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MAXIMAL REGULARITY FOR TIME-STEPPING SCHEMES ARISING FROM CONVOLUTION QUADRATURE OF NON-LOCAL IN TIME EQUATIONS

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Abstract. We study discrete time maximal regularity in Lebesgue spaces of sequences for time-stepping schemes arising from Lubich's convolution quadrature method. We show minimal properties on the quadrature weights that determines a wide class of implicit schemes. For an appropriate choice of the weights, we are able to identify the θ-method as well as the backward differentiation formulas and the $L1$-scheme. Fractional versions of these schemes, some of them completely new, are also shown, as well as their representation by means of the Grünwald–Letnikov fractional order derivative. Our results extend and improve some recent results on the subject and provide new insights on the basic nature of the weights that ensure maximal regularity.

1. **Introduction.** Our concern in this paper is the study of the maximal regularity property, in Lebesgue spaces of sequences, for the time-discretization of the following non-local in time abstract evolution equation:

$$\int_0^t a(t-s)v(s)ds = Av(t) + f(t), \quad t > 0, \tag{1}$$

where A is a closed linear operator (not necessarily bounded) with domain $D(A)$ defined on a Banach space X and $a \in L^1_{loc}(\mathbb{R}_+)$ is a scalar kernel. Typical kernels a are the standard kernel $a(t) = g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, which produces the classical time-fractional case, the fractional case with exponential weight $a(t) = g_\beta(t)e^{-\gamma t}$ where $\gamma > 0$ and $0 < \beta < 1$, and the distributed order case (or ultraslow diffusion) see e.g. [33, Section 6]. The operator A typically denotes the negative Laplacian in $X = L^2(\Omega)$, or the elasticity operator, the Stokes operator, or the biharmonic Δ^2,

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equipped with suitable boundary conditions. Problems of the form (1) have experimented a great interest during the last years. This kind of equations models anomalous diffusion. See for instance [17] where these equations appear in the modeling of ultraslow diffusion and renewal processes. In [32], the authors consider fractional models that label into this scheme for modeling anomalous diffusive processes in physics and engineering. In [33], optimal decay estimates for time-fractional and other non-local subdiffusion equations using energy methods are obtained. Another context where these equations and nonlinear versions of them arise is the pattern of dynamic processes in materials with memory. See [27] where heat conduction with memory is analyzed and [8] where these equations model the diffusion of fluids in porous media with memory.

The study of discrete *time* maximal regularity, or more precisely discrete maximal regularity estimates, has apparently its origins in the works by S. Blunck [6, 7], who characterizes this property in terms of R-boundedness of an operator-valued symbol. Then Portal [26] studied different notions of discrete maximal regularity and generalized Blunck's result to the case of some discrete time scales (discrete problems with nonconstant step size). Also, maximal regularity of evolution equations with operator-valued Fourier multipliers in L^p-spaces was studied in [34]. A review of these results as well as applications are given in the monograph [1]. After that, Kovács, Li and Lubich [18] gave an important step further studying under which conditions the property of maximal regularity can be preserved from the continuous to the discrete time setting in Lebesgue spaces. They showed that this property holds for A-stable multistep and Runge–Kutta methods under minor additional conditions. In particular, the implicit Euler method, the Crank–Nicolson method, the second-order backward difference formula, and higher-order A-stable implicit Runge–Kutta methods such as the Radau IIA and Gauss methods preserve maximal regularity. By the same time, Kemmochi [15] analyzed the notion of discrete maximal regularity for the finite difference method (θ-method) still under the hypothesis of continuous maximal regularity. The θ-method represents the most simple Runge–Kutta method (and also linear multistep method) and different values of θ are of significance in some contexts.

It should be noted that discrete maximal regularity has several applications in the analysis of stability, error bounds and convergence of numerical methods for nonlinear problems, including quasi-linear parabolic equations [3] even with nonsmooth coefficients [20], optimal control and inverse problems [19].

The study of maximal regularity of time-stepping schemes of approximation in Lebesgue spaces of sequences for *non-local* time-stepping schemes, more precisely of fractional order $\alpha \in (0, 1)$, has its origin in the paper [22] where Blunck's Theorem on operator-valued multipliers [7] is the basis of the analysis. An important step forward was given in the seminal paper [13] by Jin, Lazarov and Zhou where the idea outlined in [22] was widely refined and applied to an important number of time-stepping schemes of fractional order that can be obtained from Lubich's quadrature method, including implicit and explicit Euler methods, backward difference formulas, the $L1$-scheme and the Crank–Nicolson method [24].

More recently, in [23], maximal regularity for discrete time Volterra equations in the form $u^n = c * Au^n + f^n$ was studied. The main theorem in [23] includes the analysis of several non-local time-stepping schemes.

However, such model requires some assumptions that apparently restrict the analysis to some classes of kernels. In fact, the kernel c must belong to a certain class of sequence kernels that resembles the Sonine class [28] (originated from a paper of N. Sonine, published in 1884). This restriction does not allow to deduce - at least in a first glance - discrete time maximal regularity results for a wider variety of non local time-stepping methods like e.g. the $L1$-scheme which has been recently studied in the literature, see [31]. We note that the $L1$-scheme is one of the most popular and successful numerical methods for discretizing the Caputo fractional derivative in time [13]. In addition to the $L1$-scheme - which is not local by its own nature - there exist other non-local schemes that have not been studied in the literature yet, as the fractional version of the θ-method whose maximal regularity has not been analyzed, except in the cases $\theta \in \{0, 1, 1/2\}$. In synthesis, we ask ourself the following question: *Which are the most crucial properties on the kernel in order to obtain maximal regularity for (local and non local) time-stepping schemes?*

In this paper, we want to give an important step further in this line of research. More concretely, we consider time-stepping schemes arising from convolution quadrature

$$\sum_{0\le j\tau\le t} b_j v(t-j\tau), \quad t=\tau, 2\tau, 3\tau, ...$$

where τ is the time step size, and the quadrature weights b_j are determined by their generating power series

$$F\left(\frac{\delta(\zeta)}{\tau}\right)=\sum_{j=0}^{\infty} b_j \zeta^j.$$

Here, $F(\cdot)$ denotes the Laplace transform of the scalar kernel a in (1) and $\delta(\zeta)$ is a rational function, chosen as the quotient of the generating polynomials of a linear multistep method. Convolution quadrature was first proposed by Lubich in a series of works [24] for discretizing Volterra integral equations.

In this paper, our main goal is to determine under which conditions on the kernel b the following time-discretization of (1)

$$\sum_{j=0}^{n} b_{n-j}u^j = Au^n + f^n, \quad n\in\mathbb{N}_0, \tag{2}$$

has maximal regularity, identifying those kernels that correspond to known time-stepping schemes of approximation, revealing their nature, and proposing new schemes through a suitable choice of these kernels.

As we will reveal along this research, by means of the abstract model (2), we are able to capture a wide variety of classical implicit time-stepping schemes such as the Backward Euler scheme (BE-scheme) or the second order backward differentiation formula for time-stepping schemes (BDF-scheme).

For instance, choosing $b_n=\frac{1}{\tau}(\delta_0-\delta_1)(n)$, where $\delta_i(j)$ is the Kronecker delta, we obtain the classical Backward Euler scheme (BE-scheme) and when $b_n=\frac{1}{\tau}[(\delta_0-\delta_1)](n)+\frac{1}{2\tau}[(\delta_0-\delta_1)*(\delta_0-\delta_1)](n)$, we get the second order backward differentiation formula for time-stepping schemes (BDF-scheme). This choice of b reveals the local character of each of the aforementioned methods.

More interesting is to observe that when we choose $b_j=\frac{1}{\tau^\alpha}k_j^{-\alpha}$ we obtain the *fractional* Backward Euler scheme [14, Section 3.1] of order $0<\alpha<1$, where

$$(1-z)^\alpha=\sum_{j=0}^{\infty} k_j^{-\alpha} z^j, \tag{3}$$

which corresponds to time-discretization of (1) with $a(t) = g_\alpha(t)$. We note that the sequence $k_j^\beta, \beta \in \mathbb{R}$ has the explicit form

$$k_j^\beta = \begin{cases} \dfrac{\Gamma(\beta+j)}{\Gamma(\beta)\Gamma(j+1)} & j \in \mathbb{N}_0,\ \beta \in \mathbb{R}\setminus\{-1,-2,..\}; \\ \\ (\delta_0-\delta_1)^{*(-\beta)}(j) & j \in \mathbb{N}_0,\ \beta \in \{-1,-2,...\}, \end{cases} \tag{4}$$

where Γ is the Euler gamma function and $p^{*n} = \underbrace{p*p*\ldots*p}_{n\text{-times}}$ where $*$ denotes the convolution of sequences given by $(u*v)(n) = \sum_{j=0}^{n} u(n-j)v(j)$, see [12].This sequence has been recently used in a variety of papers in connection with time fractional discrete systems in several areas of research. For instance, Sun and Phillips in [30, Section 2.1] (and references therein) used this sequence for the definition of fractional difference filters in finance and macroeconomics where long memory is relevant. Moreover, in [21] (see also [23] and references therein) it was used to define a new fractional order difference operator which is linked - by means of the so called transference principle [12, Section 4] - to the most common fractional forward difference operator of order $\alpha > 0$ defined in the classical paper [5] by Atici and Eloe. It is remarkable that the sequence k_n^β plays the same role than the kernel $g_\beta(t)$ in continuous fractional calculus, see [21].

Continuing the discussion on fractional order schemes, another possibility is to choose

$$b_n = \frac{1}{\tau^\alpha}\frac{1}{(1+\alpha(1-\theta))}\sum_{j=0}^{n} k_{n-j}^{-\alpha}\frac{(\alpha(\theta-1))^j}{(1+\alpha(1-\theta))^j}, \quad n \in \mathbb{N}_0, \tag{5}$$

where $0 < \alpha < 2,\ \alpha \neq 1$ and $\frac{1}{2} \leq \theta < 1$. Note that when $\theta = \frac{1}{2}$ it corresponds to the fractional Crank–Nicolson scheme studied in [14, Section 6]. We will see that this kernel gives rise to a fractional version of the θ-scheme that will be studied in this paper for the first time. Note that this new scheme is not related to the fractional step θ-scheme, studied in older references [16].

