BOUNDED MILD SOLUTIONS FOR SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES.

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ABSTRACT. In this paper we study the structure of various classes of spaces of vector-valued functions $\mathcal{M}(\mathbb{R};X)$ ranging between periodic functions and bounded continuous functions. Some of these functions are introduced here for the first time. We propose a general operator theoretical approach to study a class of semilinear integro-differential equations. The results obtained are new and they recover, extend or improve variety of recent works.

1. INTRODUCTION

The rapid development of the theory of integro-differential equations in infinite - dimensional spaces has been strongly promoted by the large number of applications in physics, engineering and biology. Abstract integro-differential equations are still in a state of flux, with new basic results continuously emerging. Questions like existence of solutions, continuous dependence, perturbations, and general asymptotic behavior are at present an active area of research.

In this paper, we consider the following integro-differential equation

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t,u(t)), \tag{1.1}$$

where A is a closed linear operator defined in a Banach space X and $a \in L^1_{loc}(\mathbb{R}_+)$ is an scalar-valued kernel. Equations of this kind arise, for example, in the study of heat flow in materials of fading memory type as well as some equations of population dynamics. For more information on this subject see the papers [10, 37, 38, 18] and the monograph [40] (particularly Chapter II, Section 9) and the references therein.

Suppose that we know something about the asymptotic behavior of the forcing function $f(t,x)$. For example, f could be bounded or asymptotically periodic. What conditions do we need on the operator A and the kernel a in order to conclude that the solution u of (1.1) exists and has the same asymptotic behavior as f?.

Among others, Da Prato and Lunardi [38] studied this problem for equation (1.1) in the linear case (see also Da Prato-Lunardi [39], Clément-Da Prato [9]) under several conditions on A and $a(\cdot)$. The results of Da Prato et.al. were then used by Sforza [42] to derive global existence and uniqueness results for the associated semilinear problem. A key assumption in all the above mentioned works is that A generates an analytic semigroup (not necessarily strongly continuous). However, they also treat more general operator valued kernels.

Key words and phrases. Linear and semilinear integro-differential equations; regularized operator families; bounded mild solutions.

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In this paper, we will answer the stated problem in various spaces of vector-valued functions $\mathcal{M}(\mathbb{R};X)$ ranging between periodic functions and bounded continuous functions. Our approach provide a unified treatment for most of the more important classes of vector-valued functions that have recently appeared in the literature like almost automorphic, pseudo-almost automorphic, asymptotically periodic and almost periodic, to mention a few examples of the spaces to be considered.

We first present and analyze the structure of a hierarchy of spaces of functions $\mathcal{M}(\mathbb{R};X)$ between periodic functions and bounded continuous functions. We note that some of these spaces, which appears naturally in our study, are introduced here for the first time. Then we propose an operator theoretical approach to study the structure of bounded solutions for (1.1) in these function spaces. It reveals the way in that regularized families of bounded linear operators [29] can be used in order to produce ad-hoc concepts of mild solutions for abstract integro-differential equations. Finally, we apply the proposed method to the semilinear history value problem (1.1) giving new results as well as recovering, improving and extending a vast class of recent studies.

We hope that the methods outlined in this paper can serve as guidelines to obtain similar results for other abstract differential equations that are of recent interest, like e.g. fractional differential equations, delay equations, etc.

2. Preliminaries: The function spaces

Let X be a Banach space. We denote

$$BC(X) := \{f : \mathbb{R} \to X : f \text{ is continuous, } ||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)|| < \infty\}.$$

Let $P_\omega(X) := \{f \in BC(X) : \exists\, \omega > 0, \quad f(t+\omega) = f(t)\ \forall t \in \mathbb{R}\}$ be the space of all vector-valued periodic functions. For the space of almost periodic functions (in the sense of Bohr), we set $AP(X)$ which consists of all functions $f \in BC(X)$ such that for each $\epsilon > 0$ there exists a $\omega > 0$ such that every subinterval of $\mathbb{R}$ of length ω contains at least one point τ such that $||f(t+\tau) - f(t)||_\infty \leq \epsilon$. This definition is equivalent to the so-called Bochner's criterion (cf. [32] Theorem 3.1.8), namely, $f \in AP(X)$ if and only if for every sequence of reals (s'_n) there exists a subsequence (s_n) such that $(f(\cdot + s_n))$ is uniformly convergent on $\mathbb{R}$. Almost periodic functions are uniformly continuous on $\mathbb{R}$. (cf. [32] Theorem 3.1.4). A simple example is $f(t) = 2 + \sin(t) + \sin(\sqrt{2}t)$. Observe that $AP(X)$ is a Banach space with the norm $||\cdot||_\infty$ and

$$P_\omega(X) \subset AP(X).$$

Note that the function $f(t)$ above is not periodic, thus the inclusion is strict.

The space of compact almost automorphic functions will be denoted by $AA_c(X)$. Recall that a continuous bounded function f belongs to $AA_c(X)$ if and only if for all sequence (s'_n) of real numbers there exists a subsequence $(s_n) \subset (s'_n)$ such that $\lim_{t\to\infty} f(t+s_n) =: \overline{f}(t)$ and $\lim_{t\to\infty} \overline{f}(t-s_n) = f(t)$ uniformly over compact subsets of $\mathbb{R}$. Clearly the function $\overline{f}$ above is continuous on $\mathbb{R}$. Therefore f is uniformly continuous [34]. In other other words compact almost automorphic functions are uniformly continuous on $\mathbb{R}$. We have that $AA_c(X)$ is a Banach space under the norm $||\cdot||_\infty$ and

$$P_\omega(X) \subset AP(X) \subset AA_c(X) \subset BC(X).$$

The space of almost automorphic functions is defined as follows
$AA(X) := \{f \in BC(X) :$ for all (s'_n), exists $(s_n) \subset (s'_n)$ such that $\lim_{t\to\infty} f(t+s_n) =: \overline{f}(t)$ and $\lim_{t\to\infty} \overline{f}(t-s_n) = f(t) \forall t \in \mathbb{R}\}$, provided with the norm $||\cdot||_\infty$.

