ON THE EXISTENCE OF CHAOS FOR THE VISCOUS VAN WIJNGAARDEN–ERINGEN EQUATION

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Abstract. We study the viscous van WijngaardenEringen equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = (\mathrm{Re}_d)^{-1} \frac{\partial^3 u}{\partial t \partial x^2} + a_0^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} \tag{1}$$

which corresponds to the linearized version of the equation that models the acoustic planar propagation in bubbly liquids. We show the existence of an explicit range, solely in terms of the constants a_0 and Re_d, in which we can ensure that this equation admits an uniformly continuous, Devaney chaotic and topologically mixing semigroup on Herzog's type Banach spaces.

1. Introduction

In the 1940's and 50's, the interest in studying the propagation of pressure waves of small amplitude in bubbly liquids appeared. The reason was to determine whether it was possible to take advantage of these acoustical properties to control the sound produced by propellers, both of surface ships and submerged ship. A vast literature on the subject deals with theoretical and experimental studies of the various aspects of propagation of pressure waves of small amplitude in bubbly liquids, see for instance [Wij72].

The acoustic planar propagation perpendicular to and along the x-axis (i.e., 1D flow) in bubbly liquids is given by the following equation [JKS14, Eq. 4.14]

$$\frac{\partial^2 u}{\partial t^2} - (1 - 2\varepsilon(\beta - 1)) \frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 = (\mathrm{Re}_d)^{-1} \frac{\partial^3 u}{\partial t \partial x^2} + a_0^2 \frac{\partial^4 u}{\partial t^2 \partial x^2}, \quad t \geq 0, \quad x \in \mathbb{R}, \tag{2}$$

being $\mathrm{Re}_d = c_e L/\delta$ a *Reynolds number*, where $c_e (> 0)$ is the adiabatic sound speed, δ is the diffusivity of sound [Tho72], and L is a characteristic (macroscopic length). The constant a_0 is a *Knudsen number* corresponds to the dimensionless bubble radius. In addition, γ stands for the adiabatic index of the liquid and $\beta (> 1)$ is known as the *coefficient of nonlinearity* [Bey97]. This coefficient is given by $\beta = (\gamma + 1)/2$ in the case of a perfect gas.

More details on the formulation of equation (2) can be found in [JKS14] and [JF06]. The linearized version, $\varepsilon = 0$, of equation (2) is known as the viscous (or dissipative) van Wijngaarden–Eringen equation, see [Wij72, Eri90].

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$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial^2 x} = (\mathrm{Re}_d)^{-1} \frac{\partial^3 u}{\partial t \partial x^2} + a_0^2 \frac{\partial^4 u}{\partial t^2 \partial x^2}, \quad t \geq 0, \quad x \in \mathbb{R}, \tag{3}$$

Some other classical models in nonlinear acoustics are the Kuznetsov equation, the Westervelt equation, and the Kokhlov-Zabolotskaya-Kuznetsov equation. Several initial boundary problems for these second order in time partial differential equations have been already solved (see for instance [KL11], [KLM11], [Roz07] and [RP08]).

Many problems in acoustics for the sound propagation are described in terms of the linear wave equation. The transport equation governing the propagation of second-sound in both metals and dielectric solids is the well known damped wave equation [Che63, Mau69] (see also [Jor15] for recent advances in the area). For these two cases, the 1D version of this PDE coincides with the hyperbolic heat transfer equation. The chaotic behaviour of the solutions to the abstract Cauchy problem stated from this equation was investigated in [CPT10], see also [GEP11]. For high wave amplitudes and intensities, phenomena such as wave distortion and formation of shocks appear, and then it is natural to study chaos for this type of equations. In [CLR15] the chaotic behaviour of the one-dimensional version of the Moore-Gibson-Thompson equation is studied. See also the seminal formulation of this equation in [Sto51, Eq. 7]. Moreover, the strong connection between acoustics and bioengineering and industry of high intensity sound waves has contributed to the improvement of the research in this area (see [Cri79], [Kuz71], [Col92]).

To the best of our knowledge, no study on chaotic behavior for equation (1) have been obtained. Hence, our aim here is to examine the van Wijngaarden-Eringen equation in the context of a dynamic, yet still analytically tractable, setting.

