MILD SOLUTIONS FOR ABSTRACT FRACTIONAL DIFFERENTIAL EQUATIONS.

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ABSTRACT. We propose a unified functional analytic approach to derive a variation of constants formula for a wide class of fractional differential equations using results on (a, k)-regularized families of bounded and linear operators, which covers as particular cases the theories of C_0-semigroups and cosine families. Using this approach we study the existence of mild solutions to fractional differential equation with nonlocal conditions. We also investigate the asymptotic behavior of mild solutions to abstract composite fractional relaxation equations. We include in our analysis the Basset and Bagley-Torvik equations.

1. INTRODUCTION

Fractional differential equations are generalizations of ordinary differential equations to an arbitrary (non-integer) order. Fractional differential equations have attracted considerable interest because of their ability to model complex phenomena. These equations capture nonlocal relations in space and time with power-law memory kernels. Due to the extensive applications of FDEs in engineering and science, research in this area has grown significantly in the past years (see, e.g., [4, 7, 11, 13, 18, 19, 23, 25, 35, 38, 41] and the references therein).

The main purpose of this paper is to establish a general procedure as to derive mild solutions to a wide class of fractional differential equations. This is a fundamental and complex problem that has been recently discussed in [15], following the publication of several papers therein cited. Our method is based on an extensive use of properties of Laplace transforms and (a, k)-regularized families, a concept introduced by C. Lizama [29]. We anticipate it can be used for more classes of fractional differential equations with various types of fractional derivatives not covered in this paper.

Our study of the variation of constant formulas for abstract fractional differential equations or, equivalently, the representation of their solution by means of families of bounded and linear operators, has been motivated by the recent paper [15], which treats the problem of the existence of solution for abstract differential equations with fractional derivatives in time. In that paper, the authors observed that the concepts of mild solutions used in several recent literature on the subject are not appropriate, because the used concept of solution is not realistic. The authors then proposed the use of the well developed theory of resolvent operators for integral equations [39]. However, it is well known that not all fractional differential equation can be formulated as an integral equation, so that the method proposed in [15] fails in the general case.

The paper is organized as follows. In Section 2, we recall the very recent facts about (a, k)-regularized families since the paper [29]. Then we present in Section 3, how to derive the variation of constants formulas for various classes of fractional differential equations with the Caputo derivative. This is the main part of the paper. We finally present some applications in Section 4, where we study the existence and asymptotic behavior of solutions of some fractional differential equations. First, using the Leray-Shauder alternative theorem, we prove the existence of a solution to the fractional differential equation with nonlocal conditions

$$(1.1) \qquad D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad t \in [0, T], \ u(0) + g(u) = u_0, \quad u'(0) = 0,$$

where $1 < \alpha < 2$ and A generates an $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-regularized family $S_\alpha(t)$.

Then we prove the existence and uniqueness of solutions to the semilinear abstract composite fractional relaxation equation

$$(1.2) \qquad u'(t) - AD_t^\alpha u(t) + u(t) = F(t, u(t)), \quad 0 < \alpha < 1, \ u(0) = x.$$

Finally, we discuss solutions for an abstract version of the Bagley- Torvik equation, proving existence of asymptotically 2π-periodic mild solutions.

2. (a, k)-REGULARIZED FAMILIES

In this section we review the main results in the literature on the theory of (a, k)-regularized families. The notion of (a, k)-regularized family was introduced in [29] and studied in subsequent papers. The definition can be now stated as follows.

Definition 2.1. *Let X be a Banach space, $k \in C(\mathbb{R}_+)$, $k \neq 0$ and let $a \in L^1_{loc}(\mathbb{R}_+)$, $a \neq 0$. Assume that A is a linear operator with domain $D(A)$. A strongly continuous family $\{R(t)\}_{t \geq 0}$ of bounded linear operators from X into X is called an (a, k)-regularized resolvent family on X (or simply (a, k)- regularized family) having A as a generator if the following hold.*

(i) *$R(0) = k(0)I$;*

(ii) *$R(t)x \in D(A)$ and $R(t)Ax = AR(t)x$ for all $x \in D(A)$ and $t \geq 0$;*

(iii) *$R(t)x = k(t)x + \int_0^t a(t-s)AR(s)x\,ds, \ t \geq 0, \ x \in D(A)$.*

Assume that a and k are both positive and one of them is non-decreasing. Let $\{R(t)\}_{t \geq 0}$ be an (a, k)-regularized family with generator A such that

$$(2.3) \qquad \|R(t)\| \leq Mk(t), \quad t \geq 0,$$

for some constant $M > 0$. Then we have

$$(2.4) \qquad Ax = \lim_{t \to 0^+} \frac{R(t)x - k(t)x}{(a * k)(t)}, \quad x \in \overline{D(A)}.$$

The above representation of A in terms of $R(t)$ was first established in [32]. The extension to the case where the domain of A is not necessarily dense in X was proved in [34].

We say that $\{R(t)\}_{t \geq 0}$ is of type (M, ω) if there exists constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|R(t)\| \leq Me^{\omega t} \qquad \text{for all } t \geq 0.$$

The next result corresponds to the generation theorem for the theory. We assume that the Laplace transform for $a(t)$ and $k(t)$ exists for all $\lambda > \omega$ and denote it by $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$ respectively.

Theorem 2.2. *([29]) Let A be a closed linear densely defined operator in a Banach space X. Then $\{R(t)\}_{t \geq 0}$ is an (a, k)-regularized family of type (M, ω) if and only if the following conditions hold:*

(1) *$\hat{a}(\lambda) \neq 0$ and $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\lambda > \omega$;*

(2) *$H(\lambda) := \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1}$ satisfies the estimates*

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda - \omega)^{n+1}}, \qquad \lambda > \omega, \ n \in \mathbb{N}_0.$$

Remark 2.3. *In the case where $k(t) \equiv 1$, Theorem 3.4 is well known. In fact, if $a(t) \equiv 1$, then it is just the Hille-Yosida theorem; if $a(t) \equiv t$, then it is the generation theorem for generators of cosine functions due to Sova and Fattorini; for arbitrary $a(t)$, it is the generation theorem for resolvent operators associated to Volterra equations, due essentially to Da Prato and Iannelli, and Prüss [39]. In the case where $k(t) = \frac{t^n}{n!}$ and $a(t) \equiv 1$, it is the generation theorem for n-times integrated semigroups [22]; if $k(t) = \frac{t^n}{n!}$ and $a(t)$ is arbitrary, it corresponds to the generation theorem for integrated solutions of Volterra equations due to Arendt and Kellermann [2].*

Let A be a closed linear operator and let $\{R(t)\}_{t \geq 0}$ be an exponentially bounded and strongly continuous operator family in $\mathcal{B}(X)$ such that the Laplace transform $\hat{R}(\lambda)$ exists for $\lambda > \omega$. It was proved in [29, Proposition 3.1] that $R(t)$ is an (a,k)-regularized family with generator A if and only if for every $\lambda > \omega$, $(I - \hat{a}(\lambda)A)^{-1}$ exists in $\mathcal{B}(X)$ and

$$(2.5) \qquad H(\lambda)x = \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}x = \int_0^\infty e^{-\lambda s} R(s)x \, ds, \quad x \in X.$$

Let $B : (D(A), \|\cdot\|) \to X$ be a linear operator. The following is the main result available in perturbation theory.

Theorem 2.4. *([32]) Assume that A generates an (a,k)-regularized family $\{R(t)\}_{t \geq 0}$ of type (M, ω) and suppose that*

(i) *there exists $b \in L^1_{loc}(\mathbb{R}_+)$ such that $\widehat{b}(\lambda) = \dfrac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}$ for all $\lambda > \omega$.*

(ii) *there exists constants $\mu > \omega$ and $\gamma \in [0,1)$ such that*

$$\int_0^\infty e^{-\mu r} \left\| B \int_0^r b(r-s)R(s)x \, ds \right\| dr \leq \gamma x, \qquad \text{for all } x \in D(A).$$

Then $A + B$ generates an (a,k)-regularized family $\{R(t)\}_{t \geq 0}$ on X such that $\|R(t)\| \leq \frac{M}{1-\gamma}e^{\mu t}$. In addition

$$R(t)x = R(t)x + \int_0^t R(t-r)B \int_0^r b(r-s)R(s)x \, ds \, dr, \qquad x \in X.$$

Note that in case that B is bounded, condition (ii) is automatically satisfied. If, moreover, we consider $k(t) \equiv 1$ and $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha > 1$, then condition (i) is satisfied with $b(t) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}$.