As a typical example, we can take $f(t) = \sin(\frac{1}{2+\sin(t)+\sin(\sqrt{2}t)})$. We have that $AA(X)$ is a Banach space with the norm $||\cdot||_\infty$ and the following inclusions hold:

$$P_\omega(X) \subset AP(X) \subset AA_c(X) \subset AA(X) \subset BC(X).$$

Note that the function $f(t) = \sin(\frac{1}{2+\sin(t)+\sin(\sqrt{2}t)})$ is not uniformly continuous, so the inclusion $AA_c(X) \subset AA(X)$ is strict.

Let $\mathcal{F}_1 = \{P_\omega(X),\, AP(X), AA_c(X), AA(X)\}$ and $\Omega \in \mathcal{F}_1$. Then we have

Theorem 2.1. *Assume $f, f_1, f_2 \in \Omega$. Then we have*

- $f_1 + f_2 \in \Omega$,
- $\lambda f \in \Omega$, *for any scalar* λ
- $f_\tau(t) := f(t+\tau) \in \Omega$ *for any* $\tau \in \mathbb{R}$

Proof. See for instance [32]. □

Theorem 2.2. *For any $f \in \Omega$, the range R_f of f is relatively compact in X.*

Proof. Let $\Omega \in \mathcal{F}_1$, then since $\Omega \subset AA(X)$, we can conclude in view of [32, Theorem 2.1.3.]. □

Theorem 2.3. *Let $(f_n) \subset \Omega$, such that $f_n \to f$ uniformly on $\mathbb{R}$. Then $f \in \Omega$.*

Proof. The case $\Omega = P_\omega$ is trivial. Indeed, let $(f_n) \subset P_\omega$, such that $f_n \to f$ uniformly on $\mathbb{R}$. Then for any $\epsilon > 0$, there exists N such that $\|f_n(t) - f(t)\| < \frac{\epsilon}{2}$ for any $n > N$ and $t \in \mathbb{R}$. Thus $\|f(t+\omega) - f(t)\| \le \|f(t+\omega) - f_n(t+\omega)\| + \|f_n(t+\omega) - f_n(t)\| + \|f_n(t) - f(t)\| < \epsilon$, for any $t \in \mathbb{R}$. Which shows that $f(t+\omega = f(t)$, for any $t \in \mathbb{R}$

For $\Omega = AP(X)$ (resp. $\Omega = AA(X)$), see [32] Theorem 3.1.4 (resp. Theorem 2.1.10). The case $\Omega = AA_c(X)$ is similar to the one of $AA(X)$. So we omit it.

□

Theorem 2.4. *Assume that $f \in \Omega$ and let $F(t) := \displaystyle\int_0^t f(s)ds$. Then $F \in \Omega$ if and only if R_f is relatively compact in X.*

Proof. For $\Omega = AP(X)$ (resp. $\Omega = AA(X)$), see [32] Theorem 3.2.6 (resp. Theorem 2.4.4). The case $\Omega = AA_c(X)$ is similar to the one of $AA(X)$. So we omit it. □

Remark 2.5. *Note that if X does not contain a subspace isomorphic to c_0 (for instance X is a uniformly convex Banach space), the above theorem is called Kadet's theorem (*[26] *Theorem 2, p. 86, and it reads:*
If f is ω-periodic (resp. almost periodic), then F is ω-periodic (resp. almost periodic) if and only if it is bounded.
Kadet's theorem is valid for all periodic, almost periodic and almost automorphic sequences ([36]*).*

Now we consider the set

$$C_0(X) := \{f \in BC(X) : \lim_{|t|\to\infty} ||f(t)|| = 0\},$$

and define the space of asymptotically periodic functions as

$$AP_\omega(X) := P_\omega(X) \oplus C_0(X).$$

Analogously, we define the space of asymptotically almost periodic functions

$$AAP(X) := AP(X) \oplus C_0(X),$$

the space of asymptotically compact almost automorphic functions,

$$AAA_c(X) := AA_c(X) \oplus C_0(X),$$

and the space of asymptotically almost automorphic functions

$$AAA(X) := AA(X) \oplus C_0(X).$$

We have the following natural proper inclusions

$$AP_\omega(X) \subset AAP(X) \subset AAA_c(X) \subset AAA(X) \subset BC(X).$$

Remark 2.6. We observe that

$$AP_\omega(X) \neq \{f \in BC(X) : \exists \omega > 0 \; ||f(t+\omega) - f(t)|| \to 0 \text{ as } t \to \infty\} =: SAP_\omega(X).$$

This fact was only recently proved in [25], providing a counterexample to the assertion given in [21, Lemma 2.1]. This way, in general we only have

$$AP_\omega(X) \subset SAP_\omega(X).$$

The class of functions in $SAP_\omega(X)$ is called S-asymptotically ω-periodic (see [25] for a discussion of qualitative properties of this class of functions).

We next consider the following set

$$P_0(X) := \{f \in BC(X) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} ||f(s)|| ds = 0\},$$

and define the following classes of spaces: The space of pseudo-periodic functions

$$PP_\omega(X) := P_\omega(X) \oplus P_0(X),$$

the space of pseudo almost periodic functions

$$PAP(X) := AP(X) \oplus P_0(X),$$

the space of pseudo compact almost automorphic functions

$$PAA_c(X) := AA_c(X) \oplus P_0(X),$$

and the space of pseudo almost automorphic functions

$$PAA(X) := AA(X) \oplus P_0(X).$$

As before, we also have the following relationship between them

$$PP_\omega(X) \subset PAP(X) \subset PAA_c(X) \subset PAA(X) \subset BC(X).$$

Since $C_0(X) \subset P_0(X)$, we have the following diagram that summarizes the different classes of subspaces defined above

$$\begin{array}{ccccc} AA(X) & \Rightarrow & AAA(X) & \Rightarrow & PAA(X) \\ \Uparrow & & \Uparrow & & \Uparrow \\ AA_c(X) & \Rightarrow & AAA_c(X) & \Rightarrow & PAA_c(X) \end{array}$$

$$
\begin{array}{ccccc}
\Uparrow & & \Uparrow & & \Uparrow \\
AP(X) & \Rightarrow & AAP(X) & \Rightarrow & PAP(X) \\
\Uparrow & & \Uparrow & & \Uparrow \\
P_\omega(X) & \Rightarrow & AP_\omega(X) & \Rightarrow & PP_\omega(X) \\
 & & \Downarrow & & \\
 & & SAP_\omega(X) & &
\end{array}
$$

