{a}(t) - a_\infty$ is non negative, non increasing and $\lim_{t\to\infty} a_1(t) = 0$.

The concept of creep function is well-known in viscoelasticity theory and corresponds to a class of functions which normally are verified in practical situations. We refer to the monograph of Prüss [20] for further information.

The following result gives us a wide class of Banach spaces which satisfies condition Z_1 with respect to (A, a). Hence, Theorem 3.6 and Theorem 4.7 together with the theorem below give us explicit conditions on a kernel $a(t)$ in order that an operator A admits multiplicative perturbation, giving an answer to the question stated at the beginning of this paper.

Theorem 5.5. *Let A be a closed linear operator with dense domain $D(A)$ defined in a Banach space X. Suppose that (A,a) generates a bounded integral resolvent $\{R(t)\}_{t\geq 0}$, and that $a \in BV(\mathbb{R}_+)$ is a creep function which satisfies $a(0^+)>0$. Then $F_{a,A}$ satisfies condition Z_1 with respect to (A,a).*

Proof. Let $T>0$. Let $\phi \in L^1([0,T],F_{a,A})$ and choose a sequence $(\phi_n) \in C^2([0,T],F_{a,A})$ such that $\phi_n \to \phi$, that is, $\int_0^T |\phi_n(s)-\phi(s)|_{F_{a,A}}\,ds \to 0$ as $n\to\infty$.

Claim 1. For all $t\geq 0$ and all $t_1 \geq 0$, $\int_0^t R(t+t_1-s)\phi_n(s)\,ds \in D(A)$.

In fact, let $\varphi_n(t) = \phi_n(t)-\phi_n(0)-t\phi_n'(0)$. Then $\varphi_n(0)=\varphi_n'(0)=0$. Define $i(t)=t$. Observe that $\phi_n(t) = (i*\varphi_n'')(t)+\phi_n'(0)i(t)+\phi_n(0)$, for all $0\leq t\leq T$. Since a is a creep function, there exists a scalar function b, such that $a*b=i$, see [20, Proposition 4.4]. Hence,

$$\begin{aligned}\int_0^t R(t-s)\phi_n(s)\,ds &= R*\phi_n(t)\\ &= R*i*\varphi_n''(t)+(i*R)(t)\phi_n'(0)+\int_0^t R(s)\phi_n(0)ds\\ &= a*R*b*\varphi_n''(t)+(a*R*b)(t)\phi_n'(0)\\ &+ \int_0^t R(s)\phi_n(0)ds,\end{aligned}$$

for all $0\leq t\leq T$. Since $\int_0^t R(s)\phi_n(0)ds \in D(A)$ (see [8, Lemma 1]) and $a*R\in D(A)$ by Theorem 2.4, we obtain that $\int_0^t R(t-s)\phi_n(s)\,ds \in D(A)$.

In a similar way to the proof of Theorem 4.7 we can extend the continuous functions ϕ_n to $[0,\infty)$ and then, taking into account the identity

$$\int_0^{t+t_1} R(t+t_1-s)\phi_n(s)ds - \int_0^{t_1} R(t_1-s)\phi_n(s+t)ds = \int_0^t R(t+t_1-s)\phi_n(s)ds,$$

valid for all $t\geq 0$, $t_1\geq 0$, we obtain the claim.

Claim 2. $\int_0^t R(t+t_1-s)\phi_n(s)\,ds \to \int_0^t R(t+t_1-s)\phi(s)\,ds$ as $n\to\infty$ for $0\leq t\leq T$ and all $t_1\geq 0$.