Our next result corresponds to the convergence theorem for the theory of (a,k)-regularized families. Note that it is the analog to the Trotter-Kato theorem for the theory of C_0-semigroups, which follows in case $a(t) \equiv k(t) \equiv 1$.

Theorem 2.5. *([30]) Let $\{k_n\}_{n \geq 0} \in L^1_{loc}(\mathbb{R}^+)$ and $\{a_n\}_{n \geq 0} \in AC_{loc}(\mathbb{R}^+)$ be of type (M, ω), $\omega \geq 0$, such that $\widehat{a}_n(\lambda) \neq 0$ for $\lambda > \omega$ and $\int_0^\infty e^{-\omega s}|a'_n(s)| \, ds < \infty$. Let A_n be closed and linear operators in X such that A_0 is densely defined. For each fixed $n \in \mathbb{N}_0$, assume that $R_n(t)$ is an (a_n, k_n)-regularized family generated by A_n in X, and that there exists constants $M > 0$ and $\omega \in \mathbb{R}$, independent of n, such that*

$$\|R_n(t)\| \leq Me^{\omega t}, \quad \text{for all } t \geq 0.$$

Suppose also $a_n(t) \to a_0(t)$ and $k_n(t) \to k_0(t)$ as $n \to \infty$. Then the following statements are equivalent:

(1) $\lim\limits_{n \to \infty} k_n(\lambda)(I - a_n(\lambda)A_n)^{-1} = k_0(\lambda)(I - a_0(\lambda)A_0)^{-1} \qquad$ *for all $\lambda \geq \omega$*

(2) $\lim\limits_{n \to \infty} R_n(t)x = R_0(t)x$ *for all $x \in X$, $t \geq 0$. Moreover the convergence is uniform in t on every compact subset of $\mathbb{R}^+$.*

In our next result, of concern are ergodic type theorems. Here the contributions to the theory are contained in the references [33], [26] and [42]. We below cite only a simple, but typical, result.

Theorem 2.6. *Let A be the generator of an (a,k)-regularized family $\{R(t)\}_{t \geq 0}$ such that*

$$\|R(t)\| \leq Mk(t) \qquad \text{for all } t \geq 0.$$

Suppose that

(i) *$a(t)$ is positive, and $k(t)$ is nondecreasing and positive as well.*

(ii) $\lim\limits_{t \to \infty} \dfrac{k(t)}{(k*a)(t)} = 0.$

(iii) $\sup\limits_{t > 0} \dfrac{k(t)(1*k)(t)}{(k*a)(t)} < \infty.$

(iv) $\lim\limits_{t\to\infty} \dfrac{(a * a * k)(t)}{(a * k)(t)} = \infty.$

Define

$$A_t x := \frac{1}{k * a(t)} \int_0^t a(t-s) R(s) x\, ds; \quad x \in X \quad t > 0.$$

Then the following hold.

(1) *The mapping* $Px := \lim\limits_{t\to\infty} A_t x$ *is a bounded linear projection with* $Ran(P) = Ker(A)$, $Ker(P) = \overline{Ran(A)}$, *and*

$$D(P) = Ker(A) \oplus \overline{Ran(A)}.$$

(2) *For* $0 < \beta \le 1$ *and* $x \in Ker(A) \oplus \overline{Ran(A)}$, *we have*

$$\|A_t x - Px\| = O\left(\left[\frac{k(t)}{a * k(t)}\right]^\beta\right) \quad as\ |t| \to \infty.$$

(3) *If* X *is reflexive then* $Ker(A) \oplus \overline{Ran(A)} = X$.

Remark 2.7. *Note that in the case* $k(t) = \frac{t^\beta}{\Gamma(\beta+1)}$, $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ $\alpha > 0$, $\beta \ge 0$ *the conditions (i)-(iv) are automatically satisfied.*

The following definition extends the concept of Favard class, which is essential in approximation theory.

Definition 2.8. *Let* $a(t)$ *and* $k(t)$ *be continuous and positive. Let* A *be the generator of an* (a, k)-*regularized family* $\{R(t)\}_{t\ge 0}$ *on* X. *The Favard class of* A *with kernels* a *and* k *is defined as the set*

$$F_{a,k} = \{x \in X : \sup_{t>0} \frac{\|R(t)x - k(t)x\|}{a * k(t)} < \infty\}$$

Observe that the Favard class $F_{a,k}$, is a Banach space with respect to the norm

$$|x|_{a,k} = \|x\| + \sup_{t>0} \frac{\|R(t)x - k(t)x\|}{a * k(t)}.$$

The following result characterizes the space $F_{a,k}$ solely in terms of A, a and k.

Theorem 2.9. *([34]) Let* A *be a linear and closed operator with dense domain* $D(A)$ *in a Banach space* X. *Suppose that* A *generates a uniformly bounded* (a, k)-*regularized family* $\{R(t)\}_{t\ge 0}$. *Assume, that*

(i) *The Laplace transform of* $a(t)$ *exists for* $\lambda > 0$ *and satisfy* $\lim\limits_{\lambda \to 0^+} \widehat{a}(\lambda) = \infty$,

(ii) $\sup\limits_{t>0} \dfrac{(1 * a)(t)}{(k * a)(t)} < \infty.$

Then

$$F_{a,k} = \{x \in X : \sup_{\lambda > 0} \|\frac{1}{\widehat{a}(\lambda)} A(\frac{1}{\widehat{a}(\lambda)} - A)^{-1} x\| < \infty\}.$$

In particular, $F_{a,k}$ *does not depend on* k.

Remark 2.10. *In the case* $k(t) = \frac{t^\beta}{\Gamma(\beta+1)}$, $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ $\alpha > 0$, $\beta \ge 0$ *we have that* $\sup\limits_{t>0} \frac{(1*a)(t)}{(k*a)(t)} < \infty$ *is satisfied only for* $\beta = 0$ *and* $\alpha > 0$. *In particular, note that* $F_{a,k} \equiv F_{\alpha,\beta} = F_{\alpha,0} \sim F_{1,0}$.

We next recall that for a closed operator A we denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_a(A)$ the spectrum, the point spectrum, the residual spectrum, and the approximate spectrum of A respectively. We denote by $s(t, \lambda)$ the unique solution of the convolution equation

$$s(t, \lambda) = a(t) + \lambda \int_0^t a(t - \tau) s(\tau, \lambda) d\tau.$$

We also define

$$r(t, \lambda) := k(t) + \lambda \int_0^t s(t - \tau, \lambda)k(\tau)d\tau.$$

The following result corresponds to the inclusion theorem in an spectral theory for (a, k)-regularized families.

Theorem 2.11. *([34]) Let $R(t)$ be an (a, k)-regularized family with generator A. Then*

(i) $\sigma(R(t)) \supset r(t, \sigma(A)), \quad t \geq 0;$

(ii) $\sigma_p(R(t)) \supset r(t, \sigma_p(A)), \quad t \geq 0;$

(iii) *If A is densely defined then* $\sigma_r(R(t)) \supset r(t, \sigma_r(A)), \quad t \geq 0;$

(iv) $\sigma_a(R(t)) \supset r(t, \sigma_a(A)), \quad t \geq 0.$

Remark 2.12. *In the case $k(t) = \frac{t^\beta}{\Gamma(\beta+1)}$, $\beta \geq 0$, $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ $\alpha > 0$, we have that*

$$r_{\alpha,\beta}(t, \lambda) = t^\beta E_{\alpha,\beta+1}(\lambda t^\alpha)$$

where $E_{\alpha,\beta+1}$ denotes the (generalized) Mittag-Leffler function, defined as follows:

$$E_{\alpha,\beta+1}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta + 1)}.$$

In particular: $\alpha = 1, \beta = 0$ gives $E_{1,1}(z) = e^z$ and then $r_{1,0}(t, \lambda) = e^{\lambda t}$. In this case $R(t)$ corresponds to the C_0-semigroup generated by A and, therefore, we recover the well known spectral inclusion [12, Chapter IV, Section 3]:

$$e^{\sigma(A)t} \subset \sigma(R(t)), \quad t > 0.$$

If $\alpha = 2, \beta = 0$ we have $E_{2,1}(z^2) = \cosh(z)$ and then $r_{1,0}(t, \lambda) = \cosh\sqrt{(\lambda)}t$. Here $R(t)$ is the cosine family generated by A and we recover the spectral inclusion corresponding to that theory [37]:

$$\cosh\sqrt{\sigma(A)}t \subset \sigma(R(t)), \quad t > 0.$$

In general, let $0 < \alpha \leq 2$ and suppose that the fractional Cauchy problem:

$$D_t^\alpha u(t) = Au(t), \quad t > 0$$

is well posed. Then A generates an $(\alpha, 0)$-regularized family $R_\alpha(t)$ and we conclude that

$$E_{\alpha,1}(\sigma(A)t^\alpha) \subset \sigma(R_\alpha(t)), \quad t > 0.$$

This result was first proved by Li and Zheng [27, Theorem 3.2].