A further option is the following

$$b_n = \left(\frac{3}{2\tau}\right)^\alpha \sum_{j=0}^{n} k_{n-j}^{-\alpha}\frac{1}{3^j}k_j^{-\alpha}, \quad n \in \mathbb{N}_0. \tag{6}$$

This sequence corresponds to the fractional second-order BDF scheme which was considered in [23]. A next interesting example that we will examine in this paper, among others, is

$$b_n = \frac{1}{\tau^\alpha\Gamma(2-\alpha)}[(n+1)^{1-\alpha} - n^{1-\alpha}],\ n \in \mathbb{N}_0, \quad 0 < \alpha < 1, \tag{7}$$

that produces the $L1$-scheme. We note that in contrast to the BE and BDF schemes, all these kernels are *non-local,* i.e. a no finite number of terms of the sequence b_n are different from zero.

Summarizing, this paper is organized as follows: Section 2 is devoted to introduce some preliminary results, most of which are taken from [23] and that can be found in earlier works on the subject. See e.g. the monograph [1]. Section 3 contains the main abstract result of this paper, namely Theorem 3.4, that states the most general sufficient conditions on the kernel b for maximal regularity of (2) in the context of UMD-spaces. Apart from the geometrical condition on the Banach space, and R-boundedness conditions of the operator-valued symbol, we provide new insights on the minimal properties on the kernel b that are crucial in order to obtain maximal regularity.

It should be noted that in the context of Hilbert spaces, the hypothesis of R-boundedness is reduced to simple uniform boundedness. This is the content of Corollary 1. Otherwise, R-sectoriality of the operator A plus a condition on the location of b_0 on the complex plane is needed. This condition can be replaced by the stronger hypothesis $0 \in \rho(A)$, the resolvent set of A, if necessary. It is interesting to observe the relation between maximal regularity and the concept of $A(\beta)$-stability. This is the content of Corollary 3. Finally, Section 4 is entirely devoted to the analysis of maximal regularity for time-stepping schemes that we identify with kernel sequences b. We begin with the θ-scheme and a new fractional version of it of order $0 < \alpha < 2$, that we call (α, θ)-scheme. We must divide the study in two cases: $0 < \theta \leq \frac{1}{2}$ and $\frac{1}{2} \leq \theta \leq 1$. See Theorems 4.2 and 4.3. It is interesting to observe that both cases are related to the generalized Grünwald–Letnikov fractional order derivative. We also improve [14, Theorem 5] and [14, Theorem 9] in Theorem 4.4. Then we take into consideration the backward differentiation formulas of order $p \leq 6$ which are given by $b_n = \sum_{j=1}^{p} \frac{1}{j}[(\delta_0 - \delta_1)^j](n)$, $n \in \mathbb{N}_0$, as they are well-known examples of $A(\alpha)$-stable linear multistep methods. Here $(\delta_0 - \delta_1)^j$ denotes convolution j-times. We compute the angle α and obtain new criteria of maximal regularity in cases $p = 2, 3, 4$. Fractional versions of each case are also included as part of our results. In particular, in case $p = 2$, we recover [14, Theorem 6]. The other cases are new. Finally, we analyze the $L1$-scheme, which is fractional in nature, and we provide their integer and fractional version that seems to be new. Our main criteria of maximal regularity in case $0 < \alpha < 1$ is contained in Theorem 4.6 which extends [14, Theorem 7].

2. **Analytical framework and notation.** In this section, we introduce some notations and preliminary results that will be necessary in the forthcoming sections.

Let X be a complex Banach space. We denote by $\mathcal{S}(\mathbb{Z}; X)$ the Schwartz space of vector-valued sequences that are rapidly decreasing, whose topology is that induced by the semi norms $p_k(f) := \sup_{n\in\mathbb{Z}} |n|^k \|f(n)\|, k \in \mathbb{N}_0$, and by $C^n_{per}(\mathbb{R}; X)$ the space of all 2π-periodic X-valued and n-times continuously differentiable functions defined in $\mathbb{R}$. We also set $\mathbb{T} := (0, 2\pi)$ and $\mathbb{T}_0 := (0, 2\pi)\setminus\{\pi\}$. The space of test functions is the space $C^\infty_{per}(\mathbb{T}; X) := \bigcap_{n\in\mathbb{N}_0} C^n_{per}(\mathbb{R}; X)$. When $X = \mathbb{R}$ we simply write $C^\infty_{per}(\mathbb{T})$ and $\mathcal{S}(\mathbb{Z})$. We recall that the discrete time Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{Z}; X) \to C^\infty_{per}(\mathbb{T}; X)$ is an isomorphism defined by $\mathcal{F}\varphi(t) \equiv \widehat{\varphi}(t) := \sum_{j=-\infty}^{\infty} e^{-ijt}\varphi_j$ and the inverse transform is given by $\mathcal{F}^{-1}\varphi_n \equiv \check{\varphi}_n := \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)e^{int}dt$.

For a sequence $b : \mathbb{N}_0 \to \mathbb{C}$ extended to negative subscripts n by 0, the *Gelfand transform* is defined by $\tilde{b}(z) := \sum_{n=0}^{\infty} b_n z^n$, $|z| < 1$. We also recall the following Lemma [23, Lemma 2.3].

Lemma 2.1. *Let $u, v : \mathbb{Z} \to X$ be given and $a : \mathbb{N}_0 \to \mathbb{C}$ which is defined by 0 for negative values of n. We assume that the series $\tilde{a}(z) := \sum_{n=0}^{\infty} a_n z^n$, converges on the complex unit disk, and that the radial limit $\widehat{a}(t) = \lim\limits_{r\nearrow 1} \tilde{a}(re^{-it})$ exists for all $t \in \mathbb{T}_0$. Suppose that $\langle u, \check{\varphi}\rangle = \langle v, (\varphi\cdot\widehat{a}_-)\check{}\,\rangle$ for all $\varphi \in C^\infty_{per}(\mathbb{T})$, where $(\varphi\cdot\widehat{a}_-)\check{}_n := \frac{1}{2\pi}\int_0^{2\pi}\widehat{a}(-t)\varphi(t)e^{int}dt$, $n \in \mathbb{Z}$ and $\langle u, \check{\varphi}\rangle := \sum_{n\in\mathbb{Z}} u(n)\check{\varphi}(n)$, $\check{\varphi} \in \mathcal{S}(\mathbb{Z})$. Then $u^n = H(v * a)_n$ for all $n \in \mathbb{Z}$, where*

$$(v * a)_n := \sum_{j=0}^{n} v_{n-j} a_j, \quad n \in \mathbb{N}_0, \tag{8}$$

denotes the finite convolution, and H denotes the Heaviside operator defined as $Hw_n = w_n$ if $n \in \mathbb{N}_0$ and by 0 otherwise.

We also recall the notion of R-bounded sets and ℓ_p-multipliers in the space $\mathcal{B}(X,Y)$ of bounded linear operators from X into Y endowed with the uniform operator topology that will play an important role in our work.

Definition 2.2. Let X and Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X,Y)$ is called R-bounded if there is a constant $c > 0$ such that for all $T_1, ..., T_n \in \mathcal{T},\ x_1, ..., x_n \in X,\ n \in \mathbb{N}$, we have $\|(T_1x_1, ..., T_nx_n)\|_R \le c\|(x_1, ..., x_n)\|_R$, where $\|(x_1, ..., x_n)\|_R := \dfrac{1}{2^n} \displaystyle\sum_{\{\epsilon_j\}_{j=1}^n \in \{-1,1\}^n} \Big\| \sum_{j=1}^{n} \epsilon_j x_j \Big\|$.

Some basic properties are preserved under R-boundedness, see [1, Section 2.2] and [10] for more information. We now recall the notion of ℓ_p-multiplier. Let X, Y be Banach spaces and $1 < p < \infty$. A function $M \in C^{\infty}_{per}(\mathbb{T}; \mathcal{B}(X,Y))$ is an ℓ_p-multiplier (from X to Y) if there exists a bounded operator $T_M : \ell_p(\mathbb{Z}; X) \to \ell_p(\mathbb{Z}; Y)$ such that

$$\sum_{n\in\mathbb{Z}} (T_M f)^n \check{\varphi}_n = \sum_{n\in\mathbb{Z}} (\varphi \cdot M_-\check{)}_n f^n \tag{9}$$

for all $f \in \ell_p(\mathbb{Z}; X)$ and all $\varphi \in C^{\infty}_{per}(\mathbb{T})$. We will need a Fourier multiplier theorem for operator - valued symbols originally given by S. Blunck [7, 1] which gives sufficient conditions to ensure when an operator-valued symbol is a multiplier, and establishes an equivalence between ℓ_p-multipliers and the notion of R-boundedness for the UMD class of Banach spaces. For more information on this class of spaces, see [4, Section III.4.3-III.4.5]. The following extended version of Blunck's theorem is essentially due to Kemmochi [15] as regards the independence of the constant involved. See [13, Theorem 4] for the precise statement.

Theorem 2.3. [7, Theorem 1.3] *Let $p \in (1, \infty)$ and let X, Y be UMD spaces. Let $M \in C^{\infty}_{per}(\mathbb{T}_0; \mathcal{B}(X,Y))$ be such that the set*

$$\{M(t) : t \in \mathbb{T}_0\} \cup \left\{(1 - e^{it})(1 + e^{it})M'(t) :\ t \in \mathbb{T}_0\right\},$$

is R-bounded, with an R-bound c_R. Then M is an ℓ_p-multiplier (from X to Y) for $1 < p < \infty$. Further, there exists a $c_{p,X} > 0$ independent of M such that the operator norm of T_M is bounded by $c_R c_{p,X}$.

Conversely, if X, Y are Banach spaces and $M \in C^{\infty}_{per}(\mathbb{T}; \mathcal{B}(X,Y))$ is an operator valued function such that there exists a bounded operator $T_M : l_p(\mathbb{Z}; X) \to l_p(\mathbb{Z}; Y)$ veryfing equality (9), then the set $\{M(t) : t \in \mathbb{T}\}$ is R-bounded.