Remark 2.7. The definition of almost periodic functions was introduced by H. Bohr [7]. Compact almost automorphic functions was introduced by A.M. Fink [19] after previous work of S. Bochner, who introduced the concept of almost automorphic functions (see [5]). Asymptotically periodic functions appears by the first time in works of N.G. de Bruijn [6] whereas asymptotically almost periodic functions was introduced by M. Fréchet [20]. The concept of asymptotically almost automorphic functions was defined by G.M. N' Guérékata [33]. Pseudo periodic functions are treated, apparently for the first time, in the article [44] by R.Yuan. Pseudo almost periodic functions are introduced in the literature by C. Y. Zhang [45]. Finally, the concept of pseudo almost automorphic functions was only recently introduced by J. Liang, J. Zhang and T.J. Xiao in the paper [28]. The concepts of asymptotically compact almost automorphic functions as well as pseudo compact almost automorphic functions appears here by the first time.

The fact that the space $PAA(X)$ is complete under the sup-norm was only recently proved, see [43].

3. The linear case

In this section we study bounded solutions for the linear integro-differential equation

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t),\ t \in \mathbb{R}. \tag{3.1}$$

Recall that a function $u \in C^1(\mathbb{R}; X)$ is called a strong solution of (3.1) on $\mathbb{R}$ if $u \in C(\mathbb{R}; D(A))$ and (3.1) holds on $\mathbb{R}$. If merely $u(t) \in X$ instead of the domain of A, we say that u is a mild solution of the linear equation (3.1). Conditions under which a mild solution implies a strong one has been studied in Prüss [40].

We shall denote by $\mathcal{M}(\mathbb{R}, X)$, or simply $\mathcal{M}(X)$, any of the spaces defined in the previous section. We define the set $\mathcal{M}(\mathbb{R} \times X; X)$ which consists of all functions $f : \mathbb{R} \times X \to X$ such that $f(\cdot, x) \in \mathcal{M}(\mathbb{R}, X)$ uniformly for each $x \in K$, where K is any bounded subset of X.

Given $f \in \mathcal{M}(\mathbb{R} \times X; X)$, we ask for conditions under which there exists a solution $u \in \mathcal{M}(\mathbb{R}, X)$.

We remark that, in general, the form in which integro-differential equations arise in applications is given by equation (3.1) on the line, and the problem

$$v'(t) = Av(t) + \int_{0}^{t} a(t-s)Av(s)ds + g(t),\ t \geq 0, \tag{3.2}$$

arises from (3.1) as a history value problem. When considering problems with forces $f \in \mathcal{M}(\mathbb{R}, X)$, the equation to consider is (3.1), since (3.2) is only time invariant but not translation invariant, only (3.1) enjoys the latter property. In this context the important question arises whether the solutions $v(t)$ of (3.2) and $u(t)$ of (3.1) are asymptotic to each other, i.e. whether $u(t) - v(t) \to 0$ as $t \to \infty$, whenever $f(t) \to g(t) \to 0$ as $t \to \infty$. Under reasonable assumptions this turns to be the case (cf. [3]), and therefore the term *limiting equation* of (3.2) makes sense for (3.1).

We recall that the Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+, X)$ is given by

$$\mathcal{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt, \quad Re\lambda > \omega,$$

where the integral is absolutely convergent for $Re\lambda > \omega$. Furthermore, we denote by $\mathcal{B}(X)$ the space of bounded linear operators from X into X endowed with the norm of operators, and the notation $\rho(A)$ stands for the resolvent set of A.

In order to give an operator theoretical approach to equation (3.1) we recall the following definition (cf. [40]) (see also Remark 3.5 below for the motivation).

Definition 3.1. *Let $b \in L^1_{loc}(\mathbb{R})$ be given. Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of a solution operator (or resolvent family) if there exists $\mu \in \mathbb{R}$ and a strongly continuous function $S : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\frac{1}{\hat{b}(\lambda)} : Re\lambda > \mu\} \subset \rho(A)$ and*

$$\frac{1}{\lambda \hat{b}(\lambda)}(\frac{1}{\hat{b}(\lambda)} - A)^{-1}x = \int_0^\infty e^{-\lambda t} S(t)x dt, \quad Re\lambda > \mu, \quad x \in X.$$

In this case, $S(t)$ is called the solution operator generated by A.

In the scalar case there is a large bibliography which studies the concept of resolvent, we refer to the monograph by Gripenberg et al. [22] , and references therein. We emphasize the fact that because of the uniqueness of the Laplace transform, in the case $b(t) \equiv 1$ the family $S(t)$ corresponds to a C_0-semigroup. We note that solution operators, as well as resolvent families, are a particular case of (b, k)-regularized families introduced in [29]. According to [27] a solution operator $S(t)$ corresponds to a $(1, b)$-regularized family.

Definition 3.2. *([40]) A strongly measurable family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called uniformly integrable (or strongly integrable) if $\int_0^\infty ||T(t)||dt < \infty$.*

In what follows, we will denote $\|T\| := \int_0^\infty ||T(t)||dt < \infty$ for any uniformly integrable family of such operators $\{T(t)\}_{t\geq 0}$.

Note that exponentially stable C_0-semigroups are examples of uniformly integrable families of operators.

The following is our main result on maximal regularity under convolution of the above mentioned spaces. It corresponds to a summary, with new, and in some cases, a slight extension and improvement of recent results given by a number of authors (cf. [1], [34], [32], [16], [3], [23], [25] and [30]).