We finish this review with the following subordination result in case of operators with dense domain. For the non-dense case, see [24].

Theorem 2.13. *[24]. Let A be the generator of an exponentially bounded (a, k)-regularized family. Let $k, a, c \in L_{loc}^1(\mathbb{R}_+)$ be such that $\int_0^\infty |a(t)|e^{-\beta t}dt < \infty$ and $\int_0^\infty |k(t)|e^{-\beta t}dt < \infty$ for some $\beta \in \mathbb{R}$. Assume*

(i) *$c(t)$ is a completely positive function, i.e. $\dfrac{1}{\lambda \hat{c}(\lambda)}$ and $-\dfrac{\hat{c}(\lambda)'}{\hat{c}(\lambda)^2}$ are completely monotonic on $(0, \infty)$.*

(ii) $\hat{a}_1(\lambda) = \hat{a}\left(\dfrac{1}{\hat{c}(\lambda)}\right)$

(iii) $\hat{k}_1(\lambda) = \dfrac{1}{\lambda \hat{c}(\lambda)}\hat{k}\left(\dfrac{1}{\hat{c}(\lambda)}\right)$

Then A is the generator of a exponentially bounded (a_1, k_1)-regularized family.

The case $k(t) \equiv 1$ gives $k_1(t) \equiv 1$ and recover [39, Theorem 4.1]. A remarkable case is $a(t) = g_\beta(t)$ and $k(t) \equiv 1$, because we have an explicit representation. Note that in order to apply the above Theorem, is enough to take $c(t) = g_{\alpha/\beta}(t)$ which is completely positive whenever $\alpha < \beta$. We restate from [6] the result.

Corollary 2.14. [6, Theorem 3.1] *Let $0 < \alpha < \beta \leq 2$, $\gamma = \alpha/\beta$. If A be the generator of an exponentially bounded $(g_\beta, 1)$-regularized family $S_\beta(t)$, then A generates an exponentially bounded $(g_\alpha, 1)$-regularized family $S_\alpha(t)$, and*

$$(2.6) \qquad S_\alpha(t) = \int_0^\infty \Phi_\gamma(s) S_\beta(st^\gamma) ds, \quad t > 0,$$

where

$$\Phi_\gamma(t) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1,$$

is the Wright function.

More results and concrete examples on the theory of (a, k) - regularized families can be obtained from the recent article [24]. There, the author introduce the more general class of (local) (a, k)-regularized C-resolvent families and discuss its basic structural properties. In particular, the analysis done in [24] covers subjects like regularity, perturbations, duality, spectral properties and subordination principles, applying they in the study of the backwards fractional diffusion-wave equation and providing several illustrative examples.

3. Variation of constants formulas: The Laplace transform method

In a recent paper, Hernández, O'Regan and Balachandran [15] have pointed out that several recent investigations treating the problem of the existence of solution for abstract differential equations with fractional derivatives are incorrect since the considered variation of constants formulas are not appropriate. In this section we indicate a general procedure on how to obtain the correct variation of constants formulas for a wide class of fractional differential equations, based on the principle of transforming the linear part of the given equation in the frequency domain and, from there and with the help of the notion of (a, k)-regularized resolvents, obtain a right formula.

We start with the fractional differential equation

$$(3.1) \qquad D_t^\alpha u(t) = Au(t) + f(t), \quad t > 0,$$

where the fractional derivative will be understood in Caputo's sense. However he procedure that we will indicate is valid also for other types of fractional derivatives, e.g. the Riemann-Liouville's one; however but the final form of the variation of constant formula may vary with the type of fractional derivative. Fractional derivatives in the sense of Riemann-Liouville seem to be more appropriate in studying the qualitative behavior of abstract fractional differential equations on $\mathbb{R}$. See e.g. [1] for an example in this direction.

Recall that the definition of Caputo's fractional derivative of order $\alpha > 0$ of a function f reads as follows:

$$D_t^\alpha f(t) = J_t^{m-\alpha} D_t^m f(t); \quad m = \lceil \alpha \rceil$$

where $J_t^\alpha f(t) = (g_\alpha * f)(t)$, and $J_t^0 f(t) := f(t)$, $g_\beta(t) := \frac{1}{\Gamma(\beta)} t^{\beta-1}$, $t > 0$, $\beta \geq 0$. Also, it is well known, that the Caputo derivative is in general a left inverse of J_t^α but in general not a right inverse. More precisely, we have $D_t^\alpha J_t^\alpha f = f$, and $J_t^\alpha D_t^\alpha f(t) = f(t) - f(0)$ for $0 < \alpha < 1$. We also recall the formula for the Laplace transform

$$(3.2) \qquad \widehat{D_t^\alpha} f(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{m-1} f^{(k)}(0) \lambda^{\alpha-1-k},$$

whenever it exists. The more useful cases are $0 < \alpha \leq 1$

$$(3.3) \qquad \widehat{D_t^\alpha} f(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1} f(0),$$

and $1 < \alpha \leq 2$

$$(3.4) \qquad \widehat{D_t^\alpha} f(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1} f(0) - \lambda^{\alpha-2} f'(0),$$

whenever all the terms make sense.

Case 1. Suppose now $0 < \alpha \leq 1$ in Equation (3.1). Taking Laplace transform to both sides of the equation, we easily arrive to the following

$$(3.5) \qquad \hat{u}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} u(0) + (\lambda^\alpha - A)^{-1} \hat{f}(\lambda),$$

whenever $\lambda^\alpha \in \rho(A)$. Hence, we need to have to our disposal a Laplace transformable and strongly continuous family of bounded and linear operators, say $S_\alpha(t)$, such that

$$(3.6) \qquad \hat{S}_\alpha(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}.$$

For example, in case $\alpha = 1$, our search corresponds exactly to the existence of a C_0-semigroup (or equivalently, well posedness of $u' = Au$), say $T(t)$, because it is well known that [3, Theorem 3.1.7]:

$$(3.7) \qquad \hat{T}(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt = (\lambda - A)^{-1}x, \text{ for all } x \in X.$$

Returning to the general case of equation (3.1), in order to known exactly for which kind of family we are looking for, we have to compare (3.6) with (2.5). In other words, we look for scalar functions $a(t)$ and $k(t)$ such that

$$(3.8) \qquad \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)}I - A\right)^{-1} = \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}.$$

It follows that we must have $\hat{a}(\lambda) = \frac{1}{\lambda^\alpha}$ and hence, necessarily, $\hat{k}(\lambda) = \frac{1}{\lambda}$ in order to have the identity (3.8). We then find by inversion of the Laplace transform $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $k(t) \equiv 1$. We conclude that the appropriate family $S_\alpha(t)$ corresponds to an (a, k)-regularized family with a and k as precisely described. From Section 2 we deduce straightforward some properties of $S_\alpha(t)$ such as: generation theorem, approximation, etc. Concerning the variation of constants formula for (3.1), if we plug (3.6) in (3.5) and invert the Laplace transform, we obtain

$$(3.9) \qquad u(t) = S_\alpha(t)u(0) + \int_0^t S_\alpha(t-s)g(s)\,ds,$$

where $g(t) := \frac{d}{dt}\int_0^t g_\alpha(s)f(t-s)\,ds$. Note that in the border situation, i.e. $\alpha = 1$, we restate the fact that $g \equiv f$. We notice that a better regularity of f gives a more precise formula. For example, assuming that f' exists, we obtain:

$$(3.10) \qquad u(t) = S_\alpha(t)u(0) + f(0)\int_0^t S_\alpha(t-s)g_\alpha(s)\,ds + \int_0^t (S_\alpha * g_\alpha)(t-s)f'(s)\,ds.$$

Note that this formula shows that for $0 < \alpha < 1$ qualitative behavior of $S_\alpha(t)$ is important as well as the behavior of $S_\alpha * g_\alpha(t)$, in order to obtain information about $u(t)$.