3. **Abstract setting: Maximal ℓ^p_0-regularity.** It is important for our analysis to observe that Lubich's convolution quadrature method implicitly considers zero-padding in the negative real axis, see [24, p.131, lines 1-2]. This notion of *causality* has been previously considered in other papers, see e.g. [23]. Therefore, in what follows, the prehistorical values of the kernel b in (2) will be assumed to be zero. According to this, we will analyze maximal regularity in the following Lebesgue space of vector-valued sequences:

$$\ell^p_0(\mathbb{N}_0; X) := \{f \in \ell_p(\mathbb{Z}; X) \ :\ f^n = 0 \text{ for all } n = -1, -2, ...\}.$$

For a given vector-valued sequence $f : \mathbb{N}_0 \to X$ we consider along this section the abstract discrete equation given by (2).

Remark 1. Note that the solvability of (2) is equivalent to the invertibility of $b_0 - A$, since (2) can be rewritten as $(b_0 - A)u^0 = f^0$ and

$$(b_0 - A)u^n = -\sum_{j=0}^{n-1} b_{n-j}u^j + f^n, \quad n \in \mathbb{N}. \tag{10}$$

In particular, if (2) is solvable, then the solution must be unique. For instance, if A is the generator of a bounded analytic semigroup on X and $b_0 > 0$ then (2) is solvable. However, in general, we must assume that b_0 belongs to $\rho(A)$, the resolvent set of A.

Definition 3.1. Let $1 < p < \infty$ be given. We say that (2) has maximal ℓ_0^p-regularity if for each $f \in \ell_0^p(\mathbb{N}_0; X)$ there exists a unique solution $u \in \ell_0^p(\mathbb{N}_0; [D(A)])$ of (2) that satisfies the estimate

$$\|Au\|_{\ell_0^p(\mathbb{N}_0;[D(A)])} + \|b * u\|_{\ell_0^p(\mathbb{N}_0;X)} \le C\|f\|_{\ell_0^p(\mathbb{N}_0;X)}, \tag{11}$$

where the constant $C > 0$ is independent of A, b and f, and $[D(A)]$ denotes the domain of A endowed with the graph norm.

We recall the following definition.

Definition 3.2. [23] Let $k \in \mathbb{N}_0$ be given. A sequence $b : \mathbb{Z} \to \mathbb{C}$ is called k-regular if there exists a constant $c > 0$ such that $|((1 + e^{it})(1 - e^{it}))^n[\widehat{b}(t)]^{(n)}| \le c|\widehat{b}(t)|$ for all $1 \le n \le k$ and all $t \in \mathbb{T}_0$.

We note that, in some sense, it is a discrete version analogous to the existing concept of 1-regularity for Volterra equations defined by Prüss in his monograph [27].

In the following result, we show the equivalence between the R-boundedness of the operator-valued symbol of the difference equation (2) given by $\widehat{b}(t)(\widehat{b}(t) - A)^{-1}$ and the fact that it is an ℓ_p-multiplier.

Theorem 3.3. *Let X be a UMD space, $1 < p < \infty$, $b : \mathbb{N}_0 \to \mathbb{C}$ such that $b_n = 0$ for all $n \in \mathbb{Z}_-$. Suppose that the Gelfand transform of b and their radial limit $\widehat{b}(t)$ exists and is 1-regular, satisfying $\widehat{b}(t) \neq 0$ for all $t \in \mathbb{T}_0$ and $\left\{\widehat{b}(t)\right\}_{t\in\mathbb{T}_0} \subset \rho(A)$. Denote $M(t) := \widehat{b}(t)(\widehat{b}(t) - A)^{-1}$, then the following assertions are equivalent:*
(i) $M(t)$ is an ℓ_p-multiplier from X to $[D(A)]$.
(ii) $\{M(t)\}_{t\in\mathbb{T}_0}$ is R-bounded.

Proof. $(ii) \implies (i)$ By Theorem 2.3 it is enough to prove that the set $\{(e^{it} - 1)(e^{it} + 1)M'(t)\}_{t\in\mathbb{T}_0}$ is R-bounded. Indeed, a computation shows that

$$M'(t) = \frac{[\widehat{b}(t)]'}{\widehat{b}(t)}M(t) - \frac{[\widehat{b}(t)]'}{\widehat{b}(t)}M(t)^2, \quad t \in \mathbb{T}_0. \tag{12}$$

Therefore, for all $t \in \mathbb{T}_0$ we have

$$(1-e^{it})(1+e^{it})M'(t) = (1-e^{it})(1+e^{it})\frac{[\widehat{b}(t)]'}{\widehat{b}(t)}M(t) - (1-e^{it})(1+e^{it})\frac{[\widehat{b}(t)]'}{\widehat{b}(t)}M(t)^2.$$

From [1, Proposition 2.2.5], the hypothesis and the 1-regularity of b we conclude that the set $\{(1-e^{it})(1+e^{it})M'(t) : \ t \in \mathbb{T}_0\}$ is R-bounded and the claim is proved.

$(i) \implies (ii)$ By hypothesis we have that there exists a bounded operator T such that (9) holds. Now, (ii) holds as a consequence of Theorem 2.3. □

With these preliminaries, we can prove the main abstract result of this work. We would like to emphasize that the main contribution in the following result is the identification of necessary properties in the kernel b of the model (2) to have existence and uniqueness with maximal regularity in the Lebesgue space of sequences ℓ_0^p for the abstract difference model (2), being the important a priori estimate (11) a consequence of the closed graph theorem, but with the notable distinction in this particular case, that the constant that appears in the estimate (11) is independent of the operator A and of the kernel b (and therefore of the step size of the scheme). This independence is essentially an application of Blunck's theorem in the form of the Theorem 2.3, after Kemmochi's crucial comments [15] on this topic.

Theorem 3.4. *Let X be a UMD space, $1 < p < \infty$, $b : \mathbb{N}_0 \to \mathbb{C}$ such that $b_n = 0$ for all $n \in \mathbb{Z}_-$. Suppose that the Gelfand transform of b and their radial limit $\widehat{b}(t)$ exists and is 1-regular, satisfying $\widehat{b}(t) \neq 0$ for all $t \in \mathbb{T}_0$. Assume that $\widehat{b} \in C^\infty_{per}(\mathbb{T})$; $0 \in \rho(A)$ or $b_0 \neq 0$; and the condition*

(MR) $\{b_0, \widehat{b}(t)\}_{t\in\mathbb{T}_0} \subset \rho(A)$ *and the set* $\{\widehat{b}(t)(\widehat{b}(t) - A)^{-1} \ : \ t \in \mathbb{T}_0\}$ *is R-bounded*

holds. Then equation (2) has maximal ℓ_0^p-regularity.

Proof. Assume first that $b_0 \neq 0$. We start showing that $b \in \mathcal{C}$ where $\mathcal{C} := \{b : \mathbb{N}_0 \to \mathbb{C} \ : \ \text{there exists } a : \mathbb{N}_0 \to \mathbb{C} \text{ such that } (a * b) = \delta_0(n), \ n \in \mathbb{N}_0\}$, see [23, Section 4]. In fact, since $b_0 \neq 0$ then from the identity $\sum_{j=0}^n a_{n-j} b_j = \delta_0(n)$, $n \in \mathbb{N}_0$, we obtain recursively

$$a_n = -\frac{1}{b_0} \sum_{j=0}^{n-1} a_{n-j-1} b_{j+1} + \frac{\delta_0(n)}{b_0}, \quad n \in \mathbb{N}.$$

Moreover, b is 1-regular if and only if a is 1-regular [23, Remark 2]. Therefore the problem (2) is equivalent to prove maximal regularity for the Volterra equation $u^n = \sum_{j=0}^n a_{n-j} A u^j + g^n$ where $g^n := (a * f)_n$, $n \in \mathbb{N}_0$. Then, the proof follows from [23, Theorem 3.6].

Suppose now that $b_0 = 0$ and $0 \in \rho(A)$, and let $f \in \ell_0^p(\mathbb{N}_0; X)$ be given and $M(t) := \widehat{b}(t)(\widehat{b}(t) - A)^{-1}$. By hypothesis and Theorem 3.3, there exists $w \in \ell_p(\mathbb{Z}; [D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} w^n \check{\varphi}_n = \sum_{n=0}^{\infty} (\varphi \cdot M_-\check{)}_n f^n, \tag{13}$$

for all $\varphi \in C^\infty_{per}(\mathbb{T})$. Assuming that $0 \in \rho(A)$ we have

$$\sum_{n\in\mathbb{Z}} A^{-1} w^n \check{\varphi}_n = \sum_{n=0}^{\infty} (\varphi \cdot A^{-1} M_-\check{)}_n f^n, \quad \varphi \in C^\infty_{per}(\mathbb{T}). \tag{14}$$

Define $N(t) := (\widehat{b}(t) - A)^{-1}$. We have the identity

$$N(t) = (\widehat{b}(t) - A)^{-1} = \widehat{b}(t) A^{-1} (\widehat{b}(t) - A)^{-1} - A^{-1} = A^{-1} M(t) - A^{-1}. \tag{15}$$

In particular, it implies that the set $\{N(t)\}_{t\in\mathbb{T}_0}$ is R-bounded. Identity (15) together with the hypothesis and the permanence properties of R-boundedness imply that

$(1-e^{it})(1+e^{it})N'(t)$ is also R-bounded and then $N(t)$ defines an ℓ_p-multiplier by Theorem 2.3. Then there exists $v \in \ell_p(\mathbb{Z};[D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} v^n \check{\psi}_n = \sum_{n=0}^{\infty} (\psi \cdot N_-)\check{}_n f^n, \quad \psi \in C^{\infty}_{per}(\mathbb{T}). \tag{16}$$