Theorem 3.3. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a uniformly integrable and strongly continuous family. If f belongs to one of the spaces $\mathcal{M}(X)$, and $w(t)$ is given by*

$$w(t) = \int_{-\infty}^t S(t-s)f(s)\, ds \tag{3.3}$$

then w belongs to the same space as f.

Proof. We first consider periodic functions. Given $f \in P_\omega(X)$, with a simple change of variables, we have

$$w(t+\omega)-w(t)=\int_{-\infty}^{t+\omega} S(t+\omega-s)f(s)ds-\int_{-\infty}^{t} S(t-s)f(s)ds=0.$$

We now consider the space of almost periodic functions $AP(X)$. So if $f \in AP(X)$, then by hypotheses, for each $\epsilon > 0$ there exists a $T > 0$ such that every subinterval of $\mathbb{R}$ of length T contains at least one point h such that $\sup_{t\in\mathbb{R}} ||f(t+h)-f(t)|| \le \epsilon$. We have

$$\begin{aligned}\sup_{t\in\mathbb{R}} ||w(t+h)-w(t)|| &= \sup_{t\in\mathbb{R}} ||\int_{-\infty}^{t} S(t-s)[f(s+h)-f(s)]ds|| \\ &\le ||S|| \sup_{t\in\mathbb{R}} ||f(t+h)-f(t)|| \le \epsilon ||S||,\end{aligned}$$

and therefore, w has the same property as f, i.e., it is almost periodic.

$AA_c(X)$: Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Since $f \in AA_c(X)$ there exist a subsequence $(s_n)_{n\in\mathbb{N}}$, and a continuous function $v \in BC(X)$ such that $f(t+s_n)$ converges to $v(t)$ and $v(t-s_n)$ converges to $f(t)$ uniformly on compact subsets of $\mathbb{R}$.

Since

$$w(t+s_n)=\int_{-\infty}^{t+s_n} S(t+s_n-s)f(s)ds=\int_{-\infty}^{t} S(t-s)f(s+s_n)ds, \tag{3.4}$$

using the Lebesgue's dominated convergence theorem, we obtain that $w(t+s_n)$ converges to $z(t)=\displaystyle\int_{-\infty}^{t} S(t-s)v(s)ds$ as $n \to \infty$ for each $t \in \mathbb{R}$.

Furthermore, the preceding convergence is uniform on compact subsets of $\mathbb{R}$. To show this assertion, we take a compact set $K=[-a,a]$. For $\varepsilon > 0$, we choose $L_\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that

$$\begin{aligned}\int_{L_\varepsilon}^{\infty} ||S(s)||ds &\le \varepsilon, \\ \|f(s+s_n)-v(s)\| &\le \varepsilon, \ \ n \ge N_\varepsilon, \ s \in [-L,L],\end{aligned}$$

where $L = L_\varepsilon + a$. For $t \in K$, we now can estimate

$$\begin{aligned}\|w(t+s_n)-z(t)\| &\le \int_{-\infty}^{t} ||S(t-s)|| \|f(s+s_n)-v(s)\|ds \\ &\le \int_{-\infty}^{-L} ||S(t-s)|| \|f(s+s_n)-v(s)\|ds \\ &\quad + \int_{-L}^{t} ||S(t-s)|| \|f(s+s_n)-v(s)\|ds \\ &\le 2\|f\|_\infty \int_{t+L}^{\infty} ||S(s)||ds + \varepsilon \int_{0}^{\infty} ||S(s)||ds \\ &\le \varepsilon \left(2\|f\|_\infty + ||S||\right),\end{aligned}$$

which proves that the convergence is independent of $t \in K$. Repeating this argument, one can show that $z(t - s_n)$ converges to $w(t)$ as $n \to \infty$ uniformly for t in compact subsets of $\mathbb{R}$. This completes the proof in case of the space $AA_c(X)$.

$AA(X)$: Let $(s'_n) \subset \mathbb{R}$ be an arbitrary sequence. Since $f \in AA(X)$ there exists a subsequence (s_n) of (s'_n) such that

$$\lim_{n\to\infty} f(t + s_n) = v(t), \text{ for all } t \in \mathbb{R}$$

and

$$\lim_{n\to\infty} v(t - s_n) = f(t), \text{ for all } t \in \mathbb{R}.$$

From (3.4), note that

$$\|w(t + s_n)\| \leq ||S||\|f\|_\infty$$

and by continuity of $S(\cdot)x$ we have $S(t - \sigma)f(\sigma + s_n) \to S(t - \sigma)v(\sigma)$, as $n \to \infty$ for each $\sigma \in \mathbb{R}$ fixed and any $t \geq \sigma$. Then by the Lebesgue's dominated convergence theorem, we obtain that $w(t + s_n)$ converges to $z(t) = \int_{-\infty}^{t} S(t - s)v(s)ds$ as $n \to \infty$ for each $t \in \mathbb{R}$. Similarly we can show that

$$z(t - s_n) \to w(t) \text{ as } n \to \infty, \text{ for all } t \in \mathbb{R},$$

and the proof is complete.

$SAP_\omega(X)$: We have

$$\begin{aligned} w(t + \omega) - w(t) &= \int_{-\infty}^{t+\omega} S(t + \omega - s)f(s)ds - \int_{0}^{t} S(t - s)f(s)ds \\ &= \int_{-\infty}^{t} S(t - s)[f(s + \omega) - f(s)]ds. \end{aligned}$$

For each $\epsilon > 0$, there is a positive constant L_ϵ such that $||f(t + \omega) - f(t)|| \leq \epsilon$, for every $t \geq L_\epsilon$. Under these conditions, for $t \geq L_\epsilon$, we can estimate

$$\begin{aligned} ||w(t + \omega) - w(t)|| &\leq \int_{-\infty}^{t} ||S(t - s)[f(s + \omega) - f(s)]||ds \\ &\leq \int_{-\infty}^{L_\epsilon} ||S(t - s)[f(s + \omega) - f(s)]||ds \\ &+ \int_{L_\epsilon}^{t} ||S(t - s)[f(s + \omega) - f(s)]||ds \\ &\leq 2||f||_\infty \int_{-\infty}^{L_\epsilon} ||S(t - s)||ds + \epsilon \int_{L_\epsilon}^{t} ||S(t - s)||ds \\ &= 2||f||_\infty \int_{t-L_\epsilon}^{\infty} ||S(s)||ds + \epsilon \int_{0}^{\infty} ||S(s)||ds \end{aligned}$$

which permits to conclude that $w(t + \omega) - w(t) \to 0$ as $t \to \infty$.