In fact, by hypothesis there exists $M>0$ such that $||R(t)||\leq M$, for all $t\geq 0$, hence

$$\begin{aligned}&||\int_0^t R(t+t_1-s)(\phi_n(s)-\phi(s))\,ds|| \leq \int_0^t ||R(t+t_1-s)||\,||(\phi_n(s)-\phi(s))||\,ds\\ &\leq \int_0^t ||R(t+t_1-s)||\;|(\phi_n(s)-\phi(s))|_{F_{a,A}}\,ds\\ &\leq M\int_0^T |(\phi_n(s)-\phi(s))|_{F_{a,A}}\,ds \;\to\; 0 \text{ as } n\to\infty.\end{aligned}$$

Claim 3. $\int_0^t R(t+t_1-s)\phi(s)\,ds \in D(A)$, for all $0 \le t \le T$ and each $t_1 \ge 0$.

In fact, first we observe that $a(t)$ is non negative, non decreasing and the condition $a(0^+) > 0$ together with the boundedness of $R(t)$ implies that $\overline{\lim_{t\to 0^+}} \frac{\|R(t)\|}{a(t)} < \infty$.

Let $\epsilon > 0$ be given. Since $\int_0^t R(t+t_1-s)\phi_n(s)ds \in D(A)$ we obtain by claim 1 and according to Theorem 2.8 that there exists $\delta > 0$ such that for $0 < h < \delta$ we have

$$\|\frac{R(h)-a(h)}{(a*a)(h)}\int_0^t R(t+t_1-s)\phi_n(s)\,ds - A\int_0^t R(t+t_1-s)\phi_n(s)\,ds\| < \epsilon,$$

equivalently,

$$\|\int_0^t R(t+t_1-s)\frac{R(h)-a(h)}{(a*a)(h)}\phi_n(s)\,ds - A\int_0^t R(t+t_1-s)\phi_n(s)\,ds\| < \epsilon.$$

Using that $\phi_n \in F_{a,A}$ and the boundedness of $R(t)$ we obtain

$$\begin{aligned}
|| \quad & A\int_0^t R(t+t_1-s)\phi_n(s)\,ds|| \\
\le\ & \epsilon + \int_0^t ||R(t+t_1-s)||||\frac{R(h)-a(h)}{(a*a)(h)}\phi_n(s)||\,ds \\
\le\ & \epsilon + \int_0^t ||R(t+t_1-s)||\sup_{h>0}||\frac{R(h)-a(h)}{(a*a)(h)}\phi_n(s)||\,ds \\
\le\ & \epsilon + M\int_0^t |\phi_n(s)|_{F_{a,A}}\,ds, \qquad \text{for all } \epsilon > 0.
\end{aligned}$$

Then we have

$$||A\int_0^t R(t+t_1-s)\phi_n(s)\,ds|| \le M\int_0^t |\phi_n(s)|_{F_{a,A}}ds. \tag{5.3}$$

Define $x_n := \int_0^t R(t+t_1-s)\phi_n(s)\,ds$. By claim 1 we have $x_n \in D(A)$ and, by claim 2, we obtain $x_n \to x := \int_0^t R(t+t_1-s)\phi(s)\,ds$ as $n \to \infty$ for all $0 \le t \le T$. Moreover by (5.3) we have

$$||Ax_m - Ax_n|| \le M\int_0^T |\phi_m(s)-\phi_n(s)|_{F_{a,A}}\,ds \ \to 0,$$

as $m, n \to \infty$. This proves that the sequence (Ax_n) is Cauchy, and hence (Ax_n) converges in X, say $Ax_n \to y \in X$. Since A is closed, we conclude that $x \in D(A)$ proving the claim. In particular, (Z_1a) is proved. Moreover, from (5.3) we deduce that

$$||A\int_0^T R(t+t_1-s)\phi(s)\,ds|| \le M\int_0^T |\phi(s)|_{F_{a,A}}\,ds$$

for all $t_1 \geq 0$, proving $(Z_1 b)$ and the theorem.

□

Combining Remark 2.6 (cf. also [20, p.46]), Theorem 5.5, Theorem 4.7, Theorem 4.4 and Proposition 2.7, we obtain the following result.

