PERIODIC SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

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ABSTRACT. In this paper we give a necessary and sufficient conditions for the existence and uniqueness of periodic solutions of inhomogeneous abstract fractional differential equations with delay. The conditions are obtained in terms of R-boundedness of operator-valued Fourier multipliers determined by the abstract model.

1. Introduction

Recent investigations in physics, engineering, biological sciences and other fields have demonstrated that the dynamics of many systems are described more accurately using fractional differential equations, and that fractional differential equations with delay are often more realistic to describe natural phenomena than those without delay (see [22], [24], [23], [3], [16] and [18]).

The aim of this paper is the study of existence of periodic solutions for the equation

$$D_t^{\alpha} u(t) = Au(t) + Fu_t + f(t), \qquad t \in [0, 2\pi], \quad 1 \leq \alpha \leq 2, \tag{1.1}$$

where $(A, D(A))$ is a (unbounded) linear operator on a Banach space X, $u_t(\cdot) = u(t + \cdot)$ on $[-r, 0], r > 0$, and the delay operator F is supposed to belong to $\mathcal{B}(L^p([-r, 0]; X), X)$ for some $1 \leq p < \infty$. The state space $L^p([-r, 0]; X)$ is a typical choice with regards to certain applications (e.g. to control theory, or to numerical methods, see [13]).

In case $\alpha = 1$, equation (1.1) with periodic boundary condition in the Lebesgue vector-valued spaces has been studied in the article [19] and, in scales of Besov and Triebel-Lizorkin spaces, by Bu and Fang [8]. The case $\alpha = 2$ has been recently treated in the article [7], simultaneously in the scale of Lebesgue, Besov and Triebel-Lizorkin vector valued spaces. Time fractional differential equations with periodic boundary conditions have recently been treated in the paper [15]. To the knowledge of the authors, time fractional evolution equations with periodic boundary conditions *and delay* have not been studied until now. One of the difficulties is to determine the right definition of fractional derivative to be used in this case. We consider here the framework of the so-called Liouville-Grünwald-Letnikov fractional derivative, studied in [6] (see also [12] and [17]) in the scalar case and used in [15] in the vector-valued case.

With the above definition, in this paper we succeed to find necessary and sufficient conditions for the existence and uniqueness of periodic solution of (1.1) in the vector-valued Lebesgue space $L^p(0, 2\pi; X), 1 < p < \infty$ (see Theorem 3.5 below).

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Considering the scalar case:

$$D_t^\alpha u(t) = \rho u(t) + u(t-\tau) + f(t), \quad t \in [0, 2\pi], \quad 1 \le \alpha \le 2, \tag{1.2}$$

where $\rho \in \mathbb{R}$, we show that if $\tau = 2\pi$ then the unique periodic solution is explicitly given by

$$u(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}((1+\rho)(t-s)^\alpha) f(s) ds \tag{1.3}$$

where $E_{\alpha,\alpha}$ denotes the Mittag- Leffler function. If $0 < \tau < 2\pi$, our characterization in the finite-dimensional case (Corollary 3.9) shows the interesting fact that the number of non-periodic solutions of (1.2), except for those in the set $\{(-1,\tau)/\tau \in [0,2\pi]\}$, is greater than 4 for $\alpha^* < \alpha < 2$, but is exactly 4 for all $1 < \alpha < \alpha^*$, where $\alpha^* \approx 1,8163$. This property reveals a distinguished behavior of fractional differential equations with delay which is not present in the case without delay (cf. [15]).

This paper is organized as follows: Section 2 collects some results about the Liouville-Grünwald-Letnikov fractional derivative of a function $f \in L^p(0,2\pi;X)$ and operator-valued Fourier multipliers in vector-valued Lebesgue spaces. Section 3 is devoted to our main abstract result (Theorem 3.5) and some important consequences, that are new even in the scalar case (Corollary 3.9). After that, we discuss periodic solutions of the scalar equation (1.2), and then we establish an abstract criteria in case X is a UMD space (Theorem 3.14).

2. Preliminaries

Let X, Y be complex Banach spaces. We denote by $\mathcal{B}(X,Y)$ be the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote its domain by $D(A)$ and its resolvent set by $\rho(A)$, and for $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1} = (\lambda - A)^{-1}$.

We shall identify the spaces of (vector or operator-valued) functions defined on $[0,2\pi]$ to their periodic extensions to $\mathbb{R}$. Thus, throughout, we consider the space $L^p(0,2\pi;X)$, $1 \le p \le \infty$ of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0,2\pi]$ is p-integrable (essentially bounded if $p = \infty$).