In this paper, we succeed in to prove the existence of a chaotic dynamics for the van Wijngaarden–Eringen equation. More precisely, we are able to show that whenever $a_0 < 1$ and

$$\frac{\sqrt{5}}{6} < a_0 \,\mathrm{Re}_b < \frac{1}{2}, \tag{4}$$

then equation (1) admits an uniformly continuous semigroup which is Devaney chaotic on an isomorphic copy of the sequence space $c_0^2(\mathbb{N}_0)$.

The paper is organized as follows: In section 2 we introduce some basic concepts related to the study of C_0- semigroups and chaos. In Section 3, chaos is also studied for the viscous van Wijngaarden–Eringen equation. Moreover, we provide a range for the bubble radius that ensures the existence of chaotic behaviour for the solutions of this equation. Finally, in Section 4, we give some physical interpretation of the results proved previously.

2. Preliminaries

Let X be a separable infinite-dimensional Banach space. We recall that $\{T_t\}_{t\geq 0}$, with $T_t : X \to X$ a continuous and linear map on X for each $t \geq 0$, is a C_0-*semigroup* if $T_0 = I$, $T_{t+s} = T_t \circ T_s$ and $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$.

Let $\{T_t\}_{t\geq 0}$ an arbitrary C_0-semigroup on X. It can be shown that

$$Ax := \lim_{t\to 0} \frac{1}{t}(T_t x - x), \tag{5}$$

exists on a dense subspace of X; the set of these x, the domain of A, is denoted by $D(A)$. Then A, or rather $(A, D(A))$, is called the *infinitesimal generator of the semigroup.*

Given the following abstract Cauchy problem on X:

$$\begin{cases} u_t(t,x) = Au(t,x), \\ u(0,x) = \varphi(x) \quad \varphi \in X, \end{cases} \tag{6}$$

the solution to (6) can be represented as a C_0-semigroup $\{T_t\}_{t\geq 0}$ on X whose infinitesimal generator is A. If $A \in L(X)$, then the operators in the C_0-semigroup can be represented as $T_t = e^{tA} = \sum_{k=0}^{\infty} (tA)^n/n!$ for all $t \geq 0$ (see for instance [EN00, Ch. I, Prop. 3.5]).

A C_0-semigroup $\{T_t\}_{t\geq 0}$ on X is said to be *hypercyclic* if there exists $x \in X$ such that the set $\{T_t x : t \geq 0\}$ is dense in X. An element $x \in X$ is called a *periodic point* for the semigroup $\{T_t\}_{t\geq 0}$ if there exists some $t > 0$ such that $T_t x = x$. A C_0-semigroup $\{T_t\}_{t\geq 0}$ is *topologically mixing* if for any pair U, V of nonempty open sets of X, there exists some $t_0 \geq 0$ such that $T_t(U) \cap V \neq \emptyset$ for all $t \geq t_0$. A C_0-semigroup $\{T_t\}_{t\geq 0}$ is called *Devaney chaotic* if it is hypercyclic and the set of periodic points is dense in X. We point out that these two requirements also yield the sensitive dependence on the initial conditions, as it was seen by Banks et al [BBC+92, GEP11]. Further information on the dynamics of C_0-semigroups can be found in [GEP11, Ch. 7]. See also [Eis10] for information regarding the stability properties of C_0-semigroups.

Another variation of the definition of chaos is the notion of *distributional chaos* introduced by Schweizer and Smítal [SS94], see also [MGOP09, Opr06] for its presentation in the infinite-dimensional linear setting. A C_0-semigroup $\{T_t\}_{t\geq 0}$ on X is said to be *distributionally chaotic* if there exists an uncountable subset $S \subset X$ and $\delta > 0$ such that, for each pair of distinct points $x, y \in S$ and for every $\varepsilon > 0$, we have $\overline{\mathrm{Dens}}(\{s \geq 0; ||T_s x - T_s y|| > \delta\}) = 1$ and $\overline{\mathrm{Dens}}(\{s \geq 0; ||T_s x - T_s y|| < \varepsilon\}) = 1$, where $\overline{\mathrm{Dens}}$ stands for the upper density of a set of real positive numbers. The set S is called the scrambled set and the C_0-semigroup is said to be *densely distributionally chaotic* if S is dense on X.