Observe that by hypothesis $\psi(t) := \varphi(t)\widehat{b}(-t) \in C^{\infty}_{per}(\mathbb{T})$. Setting ψ in (16), and taking into account that $\widehat{b}(t)N(t) = M(t)$, we get by (13) that $\langle v, (\varphi \cdot \widehat{b}_-)\check{}\rangle = \langle f, (\varphi \cdot (\widehat{b}N)_-)\check{}\rangle = \langle f, (\varphi \cdot M_-)\check{}\rangle = \langle w, \check{\varphi}\rangle$. From Lemma 2.1 we conclude from the above identity and the fact that $b_n = 0$ for negative values of n that

$$w^n = H(v * b)_n = \sum_{j=0}^{n} v_{n-j} b_j, \quad n \in \mathbb{N}_0, \tag{17}$$

and in particular $w \in \ell^p_0(\mathbb{N}_0; X)$. Since $N(t) = A^{-1}M(t) - A^{-1}$, after multiplication by $e^{int}\psi(t)$ and integration over the interval $(0, 2\pi)$, we have $(\psi \cdot N_-)\check{}_n = (\psi \cdot A^{-1}M_-)\check{}_n - A^{-1}\check{\psi}_n$, for all $\psi \in C^{\infty}_{per}(\mathbb{T})$. Then we obtain $\langle f, (\psi \cdot N_-)\check{}\rangle = \langle f, (\psi \cdot A^{-1}M_-)\check{}\rangle - \langle f, A^{-1}\check{\psi}\rangle$. By replacing (16) and (14) in the above identity and then taking into account (17) we obtain for all $\psi \in C^{\infty}_{per}(\mathbb{T})$:

$$\begin{aligned}
\sum_{n=0}^{\infty} v^n \check{\psi}_n &= \sum_{n=0}^{\infty} A^{-1} w^n \check{\psi}_n - \sum_{n=0}^{\infty} \check{\psi}(n) A^{-1} f^n \\
&= \sum_{n=0}^{\infty} A^{-1} \left[\sum_{j=0}^{n} v_{n-j} b_j \right] \check{\psi}_n - \sum_{n=0}^{\infty} \check{\psi}_n A^{-1} f^n.
\end{aligned}$$

In particular, this identity shows that the right hand side belongs to $D(A)$. Choosing $\psi(t) = e^{-ikt}$ $(k \in \mathbb{Z})$, and applying A in both sides of the above identity, we conclude that $v \in \ell^p_0(\mathbb{N}_0; [D(A)])$ and satisfies the equation (2). We have proved the existence of a solution. The proof of the uniqueness of the solution follows from [23, Theorem 3.6] and therefore we omit it. The estimate (11) and its independence of the constant involved is a consequence of Theorem 2.3. □

As a immediate consequence of the fact that R-boundedness is equivalent to boundedness in Hilbert spaces [10] we obtain the following corollary.

Corollary 1. *If X in Theorem 3.4 is a Hilbert space, then condition (MR) can be replaced by* $\sup_{t\in\mathbb{T}_0} \left\| \widehat{b}(t)(\widehat{b}(t) - A)^{-1} \right\| < \infty$.

As we rapidly observe, the given hypothesis (MR) of R-boundedness - although very general - is not easy to check in practice. Kemmochi [15] uses the ill-posedness of the θ-method in order to restrict the analysis to the case that A is bounded and in this way the study of R-boundedness is concentrated in the location of the bounded spectrum of the operator A. In contrast, the authors in [14] realized that a suitable hypothesis of sectoriality for the operator A (in Hilbert spaces), or R-sectoriality of angle θ in UMD-spaces, suffices. It turns out that this last concept is closely related to the concept of $A(\beta)$-stability, which is specially well adapted when we treat with backward differentiation formulas of order $p \le 6$. In simple words, it refers to the geometrical location of the set $\{\widehat{b}(t)\}_{t\in\mathbb{T}_0}$ which, roughly speaking, must remain within a sector $\Sigma_{\pi-\beta}$ for some β associated to the time-stepping scheme. We point out that the precise calculation of the maximum angle β for BDF-schemes has been improved very recently in [2, Theorem 1] and we will revisit this topic briefly in the next section.

Next, we briefly recall the concept of R-sectoriality. Given any $\theta \in (0,\pi)$, we denote $\Sigma_\theta := \{z \in \mathbb{C} \,:\, |\arg(z)| < \theta, \quad z \neq 0\}$ where $-\pi < \arg(z) \leq \pi$. Recall that a closed operator $A : D(A) \subset X \to X$ with dense domain $D(A)$ is said to be R-sectorial of angle θ if the following conditions are satisfied

(i) $\sigma(A) \subseteq \mathbb{C} \setminus \Sigma_\theta$;

(ii) The set $\{z(z-A)^{-1} \,:\, z \in \Sigma_\theta\}$ is R-bounded in $\mathcal{B}(X)$.

The permanence properties for R-sectorial operators are similar to those for sectorial operators. For instance, they behave well under perturbations. Sufficient conditions for R-sectoriality are studied in the monograph [10, Chapter 4]. Moreover, R-sectoriality characterizes maximal regularity of type L^p in $\mathbb{R}_+$ for the abstract Cauchy problem of first order, see [10, Theorem 4.4]. As a consequence, we obtain the following remarkable result.

Corollary 2. *Let X be a UMD space, $1 < p < \infty$ and $b : \mathbb{N}_0 \to \mathbb{C}$ such that $b_n = 0$ for all $n \in \mathbb{Z}_-$. Suppose that the Gelfand transform of b and their radial limit $\widehat{b}(t)$ exists and is 1-regular, satisfying $\widehat{b}(t) \neq 0$ for all $t \in \mathbb{T}_0$. Assume that A is R-sectorial of angle θ such that $0 \in \rho(A)$ or $b_0 \neq 0$ and the condition*

$$\{b_0, \widehat{b}(t)\}_{t\in\mathbb{T}_0} \subset \Sigma_\theta \tag{18}$$

is verified. Then equation (2) has maximal ℓ_0^p-regularity.

We recall that (2) is said to be $A(\beta)$-stable if $|\arg(\tilde{b}(z))| \leq \pi - \beta$ for $|z| < 1$ and some $0 < \beta < \pi$. See e.g. [24]. Consequently, we obtain the following corollary.

Corollary 3. *Let X be a UMD space, $1 < p < \infty$, $b : \mathbb{N}_0 \to \mathbb{C}$ such that $b_n = 0$ for all $n \in \mathbb{Z}_-$. Suppose that the Gelfand transform of b and their radial limit $\widehat{b}(t)$ exists and is 1-regular, satisfying $\widehat{b}(t) \neq 0$ for all $t \in \mathbb{T}_0$. Assume that A is R-sectorial of angle β for some $\beta \in (0,\pi)$ and $b_0 > 0$. If (2) is $A(\beta)$-stable, then the scheme given by (2) has maximal ℓ_0^p-regularity.*

Proof. Observe that under the given hypothesis we have both $0 \neq b_0 \in \Sigma_{\pi-\beta}$ and $\{\widehat{b}(t)\}_{t\in\mathbb{T}_0} \subset \Sigma_{\pi-\beta}$. □

It should be noted that, even when the set $\{\widehat{b}(t)\}_{t\in\mathbb{T}_0}$ is not contained in a sector Σ_θ for some angle $\theta < \pi$, it does not imply that the general hypothesis in Theorem 3.4 on the R-boundedness of the set $\{\widehat{b}(t)(\widehat{b}(t) - A)^{-1} \,:\, t \in \mathbb{T}_0\}$ may fail. In fact, it could happen that the portion of the set that relies outside of the region, remains R-bounded. We observe that this phenomenon was previously observed and heavily worked in the references [15, 14].

4. **Time-stepping schemes.** In the previous section, we have obtained an ℓ_0^p-maximal regularity result in Theorem 3.4 for an abstract difference equation (2) that involves a convolution term. This general result allows us to unify the theory. More concretely, we recover maximal regularity results for some time-stepping schemes already existing in the literature and we obtain new ℓ_0^p-maximal regularity results for others – mainly nonlocal – schemes.

For later use, we recall that the generalized forward Grünwald–Letnikov derivative [25, Section 3.3], is defined by:

$$u_{\nabla}^{(\alpha)}(n) := \sum_{j=0}^{n} k_{n-j}^{-\alpha} u^j, \quad \alpha > 0, \tag{19}$$

where the sequence k_j^{α} is defined in (4). Note that $u_{\nabla}^{(1)}(n) = u^n - u^{n-1}$.

4.1. **θ-scheme.** Given $0 \leq \theta \leq 1$ we consider the θ-scheme

$$\frac{1}{\tau} u_{\nabla}^{(1)}(n+1) = (1-\theta)Au^n + \theta A u^{n+1} + (1-\theta)f^n + \theta f^{n+1}, \quad n \in \mathbb{N}_0, \tag{20}$$

with initial condition $u^0 = 0$ and stepsize $\tau > 0$. The maximal regularity for this scheme was studied in [15] under the hypothesis that A has L^p-maximal regularity. It is important to point out that this method coincides with the explicit (or forward) Euler scheme when $\theta = 0$ whereas the implicit (or backward) Euler scheme is obtained for $\theta = 1$. Finally, if $\theta = \frac{1}{2}$ then the θ-scheme corresponds to the Crank–Nicolson method, see e.g. [18, Theorem 3.2] where a discrete maximal regularity estimate is established for this method under the hypothesis of L^p-maximal regularity for the operator A. Define

$$b_n = \begin{cases} \frac{1}{\tau}\frac{1}{(1-\theta)}[\frac{(-1)^n}{\theta}(\frac{1-\theta}{\theta})^n - \delta_0(n)] & \text{if } 0 < \theta < 1; \\ \frac{1}{\tau}(\delta_0(n) - \delta_1(n)) & \text{if } \theta = 1, \end{cases} \tag{21}$$

where $\delta_i(j)$ is the Kronecker delta. Observe that for $\theta = 0$ the sequence $b_n = \frac{1}{\tau}(\delta_{-1}(n) - \delta_0(n))$ that defines the explicit Euler scheme cannot be included in our study since $b_{-1} \neq 0$ and then the hypothesis on b of Theorem 3.4 is not fulfilled. This is not a surprise, because by their own nature our model (2) only includes implicit methods. As a consequence of our main theorem proved in the previous section, we obtain the following result that generalizes [15, Theorem 3.2].