In the paper [6], Butzer and Westphal studied the fractional derivative directly as a limit of a fractional difference quotient. In the case of periodic functions, it enables one to set up a fractional calculus in the L^p setting with the usual rules, as well as the connection with the classical Weyl fractional derivative (see [20]).

Let $\alpha > 0$. Given $f \in L^p(0,2\pi;X)$, $(1 \le p < \infty)$ the Riemann difference

$$\Delta_t^\alpha f(x) := \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - tj) \tag{2.1}$$

(where $\binom{\alpha}{j} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-j-1)}{j!}$ is the binomial coefficient) exists almost everywhere and

$$\|\Delta_t^\alpha f\|_{L^p(0,2\pi;X)} \le \sum_{j=0}^{\infty} |\binom{\alpha}{j}| \|f\|_{L^p(0,2\pi;X)} = O(1) \tag{2.2}$$

since $\binom{\alpha}{j} = O(j^{-j-1})$ as $j \to \infty$.

The following definition is the direct extension of [6, Definition 2.1] to the vector-valued case. See also [15] for their connection with fractional differential equations.

Definition 2.1. *Let X be a complex Banach space, $\alpha > 0$ and $1 \le p < \infty$. If for $f \in L^p(0, 2\pi; X)$ there exists $g \in L^p(0, 2\pi; X)$ such that $\lim_{t \to 0^+} t^{-\alpha} \Delta_t^\alpha f = g$ in the $L^p(0, 2\pi; X)$ norm, then g is called the α^{th} Liouville-Grünwald-Letnikov derivative of f in the mean of order p. We use the notation $g = D^\alpha f$.*

Example 2.2. *The α^{th} fractional derivative of e^{iax} for any real a is given by $(ia)^\alpha e^{iax}$. In particular, $D^\alpha \sin x = \sin(x + \frac{\pi}{2}\alpha)$ and $D^\alpha \cos x = \cos(x + \frac{\pi}{2}\alpha)$.*

We also have the following properties.

Proposition 2.3. *For $f \in L^p(0, 2\pi; X)$, $1 \le p < \infty$, α, $\beta > 0$ we have*
(i) If $D^\alpha f \in L^p(0, 2\pi; X)$, then $D^\beta f \in L^p(0, 2\pi; X)$ for all $0 < \beta < \alpha$,
(ii) $D^\alpha D^\beta f = D^{\alpha+\beta} f$ whenever one of the two sides is well defined.

Proof. The proof is the same as in the scalar case, which is given in [6, Proposition 4.1]. ■

We recall that the Fourier series of $f \in L^p(0, 2\pi; X)(1 \le p < \infty)$ is defined for $k \in \mathbb{Z}$ by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt}(t) f(t) dt.$$

In what follows we denote by $H^{\alpha,p}(0, 2\pi; X)$ the vector-valued function space

$$H^{\alpha,p}(0, 2\pi; X) := \{u \in L^p(0, 2\pi; X) : \quad \text{there exists } \; v \in L^p(0, 2\pi; X) \text{ such that } \hat{v}(k) = (ik)^\alpha \hat{u}(k), \text{ for all } \; k \in \mathbb{Z}\}.$$

By [6, Theorem 4.1] we also have

$$H^{\alpha,p}(0, 2\pi; X) = \{u \in L^p(0, 2\pi; X) : D^\alpha u \in L^p(0, 2\pi; X)\}.$$

Note that if $1 < \alpha \le 2$, then for $u \in H^{\alpha,p}(0, 2\pi; X)$, it follows that $u(0) = u(2\pi)$ and $D^{\alpha-1}u(0) = D^{\alpha-1}u(2\pi)$.

Let Φ_α be the function defined by

$$\Phi_\alpha(t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{ikt}}{(ik)^\alpha}, \quad t \in \mathbb{R} \setminus 2\pi\mathbb{Z}, \quad \alpha > 0 \tag{2.3}$$

where $(ik)^\alpha = |k|^\alpha e^{\frac{\pi i \alpha}{2} \mathrm{sgn} k}$. Note that $\Phi_\alpha \in L^1(0,\, 2\pi)$ (see [25] for more details) and hence for $u \in L^p(0, 2\pi; X), 1 \le p < \infty$ and $\alpha > 0$, we can define

$$I^\alpha u(t) = \frac{1}{2\pi} \int_0^{2\pi} u(t-s) \Phi_\alpha(s) ds. \tag{2.4}$$

The following lemma from [15], is essentially contained in [6, Theorem 4.1].