Now, we present a criterion that ensures Devaney chaos for C_0-semigroups. It is a variation of the (DSW) criterion [DSW97] which depends on verifying that the point spectrum of the infinitesimal generator of the C_0-semigroup contains "enough" eigenvalues. A first criterion stated in these terms was given for operators by Godefroy and Shapiro in [GS91]. We will use the following version of the (DSW) Criterion, see [GEP11, Th. 7.30]. It is also well known that distributional chaos holds whenever the DSW criterion can be applied [BC12, BBMGP11].

Theorem 2.1. *Let X be a complex separable Banach space, and $\{T_t\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator $(A, D(A))$. Assume that there exists an open connected subset U and a weakly holomorphic function $f : U \to X$, such that*

(1) $U \cap i\mathbb{R} \neq \emptyset$,

(2) *$f(\lambda) \in \ker(\lambda I - A)$ for every $\lambda \in U$,*
(3) *for any $x^* \in X^*$, if $\langle f(\lambda), x^* \rangle = 0$ for all $\lambda \in U$, then $x^* = 0$.*

Then the semigroup $\{T_t\}_{t\geq 0}$ is Devaney chaotic and topologically mixing.

A Borel probability measure $(\mu, \mathfrak{B})$ is said to have *full support* if for all non-empty open set $U \subset X$ we have $\mu(U) > 0$, and μ is said to be *T_t-invariant* if for all $A \in \mathfrak{B}$ we have that $\mu(A) = \mu(T_t^{-1}(A))$ for all $t \geq 0$. A C_0- semigroup $\{T_t\}_{t\geq 0}$ is *μ-strongly mixing* if $\lim_{t\to\infty} \mu(A \cap T_t^{-1}(B)) = \mu(A)\mu(B)$, for any A, B borelian sets of X. The existence of these measures are ensured if hypothesis of the DSW criterion are satisfied [MAP15].

Finally, we recall the definition of the space of analytic functions of Herzog type. Given $\rho > 0$, let:

$$X_\rho = \left\{ f : \mathbb{R} \to \mathbb{C}; f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, (a_n)_{n\geq 0} \in c_0(\mathbb{N}_0) \right\}$$

endowed with the norm $\|f\|_\rho = \sup_{n\geq 0} |a_n|$. This space is isometrically isomorphic to $c_0(\mathbb{N}_0)$. For examples and references on Herzog spaces, we refer the reader to [CLR15].

3. Existence of chaotic behavior

In this section, we will study the chaotic behavior of the viscous van Wijngaarden–Eringen equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) = (\mathrm{Re}_b)^{-1} \frac{\partial^3 u}{\partial t \partial x^2} u(t,x) + a_0^2 \frac{\partial^4 u}{\partial t^2 \partial x^2}(t,x), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{7}$$

in the space of analytic functions of Herzog type. We recall that $a_0 > 0$ denotes the dimensionless bubble radius and Re_b is a Reynolds number. Since the second order differential operator ∂_{xx} turns out to be a bounded operator on X_ρ, assuming the condition

$$a_0 < 1, \tag{8}$$

we obtain $\|a_0^2 \partial_{xx}\|_\rho < 1$ and, consequently, the inverse operator $(1 - a_0^2 \partial_{xx})^{-1}$ exists on X_ρ. Then we can express (7) as a first order equation on the product space $X := X_\rho \oplus X_\rho$. Setting $u_1 = u$ and $u_2 = \frac{\partial u}{\partial t}$ we can pose the following abstract Cauchy problem:

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ (1 - a_0^2 \partial_{xx})^{-1} \partial_{xx} & (\mathrm{Re}_b)^{-1} (1 - a_0^2 \partial_{xx})^{-1} \partial_{xx} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}; \\ \\ \qquad \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{array} \right. \tag{9}$$

Then, the operator-valued matrix

$$A := \begin{pmatrix} 0 & I \\ -(1 - a_0^2 \partial_{xx})^{-1} \partial_{xx} & (\mathrm{Re}_b)^{-1} (1 - a_0^2 \partial_{xx})^{-1} \partial_{xx} \end{pmatrix} \tag{10}$$

defines a bounded operator on X and, consequently, we have that $\{e^{tA}\}_{t\geq 0}$ is the solution C_0-semigroup of (9). See also [CPT10, Sec. 2.2] for a similar representation of the solution in the case of the wave equation, and [CLR15, Sec. 3] for the case of the Moore-Gibson-Thompson equation.

The following theorem is the main result in this paper.