Now we will study the **asymptotic behavior** of the solutions. Let $h \in C_0(X)$ and define

$$H(t) = \int_{-\infty}^{t} S(t - s)h(s)ds. \tag{3.5}$$

Let $\epsilon > 0$ be given. There exist $T > 0$ such that $||h(s)|| < \epsilon$ for all $s > T$ and hence we can write

$$H(t) = \int_{-\infty}^{T} S(t-s)h(s)\,ds + \int_{T}^{t} S(t-s)h(s)\,ds.$$

Then

$$\begin{aligned} ||H(t)|| &\le \int_{-\infty}^{T} ||S(t-s)||||h(s)||\,ds + \int_{T}^{t} ||S(t-s)||\epsilon\,ds \\ &\le ||h||_\infty \int_{t-T}^{\infty} ||S(v)||dv + ||S||\epsilon, \end{aligned}$$

and we conclude that $H(t) \to 0$ as $t \to \infty$. It permits us to infer the conclusion of the theorem for the spaces $AP(X), AAP(X), AAA_c(X)$ and $AAA(X)$.

Vanishing mean value: Let $h \in P_0(X)$ and define $H(t)$ as in (3.5). For $R > 0$ we have

$$\begin{aligned} \frac{1}{2R}\int_{-R}^{R} ||H(t)||dt &\le \frac{1}{2R}\int_{-R}^{R}[\int_{-\infty}^{t} ||S(t-s)||||h(s)||ds]dt \\ &\le \frac{1}{2R}\int_{-R}^{R}[\int_{0}^{\infty} ||S(s)||||h(t-s)||ds]dt \\ &= \int_{0}^{\infty} ||S(s)||[\frac{1}{2R}\int_{-R}^{R} ||h(t-s)||dt]ds \end{aligned}$$

Note that the set $P_0(X)$ is translation-invariant. Hence, using the Lebesgue's dominated convergence theorem, we obtain from the above inequality that $\frac{1}{2R}\int_{-R}^{R} ||H(t)||dt \to 0$ as $R \to \infty$. We conclude that the spaces $PP_\omega(X), PAP(X), PAA_c(X)$ and $PAA(X)$ have the maximal regularity property under the convolution defined by (3.3).

□

The following consequence is the main result of this section.

Theorem 3.4. *Assume that A generates a uniformly integrable $(1, 1-(1*a))$-regularized family $S(t)$ on the Banach space X. Then for each $f \in \mathcal{M}(X)$ there is a unique mild solution $u \in \mathcal{M}(X)$ of equation (3.1).*

Proof. Let $b(t) := 1 - \int_0^t a(s)ds$ and define

$$u(t) := \int_{-\infty}^{t} S(t-s)f(s)ds,$$

where S satisfies the resolvent equation

$$S(t)x = x + \int_0^t b(t-s)AS(s)xds, \quad x \in X, t \ge 0. \tag{3.6}$$

In particular, since $b(t)$ is differentiable, the above equation shows that for each $x \in X$, $S'(t)x$ exists and

$$S'(t)x = AS(t)x - \int_0^t a(t-s)AS(s)xds, \quad t \geq 0. \tag{3.7}$$

It remains to prove that u defined as above is a mild solution for equation (3.1). In fact, since $S'(t)x$ exists and A is closed, using Fubini's theorem we obtain

$$\begin{aligned} u'(t) &= S(0)f(t) + \int_{-\infty}^t S'(t-s)f(s)ds \\ &= f(t) + \int_{-\infty}^t AS(t-s)f(s)ds - \int_{-\infty}^t \int_0^{t-s} a(t-s-\tau)AS(\tau)f(s)d\tau ds \\ &= f(t) + Au(t) - \int_{-\infty}^t \int_s^t a(t-v)AS(v-s)f(s)dvds \\ &= f(t) + Au(t) - \int_{-\infty}^t \int_{-\infty}^v a(t-v)AS(v-s)f(s)dsdv \\ &= f(t) + Au(t) - \int_{-\infty}^t a(t-v) \int_{-\infty}^v AS(v-s)f(s)dsdv \\ &= f(t) + Au(t) - \int_{-\infty}^t a(t-v)Au(v)dv. \end{aligned}$$

□

Remark 3.5. The idea behind of the above theorem and their proof is the following: Given an abstract linear equation (in this case the equation (3.2) as limiting equation of (3.1)) we take formally the Laplace transform and obtain

$$F(\lambda)\hat{u}(\lambda) = \hat{f}(\lambda) + \text{initial conditions}.$$

For example, $F(\lambda) = (\lambda - A - \hat{a}(\lambda)A)$ in this case. Then, we define an ad-hoc strongly continuous family of bounded and linear operators $S(t)$ for the given abstract linear equation as those that satisfy

$$F(\lambda)\hat{S}(\lambda) = I,$$

and

$$\hat{S}(\lambda)F(\lambda) = I.$$

Then, we directly prove that the (mild) solution of equation (3.1) have the convolution structure $u(t) = \int_{-\infty}^t S(t-s)f(s)ds$. For instance, in case of (3.1) we find that $S(t)$ should formally satisfy

$$\hat{S}(\lambda) = (\lambda - A - \hat{a}(\lambda)A)^{-1} = \frac{1}{\lambda(1+\hat{a}(\lambda))}(\frac{1}{1+\hat{a}(\lambda)} - A)^{-1}.$$

Comparing with Definition 3.1 (or more generally the definition of (b,k)-regularized families, cf. [29]) we find $b(t) = 1 + a(t)$ (and $k(t) = 1$, resp.), so that the right ad-hoc family to consider in this case, corresponds to those of resolvent families [40]. After this procedure, we simply check directly that the proposed "mild" solution solves, in fact, the abstract linear evolution equation under study. The definition of "mild" solution for the nonlinear case is then straightforward (see e.g. Definition 4.2 below).