Lemma 2.4. *Let $1 \le p < \infty$ and let $u, w \in L^p(0, 2\pi; X)$. The following statements are equivalent* *(i)* $\int_0^{2\pi} w(t)dt = 0$ *and there exists* $x \in X$ *such that*

$$u(t) = x + \frac{1}{2\pi}\int_0^{2\pi} w(t-s)\Phi_\alpha(s)ds \text{ a.e. on } [0,2\pi], \tag{2.5}$$

(ii) $\hat{w}(k) = (ik)^\alpha \hat{u}(k)$ *for all* $k \in \mathbb{Z}$.

We will need the following definition of operator-valued Fourier multipliers.

Definition 2.5. *For* $1 \le p \le \infty$, $\alpha \ge 0$ *we say that a sequence* $\{M_k\}_{k\in\mathbf{Z}} \subset \mathcal{B}(X,Y)$ *is an* $(L^p, H^{\alpha,p})$*-multiplier, if for each* $f \in L^p(0,2\pi;X)$ *there exists* $u \in H^{\alpha,p}(0,2\pi;Y)$ *such that*

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

In particular, in case $\alpha = 0$ (therefore $H^{\alpha,p} = H^{0,p} = L^p$) the definition coincides with the one contained in [2, Proposition 1.1]. The proof of the following lemma is similar to that of [2, Lemma 2.2] taking into account Lemma 2.4 above.

Lemma 2.6. *Let* $1 \le p < \infty$, $\alpha > 0$ *and* $(M_k)_{k\in\mathbf{Z}} \subset \mathcal{B}(X)$. *The following assertions are equivalent*

(i) $(M_k)_{k\in\mathbb{Z}}$ *is an* $(L^p, H^{\alpha,p})$*-multiplier;*

(ii) $((ik)^\alpha M_k)_{k\in\mathbb{Z}}$ *is an* (L^p, L^p)*- multiplier.*

A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R};X)$ for some (and then all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R};X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f.$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain[4] and Burkholder [5].

Definition 2.7. *Let* X *and* Y *be Banach spaces. A family of operators* $\mathcal{T} \subset \mathcal{B}(X,Y)$ *is called* R*-bounded, if there is a constant* $C > 0$ *and* $p \in [1,\infty)$ *such that for each* $N \in \mathbb{N}, T_j \in \mathcal{T}, x_j \in X$ *and for all independent, symmetric,* $\{-1,1\}$*-valued random variables* r_j *on a probability space* $(\Omega, \mathcal{M}, \mu)$ *the inequality*

$$||\sum_{j=1}^N r_j T_j x_j||_{L^p(\Omega,Y)} \le C||\sum_{j=1}^N r_j x_j||_{L^p(\Omega,X)} \tag{2.6}$$

is valid. The smallest such C *is called* R*-bound of* $\mathcal{T}$, *we denote it by* $R_p(\mathcal{T})$.

Several properties of R-bounded families can be founded in the recent monograph of Denk-Hieber-Prüss [10, Section 3].

We remark that large classes of classical operators are R-bounded (cf. [11] and references therein). Hence, this assumption is not too restrictive for the applications that we consider in this article.

The following theorem, due to Arendt and Bu [2, Theorem 1.3], is the discrete analogue of the operator-valued version of Mikhlin's theorem due to Weis [21] and play an important role in the following sections.

Theorem 2.8. *Let X, Y be UMD spaces and let $\{M_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{B}(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1}-M_k)\}_{k\in\mathbb{Z}}$ are R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.*

3. Periodic solutions

We consider in this section the equation

$$D^\alpha u(t) = Au(t) + Fu_t + f(t), \qquad t \in [0, 2\pi],\ 1 < \alpha \leq 2, \tag{3.1}$$

where $A : D(A) \subseteq X \to X$ is a linear, closed operator; $f \in L^p(0, 2\pi; X), p \geq 1$. Setting $r_{2\pi} := 2\pi N$, for some $N \in \mathbb{N}$, $F : L^p([-r_{2\pi}, 0]; X) \to X$ is a linear, bounded operator and u_t is an element of $L^p([-r_{2\pi}, 0]; X)$ which is defined as $u_t(\theta) = u(t+\theta)$ for $-r_{2\pi} \leq \theta \leq 0$.

Next, we define the notion of strong solution of the fractional differential equation with delay (1.1) and the associated concept of well-posedness.