Theorem 3.1. *Suppose that* $a_0 < 1$ *and*

$$0.3726 \approx \frac{\sqrt{5}}{6} < a_0 \operatorname{Re}_b < \frac{1}{2}, \tag{11}$$

then for each ρ *satisfying*

$$\rho > \frac{r_0}{(\frac{1}{2a_0^2 \operatorname{Re}_b} - 3r_0)a_0}, \tag{12}$$

where $r_0 := \frac{1}{2}\frac{\sqrt{1-4a_0^2 \operatorname{Re}_b^2}}{2a_0^2 \operatorname{Re}_b}$, *the operator* A *generates a uniformly continuous semigroup which is Devaney and distributionally chaotic, topologically mixing and admits a strongly mixing measure with full support on* $X_\rho \oplus X_\rho$.

Proof. Fix an arbitrary $\rho > 0$ satisfying (12). Our purpose is to apply Theorem 2.1. Firstly, we define

$$U := \{z \in \mathbb{C} \,:\, |z| < r_0\}. \tag{13}$$

Note that $1 - 4a_0^2 \operatorname{Re}_b^2 > 0$ due to the second inequality in (11). Hence $r_0 > 0$ and $U \cap i\mathbb{R} \neq \emptyset$. This proves condition (1) in Theorem 2.1.

Secondly, for each $\lambda \in U$ we define

$$R_\lambda := \frac{\lambda^2}{1 + (\operatorname{Re}_b)^{-1}\lambda + a_0^2\lambda^2}, \tag{14}$$

and weakly analytic functions $f_{z_0,z_1} : U \to X_\rho \oplus X_\rho$ by

$$f_{z_0,z_1}(\lambda) := \begin{pmatrix} \varphi_\lambda \\ \lambda\varphi_\lambda \end{pmatrix}, \tag{15}$$

where $\varphi_\lambda(x) := z_0 \cosh(\sqrt{R_\lambda}x) + z_1 \sinh(\sqrt{R_\lambda}x)$, with $z_0, z_1 \in \mathbb{C}$.

It is easy to verify that

$$\varphi_\lambda''(x) + (\operatorname{Re}_b)^{-1}\lambda\varphi_\lambda''(x) + a_0^2\lambda^2\varphi_\lambda''(x) = \lambda^2\varphi_\lambda(x), \quad x \in \mathbb{R}, \tag{16}$$

and therefore $Af_{z_0,z_1}(\lambda) = \lambda f_{z_0,z_1}(\lambda)$. We will show that $f_{z_0,z_1}(\lambda) \in X_\rho \oplus X_\rho$ for all $\lambda \in U$. Indeed, first note that we can rewrite φ_λ as follows:

$$\varphi_\lambda(x) = \cosh\left(\rho x\sqrt{\frac{R_\lambda}{\rho^2}}\right) z_0 + \sinh\left(\rho x\sqrt{\frac{R_\lambda}{\rho^2}}\right) z_1 = \sum_{n=0}^{\infty} a_n(\lambda)\frac{(\rho x)^n}{n!},\ x \in \mathbb{R}, \tag{17}$$

where $a_n(\lambda) = z_0\frac{R_\lambda^{n/2}}{\rho^n}, n = 0, 2, 4, \dots$ and $a_n(\lambda) = z_1\sqrt{R_\lambda}\frac{R_\lambda^{(n-1)/2}}{\rho^n}, n = 1, 3, 5, \dots$

Therefore, by definition, it is enough to prove that $|\frac{R_\lambda}{\rho^2}| < 1$. Indeed, observe that

$$a_0^2\lambda^2 + (\mathrm{Re}_b)^{-1}\lambda + 1 = a_0^2\left[\left(\lambda + \frac{1}{2a_0^2\,\mathrm{Re}_b}\right) + 2r_0\right]\left[\left(\lambda + \frac{1}{2a_0^2\,\mathrm{Re}_b}\right) - 2r_0\right]. \tag{18}$$

Define $\alpha := \dfrac{1}{2a_0^2\,\mathrm{Re}_b} + 2r_0$ and $\beta := \dfrac{1}{2a_0^2\,\mathrm{Re}_b} - 2r_0$. Then $0 < \beta < \alpha$ and for all $\lambda \in U$ we have

$$|\lambda + \alpha| \geq \alpha - |\lambda| \geq \beta - r_0 \quad \text{and} \quad |\lambda + \beta| \geq \beta - |\lambda| \geq \beta - r_0. \tag{19}$$