Using the subordination results for (a, k)-regularized families, i.e. Corollary 2.14 with $\beta = 1$, we immediately obtain the following result.

Theorem 3.1. *Let A be the generator of a C_0-semigroup $T(t)$. Then A generates $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-regularized family $S_\alpha(t)$ given by*

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s)T(st^\alpha)\,ds, \quad t > 0, \quad 0 < \alpha < 1.$$

We note that such representation and variation of parameters formula have been used recently by some authors to study nonlinear abstract fractional differential equations. In that applications, it is assumed compactness of $T(t)$ in order to have compactness for $S_\alpha(t)$ and hence several types of fixed point theorems can be applied. See [10] and [43].

Case 2. We consider equation (3.1) for $1 < \alpha \le 2$. Then

$$(3.11) \qquad \hat{u}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}u(0) + \lambda^{\alpha-2}(\lambda^\alpha - A)^{-1}u'(0) + (\lambda^\alpha - A)^{-1}\hat{f}(\lambda),$$

whenever $\lambda^\alpha \in \rho(A)$. As in the first case, we look for a strongly continuous family $S_\alpha(t) \in \mathcal{B}(X)$ such that

$$(3.12) \qquad \widehat{S}_\alpha(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}.$$

Proceeding analogously as in the first case we note that it corresponds to an $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-resolvent family generated by A (compare (3.12) with (2.5) and find $a(t)$ and $k(t)$). For example, if $\alpha = 2$ we have $S_2(t)$, which corresponds to a cosine family generated by A (see [3, Proposition 3.14.4]). In the general case, we first set (3.12) in (3.11), then invert the Laplace transform, and obtain the following variation of constant formula:

$$(3.13) \qquad u(t) = S_\alpha(t)u(0) + \int_0^t S_\alpha(s)u'(0)ds + \int_0^t S_\alpha(t-s)g(s)ds,$$

where $g(t) = \frac{d}{dt}\int_0^t g_{\alpha-1}(t-s)f(s)ds$.

It is worthwhile to obtain an explicit representation of $S_\alpha(t)$, at least in the scalar case. We recall that the Mittag-Leffler function can be represented by

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{Ha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z}d\mu, \quad \alpha, \beta > 0, \quad z \in \mathbb{C},$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \le |z|^{1/\alpha}$ counter-clockwise.

Taking into account the following Laplace integral [38]

$$(3.14) \qquad \int_0^t e^{-\lambda t}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm\omega t^\alpha)dt = \frac{k!\lambda^{\alpha-\beta}}{(\lambda^\alpha \mp \omega)^{k+1}}, \quad Re(\lambda) > |\omega|^{1/\alpha},$$

and comparing with (3.12) we deduce the representation $S_\alpha(t) = E_{\alpha,1}(\omega t^\alpha)$ in the scalar case $A = \omega I$. In particular, $S_1(t) = E_{1,1}(\omega t) = e^{\omega t}$ and $S_2(t) = E_{2,1}(\omega t^2) = \cosh\sqrt{\omega}t$ as expected.

In analogy with the case 1, using the subordination principle (Corollary 2.14) with $\beta = 2$, we obtain the following criteria for existence of $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-regularized families in case $0 < \alpha < 2$.

Theorem 3.2. *Let A be the generator of a strongly continuous cosine family $C(t)$. Then A generates $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-regularized family $S_\alpha(t)$ given by*

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(s)C(st^{\alpha/2})ds, \quad t > 0, \quad 0 < \alpha < 2.$$

We consider now the *composite fractional relaxation equation*

$$(3.15) \qquad u'(t) - AD_t^\alpha u(t) + u(t) = f(t), \quad u(0) = x, \quad 0 < \alpha < 1,$$

where A is a closed linear operator and D_t^α denotes Caputo's fractional derivative.

In the scalar case, the fractional differential equation in (3.15) with $\alpha = 1/2$ corresponds to the Basset problem, a classical problem in fluid dynamics concerning the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity. The abstract version (3.15) has been studied in [34] and [20].

Taking Laplace transform, we obtain

$$(3.16) \qquad \hat{u}(\lambda) = \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}u(0) - \frac{1}{\lambda}A(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}u(0) + \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}\hat{f}(\lambda),$$

whenever $\frac{\lambda+1}{\lambda^{\alpha}} \in \rho(A)$. Note the unpleasant fact that A is involved in the above formula. We then replace the identity $\frac{1}{\lambda}I = \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1} - \frac{1}{\lambda}A(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1} + \frac{1}{\lambda^{\alpha+1}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}$ in (3.16) to obtain the equivalent form

$$(3.17) \qquad \hat{u}(\lambda) = \frac{1}{\lambda}u(0) - \frac{1}{\lambda}\frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}u(0) + \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}} - A)^{-1}\hat{f}(\lambda).$$

As before, we look for an strongly continuous family $R_{\alpha}(t) \subset \mathcal{B}(X)$ such that

$$(3.18) \qquad \hat{R}_{\alpha}(\lambda) = \frac{1}{\lambda^{\alpha}}(\frac{\lambda+1}{\lambda^{\alpha}}I - A)^{-1}.$$

Comparing equation (3.17) with equation (2.5) we note, as in the previous examples, that we need to find two Laplace transformable functions, $a(t)$ and $k(t)$, such that $\hat{a}(\lambda) = \frac{\lambda^{\alpha}}{\lambda+1}$ and $\hat{k}(\lambda) = \frac{1}{\lambda+1}$. It happens when we choose $k(t) = e^{-t}$ and $a(t) = t^{\alpha}E_{1,1-\alpha}(-t)$, the Mittag-Leffler function (compare with (3.14)).

Hence, in this case it is enough to require that equation (3.15) has A as the generator of an (a, k)-regularized family $S_{\alpha}(t)$, with a, k as defined before. The properties of such particular strongly continuous family can be obtained from Section 2. Setting now $\hat{R}_{\alpha}(\lambda)$ in equation (3.17) we obtain

$$(3.19) \qquad \hat{u}(\lambda) = \frac{1}{\lambda}u(0) - \frac{1}{\lambda}\hat{R}_{\alpha}(\lambda)u(0) + \hat{R}_{\alpha}(\lambda)\hat{f}(\lambda).$$

Inverting the Laplace transform, we obtain as variation of constants formula for equation (3.15) the following expression

$$(3.20) \qquad u(t) = u(0) - \int_0^t R_{\alpha}(s)u(0)ds + \int_0^t R_{\alpha}(t-s)f(s)ds.$$

Our next example consists in the *composite fractional oscillation equation*

$$(3.21) \qquad u''(t) - AD_t^{\alpha}u(t) + u(t) = f(t), \quad u(0) = x, \ u'(0) = y \quad 0 < \alpha < 2,$$

where A is a closed linear operator with domain $D(A) \subset X$. The fractional differential equation in (3.21) with $0 < \alpha < 2$ models an oscillation process with fractional damping term. It was formerly treated by Caputo, who provided a preliminary analysis by the Laplace transform. The special cases $\alpha = 1/2$ and $\alpha = 3/2$ have been discussed by Bagley and Torvik [5]. We note that the Bagley- Torvik equation was originally derived to study the motion of a rigid plate in a Newtonian fluid (see, e.g., [38], [5], [40]). The abstract form have been studied in [21].