Definition 3.1. *Let $1 \leq p < \infty$. A function u is called a strong L^p-solution of (1.1) if $u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A))$ and equation (1.1) holds for almost all $t \in [0, 2\pi]$.*

Definition 3.2. *Let $1 \leq p < \infty$. We say that problem (1.1) is strongly L^p-well posed (or has maximal regularity) if for every $f \in L^p(0, 2\pi; X)$ there exists a unique strong L^p-solution of (1.1).*

The concept of maximal regularity has received much attention in recent years. It is connected to the question of closedness of the sum of two closed operators. It has proven very efficient in the treatment of non linear problems in partial differential equations, especially semilinear and quasilinear ones (see for example [9]).

Denote by $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{B_\lambda\}_{\lambda\in\mathbb{R}} \subseteq \mathcal{B}(X)$ by

$$B_\lambda x = F(e_\lambda x),\ \text{for all } \lambda \in \mathbb{R} \text{ and } x \in X. \tag{3.2}$$

Defining the *real spectrum* of (3.1) by

$$\sigma(\Delta) = \{s \in \mathbb{R} : (is)^\alpha I - B_s - A \in \mathcal{B}(D(A), X) \text{ is not invertible }\},$$

and denote $\rho(\Delta) = \mathbb{R} \setminus \sigma(\Delta)$. We prove the following result.

Proposition 3.3. *Lets A be a closed linear operator defined on a UMD space X and $1 < \alpha \leq 2$. Suppose that $\mathbb{Z} \subset \rho(\Delta)$. Then the following assertions are equivalent.*

(i) $\{(ik)^\alpha((ik)^\alpha I - B_k - A)^{-1}\}_{k\in\mathbb{Z}}$ *is an (L^p, L^p)-multiplier for $1 < p < \infty$.*

(ii) $\{(ik)^\alpha((ik)^\alpha I - B_k - A)^{-1}\}_{k\in\mathbb{Z}}$ *is R-bounded.*

Proof. By [2, Proposition 1.11] it follows that (i) implies (ii). Conversely, define $M_k = (ik)^\alpha(N_k - A)^{-1}$, where $N_k := (ik)^\alpha I - B_k$. By Theorem 2.8 is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. From the proof of [19, Proposition 3.2] we have that the set $\{B_k\}_{k\in\mathbb{Z}}$ is R-bounded.

Let $a_k = 1/(ik)^\alpha$, $k \neq 0$. Next we note the following identity

$$
\begin{aligned}
k[M_{k+1}-M_k] &= ka_{k+1}M_{k+1}B_{k+1}M_k - ka_kM_{k+1}B_kM_k + k\frac{a_{k+1}-a_k}{a_k}M_{k+1}[M_k-I] \\
&\quad -k(a_{k+1}-a_k)M_{k+1}B_kM_k.
\end{aligned}
$$

Observe that for $\gamma > 0$ we have that $|(i(k+1))^\gamma - (ik)^\gamma|$ can be estimated by $(ik)^{\gamma-1}$ uniformly in k according to the definition of $|(ik)^\gamma|$ and the mean value theorem. This implies that $\{k(a_{k+1}-a_k)\}$ and $\{k\frac{a_{k+1}-a_k}{a_k}\}$ are bounded sequences. Since $\{ka_k\}$ also is bounded for $\alpha > 1$, taking into account that the products and sums of R-bounded sequences is R-bounded (see [10]), the proof is finished.

■

Proposition 3.4. *Let X be a Banach space and let $A : D(A) \subset X \to X$ be a closed linear operator. Suppose that for every $f \in L^p(0, 2\pi; X)$, there exists a unique strong L^p-solution of (3.1) for $1 < p < \infty$. Then*

(i) $\mathbb{Z} \subset \rho(\Delta)$,

(ii) $\{(ik)^\alpha((ik)^\alpha I - B_k - A)^{-1}\}_{k\in\mathbb{Z}}$ *is R-bounded.*

Proof. Follows the same lines of [19, Proposition 3.3]. ■

Our main result in this paper, establish that the converse of Proposition 3.4 is true, provided X is an UMD space.