Theorem 4.1. *On UMD-spaces, and for $0 < \theta \leq 1$, the abstract difference equation (20) (or the θ-scheme) has ℓ_0^p-maximal regularity if $\left\{\frac{1}{\tau\theta}, \frac{(1-e^{-it})}{\tau(1-\theta)e^{-it}+\tau\theta}\right\}_{t\in\mathbb{T}_0} \subset \rho(A)$ and the set*

$$\left\{ \frac{(1-e^{-it})}{\tau(1-\theta)e^{-it}+\tau\theta}\left(\frac{(1-e^{-it})}{\tau(1-\theta)e^{-it}+\tau\theta} - A\right)^{-1} \ : \ t \in \mathbb{T}_0 \right\}$$

is R-bounded.

Proof. Let $\theta \neq 1$ be given, then $\tilde{b}(z) = \frac{(1-z)}{\tau(1-\theta)z+\tau\theta}$ is exactly the characteristic symbol corresponding to the θ-scheme [15, Section 2.3]. Observe that the radial limit for such symbol exists for all $t \in \mathbb{T}_0$ and is given by $\widehat{b}(t) = \frac{(1-e^{-it})}{\tau(1-\theta)e^{-it}+\tau\theta}$. Since $|(1-\theta)e^{-it}+\theta|^2 = (1-\theta)^2 + 2\theta(1-\theta)\cos(t) + \theta^2 \geq (1-2\theta)^2 + 2\theta^2(1-\theta) > 0$ it follows that

$$(1+e^{it})(1-e^{it})\frac{[\widehat{b}(t)]'(t)}{\widehat{b}(t)} = \frac{-i(1+e^{it})(1-e^{-it})((1-\theta)e^{-it}+1)}{(1-\theta)e^{-it}+\theta}$$

is bounded. Hence, we conclude that b is 1-regular. If $\theta = 1$, then $\tilde{b}(z) = \frac{1}{\tau}(1-z)$ is the characteristic symbol corresponding to the implicit Euler scheme of approximation (see e.g [14, Section 3.1]). Now, observe that in this last case the radial limit $\widehat{b}(t) = \frac{1}{\tau}(1-e^{-it})$ exists for all $t \in \mathbb{T}_0$ and

$$(1+e^{it})(1-e^{it})\frac{[\widehat{b}(t)]'(t)}{\widehat{b}(t)} = -i(1+e^{it})$$

is clearly bounded. □

In [15, Lemma 2.9] the authors showed that the symbol $\widehat{b}(t)$ for the θ-method corresponds to a circumference located in the left half complex plane for $0 \leq \theta < \frac{1}{2}$. In contrast, when $\frac{1}{2} < \theta \leq 1$ it is a circumference located in the right half plane. In the limit case $\theta = \frac{1}{2}$, we have $\widehat{b}(t) = i\mathbb{R} \setminus \{0\}$. As an immediate consequence of Corollary 2, and taking into account that $b_0 = \frac{1}{\tau\theta}$ in case $0 < \theta \leq 1$ we obtain the following result:

Corollary 4. *Let X be a UMD space, $1/2 < \theta \leq 1$, and let A be an R-sectorial operator of angle $\pi/2$. Then the θ-scheme has ℓ_0^p-maximal regularity.*

Proof. It follows immediately from the observation given in [15, Lemma 2.9], where the authors show that for $1/2 < \theta \leq 1$ the set $\widehat{b}(\mathbb{T})$ corresponds exactly to the circumference of center and radius the same point $\frac{1}{(2\theta-1)\tau}$. Moreover, it is clear that $\frac{1}{\tau\theta} \in \rho(A)$. □

Remark 2. Observe that if A is R-sectorial operator of angle $\delta + \pi/2$ for any $\delta > 0$, then we can ensure ℓ_0^p-maximal regularity for the Crank–Nicolson scheme.

4.2. (α, θ)**-scheme.** In this section, our purpose is to define a new fractional order version of the θ-scheme, that we call the (α, θ)-scheme, where $0 < \alpha < 2$. We recall that the qualitative analysis of the θ-method is investigated in several works, mainly, for obtaining the numerical solution of some semidiscretized linear parabolic problems, see e.g. [29]. In what follows, we consider the following two cases:

Case 1: $0 < \theta \leq \frac{1}{2}$. We consider the scheme

$$\frac{1}{\tau^\alpha} u_\nabla^{(\alpha)}(n+1) = ((1-\theta)-(1-\alpha)\theta)(Au^n+f^n)+(\theta+(1-\alpha)\theta)(Au^{n+1}+f^{n+1}),\ n \in \mathbb{N}_0, \tag{22}$$

with initial condition $u^0 = 0$, stepsize $\tau > 0$. When $\theta = 1/2$, the $(\alpha, 1/2)$-scheme given in (22) corresponds to the fractional Crank–Nicolson scheme that has been previously analyzed in [14, Section 6], namely

$$\frac{1}{\tau^\alpha} u_\nabla^{(\alpha)}(n+1) = \left(1 - \frac{\alpha}{2}\right) Au^{n+1} + \frac{\alpha}{2} Au^n + \left(1 - \frac{\alpha}{2}\right) f^{n+1} + \frac{\alpha}{2} f^n, \quad n \in \mathbb{N}.$$

In case $\alpha = 1$, the $(1, \theta)$-scheme corresponds to the θ-scheme whose ℓ_0^p-maximal regularity was obtained in the previous subsection, and where the kernel sequence b_n is defined by (21). For $\alpha \neq 1$ and $0 < \theta \leq \frac{1}{2}$ we define

$$b_n = \frac{1}{\tau^\alpha} \frac{1}{(2\ -\alpha)\theta} \sum_{j=0}^{n} k_{n-j}^{-\alpha} \frac{(1-(2-\alpha)\theta)^j}{(\theta(\alpha-2))^j}, \quad n \in \mathbb{N}_0.$$

Theorem 4.2. *Let X be a UMD space, $0 < \alpha < 2, \alpha \neq 1$ and $0 < \theta \leq \frac{1}{2}$. Assume that $\left\{\frac{1}{\tau^\alpha(2-\alpha)\theta}, \frac{1}{\tau^\alpha} \frac{(1-e^{-it})^\alpha}{(2\theta-\alpha\theta)+(1-2\theta+\alpha\theta)e^{-it}}\right\}_{t \in \mathbb{T}_0} \subset \rho(A)$ and the set*

$$\left\{\frac{(1-e^{-it})^\alpha}{\tau^\alpha\theta(2-\alpha)+\tau^\alpha(1-\theta(2-\alpha))e^{-it}} \left(\frac{(1-e^{-it})^\alpha}{\tau^\alpha\theta(2-\alpha)+\tau^\alpha(1-\theta(2-\alpha))e^{-it}} - A\right)^{-1}\right\}_{t \in \mathbb{T}_0}$$

is R-bounded, Then the (α, θ)-scheme (22) has maximal ℓ_0^p-regularity.

Proof. Using (3) it follows that $\tilde{b}(z) = \frac{1}{\tau^\alpha}\frac{(1-z)^\alpha}{(2\theta-\alpha\theta)+(1-2\theta+\alpha\theta)z}$ and hence $\widehat{b}(t)$ exists for all $t \in \mathbb{T}_0$. Moreover, the following identity holds for all $t \in \mathbb{T}_0$:

$$(1+e^{it})(1-e^{it})\frac{[\widehat{b}(t)]'}{\widehat{b}(t)} = -i\alpha(1+e^{it}) - \frac{i(1+e^{it})(1-e^{-it})(1-2\theta+\alpha\theta)}{(2\theta-\alpha\theta)+(1-2\theta+\alpha\theta)e^{-it}}. \quad (23)$$

Since $|(2\theta-\alpha\theta)+(1-2\theta+\alpha\theta)e^{-it}|^2 \geq (4\theta-2\alpha\theta-1)^2$, it is clear that if $\theta \neq \frac{1}{2(2-\alpha)}$, then b is 1-regular. Meanwhile, if $\theta = \frac{1}{2(2-\alpha)}$, (23) reduces to $-i\alpha(1+e^{it})-i(e^{it}-1)$ and then b is again 1-regular. Finally, taking into account that $b_0 = \frac{1}{\tau^\alpha(2-\alpha)\theta}$ the conclusion follows from Theorem 3.4. □

Remark 3. We observe that choosing $\alpha_0 = 2 - \frac{1}{2\theta}$ for any fixed θ in the interval $(1/4, 1/2]$ we obtain the scheme

$$\frac{1}{\tau^{\alpha_0}} u_{\nabla}^{(\alpha_0)}(n+1) = \frac{1}{2}(Au^n + f^n) + \frac{1}{2}(Au^{n+1} + f^{n+1}), \quad n \in \mathbb{N}_0, \quad (24)$$

which should be compared with the Crank–Nicolson scheme.

Case 2: $\frac{1}{2} \leq \theta \leq 1$. We consider the scheme

$$\frac{1}{\tau^\alpha} u_{\nabla}^{(\alpha)}(n+1) = ((1-\theta)\alpha)(Au^n+f^n)+(1-\alpha(1-\theta))(Au^{n+1}+f^{n+1}),\ n \in \mathbb{N}_0, \quad (25)$$

with initial condition $u^0 = 0$, stepsize $\tau > 0$ and where $u_{\nabla}^{(\alpha)}$ denotes the generalized forward Grünwald–Letnikov derivative defined in (19). Note that when $\theta = 1/2$, it corresponds again to the fractional Crank–Nicolson scheme. For $\alpha \neq 1$ and $\theta \neq 1$ we define the sequence b_n as in (5) and for $\alpha \neq 1$ and $\theta = 1$, we set $b_n := \frac{1}{\tau^\alpha} k_n^{-\alpha}$. Observe that in this last case the model (25) corresponds to the fractional Backward Euler scheme [13] and is given by:

$$\frac{1}{\tau^\alpha} u_{\nabla}^{(\alpha)}(n) = Au^n + f^n, \quad n \in \mathbb{N}. \quad (26)$$

We obtain the following maximal ℓ_0^p-regularity result for the (α,θ)-scheme.