Corollary 5.6. *Let A be a linear and closed operator with dense domain $D(A)$ defined in a Banach space X. Suppose (A,a) generates a resolvent family and that $a \in AC(\mathbb{R}_+)$ is a creep function which satisfies $a(0^+) > 0$. Then, for all $B \in \mathcal{B}(X, F_{a,A})$, the pair $(A(I+B),a)$ is the generator of a resolvent family.*

We remark that the above corollary has been recently proved in [2, Theorem 3.5].

A direct consequence of Corollary 5.6 and [20, theorem 4.3] is the following substantial result.

Corollary 5.7. *Suppose A generates a strongly continuous cosine family in X and let $a \in AC(\mathbb{R}_+)$ be a creep function with $a_1(t)$ log-convex and $a(0^+) > 0$. Then the pair $(A(I+B),a)$ is the generator of a resolvent family for each $B \in \mathcal{B}(X, F_{a,A})$.*

The following result characterizes the Favard class with kernel $a(t)$ solely in terms of $a(t)$ and the operator A.

Theorem 5.8. *Let A be a linear and closed operator with densely defined domain $D(A)$ in a Banach space X. Suppose that (A,a) generates a bounded integral resolvent $\{R(t)\}_{t\geq 0}$. Let $a(t) > 0$ such that $a(t) \leq C\, e^{\omega t}$, for some $C > 0$, $\omega > 0$ satisfying $\sup_{t>0} \dfrac{(1*a)(t)}{(a*a)(t)} < \infty$ and $\hat{a}(0) = \infty$. Then*

$$F_{a,A} = \{x \in X \quad : \quad \sup_{\lambda > \omega} || \frac{1}{\hat{a}(\lambda)}\, A\, (\frac{1}{\hat{a}(\lambda)} - A)^{-1} x|| < \infty \ \}$$

Proof. Define $K(\lambda) := \displaystyle\int_0^\infty e^{-\lambda s} R(s)\, ds \;=\; \hat{a}(\lambda)\,(I - \hat{a}(\lambda)A)^{-1}$, for $\lambda > 0$. Let $x \in F_{a,A}$, be given, then $\sup_{t>0} \dfrac{||R(t)x - a(t)x||}{(a*a)(t)} := J < \infty$. For each $\lambda > 0$

$$\begin{aligned} AK(\lambda)x &= \hat{a}(\lambda)^{-1} K(\lambda)x - x \\ &= \hat{a}(\lambda)^{-1} \int_0^\infty e^{-\lambda s} R(s)x\, ds \;-\; x \\ &= \hat{a}(\lambda)^{-1} \int_0^\infty e^{-\lambda s} R(s)x\, ds \;-\; \hat{a}(\lambda)^{-1} \int_0^\infty e^{-\lambda s} a(s)x\, dsx \\ &= \hat{a}(\lambda)^{-1} \int_0^\infty e^{-\lambda s} (R(s)x - a(s)x)\, ds \\ &= \hat{a}(\lambda)^{-1} \int_0^\infty e^{-\lambda s} \frac{(R(s)x - a(s)x)}{(a*a)(s)}\, (a*a)(s)\, ds\,, \end{aligned}$$

hence

$$\begin{aligned} ||AK(\lambda)x|| &\leq \hat{a}(\lambda)^{-1}\int_0^\infty e^{-\lambda s}||\frac{(R(s)x-a(s)x)}{(a*a)(s)}||\,(a*a)(s)\,ds \\ &\leq J\hat{a}(\lambda)^{-1}\int_0^\infty e^{-\lambda s}\,(a*a)(s)\,ds \\ &\leq J\hat{a}(\lambda)^{-1}\,\hat{a}(\lambda)\,\hat{a}(\lambda) = J\hat{a}(\lambda). \end{aligned}$$

Therefore $\sup\limits_{\lambda>\omega}||\hat{a}(\lambda)^{-1}\,A\,K(\lambda)x|| < J < \infty\,.$