Theorem 3.5. *Let X be a UMD space and let $A : D(A) \subset X \to X$ be a closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$ and $1 < \alpha \le 2$.*

(i) For every $f \in L^p(0, 2\pi; X)$, there exists a unique strong L^p-solution of (3.1);

(ii) $\mathbb{Z} \subset \rho(\Delta)$ *and* $\{(ik)^\alpha((ik)^\alpha I - B_k - A)^{-1}\}_{k\in\mathbb{Z}}$ *is R-bounded.*

Proof. Let $f \in L^p(0, 2\pi; X)$. Define $N_k = ((ik)^\alpha I - B_k - A)^{-1}$. By Proposition 3.3, the family $\{M_k := (ik)^\alpha N_k\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. By Lemma 2.6, it is equivalent to the fact that the family $\{N_k\}_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha,p})$-multiplier, i.e. there exists $u \in H^{\alpha,p}(0, 2\pi; X)$ such that

$$\hat{u}(k) = N_k\hat{f}(k) = ((ik)^\alpha I - B_k - A)^{-1}\hat{f}(k). \tag{3.3}$$

In particular, $u \in L^p(0, 2\pi; X)$ and there exists $v \in L^p(0, 2\pi; X)$ such that

$$\hat{v}(k) = (ik)^\alpha\hat{u}(k), \tag{3.4}$$

Moreover, $D^\alpha u \in L^p(0, 2\pi; X)$ and $\widehat{D^\alpha u}(k) = \hat{v}(k)$.

We claim that the family $\{B_kN_k\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. In fact, it is clear that $\{B_kN_k\}_{k\in\mathbb{Z}}$ is R-bounded. On the other hand, since $\{B_k\}_{k\in\mathbb{Z}}$ is R-bounded (see the proof of [19, Proposition 3.2]) the identity

$$k(B_{k+1}N_{k+1} - B_kN_k) = ka_{k+1}B_{k+1}M_{k+1} - ka_kB_kM_k$$

shows that $\{k(B_{k+1}N_{k+1} - B_kN_k)\}_{k\in\mathbb{Z}}$ is also R-bounded. Then the claim follows from Theorem 2.8.

By Fejer's theorem (see [14]) one has in $L^p([-r_{2\pi},0];X)$

$$u_t(\theta) = u(t+\theta) = \lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{ikt}e^{ik\theta}\hat{u}(k).$$

Hence in $L^p(0,2\pi;X)$ we obtain

$$u_t = \lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{ikt}e_k\hat{u}(k).$$

Then, since F is linear and bounded

$$Fu_t = \lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{ikt}F(e_k\hat{u}(k)) = \lim_{n\to\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m} e^{ikt}B_k\hat{u}(k)$$

By (3.3) and (3.4) we have

$$\widehat{D^\alpha u}(k) = (ik)^\alpha\hat{u}(k) = A\hat{u}(k) + B_k\hat{u}(k) + \hat{f}(k)$$

for all $k\in\mathbb{Z}$. Then using that A is closed we conclude that $u(t)\in D(A)$ (cf. [2, Lemma 3.1]) and, from the uniqueness theorem of Fourier coefficients, that (3.1) is valid for a.a. $t\in[0,2\pi]$.

To show uniqueness, let $u\in L^p(0,2\pi;D(A))\cap H^{\alpha,p}(0,2\pi;X)$ be such that $D^\alpha u(t) = Au(t)+Fu_t,\quad t\in[0,2\pi]$, then $\hat{u}(k)\in D(A)$ and $(ik)^\alpha\hat{u}(k) = A\hat{u}(k)+B_k\hat{u}(k)$. Since $\mathbb{Z}\cap\sigma(\Delta)=\emptyset$ this implies that $\hat{u}(k)=0$ for all $k\in\mathbb{Z}$ and thus $u=0$.

■

The solution $u(\cdot)$ given in Theorem 3.5 actually satisfies the following maximal regularity property.

Corollary 3.6. *In the context of Theorem 3.5, if condition (ii) is fulfilled, we have* $D^\alpha u, Au, Fu_{(\cdot)}\in L^p(0,2\pi;X)$. *Moreover, there exists a constant* $C>0$ *independent of* $f\in L^p(0,2\pi;X)$ *such that*

$$||D^\alpha u||_{L^p(0,2\pi;X)} + ||Au||_{L^p(0,2\pi;X)} + ||Fu_{(\cdot)}||_{L^p(0,2\pi;X)} \le C||f||_{L^p(0,2\pi;X)}. \tag{3.5}$$

Remark 3.7.