Theorem 4.3. *Let X be a UMD space, $0 < \alpha < 2, \alpha \neq 1$ and $\frac{1}{2} \leq \theta \leq 1$. Assume that*

$$\left\{\frac{1}{\tau^\alpha(1+\alpha(1-\theta))}, \frac{1}{\tau^\alpha}\frac{(1-e^{-it})^\alpha}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)e^{-it}}\right\}_{t\in\mathbb{T}_0} \subset \rho(A)$$

and the set

$$\left\{\frac{1}{\tau^\alpha}\frac{(1-e^{-it})^\alpha}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)e^{-it}}\left(\frac{1}{\tau^\alpha}\frac{(1-e^{-it})^\alpha}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)e^{-it}} - A\right)^{-1}\right\}_{t\in\mathbb{T}_0}$$

is R-bounded, Then the (α,θ)-scheme (25) has maximal ℓ_0^p-regularity.

Proof. Using (3) it follows that if $\theta \neq 1$ then $\tilde{b}(z) = \frac{1}{\tau^\alpha}\frac{(1-z)^\alpha}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)z}$ and hence $\widehat{b}(t)$ exists for all $t \in \mathbb{T}_0$. Moreover, it follows that:

$$(1+e^{it})(1-e^{it})\frac{[\widehat{b}(t)]'(t)}{\widehat{b}(t)} = -i\alpha(1+e^{it}) - \frac{i(1+e^{it})(1-e^{-it})(\alpha-\alpha\theta)}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)e^{-it}}$$

where $1-\alpha(1-\theta) \geq 1 - \frac{\alpha}{2} > 0$ since $\alpha < 2$, due to the hypothesis on θ and α. It implies that b is 1-regular. Otherwise, if $\theta = 1$, then $\tilde{b}(z) = \frac{1}{\tau^\alpha}(1-z)^\alpha$ and therefore

$\widehat{b}(t) = \frac{1}{\tau^\alpha}(1-e^{-it})^\alpha$ exists for all $t \in \mathbb{T}_0$. We note that

$$(1-e^{it})(1+e^{it})\frac{d}{dt}[(1-e^{-it})^\alpha]\frac{1}{(1-e^{-it})^\alpha} = -i\alpha(1+e^{it}),$$

is bounded, and therefore b is 1-regular. Taking into account that $b_0 = \frac{1}{\tau^\alpha(1+\alpha(1-\theta))}$ for $\frac{1}{2} \le \theta \le 1$, the conclusion follows from Theorem 3.4. □

Remark 4. It is interesting to observe that if we choose $\alpha_0 = \frac{1}{2(1-\theta)}$ which is possible whenever $1/2 \le \theta < 3/4$ then we obtain the scheme (24) again.

The following result recovers and extends [14, Theorem 5] and [14, Theorem 9].

Theorem 4.4. *Let X be a UMD space, $\frac{1}{2} \le \theta \le 1$, $0 < \alpha < 1$ and A be a R-sectorial operator of angle $\frac{\alpha\pi}{2}$. Then the (α,θ)-scheme (22) has maximal ℓ_0^p-regularity.*

Proof. It suffices to prove that the set

$$\left\{\frac{1}{\tau^\alpha}\frac{(1-e^{-it})^\alpha}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)e^{-it}}\left(\frac{1}{\tau^\alpha}\frac{(1-e^{-it})^\alpha}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)e^{-it}} - A\right)^{-1}\right\}_{t\in\mathbb{T}_0}$$

is R-bounded. In fact, following an idea of [14] we observe that

$$\widehat{b}(t) = \frac{1}{\tau^\alpha}\frac{(1-e^{-it})^\alpha}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)e^{-it}} = \frac{2^{\frac{3\alpha}{2}}[\sin(t/2)]^\alpha e^{i(\frac{\alpha\pi}{2}-\frac{\alpha t}{2}-\psi(t))}}{\tau^\alpha\rho(t)}$$

where $t \in \mathbb{T}$ and $\rho(t) := \sqrt{(1-\alpha+\alpha\theta)^2+(\alpha-\alpha\theta)^2+2(1-\alpha+\alpha\theta)(\alpha-\alpha\theta)\cos t}$ > 0 and $\psi(t) := \arg((1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)e^{-it}) = \arctan\left(\frac{(\alpha\theta-\alpha)\sin t}{(1-\alpha+\alpha\theta)+(\alpha-\alpha\theta)\cos t}\right)$. On the other hand, since $1 < \frac{1}{2-2\theta}$ and $\alpha < \min\{\frac{\theta-2}{\theta-1}, \frac{1}{1-\theta}\}$ we obtain that:

$$\begin{aligned}\frac{-\alpha}{2} - \psi'(t) &= -\frac{\alpha[(2\alpha^2(\theta-1)^2 - 2\alpha(\theta-2)(\theta-1))(\cos t - 1) + (2-2\theta)\cos t - 1]}{2[(2\alpha^2(\theta-1)^2 + 2\alpha(\theta-1))(\cos t - 1) - 1]} \\ &\le 0.\end{aligned}$$

As a consequence, the function $-\frac{\alpha t}{2} - \psi(t)$ is decreasing from 0 to $-\alpha\pi$ as t changes from 0 to 2π and then the symbol $\widehat{b}(t) \subset \Sigma_{\alpha\pi/2}$. From the sectoriality of the operator A, we obtain the conclusion using Corollary 2. □

4.3. **Second-order BDF scheme.** Let us consider the scheme

$$\frac{3}{2\tau}u^{n+2} - \frac{1}{\tau}u^{n+1} - \frac{1}{2\tau}u^n = Au^{n+2} + f^{n+2}, \quad n \in \mathbb{N}_0, \tag{27}$$

with initial conditions $u^0 = u^1 = 0$. This scheme is well-known in the literature as the second order backward differentiation formula. If we define

$$b_n := \frac{1}{\tau}[(\delta_0(n) - \delta_1(n))] + \frac{1}{2\tau}[(\delta_0-\delta_1)*(\delta_0-\delta_1)](n) = \frac{1}{\tau}\left(\frac{3}{2}\delta_0(n) - 2\delta_1(n) + \frac{1}{2}\delta_2(n)\right),$$

then $\tilde{b}(z) = \frac{1}{\tau}(\frac{3}{2} - 2z + \frac{1}{2}z^2)$, the radial limit exists, and is given by:

$$\widehat{b}(t) = \frac{1}{\tau}\left(\frac{3}{2} - 2e^{-it} + \frac{1}{2}e^{-2it}\right) = \frac{3}{2\tau}\left(1 - \frac{1}{3}e^{-it}\right)(1-e^{-it}), \quad t \in \mathbb{T}_0.$$

We observe that the above symbol corresponds exactly to the second order backward differentiation formula for time-stepping schemes [24, p.131]. The maximal regularity of this scheme has been studied in [18, Theorem 4.1 and 4.2] and extended to the fractional order case ($\alpha \in (0,2)\setminus\{1\}$) in the recent reference [14, Theorem 6]. We can obtain the same result easily, only verifying that b is 1-regular. Indeed, we conclude that the scheme (27) has ℓ_0^p-maximal regularity whenever the following conditions are verified: $\{\frac{3}{2\tau}, \frac{1}{\tau}\left(\frac{3}{2} - 2e^{-it} + \frac{1}{2}e^{-2it}\right)\}_{t\in\mathbb{T}_0} \subset \rho(A)$ and the set

$$\{\frac{1}{\tau}\left(\frac{3}{2} - 2e^{-it} + \frac{1}{2}e^{-2it}\right)(\frac{1}{\tau}\left(\frac{3}{2} - 2e^{-it} + \frac{1}{2}e^{-2it}\right) - A)^{-1} \;:\; t \in \mathbb{T}_0\}$$

is R-bounded.

Corollary 5. *Let X be a UMD space, $1 < p < \infty$. Assume that A is R-sectorial of angle $\pi/2$. Then equation (27) has maximal ℓ_0^p-regularity.*

Proof. It is enough to observe that $\Re(\frac{3}{2} - 2e^{-it} + \frac{1}{2}e^{-2it}) = (1-\cos t)^2 > 0$ for all $t \in \mathbb{T}_0$ and that $b_0 = \frac{3}{2\tau}$. $\square$

Now, given $\alpha > 0$ and $a \in \mathbb{C}$ we define the sequence $k_a^\alpha(n) := a^n k_n^\alpha$. We consider the scheme

$$\left(\frac{3}{2\tau}\right)^\alpha u_\nabla^{(\alpha)}(n) = A(k_{1/3}^\alpha * u)_n + (k_{1/3}^\alpha * f)_n, \quad n \in \mathbb{N}, \tag{28}$$

and define the sequence b_n as:

$$b_n = \left(\frac{3}{2\tau}\right)^\alpha \sum_{j=0}^{n} k_{n-j}^{-\alpha}\frac{1}{3^j}k_j^{-\alpha}, \quad n \in \mathbb{N}_0. \tag{29}$$

This sequence corresponds to the fractional second-order BDF scheme which was considered in [23] and previously in the paper [9, Formula (4.6)]. It is not difficult to see, using (3) and the rule for the product of convolution, that $\tilde{b}(z) = \frac{1}{\tau^\alpha}(\frac{1}{2}z^2 - 2z + \frac{3}{2})^\alpha$ and hence $\widehat{b}(t) = \frac{1}{\tau^\alpha}(\frac{1}{2}e^{-2it} - 2e^{-it} + \frac{3}{2})^\alpha$ exists for all $t \in \mathbb{T}_0$. Moreover, $\widehat{b}(t) \neq 0$ for all $t \in \mathbb{T}_0$. It is straightforward to compute

$$(1-e^{it})(1+e^{it})\frac{[\widehat{b}(t)]'}{\widehat{b}(t)} = \frac{-\alpha(1+e^{it})(2i - ie^{-it})}{\frac{3}{2}\left(1 - \frac{e^{-it}}{3}\right)}$$

showing that the left hand side is bounded for all $t \in \mathbb{T}_0$. It implies that b is 1-regular for any $\alpha > 0$. Moreover, note that $b_0 = (\frac{3}{2\tau})^\alpha$. Summarizing, we obtain the following new result which extends [14, Theorem 6].