Taking Laplace transform, we obtain

$$(3.22)$$
$$\hat{u}(\lambda) = \frac{1}{\lambda^{\alpha-1}}(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1}u(0) - \frac{1}{\lambda}A(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1}u(0) + \frac{1}{\lambda^{\alpha}}(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1}u'(0) + \frac{1}{\lambda^{\alpha}}(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1}\hat{f}(\lambda),$$

whenever $\frac{\lambda^2+1}{\lambda^{\alpha}} \in \rho(A)$. Taking into account the identity $\frac{1}{\lambda}I = \frac{1}{\lambda^{\alpha-1}}(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1} - \frac{1}{\lambda}A(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1} + \frac{1}{\lambda^{\alpha+1}}(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1}$ we obtain the equivalent form

$$(3.23) \qquad \hat{u}(\lambda) = \frac{1}{\lambda}u(0) - \frac{1}{\lambda}\frac{1}{\lambda^{\alpha}}(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1}u(0) + \frac{1}{\lambda^{\alpha}}(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1}u'(0) + \frac{1}{\lambda^{\alpha}}(\frac{\lambda^2+1}{\lambda^{\alpha}} - A)^{-1}\hat{f}(\lambda).$$

We now look for an strongly continuous family $T_{\alpha}(t) \subset \mathcal{B}(X)$ such that

$$(3.24) \qquad \hat{T}_{\alpha}(\lambda) = \frac{1}{\lambda^{\alpha}}(\frac{\lambda^2+1}{\lambda^{\alpha}}I - A)^{-1}.$$

Comparing equation (3.24) with equation (2.5) we note that we need to find two Laplace transformable functions, $a(t)$ and $k(t)$, such that $\hat{a}(\lambda) = \frac{\lambda^\alpha}{\lambda^2+1}$ and $\hat{k}(\lambda) = \frac{1}{\lambda^2+1}$. It holds when we choose $k(t) = \sin(t)$ and $a(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2)$(compare with (3.14)).

Putting now $\hat{T}_\alpha(\lambda)$ in equation (3.24) we have

$$(3.25) \qquad \hat{u}(\lambda) = \frac{1}{\lambda}u(0) - \frac{1}{\lambda}\hat{T}_\alpha(\lambda)u(0) + \hat{T}_\alpha(\lambda)u'(0) + \hat{T}_\alpha(\lambda)\hat{f}(\lambda).$$

From inversion of the Laplace transform, we obtain the following variation of constants formula for the composite fractional oscillation equation (3.21):

$$(3.26) \qquad u(t) = u(0) - \int_0^t T_\alpha(s)u(0)ds + T_\alpha(t)u'(0) + \int_0^t T_\alpha(t-s)f(s)ds.$$

The above procedure is not restricted only to abstract fractional differential equations. As an example, we consider now the well known Volterra equation of convolution type:

$$(3.27) \qquad u(t) = h(t) + A\int_0^t a(t-s)u(s)ds.$$

Taking formally the Laplace transform, we obtain

$$(3.28) \qquad \hat{u}(\lambda) = (I - \hat{a}(\lambda)A)^{-1}\hat{h}(\lambda),$$

whenever $(I - \hat{a}(\lambda)A)^{-1}$ exists in $\mathcal{B}(X)$. We now look for a strongly continuous family of bounded and linear operators $S(t)$ such that

$$(3.29) \qquad \hat{S}(\lambda) = \frac{1}{\lambda}(I - \hat{a}(\lambda)A)^{-1} = \frac{1}{\lambda\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)} - A\right)^{-1}.$$

The reason of this choice, instead of the most natural

$$\hat{S}(\lambda) = (I - \hat{a}(\lambda)A)^{-1}$$

is that we are searching that $S(t)$ coincides with the semigroup theory and cosine family theory in cases $a(t) \equiv 1$ and $a(t) \equiv t$ respectively. It was the argument used in the book (see [39]), originating the theory of resolvent families. Comparing equation (3.29) with equation (2.5) we note that if we choose $\hat{k}(\lambda) = \frac{1}{\lambda}$ then is enough to require that equation (3.27) admits A as the generator of an $(a, 1)$-regularized family $S(t)$ (or, equivalently, resolvent family; see [39]). Setting now (3.29) in equation (3.28) we obtain from the properties of the Laplace transform

$$(3.30) \qquad \hat{u}(\lambda) = \lambda\hat{S}(\lambda)\hat{h}(\lambda) = \widehat{(S*h)}'(\lambda).$$

Then, a variation of constants formula for equation (3.27) can be defined by

$$(3.31) \qquad u(t) = \frac{d}{dt}\int_0^t S(t-s)h(s)ds.$$

From the above formula, we obtain two possible expressions for variation of constants formula, depending on the hypothesis. If we suppose that $S(t)$ is differentiable, we have

$$(3.32) \qquad u(t) = h(t) + \int_0^t S'(t-s)h(s)ds,$$

but if we suppose that h is differentiable, we have

$$(3.33) \qquad u(t) = S(t)h(0) + \int_0^t S(t-s)h'(s)ds.$$

The following example illustrate the application of the above variation of constants formula for abstract fractional semilinear equations.

Example 3.3. *Consider the abstract fractional functional differential equation*

$$(3.34) \qquad D_t^\alpha(u(t) + g(t, u(t))) = Au(t) + f(t, u(t)), \quad u(0) = x_0, \ 0 < \alpha \le 1,$$

where D_t^α denotes Caputo's fractional derivative. Integrating, we have the equivalent equation

$$(3.35) \qquad u(t) = x_0 + g(0, x_0) - g(t, u(t)) + \int_0^t g_\alpha(t - s)f(s, u(s))ds + A\int_0^t g_\alpha(t - s)u(s)ds$$

which is of the form (3.27) with $a(t) = g_\alpha(t) := \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ and

$$h(t) = x_0 + g(0, x_0) - g(t, u(t)) + \int_0^t g_\alpha(t - s)f(s, u(s))ds$$

In fact: Suppose that equation (3.34) holds. Then applying J_t^α to both sides of the equation, we obtain:

$$(u(t) + g(t, u(t))) - (u(0) + g(0, u(0))) = AJ_t^\alpha u(t) + J_t^\alpha f(t, u(t)),$$

or

$$(u(t) + g(t, u(t))) - (x_0 + g(0, x_0)) = Ag_\alpha * u(t) + g_\alpha * f(t, u(t)),$$

which is exactly equation (3.35). Conversely, suppose that equation (3.35) holds. Then, applying D_t^α to both sides of the equation, we get

$$D_t^\alpha(u(t) + g(t, u(t))) - D_t^\alpha(x_0 + g(0, x_0)) = AD_t^\alpha J_t^\alpha u(t) + D_t^\alpha J_t^\alpha f(t, u(t)),$$

or

$$D_t^\alpha(u(t) + g(t, u(t))) - D_t^\alpha(x_0 + g(0, x_0)) = Au(t) + f(t, u(t)).$$

Finally, since $D_t^\alpha(\mathbf{1}) = 0$, we obtain (3.34), proving the claim.

  Suppose that A generates a differentiable (a,1)-regularized family $S(t)$. Then we have that a mild solution of equation (3.34) is well and consistently defined as the solution of the following integral equation

$$(3.36) \qquad u(t) = G_u(t) + F_u(t) + \int_0^t S'(t - s)[G_u(s) + F_u(s)]ds,$$

where $G_u(t) := x_0 + g(0, x_0) - g(t, u(t))$ and $F_u(t) := \int_0^t g_\alpha(t - s)f(s, u(s))ds$. Compare it with the definition of Γ in the proof of the main Theorem 2.1 in [15].

Example 3.4. *Consider the semilinear abstract composite fractional relaxation equation*

$$(3.37) \qquad u'(t) - AD_t^\alpha u(t) + u(t) = f(t, u(t)), \quad 0 < \alpha < 1, \ u(0) = x,$$

where A is the generator of an (a, k)-regularized family $R_\alpha(t)$, with $k(t) = e^{-t}$ and $a(t) = t^\alpha E_{1,1-\alpha}(-t)$. Then a mild solution of the semilinear equation (3.37) is well and consistently defined as a solution of the following integral equation

$$(3.38) \qquad u(t) = x - \int_0^t R_\alpha(s)xds + \int_0^t R_\alpha(t - s)f(s, u(s))ds.$$

Example 3.5. *Consider the semilinear abstract composite fractional oscillation equation*

$$(3.39) \qquad u''(t) - AD_t^\alpha u(t) + u(t) = f(t, u(t), u'(t)), \quad 0 < \alpha < 2, \ u(0) = x, \ u'(0) = y$$

where A is the generator of an (a, k)-regularized family $T_\alpha(t)$, with $k(t) = \sin(t)$ and $a(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2)$. Then a mild solution of the semilinear equation (4.1) is well and consistently defined as a solution of the following integral equation

$$(3.40) \qquad u(t) = u(0) - \int_0^t T_\alpha(s)u(0)\, ds + T_\alpha(t)u'(0) + \int_0^t T_\alpha(t - s)f(s, u(s), u'(s))ds.$$

4. Applications

In this section we will study some existence results and asymptotic behavior of solutions of some fractional differential equations using the variation of constants formulas developed earlier.