From the inequality (3.5) we deduce that the operator L defined by:

$$(Lu)(t) = D^\alpha(t) - Au(t) - Fu_t \text{ with domain } D(L) = H^{\alpha,p}(0,2\pi;X)\cap L^p(0,2\pi;D(A)),$$

is an isomorphism onto. Indeed, since A is closed, the space $H^{\alpha,p}(0,2\pi;X)\cap L^p(0,2\pi;D(A))$ becomes a Banach space under the norm

$$|||u||| := ||u||_p + ||D^\alpha u||_p + ||Au||_p.$$

We remark that such isomorphisms are crucial for the handling of nonlinear evolution equations (see [1]).

In the case of a Hilbert space, Theorem 3.5 takes a particularly simple form.

Corollary 3.8. *Let H be Hilbert space and let $A : D(A) \subset H \to H$ be a closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$ and $1 < \alpha \leq 2$.*

(i) For every $f \in L^p(0, 2\pi; H)$, there exists a unique strong L^p-solution of (3.1);

(ii) $\mathbb{Z} \subset \rho(\Delta)$ and

$$\sup_{k\in\mathbb{Z}} ||(ik)^{\alpha}((ik)^{\alpha}I - B_k - A)^{-1}|| < \infty. \tag{3.6}$$

Proof. This is a consequence of Plancherel's Theorem. ■

For future reference, we state separately the finite dimensional case, i.e. $H = \mathbb{C}^n$.

Corollary 3.9. *Let $1 < p < \infty$, $1 < \alpha \leq 2$ and $f \in L^p(0, 2\pi; \mathbb{C}^n)$. A necessary and sufficient condition for the existence of a unique strong L^p-solution of (3.1) is that*

$$det((ik)^{\alpha}I - B_k - A) \neq 0, \qquad \text{for all } k \in \mathbb{Z}$$

where A is a $n \times n$ matrix and $(B_k)_k$ is a sequence of $n \times n$ matrices.

Example 3.10.

Set $X = \mathbb{C}$ and $1 < \alpha < 2$. For $\rho \in \mathbb{R} \setminus \{-1\}$ consider the fractional differential equation with delay

$$D_t^{\alpha}x(t) = \rho x(t) + x(t - 2\pi) + f(t), \quad t \in [0, 2\pi]. \tag{3.7}$$

Defining $Fx := x(2\pi)$ we obtain the abstract form (1.1) with $A = \rho$. Note that $B_k = e^{2\pi ik} = 1$ for all $k \in \mathbb{Z}$, and hence we only have to examine the sequence $s_k := (ik)^{\alpha} - 1 - \rho$. Since $\rho \in \mathbb{R} \setminus \{-1\}$, we observe that $s_k \neq 0$ for all $k \in \mathbb{Z}$ and hence Corollary 3.9 implies the existence of a unique solution $x \in L^p(0, 2\pi)$. In this case we can show explicitly that solution, noting that the Laplace transform of the function

$$M(t) := t^{\alpha-1}E_{\alpha,\alpha}((1 + \rho)t^{\alpha}),$$

where $E_{\alpha,\alpha}$ denotes the Mittag-Leffler function, can be extended to the imaginary axis and is given by

$$\mathcal{L}(M)(ik) = \frac{1}{(ik)^{\alpha} - (1 + \rho)}, \qquad k \in \mathbb{Z}.$$

It follows, that the explicit form of the periodic solution is

$$x(t) = \int_{-\infty}^{t} M(t - s)f(s)ds. \tag{3.8}$$

Indeed, note that the Fourier transform of (3.8) is given by the product of the Laplace transform of M (evaluated in the imaginary axis) and the Fourier transform of f. Then, an straightforward calculation, shows that it coincides with the Fourier transform of the given equation (3.9). The claim then follows from the uniqueness of Fourier coefficients.

Example 3.11.

Again, set $X = \mathbb{C}$ and $1 < \alpha < 2$. We consider the fractional differential equation with delay

$$D_t^{\alpha}x(t) = ax(t) + x(t - \tau) + f(t), \quad t \in [0, 2\pi], \tag{3.9}$$

where $\tau \in [0, 2\pi]$ and $a \in \mathbb{R}$. Defining $Fx := x(\tau)$ we obtain the abstract form (1.1) with $A = a$. Note that $B_k = e^{i\tau k}$ for all $k \in \mathbb{Z}$, and hence we have to examine the zeroes of the sequence $f_\alpha(m) := (im)^\alpha - e^{i\tau m} - a, \quad m \in \mathbb{Z}$. Define the set

$$\mathcal{M}_\alpha = \{(a, \tau) \in \mathbb{R} \times [0, 2\pi] \, / \, f_\alpha(m) = 0 \text{ for some } m \in \mathbb{Z}\}. \tag{3.10}$$

Note that $(-1, \tau) \in \mathcal{M}_\alpha$ for all $\tau \in [0, 2\pi]$, since in such case $f_\alpha(0) = 0$.