Theorem 4.5. *Let X be a UMD space, $\alpha > 0$, assume that*

$$\{(\frac{3}{2\tau})^\alpha, [\frac{1}{\tau}\left(\frac{1}{2}e^{-2it} - 2e^{-it} + \frac{3}{2}\right)]^\alpha\}_{t\in\mathbb{T}_0} \subset \rho(A)$$

and that the set

$$\{[\frac{1}{\tau}(\frac{1}{2}e^{-2it} - 2e^{-it} + \frac{3}{2})]^\alpha([\frac{1}{\tau}(\frac{1}{2}e^{-2it} - 2e^{-it} + \frac{3}{2})]^\alpha - A)^{-1}\}_{t\in\mathbb{T}_0} \text{ is } R\text{-bounded.}$$

Then the fractional second-order BDF scheme given by (28) has maximal ℓ_0^p-regularity.

If we add the condition of R-sectoriality for the operator A we obtain as a corollary [14, Theorem 6].

Corollary 6. *Let X be a UMD space, $0 < \alpha \leq 2$, and let A be an R-sectorial operator of angle $\alpha\pi/2$. Then all the hypothesis of Theorem 4.5 are fulfilled.*

Proof. Since $\tau^{-1}(\frac{1}{2}e^{-2it} - 2e^{-it} + \frac{3}{2}) \in \Sigma_{\pi/2}$ for all $t \in \mathbb{T}_0$, we have $\tau^{-\alpha}(\frac{1}{2}e^{-2it} - 2e^{-it} + \frac{3}{2})^\alpha \in \Sigma_{\alpha\pi/2}$. The conclusion follows from Corollary 2. $\square$

4.4. **Third-order BDF scheme.** Let us consider the following abstract difference equation

$$\frac{1}{\tau}\left[\frac{11}{6}u^{n+3}-2u^{n+2}+\frac{3}{2}u^{n+1}-\frac{1}{3}u^{n}\right]=Au^{n+3}+f^{n+3},\quad n\in\mathbb{N}_0,\tag{30}$$

with initial conditions $u^0=u^1=u^2=0$. This time-stepping scheme has symbol $\frac{1}{\tau}(\frac{11}{6}-3z+\frac{3}{2}z^2-\frac{1}{3}z^3)$. It corresponds to the backward differentiation formula of order 3, see e.g. [24], which can be seen by defining

$$\begin{aligned}b_n&:=\frac{1}{\tau}(\delta_0-\delta_1)(n)+\frac{1}{2\tau}[(\delta_0-\delta_1)*(\delta_0-\delta_1)](n)\\&\quad+\frac{1}{3\tau}[(\delta_0-\delta_1)*(\delta_0-\delta_1)*(\delta_0-\delta_1)](n)\\&=\frac{1}{\tau}\left(\frac{11}{6}\delta_0(n)-3\delta_1(n)+\frac{3}{2}\delta_2(n)-\frac{1}{3}\delta_3(n)\right).\end{aligned}$$

It follows that $\tilde{b}(z)=\frac{1}{\tau}\left(\frac{11}{6}-3z+\frac{3}{2}z^2-\frac{1}{3}z^3\right)$, the radial limit exists, and is given by:

$$\widehat{b}(t)=\frac{1}{\tau}\left(\frac{11}{6}-3e^{-it}+\frac{3}{2}e^{-2it}-\frac{1}{3}e^{-3it}\right)=\frac{1}{6}(1-e^{-it})(11-7e^{-it}+2e^{-2it}),\ t\in\mathbb{T}_0.$$

We can easily verify that b is 1-regular. In fact, a calculation shows that

$$(1-e^{it})(1+e^{it})\frac{[\widehat{b}(t)]'}{\widehat{b}(t)}=\frac{-6i(1+e^{-it})(3-3e^{-it}+e^{-2it})}{11-7e^{-it}+2e^{-2it}}$$

where $11-7e^{-it}+2e^{-2it}=2(a+e^{-it})(\overline{a}+e^{-it})$ with $a:=7/4+i\sqrt{39}/4$. Note that $|a|^2=11/2$ and $b_0=\frac{11}{6\tau}$. We can conclude from Theorem 3.3 that the abstract difference equation (30), that originates from the third-order BDF-scheme, has ℓ_0^p-maximal regularity whenever the following conditions are verified:

$$\left\{\frac{11}{6\tau},\frac{1}{\tau}\left(\frac{11}{6}-3e^{-it}+\frac{3}{2}e^{-2it}-\frac{1}{3}e^{-3it}\right)\right\}_{t\in\mathbb{T}_0}\subset\rho(A)$$

and the set

$$\left\{\frac{1}{\tau}\left(\frac{11}{6}-3e^{-it}+\frac{3}{2}e^{-2it}-\frac{1}{3}e^{-3it}\right)\left(\frac{1}{\tau}\left(\frac{11}{6}-3e^{-it}+\frac{3}{2}e^{-2it}-\frac{1}{3}e^{-3it}\right)-A\right)^{-1}\right\}_{t\in\mathbb{T}_0}$$

is R-bounded. We remark that this ℓ_0^p-maximal regularity result is completely new.

For a third-order BDF scheme it is well-known that the set $\{\widehat{b}(t)\}_{t\in\mathbb{T}_0}$ is contained in a sector Σ_{θ_0} for some $\theta_0>\pi/2$. We note that recently, in the reference [2], the exact angles for the BDF schemes from order 3 to 6 have been computed. Since the precise computation of these angles is needed, we have included a short proof of the computation of the exact angle of order 3 in the following result:

Corollary 7. *Let X be a UMD space, and let A be an R-sectorial operator of angle $\pi-\arctan\left(\frac{329\sqrt{\frac{7}{5}}}{27}\right)$. Then equation (30) has maximal ℓ_0^p-regularity.*

Proof. Let $\Phi(t)=\arctan\left(\frac{7\sin t-2\sin 2t}{11-7\cos t+2\cos 2t}\right)+\arctan\left(\frac{\sin t}{1-\cos t}\right)$ be the argument function of the symbol for the third order scheme $\widehat{b}(t)$ for $t\in\mathbb{T}$. A simple calculus shows that:

$$\Phi'(t)=\frac{6\sin^2(\frac{t}{2})(22\cos t-13)}{-91\cos t+22\cos 2t+87}.$$

As a consequence, Φ has a relative maximum at every: $t_n = 2\pi n + \arccos(13/22)$, $n \in \mathbb{Z}$. Taking $n = 0$, we get $t_0 = \arccos(13/22)$ which corresponds to the unique relative maximum which is contained in the interval $\mathbb{T}$. Then, the maximum of $\Phi(t)$ is given by $\Phi(t_0) = \pi - \arctan\left(\frac{329\sqrt{\frac{7}{5}}}{27}\right)$ and the third order numerical scheme is $A(\theta)$-stable for $\theta = \arctan\left(\frac{329\sqrt{\frac{7}{5}}}{27}\right)$. Since A is an R-sectorial operator of angle $\pi - \theta$, $b_0 > 0$ and we previously checked that b is 1-regular then the conclusion follows from Corollary 3. $\square$

Let $\alpha > 0$ be given and we consider the scheme

$$\left(\frac{11}{6\tau}\right)^{\alpha} u_{\nabla}^{(\alpha)}(n) = A(k_a^{\alpha} * k_{\overline{a}}^{\alpha} * u)_n + (k_a^{\alpha} * k_{\overline{a}}^{\alpha} * f)_n, \quad n \in \mathbb{N}, \tag{31}$$

where we recall that $a = 7/4 + i\sqrt{39}/4$. We define the sequence:

$$b_n = \left(\frac{11}{6\tau}\right)^{\alpha} (k^{-\alpha} * k_a^{-\alpha} * k_{\overline{a}}^{-\alpha})_n,$$

Then, it is easy to check using (3) that $\tilde{b}(z) = (\frac{11}{6\tau})^{\alpha}(1-z)^{\alpha}(1-\frac{z}{a})^{\alpha}(1-\frac{z}{\overline{a}})^{\alpha}$ and hence

$$\begin{aligned}\widehat{b}(t) &= \frac{(11)^{\alpha}}{(3\tau)^{\alpha}}(1-e^{-it})^{\alpha}(1-\frac{1}{a}e^{-it})^{\alpha}(1-\frac{1}{\overline{a}}e^{-it})^{\alpha} \\ &= \frac{1}{(3\tau)^{\alpha}}(1-e^{-it})^{\alpha}(a_1-e^{-it})^{\alpha}(a_2-e^{-it})^{\alpha} = \frac{1}{\tau^{\alpha}}(\frac{11}{6} - 3e^{-it} + \frac{3}{2}e^{-2it} - \frac{1}{3}e^{-3it})^{\alpha}.\end{aligned}$$

It is not difficult to check that b is 1-regular, $b_0 = \left(\frac{11}{6\tau}\right)^{\alpha} \neq 0$ and, therefore, an analogous result to Theorem 4.5 holds for the fractional scheme (31) as follows.

Corollary 8. *Let X be a UMD space, $0 < \alpha < \pi\theta_0^{-1}$ where $\theta_0 := \pi - \arctan\left(\frac{329\sqrt{\frac{7}{5}}}{27}\right)$ and let A be an R-sectorial operator of angle $\alpha\theta_0$. Then the third order difference fractional scheme (31) has maximal ℓ_0^p-regularity.*

Proof. Since $\widehat{b}(t) = \frac{1}{\tau}(\frac{11}{6} - 3e^{-it} + \frac{3}{2}e^{-2it} - \frac{1}{3}e^{-3it}) \in \Sigma_{\theta_0}$, then $\widehat{b}(t)^{\alpha} \in \Sigma_{\alpha\theta_0}$. The conclusion immediately holds from Corollary 2. $\square$

The following subsection illustrates how to continue the analysis of the backward differentiation formulas of order p ($p \leq 6$).