4.1. Existence results for semilinear fractional differential equations.

In this subsection, we will prove the existence of mild solutions of some fractional differential equations with nonlocal conditions. We will need the following results.

Lemma 4.1. *(Mazur's Lemma)*
If K is a compact subset of a Banach space X, then its convex closure $\overline{conv}K$ is also compact.

Theorem 4.2. *(Arzela-Ascoli's theorem)*
Let $\mathcal{T} := \{f(t)\}$ be a family of continuous mappings $f : I \to X$. If $\mathcal{T}$ is uniformly bounded and equicontinuous, and for any $t^* \in I$, the set $\{f(t^*) : f \in \mathcal{T}\}$ is relatively compact, then there exists a uniformly convergent sequence $\{f_n(t)\}$ in $\mathcal{T}$.

Theorem 4.3. *(Leray-Schauder Alternative Theorem)*
Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then either G has a fixed pint, or the set $\{x \in D : x = \lambda G(x),\ 0 < \lambda < 1\}$ is unbounded.

Consider the fractional differential equation with nonlocal conditions

$$(4.1) \qquad D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t)), \quad t \in [0,T],\ u(0) + g(u) = u_0, \quad u'(0) = 0,$$

where $1 < \alpha < 2$ and A generates an $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, 1)$-regularized family $S_\alpha(t)$ such that there exists $\omega > 0$ and $M > 0$ such that $\|S_\alpha(t)\| \le Me^{\omega t}$ for $t \in I$. Note that the existence of $S_\alpha(t)$ satisfying this condition follows, for example, if A generates a cosine function or if we assume that A is ω sectorial with angle $0 \le \theta < \pi(1 - \alpha/2)$. See [8]. We notice that several qualitative properties of equation (4.1) with initial conditions has been studied in [9] and [31].

According to the variation of constants formula (3.17), the problem above is equivalent to solving the following

$$(4.2) \qquad u(t) = S_\alpha(t)[u_0 - g(u)] + \int_0^t S_\alpha(t-s)f(s, u(s))ds.$$

We will make the following assumptions

- **H1.** f satisfies the Carathéodory condition, that is $f(\cdot, u)$ is strongly measurable for each $u \in X$ and $f(t, \cdot)$ is continuous for each $t \in I$.
- **H2.** There exists a continuous function $\mu : I \to \mathbb{R}^+$ such that
$$\|f(t, u)\| \le \mu(t)\|u\|, \forall t \in I,\ u \in X.$$
- **H3.** $g : C(I, X) \to C(I, X)$ is continuous and there exists $L_g > 0$ such that
$$\|g(u) - g(v)\| < L_g\|u - v\|,\ \forall u, v \in C(I, X).$$
- **H4** The set $\mathcal{K} = \{S_\alpha(t-s)f(s, u(s)) : u \in C(I, X), 0 \le s \le t\}$ is relatively compact for each $t \in I$.

Theorem 4.4. *Under assumptions H1-H4, Eq.(4.1) has at least a solution.*

Proof. Define the operator $\Gamma : C(I, X) \to C(I, X)$ by

$$(\Gamma u)(t) := S_\alpha(t)[u_0 - g(u)] + \int_0^t S_\alpha(t-s)f(s, u(s))ds$$

Let $B_r := \{u \in C(I, X) : \|u\| \le r\}$. The proof will be conducted into several steps

Step 1.

We first show that Γ sends bounded sets of $C(I, X)$ into bounded sets of $C(I, X)$; in other words for any given $r > 0$ there exists $\xi > 0$ such that $\Gamma B_r \subset B_\xi$. So let $u \in B_r$ and $G := \sup_{u \in B_r} \|g(u)\|$. Then

$$
\begin{aligned}
\|\Gamma u(t)\| &\leq M\|S_\alpha(t)\|(\|u_0\| + \|g(u)\|) + M \int_0^t \|S_\alpha(t-s)\|\|f(s, u(s))\|ds \\
\\
&\leq Me^{\omega t}(\|u_0\| + \|g(u)\|) + M \int_0^t \|S_\alpha(t-s)\|\|f(s, u(s))\|ds \\
\\
&\leq Me^{\omega t}(\|u_0\| + \|g(u)\|) + M \int_0^t e^{\omega(t-s)}\|f(s, u(s))\|ds \\
\\
&\leq Me^{\omega t}(\|u_0\| + \|g(u)\|) + M \int_0^t e^{\omega(t-s)}\|\mu(s)\|\|u(s)\|ds \\
\\
&\leq Me^{\omega T}(\|u_0\| + G) + Mr\|\mu\|\frac{e^{\omega T}}{\omega} = \xi.
\end{aligned}
$$

(4.3)

Thus $\Gamma B_r \subset B_\xi$.

Step 2. Let's show that Γ is a continuous operator.

Let $u_n, u \in B_r$ such that $u_n \to u$ in $C(I, X)$. Then we have

$$
\begin{aligned}
\|\Gamma u_n(t) - \Gamma u(t)\| &\leq \|S_\alpha(t)\|(\|g(u_n) - g(u)\|) + \int_0^t \|S_\alpha(t-s)\|\|f(s, u_n(s)) - f(s, u(s))\|ds \\
\\
&\leq Me^{\omega t}L_g\|u_n - u\| + M \int_0^t e^{\omega(t-s)}\|f(s, u_n(s)) - f(s, u(s))\|ds \\
\\
&\leq Me^{\omega T}L_g\|u_n - u\| + M \int_0^t e^{\omega(t-s)}\|f(s, u_n(s)) - f(s, u(s))\|ds \\
\\
&\leq Me^{\omega t}L_g\|u_n - v\| + M \int_0^t e^{\omega(t-s)}\mu(s)(\|u_n(s)\| + \|u(s)\|)ds \\
\\
&\leq Me^{\omega t}L_g\|u_n - u\| + 2rM \int_0^t e^{\omega(t-s)}\mu(s)ds.
\end{aligned}
$$

(4.4)

Choose n large enough such that $\|u_n - u\| < \epsilon$. Also note that $e^{\omega(t-s)}\mu(s)$ is integrable on I. So by the Lebesgue's Dominated Convergence Theorem, $\int_0^t e^{\omega(t-s)}\|f(s, u_n(s)) - f(s, u(s))\|ds \to 0$ as $n \to \infty$; which shows that Γ is continuous.

Step 3 Γ sends bounded sets of $C(I, X)$ into equicontinuous sets of $C(I, X)$.

Let $u \in B_r$, with $r > 0$ and take $t_1, t_2 \in I$ with $t_2 < t_1$. Then we have

$$
\begin{aligned}
\|\Gamma u(t_1) - \Gamma u(t_2)\| &\leq \|(S_\alpha(t_1) - S_\alpha(t_2))(u_0 - g(u))\| + \int_{t_2}^{t_1} \|S_\alpha(t_1 - s)f(s, u(s))\|ds \\
\\
&\quad + \int_0^{t_2} \|(S_\alpha(t_1 - s) - S_\alpha(t_2 - s))f(s, u(s))\|ds \\
\\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

(4.5)

We have

$$
I_1 \leq \|(S_\alpha(t_1) - S_\alpha(t_2))\|\|(u_0 - g(u))\|.
$$

Using the continuity of $S_\alpha(t)$, we obtain that $\lim_{t_1 \to t_2} I_1 = 0$.