For $s > 0$, $\lfloor s \rfloor$ denotes the largest integer less than or equal to s. Fixed $n := \left\lfloor \frac{1}{\sin^{1/\alpha}(\alpha\pi/2)} \right\rfloor \in \mathbb{N}$, observe that there exists numbers $\tau_1 = \tau_1(\alpha), ..., \tau_{4n} = \tau_{4n}(\alpha) \in [0, 2\pi]$ such that for each $|m| \leq n$, $m \neq 0$, we have

$$\sin(m\tau_j) = |m|^\alpha \sin(\alpha\pi/2).$$

Now define $a_j = n^\alpha \cos(\alpha\pi/2) - \cos(\tau_j n);\ j = 1, ...4n$. Then $(a_j, \tau_j) \in \mathcal{M}_\alpha \setminus \{(-1, \tau)/\tau \in [0, 2\pi]\}$. It is then easy to prove that

$$\mathcal{M}_\alpha = \{(-1, \tau)/\tau \in [0, 2\pi]\} \cup \{(a_j, \tau_j)\}_{j=1,...,4n}$$

is the set of zeroes of the function f_α in $\mathbb{Z}$. Some concrete examples are:

$$\mathcal{M}_{3/2} = \{(-1, \tau)/\tau \in [0, 2\pi]\} \cup \{(0, 3\pi/4), (0, 5\pi/4), (-\sqrt{2}, \pi/4), (-\sqrt{2}, 7\pi/4)\},$$

$$\begin{aligned}
\mathcal{M}_{5/4} &= \{(-1, \tau)/\tau \in [0, 2\pi]\} \\
&\cup \{(0, 5\pi/8), (0, 11\pi/8), (-2\cos(3\pi/8), 3\pi/8), (-2\cos(3\pi/8), 13\pi/8)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{7/4} &= \{(-1, \tau)/\tau \in [0, 2\pi]\} \\
&\cup \{(0, 7\pi/8), (0, 9\pi/8), (-2\cos(\pi/8), \pi/8), (-2\cos(\pi/8), 15\pi/8)\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{9/5} &= \{(-1, \tau)/\tau \in [0, 2\pi]\} \\
&\cup \{(0, 9\pi/10), (0, 11\pi/10), (-2\cos(\pi/10), \pi/10), (-2\cos(\pi/10), 19\pi/10)\},
\end{aligned}$$

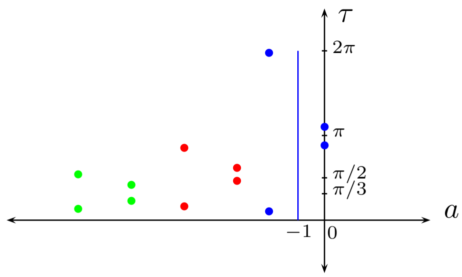
$$\begin{aligned}
\mathcal{M}_{19/10} &= \{(-1, \tau)/\tau \in [0, 2\pi]\} \\
&\cup \{(0, 19\pi/20), (0, 21\pi/20), (-2\cos(\pi/20), \pi/20), (-2\cos(\pi/20), 39\pi/20), \\
&(a_2^+, \frac{\pi}{2} - \frac{\beta_2}{2}), (a_2^+, \frac{\pi}{2} + \frac{\beta_2}{2}), (a_2^-, \pi - \frac{\beta_2}{2}), (a_2^-, \frac{\beta_2}{2})\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{195/10} &= \{(-1, \tau)/\tau \in [0, 2\pi]\} \\
&\cup \{(0, 195\pi/200), (0, 205\pi/200), (-2\cos(5\pi/200), 5\pi/200), \\
&(-2\cos(5\pi/200), 395\pi/200), (a_2^+, \frac{\pi}{2} - \frac{\beta_2}{2}), (a_2^+, \frac{\pi}{2} + \frac{\beta_2}{2}), (a_2^-, \pi - \frac{\beta_2}{2}), (a_2^-, \frac{\beta_2}{2}), \\
&(a_3^+, \frac{\pi}{3} - \frac{\beta_3}{3}), (a_3^+, \frac{\pi}{3} + \frac{\beta_3}{3}), (a_3^-, \frac{2\pi}{3} - \frac{\beta_3}{3}), (a_3^-, \frac{\beta_3}{3})\}
\end{aligned}$$

where $\beta_j = \arcsin(j^\alpha \sin(\alpha\pi/2))$ and $a_j^\pm = j^\alpha \cos(\alpha\pi/2) \pm \cos(\beta_j)$, $j = 2, 3$.