The hypothesis $\sqrt{5}/6 < a_0\,\mathrm{Re}_b$ implies that $\dfrac{1}{2a_0^2\,\mathrm{Re}_b} - 3r_0 > 0$. Therefore $\rho > 0$ and a calculation gives

$$\beta - r_0 > \frac{r_0}{\rho a_0}. \tag{20}$$

The above considerations imply,

$$\left|\frac{R_\lambda}{\rho^2}\right| = \frac{|\lambda|^2}{\rho^2 a_0^2 |\lambda + \alpha||\lambda + \beta|} < r_0^2 \frac{1}{\rho^2 a_0^2}\frac{\rho^2 a_0^2}{r_0^2} = 1,$$

for all $\lambda \in U$, proving the claim. It proves condition (2) in Theorem 2.1.

It only remains to show that for any $x^* \in X_\rho^* \oplus X_\rho^*$ the functions $\lambda \to \langle f_{z_0,z_1}(\lambda), x^*\rangle$, z_0, $z_1 \in \mathbb{C}$, are holomorphic on U, and if they all vanish on U, then $x^* = 0$. Since X_ρ is isometrically isomorphic to c_0, in what follows, we identify the dual space X_ρ^* with ℓ^1.

Let $x^* \in X_\rho^* \oplus X_\rho^*$. It can be represented in a canonical way by $x^* = (x_1^*, x_2^*) = ((x_{1,n}^*)_{n\geq 0}, (x_{2,n}^*)_{n\geq 0}) \in \ell^1 \oplus \ell^1$. Then, we have

$$0 = \langle f_{z_0,z_1}(\lambda), x^*\rangle = \langle \varphi_\lambda, x_1^*\rangle + \langle \lambda\varphi_\lambda, x_2^*\rangle, \tag{21}$$

for all $\lambda \in U, z_0, z_1 \in \mathbb{C}$. This last equation can be reformulated in the following way:

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} a_n(\lambda)x_{1,n}^* + \lambda\sum_{n=0}^{\infty} a_n(\lambda)x_{2,n}^* \\ &= z_0x_{1,0}^* + \lambda z_0 x_{2,0}^* + \frac{z_1}{\rho}\sqrt{R_\lambda}x_{1,1}^* + \frac{z_1}{\rho}\lambda\sqrt{R_\lambda}x_{2,1}^* + \frac{z_0}{\rho^2}R_\lambda x_{1,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda x_{2,2}^* + \ldots \end{aligned} \tag{22}$$

Let $\lambda_0 = 0$. It is clear that $\lambda_0 \in U$ and $R_{\lambda_0} = 0$. Evaluating (22) in λ_0, we get the following equation:

$$z_0x_{1,0}^* = 0 \tag{23}$$

for all $z_0 \in \mathbb{C}$. Therefore, $x_{1,0}^* = 0$.

Now, we divide (22) by λ and we get:

$$
\begin{aligned}
0 &= \frac{1}{\lambda}\left(\sum_{n=0}^{\infty} a_n(\lambda)x_{1,n}^* + \lambda\sum_{n=0}^{\infty} a_n(\lambda)x_{2,n}^*\right)\\
&= z_0x_{2,0}^* + \frac{z_1}{\rho}\frac{\sqrt{R_\lambda}}{\lambda}x_{1,1}^* + \frac{z_1}{\rho}\sqrt{R_\lambda}x_{2,1}^* + \frac{z_0}{\rho^2}\frac{R_\lambda}{\lambda}x_{1,2}^* + \frac{z_0}{\rho^2}R_\lambda x_{2,2}^* + \dots\\
&= z_0x_{2,0}^* + \frac{z_1}{\rho}\frac{1}{\sqrt{1+(\mathrm{Re}_b)^{-1}\lambda + a_0^2\lambda^2}}x_{1,1}^* + \frac{z_1}{\rho}\sqrt{R_\lambda}x_{2,1}^*\\
&+ \frac{z_0}{\rho^2}\frac{\lambda}{(1+(\mathrm{Re}_b)^{-1}\lambda + a_0^2\lambda^2)}x_{1,2}^* + \frac{z_0}{\rho^2}R_\lambda x_{2,2}^* + \dots
\end{aligned} \tag{24}
$$

As $R_{\lambda_0} = 0$, evaluating (24) in λ_0 we get:

$$
z_0x_{2,0}^* + \frac{z_1}{\rho}x_{1,1}^* = 0. \tag{25}
$$

for all $z_0, z_1 \in \mathbb{C}$. Therefore, $x_{2,0}^* = 0$ and $x_{1,1}^* = 0$.