Conversely, suppose that $\sup_{\lambda>\omega}||\hat{a}(\lambda)^{-1}\,A\,K(\lambda)x|| := N < \infty\,.$ Let $x = \hat{a}(\lambda)^{-1}K(\lambda)x - AK(\lambda)x := x_\lambda - y_\lambda\,.$ Then we have

$$\begin{aligned} ||R(t)x_\lambda - a(t)x_\lambda|| &= ||\int_0^t a(t-s)AR(s)x_\lambda\,ds|| \\ &\leq \int_0^t a(t-s)||R(s)||\,||Ax_\lambda||\,ds \\ &\leq M||Ax_\lambda||\int_0^t a(t-s)\,ds \\ &= M\,||\hat{a}(\lambda)^{-1}AK(\lambda)x||\int_0^t a(t-s)\,ds \\ &\leq M\,N\int_0^t a(s)\,ds\,. \end{aligned}$$

On the other hand,

$$||R(t)y_\lambda - a(t)y_\lambda|| \leq ||R(t)y_\lambda|| + ||a(t)y_\lambda|| \leq (M+a(t))N\hat{a}(\lambda)\,.$$

Dividing by $(a*a)(t)$ we have that, for all $\lambda > \omega\,,$

$$\frac{||R(t)x - a(t)x||}{(a*a)(t)} \leq MN\frac{(1*a)(t)}{(a*a)(t)} + \frac{M+a(t)}{(a*a)(t)}N\hat{a}(\lambda)\,.$$

Since $\hat{a}(0) = \infty$ we obtain that $\hat{a}(\lambda)$ is surjective, hence there exists λ_t such that $(\hat{a}(\lambda_t))^{-1} = \dfrac{M+a(t)}{(a*a)(t)}$. Applying the hypothesis we conclude that

$$\sup_{t>0}\frac{||R(t)x - a(t)x||}{(a*a)(t)} < MNK + N < \infty\,.$$

This proves the theorem.

□

Observe that $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$. Combining this with the above theorem, we obtain that the Favard class $F_{a,A}$ is independent of the kernel a, and the following result.

Corollary 5.9. *let A be a closed linear operator with densely defined domain $D(A)$ in a Banach space X. Suppose that a is a positive creep function with satisfies $a(t) \leq C\,e^{\omega t}$, for some $C > 0\,,\, \omega > 0$ and that (A,a)*

generates a bounded integral resolvent, then

$$\begin{aligned} F_{a,A} &= \{x \in X \quad : \quad \sup_{\lambda>\omega} || \frac{1}{\hat{a}(\lambda)} A \, (\frac{1}{\hat{a}(\lambda)} - A)^{-1} x|| < \infty \; \} \\ &= \{x \in X \quad : \quad \lim \sup_{h \to 0^+} ||hA \, (h - A)^{-1} x|| < \infty \; \} \end{aligned}$$

Corollary 5.10. *Let A be a closed linear operator with densely defined domain $D(A)$ in a Banach space X. Suppose that A generates a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$. Then*

$$F_{1,A} = \{x \in X \quad : \quad \sup_{\lambda>0} ||\lambda \, A \, R(\lambda, A)x|| < \infty \; \}.$$

6. Examples

In this section, we consider concrete examples to illustrate some results in the previous sections.

The following example shows a kernel $a(t)$ which satisfies the conditions of Theorem 5.5 and Theorem 5.8.

1.- Let $a(t) = b + ct^{\alpha}$, $0 < \alpha < 1$, $c > 0$, $b > 0$. We have that $a(t)$ is a creep function. Suppose that (A, a) is the generator of a bounded integral resolvent $R(t)$.