In case $a(t) \equiv 0$ we obtain the following corollary.

Corollary 3.6. *Assume that A generates an uniformly integrable semigroup $S(t)$ on the Banach space X. Then for each $f \in \mathcal{M}(X)$ there is a unique mild solution $u \in \mathcal{M}(X)$ of the equation*

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R}. \tag{3.8}$$

Remark 3.7. *The above corollary generalizes* [35, Theorem 3.1] *when $\Omega = AA(X)$ and recovers* [17, Theorem 2.7] *when $B = g = 0$; $\Omega = AAA(X)$ and the remainder cases are new results.*

Example 3.8. *Let $A = -\rho I$ where $\rho > 0$ and $a(t) \equiv 0$. Then $S(t) = e^{-\rho t}I$ and we conclude that for each $f \in \mathcal{M}(X)$ the equation*

$$u'(t) = -\rho u(t) + f(t), \quad t \in \mathbb{R}, \tag{3.9}$$

has the unique strong solution $u(t) = \displaystyle\int_{-\infty}^{t} e^{-\rho(t-s)} f(s)ds$, which belongs to $\mathcal{M}(X)$.

In practice it may not be easy to check in Theorem 3.4 the conditions of uniform integrability or even the hypothesis of A being the generator of an $(1, 1-(1*a))$-regularized family, therefore we state below a more direct criterion. Recall that a C^∞-function $g : \mathbb{R}_+ \to \mathbb{R}$ is called completely monotonic if $(-1)^n g^{(n)}(t) \geq 0$ for all $t \geq 0$ and $n \in \mathbb{Z}_+$.

Theorem 3.9. *Suppose A generates an analytic C_0-semigroup, bounded on some sector $\Sigma(0,\theta)$, and is invertible, let $a \in L^1_{loc}(\mathbb{R}_+)$ be completely monotonic and such that $\displaystyle\int_0^\infty a(s)ds < 1$. Then for each $f \in \mathcal{M}(X)$ there is a unique mild solution $u \in \mathcal{M}(X)$ of equation (3.1)..*

Proof. Direct consequence of Theorem 3.4 and [40, Corollary 10.1]. □

Remark 3.10. *Kernels $a(t)$ satisfying the condition of the above theorem can be easily found using Bernstein's theorem, which characterizes completely monotonic functions as Laplace transforms of positive measures supported on $\mathbb{R}_+$. A simple example following this method is $a(t) = \frac{1}{(t+\alpha)^2}$ for $\alpha > 1$.*

Example 3.11. *Suppose $f(\cdot, x) \in \mathcal{M}(L^2(\Omega))$ for each fixed $x \in \Omega$. Let $\alpha > 1$. The problem*

$$u_t(t,x) = \Delta u(t,x) + \int_{-\infty}^{t} \frac{1}{(t-s+\alpha)^2} \Delta u(s,x)ds + f(t,x)$$

admits a mild solution u which belongs to $\mathcal{M}(L^2(\Omega))$.

4. The semilinear problem

It is well known that the study of composition of two functions with special properties plays a key role in discussing the existence of solutions to semilinear equations. Our first main result in this section, review and give some new composition theorems on the spaces defined in the second section.

Define the Nemytskii's superposition operator

$$\mathcal{N}(\varphi)(\cdot) := f(\cdot, \varphi(\cdot))$$

for $\varphi \in \mathcal{M}(X)$. From here on, $\mathcal{M}(X)$ will denote one the following spaces $P_\omega(X), AP_\omega(X), PP_\omega(X), SAP_\omega(X), AP(X), AAP(X), PAP(X), AA(X), AAA(X), PAA(X)$. We also denote

$$C_0(\mathbb{R}\times X, X) = \{h \in BC(\mathbb{R}\times X, X) : \lim_{t\to\infty} ||f(t,x)|| = 0 \text{ uniformly on any subset of } X\}$$

and

$$P_0(\mathbb{R}\times X, X) = \{h \in BC(\mathbb{R}\times X, X) : \ h(\cdot, x) \in BC(X) \text{ for all } x \in X, \text{ and}$$
$$\lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} ||h(t,x)||dt = 0 \text{ uniformly in } x \in X\}.$$

Theorem 4.1. *Let $f \in \mathcal{M}(\mathbb{R}\times X, X)$ be given and assume that there exists a constant $L_f > 0$ such that*

$$||f(t,x) - f(t,y)|| \le L_f ||x-y||, \tag{4.1}$$

for all $t \in \mathbb{R}, x, y \in X$. Let $\varphi \in \mathcal{M}(X)$. Then $\mathcal{N}(\varphi)$ belongs to the same space as φ.

Proof. For almost periodic functions, $AP(X)$, the proof was first provided in [1, Proposition 1]. See also [4, Lemma 3.4] and references therein. For the space $AAP(X)$ our result is a consequence of [4, Lemma 8.3]. The case of $PAP(X)$ is consequence of [27, Theorem 2.1]. See also [14, Theorem 3.4]. For almost automorphic functions, $AA(X)$, a proof is given in [32, Theorem 2.2.4]. In case of $AAA(X)$ the proof is contained in [28, Theorem 2.3]. In case of $PAA(X)$ the result is consequence of [28, Theorem 2.4]. For the space $SAP_\omega(X)$ the result appear in [25, Lemma 4.1] and implicitly before in [24].