Next we have

$$I_2 \le \int_{t_2}^{t_1} e^{\omega(t_1 - s)} \mu(s) \|u(s)\| ds \le r \|\mu\| e^{\omega T}(t_1 - t_2).$$

Thus $\lim_{t_1 \to t_2} I_2 = 0$. Finally we have

$$
\begin{aligned}
I_3 \quad &\le \quad \int_0^{t_2} \|S_\alpha(t_1 - s) - S_\alpha(t_2 - s)\| \|f(s, u(s))\| ds \\
\\
(4.6) \qquad &\le \quad \int_0^{t_2} \|S_\alpha(t_1 - s) - S_\alpha(t_2 - s)\| \mu(s) \|u(s)\| ds \\
\\
&\le \quad r \int_0^{t_2} \|S_\alpha(t_1 - s) - S_\alpha(t_2 - s)\| \mu(s) ds.
\end{aligned}
$$

Now observe that

$$\|S_\alpha(t_1 - \cdot) - S_\alpha(t_2 - \cdot)\| \mu(s) \le 2 M e^{\omega T} \mu(\cdot) \in L^1(I, \mathbb{R}),$$

and $S_\alpha(t_1 - s) - S_\alpha(t_2 - s) \to 0$ in $\mathcal{L}(X)$, as $t_1 \to t_2$. Thus $\lim_{t_1 \to t_2} I_3 = 0$ by the Lebesgue's dominated convergence theorem.

Step 4. Γ maps B_r into relatively compact sets in X.

Indeed in view of Lemma 3.3, we deduce that $\overline{conv}\mathcal{K}$ is compact. Moreover, for $u \in B_r$, using the Mean-Value Theorem for the Bochner integral, we obtain

$$\Gamma(u(t)) \in t\overline{conv}\mathcal{K}, \ \forall t \in [0, T].$$

Therefore the set $\{\Gamma u(t); u \in B_r\}$ is relatively compact in X for every $t \in [0, T]$. From Steps 1-4, we deduce that Γ is continuous and compact by the Arzela-Ascoli's theorem.

Step 5. Consider the set

$$\Omega := \{u \in B_r : u = \lambda \Gamma u, \ 0 < \lambda < 1\}.$$

Clearly $\Omega \ne \emptyset$ since $0 \in \Omega$. So let $u \in \Omega$. Then we have

$$\|u(t)\| \le \lambda [M e^{\alpha t}(\|u_0\| + \|g(u)\|) + M \int_0^t e^{\omega(t-s)} \|f(s, u(s)\| ds]$$

$$\le \lambda [M e^{\alpha t}(\|u_0\| + G) + M r \int_0^t e^{\omega(t-s)} \mu(s) ds]$$

$$\le [M e^{\alpha t}(\|u_0\| + G) + M r \|\mu\| \frac{e^{\omega T}}{\omega}]$$

Thus Ω is bounded. So by the Leray-Schauder theorem Γ has a fixed point. The proof is complete.

$\square$

4.2. *S-asymptotically ω-periodic solutions.* In this subsection we will study some asymptotic properties of mild solutions of semilinear and linear abstract composite fractional relaxation equation. Let's start with the linear case.

$$(4.1) \qquad\qquad u'(t) - A D_t^\alpha u(t) + u(t) = f(t), \quad 0 < \alpha < 1, \ u(0) = x,$$

where A is the generator of an (a, k)-regularized family $R_\alpha(t)$, with $k(t) = e^{-t}$ and $a(t) = t^\alpha E_{1,1-\alpha}(-t)$.

From now on we let $Y := ker(A)$.

We recall some definitions.

Definition 4.1. *([16, 17]) A function $f \in BC(\mathbb{R}_+, X)$ such that there exists $\omega > 0$ such that $\lim_{t \to \infty}(f(t + \omega) - f(t)) = 0$ is called S-asymptotically ω-periodic.*

We will denote by $SAP_\omega(X)$, the space of all S-asymptotically ω-periodic functions $f \in BC(\mathbb{R}_+, X)$.

Definition 4.2. *([16, 17]) A function $f \in BC(\mathbb{R}_+, X)$ is called asymptotically ω-periodic for $\omega > 0$ if there exist an ω-periodic function ϕ and and a $h \in BC(\mathbb{R}_+, X)$ with $\lim_{t \to \infty} h(t) = 0$ such that $f = \phi + h$.*

We will denote by $AP_\omega(X)$, the space of all asymptotically ω-periodic functions $f \in BC(\mathbb{R}_+, X)$.

Remark 4.3. *It is known that the space of all asymptotically ω-periodic functions is strictly contained in the space of S-asymptotically ω-periodic functions.*

Lemma 4.4. *Let f be an Y-valued S-asymptotically ω-periodic function; then the function $\zeta(t)$ defined by*

$$\zeta(t) := \int_0^t R_\alpha(t - s) f(s) ds. \ t > 0$$

is S-asymptotically ω-periodic.

Proof. It is clear that $\zeta \in BC(\mathbb{R}_+, Y)$. Notice that by (ii)-(iii) of Definition 2.1 we have $R_\alpha(s)f(s) = e^{-s}f(s)$ for all $s > 0$. We have

$$
\begin{aligned}
\zeta(t + \omega) - \zeta(t) &= \int_0^{t+\omega} e^{-(t+\omega-s)} f(s) ds - \int_0^t e^{-(t-s)} f(s) ds \\
&= \int_0^\omega e^{-(t+\omega-s)} f(s) ds + \int_{\omega}^{t+\omega} e^{-(t+\omega-s)} f(s) ds - \int_0^t e^{-(t-s)} f(s) ds \\
&= \int_0^\omega e^{-(t+\omega-s)} f(s) ds + \int_0^t e^{-(t-s)} f(s+\omega) ds - \int_0^t e^{-(t-s)} f(s) ds
\end{aligned}
$$

For each $\epsilon > 0$, there is a positive constant L_ϵ such that $||f(t+\omega) - f(t)|| \leq \epsilon$, for every $t \geq L_\epsilon$. Under these conditions, for $t \geq L_\epsilon$, we can estimate

$$
\begin{aligned}
||\zeta(t+\omega) - \zeta(t)|| &\leq \int_0^\omega ||e^{-(t+\omega-s)} f(s)|| ds \\
&+ \int_0^{L_\epsilon} e^{-(t-s)} ||f(s+\omega) - f(s)|| ds \\
&+ \int_{L_\epsilon}^t e^{-(t-s)} ||f(s+\omega) - f(s)|| ds \\
&\leq ||f||_\infty \int_0^\omega e^{-(t+\omega-s)} ds + 2||f||_\infty \int_0^{L_\epsilon} e^{-(t-s)} ds \\
&+ \epsilon \int_{L_\epsilon}^t e^{-(t-s)} ds \\
&= ||f||_\infty \int_t^{t+\omega} e^{-s} ds + 2||f||_\infty \int_{t-L_\epsilon}^t e^{-s} ds \\
&+ \epsilon \int_0^{t-L_\epsilon} e^{-s} ds \\
&\leq ||f||_\infty \int_t^\infty e^{-s} ds + 2||f||_\infty \int_{t-L_\epsilon}^\infty e^{-s} ds \\
&+ \epsilon \int_0^\infty e^{-s} ds
\end{aligned}
$$

which shows that $\lim_{t \to \infty}(\zeta(t+\omega) - \zeta(t)) = 0$ and ends the proof. $\qquad\square$

Our main result on the linear composite fractional relaxation equation is the following theorem.

Theorem 4.5. *If $x \in Y$ and f is an Y-valued S-asymptotically ω-periodic function, then every mild solution of (4.1) is S-asymptotically ω-periodic.*

Proof. Let $u(t)$ be a mild solution of (4.1). Then we have (cf. Example 3.4):

$$(4.2) \qquad u(t) = x - \int_0^t R_\alpha(s)x\,ds + \int_0^t R_\alpha(t-s)f(s)\,ds.$$

By Definition 2.1, if $u \in Ker(A)$ then we have $R_\alpha(t)u = e^{-t}u$. Thus under the assumptions of the theorem we have

$$(4.3) \qquad u(t) = x - \int_0^t e^{-s}x\,ds + \int_0^t e^{-(t-s)}f(s)\,ds.$$

Therefore

$$\|u(t+\omega) - u(t)\| \le I_1 + I_2$$

where

$$I_1 = \| \int_0^{t+\omega} e^{-s}x\,ds - \int_0^t e^{-s}x\,ds \| = \| \int_t^{t+\omega} e^{-s}x\,ds \| \le e^{-t}(1 - e^{-\omega})\|x\|$$

which shows that $\lim_{t\to\infty} I_1 = 0$, and $I_2 = \|\zeta(t)\|$ where $\zeta(t)$ is the S-asymptotically ω-periodic function in the lemma above. The proof is complete.