Now, we divide (22) by $\lambda\sqrt{R_\lambda}$ and we get:

$$
0 = \frac{1}{\lambda\sqrt{R_\lambda}}\left(\sum_{n=0}^{\infty} a_n(\lambda)x_{1,n}^* + \lambda\sum_{n=0}^{\infty} a_n(\lambda)x_{2,n}^*\right). \tag{26}
$$

So that equation (26) can be reduced to:

$$
\begin{aligned}
0 &= \frac{z_1}{\rho}x_{2,1}^* + \frac{z_0}{\rho^2}\frac{\sqrt{R_\lambda}}{\lambda}x_{1,2}^* + \frac{z_0}{\rho^2}\sqrt{R_\lambda}x_{2,2}^* + \dots\\
&= \frac{z_1}{\rho}x_{2,1}^* + \frac{z_0}{\rho^2}\frac{1}{\sqrt{1+(\mathrm{Re}_b)^{-1}\lambda + a_0^2\lambda^2}}x_{1,2}^* + \frac{z_0}{\rho^2}\frac{\lambda}{\sqrt{1+(\mathrm{Re}_b)^{-1}\lambda + a_0^2\lambda^2}}x_{2,2}^* + \dots
\end{aligned} \tag{27}
$$

Evaluating (27) in λ_0, we get:

$$
\frac{z_1}{\rho}x_{2,1}^* + \frac{z_0}{\rho^2}x_{1,2}^* = 0 \tag{28}
$$

for all $z_0, z_1 \in \mathbb{C}$. Therefore, $x_{2,1}^* = 0$ and $x_{1,2}^*$.

Proceeding inductively, we will get that $x_{i,n}^* = 0$ for $i = 1, 2$ and $n \in \mathbb{N}$. We finally have $x^* = 0$ and we conclude the result by applying Theorem 2.1. □

Remark 3.2. *Recalling that $a_0 > 0$ denotes the bubble radius and $\mathrm{Re}_b = c_e L/\delta$ is a Reynolds number, where L denote a characteristic length, c_e the adiabatic sound speed and the positive constant $\delta > 0$ is known as the diffusivity of sound, we observe that the condition* (11)*, namely*

$$
0.3726 \approx \frac{\sqrt{5}}{6} < a_0\,\mathrm{Re}_b < \frac{1}{2}, \tag{29}
$$

together with the general condition $a_0 < 1$*, give an explicit range for the bubble radius in order to obtain chaotic behavior in the given model.*

This estimation can be compared with the physical one of

$$0.4899 \approx \frac{\sqrt{6}}{5} < a_0 \,\mathrm{Re}_b, \tag{30}$$

given in [JKS14, Eq. 4.20] *that has to be satisfied by the Reynolds number.*

4. Conclusions

These results can be understood in the following sense. For the hypercyclicity, given an arbitrary acoustic planar wave close to the origin one can determine a wave far away enough in order that, as time goes by, we can have that its propagation through a bubbly liquid gives us a wave at the origin as close as we want to the initial one. The Devaney chaos also yields the existence of periodic waves similar as much as we want to a prescribed one close to the origin.

These theoretical results present some limitations: The amplitudes required to resemble a prescribed wave at the origin can be so wide that cannot even be easily generated.

Furthermore, we recall that the existence of Devaney chaos yields the sensitive dependence of the solutions to the problem respect to the initial conditions. This affirms that given some planar wave, one can find a small perturbation, such that after a long enough period of time the behavior of both waves is completely different.

One last comment, the existence of distributional chaos asserts, roughly speaking, that there is an uncountable set of initial conditions such that we can pick a pair of initial conditions from this set and, as time goes by, there will be long time intervals in which the behaviour of the waves close to the origin are very similar for both initial waves. On the other hand, there will be also intervals as long as the previous ones in which the waves are quite different depending on which one of these two initial conditions we have chosen, see [BCMASS15].

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