The remaining cases $P_\omega(X); AP_\omega(X); PP_\omega(X)$; are proved as follows:
Let $\Omega = P_\omega(X)$ and assume that $\varphi \in \Omega$ and $f(\cdot, x) \in \Omega$ uniformly for each $x \in K$, where K is any bounded subset of X. Then for all $t \in \mathbb{R}$ we have

$$\begin{aligned}
\|f(t+\omega, \varphi(t+\omega)) - f(t, \varphi(t))\| &= \|f(t+\omega, \varphi(t+\omega)) - f(t+\omega, \varphi(t)) \\
&+ f(t+\omega, \varphi(t) - f(t, \varphi(t))\| \\
&\le \|f(t+\omega, \varphi(t+\omega)) - f(t+\omega, \varphi(t))\| \\
&+ \|f(t+\omega, \varphi(t) - f(t, \varphi(t))\| \\
&\le L\|\varphi(t+\omega) - \phi(t)\| \\
&= 0.
\end{aligned}$$

Thus $\mathcal{N}(\varphi) \in \Omega$.

Let $\Omega = AP_\omega(X)$ and assume that $\varphi \in \Omega$ and $f(\cdot, x) \in \Omega$ uniformly for each $x \in K$, where K is any bounded subset of X. We can write $f = g + h$ where $g \in P_\omega(\mathbb{R}\times X, X)$, $h \in C_0(\mathbb{R}^+ \times X, X)$ and $\varphi = \alpha + \beta$ where $\varphi \in P_\omega(X)$ and $\beta \in C_0(\mathbb{R}^+, X)$.
Now we write

$$f(t, \varphi(t)) = f(t, \varphi(t)) - f(t, \alpha(t)) + g(t, \alpha(t)) + h(t, \alpha(t)).$$

Observe that by the above $g(\cdot, \alpha(\cdot)) \in P_\omega(X)$. Now

$$I(t) := \|f(t, \varphi(t)) - f(t, \alpha(t))\| \le L_f\|\varphi(t) - \alpha(t)\| = L_f\|\beta(t)\|$$

which shows that

$$\lim_{t\to\infty} I(t) = 0.$$

Also if we let $K = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ which is compact and bounded, then we obtain that

$$\lim_{t\to\infty} \|h(t, \alpha(t))\| = 0.$$

Thus $\mathcal{N}(\varphi) \in \Omega$.

Let $\Omega = PP_\omega(X)$ and write

$$f(t, \varphi(t)) = g(t, \alpha(t)) + f(t, \varphi(t)) - g(t, \alpha(t)) = g(t, \alpha(t)) + f(t, \varphi(t)) - f(t, \alpha(t)) + h(t, \alpha(t)).$$

As above $g(\cdot, \alpha(\cdot)) \in P_\omega(X)$. Now as in the Proof of Theorem 2.1 $f(t, \varphi(t)) - f(t, \alpha(t))$, and $h(t, \alpha(t))$ are in $P_0(X)$.

□

In what follows we study existence and uniqueness of solutions in $\mathcal{M}(X)$ for the semilinear integro-differential equation given in the introduction, namely

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t, u(t)). \tag{4.2}$$

Definition 4.2. *A function $u : \mathbb{R} \to X$ is said to be a mild solution to equation (4.2) if there exists a strongly continuous family of bounded and linear operators on X such that the function $s \to S(t-s)f(s, u(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and*

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s, u(s))ds, \tag{4.3}$$

for each $t \in \mathbb{R}$.

We next give several theorems on existence of mild solutions for the semilinear problem. We begin with the following simple result.

Theorem 4.3. *Assume that A generates an uniformly integrable $(1, 1-(1*a))$-regularized family $S(t)$ on the Banach space X. Let $f \in \mathcal{M}(\mathbb{R} \times X, X)$ be given and assume that f satisfy*

$$||f(t, x) - f(t, y)|| \leq L_f ||x - y||, \tag{4.4}$$

for all $t \in \mathbb{R}$. Then equation (4.2) has a unique mild solution $u \in \mathcal{M}(X)$ whenever $L_f < ||S||^{-1}$.

Proof. We define the operator $F : \mathcal{M}(X) \to \mathcal{M}(X)$ by

$$(F\varphi)(t) := \int_{-\infty}^{t} S(t-s)f(s, \varphi(s))\, ds, \quad t \in \mathbb{R}. \tag{4.5}$$

In view of Theorems 3.3 and 4.1, F is well defined. Then for $\varphi_1, \varphi_2 \in \mathcal{M}(X)$ we have:

$$\begin{aligned}
\|F\varphi_1 - F\varphi_2\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \int_{-\infty}^{t} S(t-s)[f(s, \varphi_1(s)) - f(s, \varphi_2(s))]ds \right\| \\
&\leq L \sup_{t\in\mathbb{R}} \int_0^\infty \|S(\tau)\| \|\varphi_1(t-\tau) - \varphi_2(t-\tau)\| d\tau \\
&\leq L \|\varphi_1 - \varphi_2\|_\infty \|S\|.
\end{aligned}$$

This proves that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in \mathcal{M}(X)$, such that $Fu = u$, that is $u(t) = \int_{-\infty}^{t} S(t-s)f(s,u(s))ds$. Since clearly u is a mild solution of (4.2) (cf. also the proof of Theorem 3.4), the proof is complete. □

The following consequence is immediate.

Corollary 4.4. *Suppose A generates an analytic C_0-semigroup, bounded on some sector $\Sigma(0,\theta)$, and is invertible, let $a \in L^1_{loc}(\mathbb{R}_+)$ be completely monotonic and such that $\int_0^\infty a(s)ds < 1$. Let $f \in \mathcal{M}(\mathbb{R} \times X, X)$ be given and assume that f satisfy*

$$||f(t,x) - f(t,y)|| \leq L_f ||x-y||, \tag{4.6}$$

for all $t \in \mathbb{R}$. Then there exists $\eta > 0$ such that equation (4.2) has a unique mild solution $u \in \mathcal{M}(X)$ whenever $L_f < \eta$.