$\square$

Now we consider the semilinear abstract composite fractional relaxation equation

$$(4.4) \qquad u'(t) - AD_t^\alpha u(t) + u(t) = F(t, u(t)), \quad 0 < \alpha < 1, \ u(0) = x,$$

where A is as above and $\{F(t,u) : t \in \mathbb{R}_+, \ u \in Y\} \subset Y$.

We recall the following definitions and results.

Definition 4.6. [16] *A continuous function $f : [0, \infty) \times X \to X$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set $K \subset X$, the set $\{f(t,x) : t \ge 0, x \in K\}$ is bounded and $\lim_{t\to\infty} (f(t,x) - f(t+\omega, x)) = 0$ uniformly on $x \in K$.*

Definition 4.7. [16] *A continuous function $f : [0, \infty) \times X \to X$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded set $K \subset X$, there exist $L_{\epsilon,K} \ge 0$ and $\delta_{\epsilon,K} > 0$ such that $\|f(t,x) - f(t,y)\| < \epsilon$ for all $t \ge L_{\epsilon,K}$ and all $x, y \in K$ with $\|x - y\| < \delta_{\epsilon,K}$.*

Theorem 4.8. [16] *Let $f : [0, \infty) \times X \to X$ be a function which is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u : [0, \infty)$ be an S-asymptotically ω-periodic function. Then the Nemytskii function $\phi(\cdot) := f(\cdot, u(\cdot))$ is S-asymptotically ω-periodic.*

We make the following assumptions

H$_1$ F is uniformly S-asymptotically ω-periodic on bounded sets.

H$_2$ F satisfies a Lipschitz condition in the second variable uniformly with respect to the first variable, i.e. there exists $L > 0$ such that

$$\|F(t,u) - F(t,v)\| \le L\|u - v\|, \quad u, v \in Y, \ t \ge 0.$$

We have the following first main result on the semilinear composite fractional relaxation equation.

Theorem 4.9. *Under assumptions $H1 - H2$, if $x \in Y$ and $L < 1$, then the equation (4.4) has a unique S-asymptotically ω-periodic solution.*

Proof. Let

$$\Upsilon : SAP_\omega(Y) \to SAP_\omega(Y)$$

defined by

$$\Upsilon(u)(t) = x - \int_0^t R_\alpha(s)x\,ds + \int_0^t R_\alpha(t-s)F(s, u(s))\,ds,$$

equivalently

$$\Upsilon(u)(t) = e^{-t}x + \int_0^t e^{-(t-s)}F(s,u(s))ds.$$

Let $f(\cdot) := F(\cdot, u(\cdot))$ where $u \in AAP(Y)$. By hypothesis, it implies $f \in SAP_\omega(Y)$. By Lemma 4.4 we deduce that $\Upsilon(u)$ well defined. Let $u, v \in SAP_\omega(Y)$. Then we have

$$\begin{aligned}
\|(\Upsilon u)(t) - (\Upsilon v)(t)\| &\leq \int_0^t e^{-(t-s)}(F(s,u(s)) - F(s,v(s))ds\| \\
&\leq L\|u-v\|_\infty \int_0^t e^{-(t-s)}ds.
\end{aligned}$$

Therefore

$$\|(\Upsilon u) - (\Upsilon v)\|_\infty \leq L\|u-v\|_\infty.$$

We conclude by using the Banach contraction mapping principle. □

Now we consider the following assumption.

H₃ There exists $L \in L^1(0,\infty)$ such that for all $u, v \in Y$ and $t \geq 0$

$$\|F(t,u) - F(t,v)\| \leq L(t)\|u-v\|.$$

We prove the following result.

Theorem 4.10. *Under $H1 - H3$ the equation (4.4) has a unique mild solution in $SAP_\omega(Y)$.*

Proof. We consider the following norm on the space $BC(\mathbb{R}_+, Y)$: $\|f\|_c := \sup_{t\geq 0}(\|f(t)\|e^{(-c\int_0^t L(s)\,ds)})$ ($c > 0$). The two norms $\|\cdot\|_c$ and $\|\cdot\|_\infty$ are equivalent, since $L \in L^1(0,\infty)$ and $L(t) \geq 0$. Let $u, v \in SAP_\omega(Y)$ and consider Υ as defined above. Then we have

$$\begin{aligned}
\|\Upsilon u(t) - \Upsilon v(t)\| &\leq \int_0^t \|R_\alpha(t-s)\|_{\mathcal{B}(Y)}\|F(s,u(s)) - F(s,v(s))\|\,ds \\
&\leq \int_0^t e^{-(t-s)}L(s)\|u(s) - v(s)\|\,ds.
\end{aligned}$$

From $\|u(s) - v(s)\| \leq \|u-v\|_c e^{(c\int_0^s L(\sigma)\,d\sigma)}$, it follows

$$\begin{aligned}
\|\Upsilon u(t) - \Upsilon v(t)\| &\leq \|u-v\|_c \int_0^t L(s)e^{(c\int_0^s L(\sigma)\,d\sigma)\,ds)} \\
&\leq \frac{1}{c}\|u-v\|_c e^{(c\int_0^t L(s)\,ds)},
\end{aligned}$$

thus

$$\|\Upsilon u - \Upsilon v\|_c \leq \frac{1}{c}\|u-v\|_c.$$

We can choose $c := 2$, then

$$\|\Upsilon u - \Upsilon v\|_2 \leq \frac{1}{2}\|u-v\|_2,$$

thus by the contraction mapping principle, Γ has a fixed point in $SAP_\omega(Y)$ which is a solution to equation (4.4). □

4.3. **Asymptotically periodic solutions.** Consider the linear abstract composite fractional oscillation equation

(4.1) $$u''(t) - AD_t^\alpha u(t) + u(t) = f(t), \quad 0 < \alpha < 2, \ u(0) = x, \ u'(0) = y$$

where A is the generator of an (a, k)-regularized family $T_\alpha(t)$, with $k(t) = \sin(t)$ and $a(t) = t^{1-\alpha}E_{2,2-\alpha}(-t^2)$. By Example 3.5, a mild solution $u(t)$ of (4.1) is given by the following variation of constant formula

(4.2) $$u(t) = x - \int_0^t T_\alpha(s)x\, ds + T_\alpha(t)y + \int_0^t T_\alpha(t-s)f(s)ds.$$

As in the previous section, we denote $Y := Ker(A)$.

We prove the following result.

Theorem 4.1. *If $x, y \in Y$ and $f \in L^1(\mathbb{R}_+; Y)$, then every mild solution of (4.1) is asymptotically 2π-periodic.*

Proof. Note that, by Definition 2.1, in this case $T_\alpha(t)w = \sin(t)w$ for all $t > 0$ and $w \in Y$. Then we have

$$
\begin{aligned}
u(t) &= x - \int_0^t \sin(s)x\, ds + \sin(t)y + \int_0^t \sin(t-s)f(s)ds \\
&= \cos(t)x + \sin(t)y + \int_0^t \sin(t-s)f(s)ds \\
&= \cos(t)x + \sin(t)y + \int_0^\infty \sin(t-s)f(s)ds - \int_t^\infty \sin(t-s)f(s)ds \\
&= \phi(t) - \psi(t)
\end{aligned}
$$

where $\phi(t) := \cos(t)x + \sin(t)y + \int_0^\infty \sin(t-s)f(s)ds$ is clearly 2π-periodic and

$$\|\psi(t)\| \leq \int_t^\infty \|\sin(t-s)f(s)\|ds \leq \int_t^\infty \|f(s)\|ds$$

which shows that $\lim_{t \to \infty} \psi(t) = 0$. It proves the Theorem.

$\square$

Remark 4.2. *Note that the above result essentially coincides, in the scalar case, with the analytic study in the paper* [36] *for equation (4.1) with $0 < \alpha < 1$ and $f \equiv 0$.*

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