Since $\hat{a}(\lambda) = \dfrac{b}{\lambda} + \dfrac{c}{\lambda^{\alpha+1}}\Gamma(\alpha+1)$ we have that $\hat{a}(0) = \infty$ and, for $\lambda > 0$, the Favard class of A is given by

$$F_{a,A} = \{ \; x \in X \quad : \quad \sup_{\lambda>0} ||A(I - (\frac{b}{\lambda} + \frac{c}{\lambda^{\alpha+1}}\Gamma(\alpha+1))A)^{-1}x|| < \infty \; \}.$$

On the other hand, we have that $a(0^+) > 0$. We conclude form Theorem 5.5 that $F_{a,A}$ satisfies condition Z_1 with respect to (A, a), and, according to Theorem 4.7 we can conclude that condition (Z) is satisfied. Applying Theorem 4.2 or Corollary 5.6, we obtain that each operator $C \in I + \mathcal{B}(X, F_{a,A})$ is in the class $M(A, R)$ of multiplicative perturbations for the generator (A, a) and (AC, a) is the generator of a resolvent family.

Note that, in the particular case of $\alpha = 1$, $a(t) = b + ct$, equation (2.1) corresponds to the model of a solid of Kelvin-Voigt (see [20] p. 131).

2.- Let (A, a) be the generator of an integral resolvent $\{R(t)\}_{t\geq 0}$ of type (M, ω). An standard example of space Z which satisfies condition (Z) with respect to A, is the domain of A, $[D(A)]$, equipped with the graph norm $|x|_A = ||x|| + ||Ax||$.

In fact, clearly $[D(A)]$ is continuously embedded in X.

Let $\phi \in C([0, \infty), [D(A)])$. For all $h, t \geq 0$, we have that $AR(t-s)\phi(s)$ is well defined and continuous and hence is integrable on $[0, h]$. Since A is closed we conclude that

$$\int_0^h R(h+t-s)\phi(s)\,ds \in D(A) \text{ and}$$

$$\int_0^h AR(h+t-s)\phi(s)\,ds = A\int_0^h R(h+t-s)\phi(s)\,ds\,.$$

Since A commutes with R we obtain that

$$\begin{aligned} ||A\int_0^h R(h+t_1-s)\phi(s)\,ds|| &\leq Me^{\omega(h+t_1)}\int_0^h |\phi(s)|_A\,ds \\ &\leq Me^{\omega t_1}\, he^{\omega h} \sup_{0\leq s\leq t} |\phi(s)|_A\,ds\,. \end{aligned}$$

for all $h\,, t_1 \geq 0$.

3.- Under weak assumptions we can rewrite an additive perturbation problem $A+B$ as an multiplicative perturbation problem.

Theorem 6.1. *Let $B : X \to X$ be a bounded linear operator. Suppose that (A, a) is the generator of an integral resolvent $\{R(t)\}_{t\geq 0}$ such that $\rho(A) \neq \emptyset$. Then $(A + B, a)$ is the generator of an integral resolvent $\{S(t)\}_{t\geq 0}$ on X.*

Proof. Let $-c \in \rho(A)$ then $0 \in \rho(A + cI)$. By [16], $(A + cI, a)$ generates an integral resolvent R_c. Since

$$A + B = (A + cI) + (B - cI)$$

we obtain that $(A + B, a)$ generates an integral resolvent if and only if $(A_c + B_c, a)$ generates an integral resolvent, where $A_c = A + cI$ and $B_c = B - cI$.

In fact, clearly $A_c^{-1}B_c \in \mathcal{B}(X)$ and so

$$|A_c^{-1}B_c x|_{Ac} = ||A_c^{-1}B_c|| + ||A_c A_c^{-1}B_c x|| \leq (||A_c^{-1}B_c||_{\mathcal{B}(X)} + ||B_c||_{\mathcal{B}(X)})||x||$$

for all $x \in X$. Hence $A_c^{-1}B_c \in \mathcal{B}(X, [D(A_c)])$ with $||A_c^{-1}B_c|| \leq ||A_c^{-1}B_c||_{\mathcal{B}(X)} + ||B_c||_{\mathcal{B}(X)}$. Since $[D(A_c)]$ satisfies condition (Z) with respect to A_c, the conclusion follows from Theorem 4.4.

□

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References

[1] J.-C. Chang and S.-Y. Shaw. *Perturbation theory of abstract Cauchy problems and Volterra equations.* Nonlinear Analysis, TMA **30** (1997), 3521-3528.