A different Lipschitz condition is considered in the following result. Recall that an strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be uniformly bounded if there exists a constant $M > 0$ such that $\|S(t)\| \leq M$ for all $t \geq 0$.

Theorem 4.5. *Assume that A generates a bounded and uniformly integrable* $(1, 1-(1*a))$*-regularized family $S(t)$ on the Banach space X. Let $f \in \mathcal{M}(\mathbb{R}\times X, X)$ be given and assume that f satisfy*

$$||f(t,x) - f(t,y)|| \leq L_f(t)||x-y||, \quad t \in \mathbb{R}, \tag{4.7}$$

where $L_f \in L^1(\mathbb{R}) \cap BC(\mathbb{R})$. Then equation (4.2) has a unique mild solution $u \in \mathcal{M}(X)$.

Proof. We define the operator F as in (4.5). Clearly under conditions on L_f, $F\varphi \in \mathcal{M}(X)$ if $\varphi \in \mathcal{M}(X)$. Now let φ_1, φ_2 be in $\mathcal{M}(X)$. We can estimate

$$\begin{aligned} ||(F\varphi_1)(t) - (F\varphi_2)(t)|| &= \left\| \int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s)) - f(s,\varphi_2(s))]ds \right\| \\ &\leq M \int_{-\infty}^{t} L_f(s)\|\varphi_1(s) - \varphi_2(s)\|ds \end{aligned}$$

Repeating the argument, we get

$$\begin{aligned} & ||(F^n\varphi_1)(t) - (F^n\varphi_2)(t)|| \\ \leq\ & M^n \int_{-\infty}^{t}\int_{-\infty}^{s}\cdots\int_{-\infty}^{s_{n-2}} L_f(s)L_f(s_1)\cdots L_f(s_{n-1})||\varphi_1(s_{n-1}) - \varphi_2(s_{n-1})||ds_{n-1}\cdots ds_1 ds \\ \leq\ & \frac{M^n}{n!}\left(\int_{-\infty}^{t} L_f(\tau)d\tau\right)^n ||\varphi_1 - \varphi_2||_\infty \\ \leq\ & \frac{(M||L_f||_1)^n}{n!}||\varphi_1 - \varphi_2||_\infty. \end{aligned}$$

Since $\frac{(M||L_f||_1)^n}{n!} < 1$ for n sufficiently large, applying the contraction principle we conclude that F has a unique fixed point $u \in \mathcal{M}(X)$ which completes the proof. □

Of course, an immediate consequence under the condition that A generates a bounded analytic semigroup, like Corollary 4.4, also holds. The particular case $a(t) \equiv 0$ reads as follows.

Corollary 4.6. *Suppose A generates an analytic C_0-semigroup, bounded on some sector $\Sigma(0,\theta)$, and invertible. Let $f \in \mathcal{M}(\mathbb{R}\times X, X)$ be given and assume that f satisfy*

$$||f(t,x)-f(t,y)|| \le L_f(t)||x-y||, \quad t \in \mathbb{R}, \tag{4.8}$$

where $L_f \in L^1(\mathbb{R}) \cap BC(\mathbb{R})$. Then equation

$$u'(t) = Au(t) + f(t,u(t)), \tag{4.9}$$

has a unique mild solution $u \in \mathcal{M}(X)$.

We note that conditions of type (4.8) has been previously considered in the literature (see [11] and references therein). Now we consider a more general case of equations introducing a new class of functions L which do not necessarily belong to $L^1(\mathbb{R})$. We have the following result.

Theorem 4.7. *Assume that A generates a bounded and uniformly integrable $(1, 1-(1*a))$-regularized family $S(t)$ on the Banach space X. Let $f \in \mathcal{M}(\mathbb{R}\times X, X)$ be given and assume that f satisfy the Lipschitz condition (4.8) where $L_f \in BC(\mathbb{R})$ and the integral $\displaystyle\int_{-\infty}^{t} L_f(s)ds$ exists for all $t \in \mathbb{R}$. Then equation (4.2) has a unique mild solution $u \in \mathcal{M}(X)$.*

Proof. Define a new norm $|||\varphi|| := \sup_{t\in\mathbb{R}}\{v(t)||\varphi(t)||\}$, where $v(t) := e^{-k\int_{-\infty}^{t} L_f(s)ds}$ and k is a fixed positive constant greater than $M := \sup_{t\in\mathbb{R}}||S(t)||$. Let φ_1, φ_2 be in $\mathcal{M}(X)$, then we have

$$v(t)||(F\varphi_1)(t)-(F\varphi_2)(t)|| = v(t)\left\|\int_{-\infty}^{t} S(t-s)[f(s,\varphi_1(s))-f(s,\varphi_2(s))]ds\right\|$$

$$\begin{aligned}
&\le M\int_{-\infty}^{t} v(t)L_f(s)\|\varphi_1(s)-\varphi_2(s)\|ds \le M\int_{-\infty}^{t} v(t)v(s)^{-1}L_f(s)v(s)\|\varphi_1(s)-\varphi_2(s)\|ds\\
&\le M|||\varphi_1(s)-\varphi_2(s)|||\int_{-\infty}^{t} v(t)v(s)^{-1}L_f(s)ds = \frac{M}{k}|||\varphi_1(s)-\varphi_2(s)|||\int_{-\infty}^{t} ke^{k\int_t^s L_f(\tau)d\tau}L_f(s)ds\\
&= \frac{M}{k}|||\varphi_1(s)-\varphi_2(s)|||\int_{-\infty}^{t}\frac{d}{ds}\left(e^{k\int_t^s L_f(\tau)d\tau}\right)ds = \frac{M}{k}[1-e^{-k\int_{-\infty}^{t} L_f(\tau)d\tau}]|||\varphi_1(s)-\varphi_2(s)|||\\
&\le \frac{M}{k}|||\varphi_1(s)-\varphi_2(s)|||.
\end{aligned}$$

Hence, since $M/k < 1$, F has a unique fixed point $u \in \mathcal{M}(X)$. □

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