We then conclude from the above and Corollary 3.9 that for all $(a, \tau) \notin \mathcal{M}_\alpha$ there exists a unique periodic solution of equation (3.9).

Remark 3.12. *It is remarkable that the number of points in the set* $\mathcal{M}^* = \mathcal{M}_\alpha \setminus \{(-1, \tau)/\tau \in [0, 2\pi]\}$ *is exactly the same (=4) until the value approximate* $\alpha^* \approx 1.816373004$ *corresponding to the unique root of* $2^\alpha \sin(\alpha\pi/2) - 1 = 0$ *in the open interval* $1 < \alpha < 2$*, and increases* *as α approach to 2. It reflects the surprising fact that the probability, in some sense, to find periodic solutions of the equation (3.9) decreases for α ($> \alpha^*$) near to 2 but, however, is the same for $\alpha \in (1, \alpha^*)$. In the following figure shows the pairs (a, τ) in the case $\alpha = 1.95$.*



Example 3.13.

Let A be a closed linear operator defined on a Hilbert space H and suppose that $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_k ||A((ik)^\alpha - A)^{-1}|| =: M < \infty$. From the identity

$$(ik)^\alpha I - A - B_k = (I - B_k((ik)^\alpha - A)^{-1})((ik)^\alpha - A)$$

it follows that $(ik)^\alpha I - A - B_k$ is invertible whenever $||B_k((ik)^\alpha - A)^{-1}|| < 1$.

Next observe that $||B_k|| \leq {r_{2\pi}}^{1/p}||F||$. Hence

$$||B_k((ik)^\alpha - A)^{-1}|| = ||B_k A^{-1} A((ik)^\alpha - A)^{-1}|| \leq {r_{2\pi}}^{1/p}||F||||A^{-1}||M =: \xi$$

Therefore, under the condition

$$||F|| < \frac{1}{||A^{-1}||M{r_{2\pi}}^{1/p}} \tag{3.11}$$

we obtain that $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$, and the identity

$$\begin{aligned} ((ik)^\alpha I - A - B_k)^{-1} &= ((ik)^\alpha - A)^{-1}(I - B_k((ik)^\alpha - A)^{-1}) \\ &= ((ik)^\alpha - A)^{-1}\sum_{n=0}^{\infty}[B_k((ik)^\alpha - A)^{-1}]^n. \end{aligned} \tag{3.12}$$

It follows that

$$||(ik)^\alpha((ik)^\alpha I - A - B_k)^{-1}|| \leq ||(ik)^\alpha((ik)^\alpha - A)^{-1}||\sum_{n=0}^{\infty}||B_k((ik)^\alpha - A)^{-1}||^n \leq \frac{1+M}{1-\xi},$$

and hence condition (ii) in Corollary 3.8 is satisfied.

The above example can be adapted to obtain the following criterion in case of UMD spaces.

Theorem 3.14. *Let X be a UMD space and let $A : D(A) \subset X \to X$ be a closed linear operator such that $\{(ik)^\alpha\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $R_p(\{A((ik)^\alpha - A)^{-1}\}_{k\in\mathbb{Z}}) =: M < \infty$. Suppose* *that*

$$||F|| < \frac{1}{(2r_{2\pi})^{1/p}||A^{-1}||M}. \tag{3.13}$$

Then for every $f \in L^p(0, 2\pi; X)$, there exists a unique strong L^p-solution of (3.1).

Proof. Follows the same lines of [19, Theorem 3.9]. ∎

To close this paper, and as an application, we want to compare the periodic problem

$$D^\alpha u(t) = Au(t) + f(t), \qquad t \in [0, 2\pi) \tag{3.14}$$

with the delay equation (3.1). As a direct consequence of Theorem 3.14 and [15, Theorem 3.1] we have the following result.

Corollary 3.15. *Assume that X is a UMD space. Let $1 < p < \infty$. If for each $f \in L^p(0, 2\pi; X)$ there is a unique strong L^p-solution of equation (3.14) and condition (3.13) is satisfied, then for all $f \in L^p(0, 2\pi; X)$ there is a unique strong L^p-solution of equation (3.1).*

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