[2] J.-C. Chang and S.-Y. Shaw. *Multiplicative and additive perturbations of resolvent families.* Intern. Math. J. **2** (2002), 841-853.

[3] P.Clément, O. Diekmann, M. Gyllenberg, H.J.A.M Heijmans and H.R.Thieme. *Perturbation theory for dual semigroups: The sun-reflexive case.* Math Ann. 277 (1987), 709-725.

[4] W. Desch and W. Schappacher. *Some generation results for perturbed semigroups.* Semigroup theory and applications, Proc. Conf., Trieste/Italy 1987, Eds. Ph. Clement et. al., Lect. Notes Pures Appl. Math. 116, Marcel Dekker, New York, 1989, 125–152.

[5] J.R. Dorroh and A. Holderrieth. *Multiplicative perturbation of semigroups generators.* Boll. Un. Mat. Ital. A. (7) 7 (1993), 47-57.

[6] B. Dorroh. *Contraction semigroups in a function space.* Pacific J. Math. 19 (1966), 35-38.

[7] G. Greiner. *Perturbing the boundary conditions of a generator.* Houston J. Math. 13 (2) (1987), 213-229.

[8] R. Grimmer, J. Prüss. *On linear Volterra equations in Banach spaces.* Comp. & Maths. with Appls. 11 (1-3) (1985), 189-205.

[9] G. Gripenberg, S-O Londen, O. Staffans. *Volterra Integral and Functional Equations.* Encyclopedia of Mathematics and Applications, vol. 34, Cambridge University Press, Cambridge-New York, 1990.

[10] K.G. Gustafson. *A note on left multiplication of semigroup generators.* Pacific J. Math. 24 (1968), 463-465.

[11] K. Gustafson, G. Lumer.*Multiplicative perturbation of semigroup generators.* Pacific J. Math. 41 (1972), 731-742.

[12] K-J Engel, R. Nagel. *One Parameter Semigroups for Linear Evolution Equations.* Graduate Texts in Mathematics. 194. Berlin: Springer, 2000.

[13] C. Lizama. *Regularized solutions for abstract Volterra equations.* J. Math. Anal. Appl. 243 (2000), 278-292.

[14] C. Lizama *On approximation and representation of k-regularized resolvent families.* Integral Equations and Operator Theory 41 (2), (2001), 223-229.

[15] C. Lizama and H. Prado *Rates of approximation and ergodic limits of regularized operator families.* J. Approx. Theory 122 (1), (2003), 42-61.

[16] C. Lizama and J. Sánchez. *On perturbation of k-regularized resolvent families.* Taiwanese Journal of Mathematics 7(2),(2003), 217-227.

[17] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences 44, Springer–Verlag,1983.

[18] S. Piskarev and S.-Y. Shaw. *Multiplicative perturbations of* C_0-*semigroups and applications to step responses and cumulative outputs.* J. Funct. Anal. 128 (1995), 315-340.

[19] S. Piskarev and S.-Y. Shaw, *Perturbation and comparison of cosine functions.* Semigroup Forum **51** (1995), 225-246.

[20] J. Pruss. *Evolutionary Integral Equations and Applications*. Monographs Math., vol 87, Birkhäuser Verlag, 1993.

[21] A. Rhandi. *Multiplicative perturbation of linear Volterra equations*. Proc. Amer. Math. Soc. 119(2), (1993).

[22] C.C. Travis and G.F. Webb *Cosine families and abstract nonlinear second order differential equations*. Acta Math. Acad. Sci. Hung. 32 (1978), 75-96.

[23] T.-J. Xiao, J. Liang, and J. Van Casteren, *Time dependent Desch-Schappacher type perturbations of Volterra integral equations.* Integr. Equ. Oper. Theory **44**(2002), 494-506.

[24] J. Zabczyk *On descomposition of generators*. SIAM J. Control and Optimization, Vol. 16, 4 (1978).

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