4.5. **Fourth-order BDF scheme.** Let us consider the following abstract difference equation

$$\frac{1}{\tau}\left[\frac{25}{12}u^{n+4} - 4u^{n+3} + 3u^{n+2} - \frac{4}{3}u^{n+1} + \frac{1}{4}u^{n}\right] = Au^{n+4} + f^{n+4}, \quad n \in \mathbb{N}_0, \tag{32}$$

with initial conditions $u^0 = u^1 = u^2 = u^3 = 0$. This time-stepping scheme has symbol $\frac{1}{\tau}\left(\frac{25}{12} - 4z + 3z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4\right)$ and corresponds to the fourth order backward difference scheme, see e.g. [24]. As before, we define $b_n := \frac{1}{\tau}(\delta_0 - \delta_1)(n) + \frac{1}{2\tau}[(\delta_0 - \delta_1) * (\delta_0 - \delta_1)](n) + \frac{1}{3\tau}[(\delta_0 - \delta_1) * (\delta_0 - \delta_1) * (\delta_0 - \delta_1)](n) + \frac{1}{4\tau}[(\delta_0 - \delta_1) * (\delta_0 - \delta_1) * (\delta_0 - \delta_1) * (\delta_0 - \delta_1)](n) = \frac{1}{\tau}\left[\frac{25}{12}\delta_0(n) - 4\delta_1(n) + 3\delta_2(n) - \frac{4}{3}\delta_3(n) + \frac{1}{4}\delta_4(n)\right]$.

It follows that $\tilde{b}(z) = \frac{1}{\tau}\left(\frac{25}{12} - 4z + 3z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4\right)$, the radial limit exists, and is given by:

$$\widehat{b}(t) = \frac{1}{\tau}\left(\frac{25}{12} - 4e^{-it} + 3e^{-2it} - \frac{4}{3}e^{-3it} + \frac{1}{4}e^{-4it}\right), \quad t \in \mathbb{T}_0.$$

We arrive at the following result.

Corollary 9. *Let X be a UMD space, and let A be an R-sectorial operator of angle $\pi - \arctan\left(\frac{699\sqrt{\frac{3}{2}}}{256}\right)$. Then equation (32) has maximal ℓ_0^p-regularity.*

Proof. Let $\Phi(t) = \arctan\big(\frac{-3\sin 3t+13\sin 2t-23\sin 3t}{3\cos 3t-13\cos 2t+23\cos t-25}\big) + \arctan\big(\frac{\sin t}{1-\cos t}\big)$ be the argument function of the symbol for the fourth order scheme $\widehat{b}(t)$ for $t \in \mathbb{T}$. A computation show that $\Phi'(t) = \frac{-480\sin^4(\frac{t}{2})(5\cos t-1)}{913\cos t-394\cos 2t+75\cos 3t-666}$. Then, Φ has a relative maximum at the points: $t_n = 2\pi n + \arccos(1/5), n \in \mathbb{Z}$. The maximum of $\Phi(t)$ is given by $\Phi(t_0) = \pi - \arctan\left(\frac{699\sqrt{\frac{3}{2}}}{256}\right)$ and then the scheme is $A(\theta)$-stable with $\theta = \arctan\left(\frac{699\sqrt{\frac{3}{2}}}{256}\right)$. A calculus shows the 1-regularity of the kernel sequence b. Moreover, $b_0 = \frac{25}{12\tau} > 0$. The conclusion then follows from Corollary 3. $\square$

In view of the preceding subsections, an analogous analysis for the fractional version of this scheme can be easily carry on. We omit the details.

4.6. **L1-Scheme.** We now consider one time-stepping scheme of finite difference type for simulating subdiffusion and superdiffusion, the L1-scheme, which has been recently studied in the literature, see [31]. The L1-scheme is one of the most popular and successful numerical methods for discretizing the Caputo fractional derivative in time [13]. Our approach here begins establishing the linear difference equation associated with the scheme, which seems to be new. In order to do that, we recall the function $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$, $t > 0$, $\beta > 0$, that we will consider evaluated on the set $\mathbb{N}$, that is $(g_\beta)_n := \frac{n^{\beta-1}}{\Gamma(\beta)}$, $n \in \mathbb{N}$. Let $0 < \alpha < 1$ be fixed and we define the difference scheme

$$\frac{(g_{2-\alpha} * u)_{n+1} - (g_{2-\alpha} * u)_n}{\tau^\alpha} = Au^n + f^n, \quad n \in \mathbb{N}. \tag{33}$$

It is interesting to observe that (33) is a nonlocal version of the difference equation $\frac{(u^{n+1}-u^n)}{\tau} = Au^n + f^n, n \in \mathbb{N}$, that appears from (33) in the limit case $\alpha = 1$. Moreover, it is remarkable that because $0 < \alpha < 1$ we have $g_{2-\alpha}(0) = 0$. Therefore, using the identity [12, Proposition 2.9]-(v), we get that the nonlocal difference equation (33) is equivalent to:

$$(b * u)(n) = Au^n + f^n, \quad n \in \mathbb{N}. \tag{34}$$

where $b_n = \frac{1}{\tau^\alpha}(g_{2-\alpha})^{(1)}_{\nabla}(n+1)$ which - after a computation - coincides with

$$b_n = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}[(n+1)^{1-\alpha} - n^{1-\alpha}], \;\; n \in \mathbb{N}_0, \quad 0 < \alpha < 1, \tag{35}$$

obtaining that (34) fits in the abstract model (2). Moreover, a calculation shows that

$$\widehat{b}(t) = \frac{(1-e^{-it})^2}{\tau^\alpha e^{-it}\Gamma(2-\alpha)} Li_{\alpha-1}(e^{-it}),$$

where $Li_{\alpha-1}(z)$ is the polylogarithmic function [11], defined by $Li_p(z) := \sum_{j=1}^\infty \frac{z^j}{j^p}$ such that $Li_{\alpha-1}(e^{-it})$ is defined via analytic continuation. We observe that a simple condition involving R-sectoriality of the operator A is not possible here, since there is no sector Σ_θ including completely the set $\{\widehat{b}(t)\}_{t\in\mathbb{T}_0}$. However, as before, one can provide a general result which is independent of this geometrical restriction, as follows:

Theorem 4.6. *Let X be a UMD space and $0<\alpha<1$. Assume that*

$$\left\{\frac{1}{\tau^\alpha\Gamma(2-\alpha)}, \frac{(1-e^{-it})^2}{\tau^\alpha e^{-it}\Gamma(2-\alpha)}Li_{\alpha-1}(e^{-it})\right\}_{t\in\mathbb{T}_0}\subset\rho(A)$$

and that the set

$$\left\{\frac{(1-e^{-it})^2}{\tau^\alpha e^{-it}\Gamma(2-\alpha)}Li_{\alpha-1}(e^{-it})\left(\frac{(1-e^{-it})^2}{\tau^\alpha e^{-it}\Gamma(2-\alpha)}Li_{\alpha-1}(e^{-it})-A\right)^{-1}\right\}_{t\in\mathbb{T}_0}$$

is R-bounded. Then the L1-scheme (33) has maximal ℓ_0^p-regularity.

Proof. The theorem follows directly from Theorem 3.4 if we show that the sequence $\{b_n\}_{n\in\mathbb{N}_0}$ defined in (35) is a 1-regular sequence. Indeed, a simple calculus shows that the following equality holds:

$$(1-e^{it})(1+e^{it})\frac{[\widehat{b}(t)]'}{\widehat{b}(t)}=-i(1+e^{it})^2-i(1-e^{2it})\frac{Li_{\alpha-2}(e^{-it})}{Li_{\alpha-1}(e^{-it})}. \tag{36}$$

We have the following expansion formula [35, Formula (13.1)]

$$\begin{aligned} &\frac{Li_{\alpha-1}(e^{-it})}{\Gamma(2-\alpha)} &(37)\\ = \;& (-2\pi i)^{\alpha-2}\sum_{k=0}^{\infty}(k+1-\frac{t}{2\pi})^{\alpha-2}+(2\pi i)^{\alpha-2}\sum_{k=0}^{\infty}(k+\frac{t}{2\pi})^{\alpha-2}\\ = \;& (2\pi)^{\alpha-2}(\cos((2-\alpha)\frac{\pi}{2})(A(t)+B(t))-i(2\pi)^{\alpha-2}(\sin((2-\alpha)\frac{\pi}{2})(A(t)-B(t))\end{aligned}$$

where $A(t)=\sum_{k=0}^{\infty}(k+\frac{t}{2\pi})^{\alpha-2}$ and $B(t)=\sum_{k=0}^{\infty}(k+1-\frac{t}{2\pi})^{\alpha-2}$ are both convergent series for $0<\alpha<1$. As a consequence, there exist positive constants $C,C'>0$ such that $|Li_{\alpha-2}(e^{-it})|<C$ and $|Li_{\alpha-1}(e^{-it})|>C'$ when $t\in\mathbb{T}_0$ is far away from 0. Meanwhile, using the asymptotic expansion [35, Equation (9.3)] given by $Li_p(e^{-it})=\Gamma(1-p)(it)^{p-1}+o(t^p)$ as $t\to 0$ we can conclude that $(1-e^{2it})\frac{Li_{\alpha-2}(e^{-it})}{Li_{\alpha-1}(e^{-it})}$ is also bounded when $t\to 0$, and hence, for all $t\in\mathbb{T}_0$. Therefore, (36) is bounded for all $t\in\mathbb{T}_0$ and the proof is finished. □

In case $1<\alpha<2$, we consider the abstract difference equation

$$\frac{1}{\tau^\alpha}(g_{3-\alpha}*u)^{(1)}_{\nabla}(n+1)=Au^n+f^n,\quad n\in\mathbb{N}. \tag{38}$$

We observe that in the limit case $\alpha=2$ the above equation reduces to $\frac{1}{\tau^2}u^{n+1}=Au^n+f^n$. In contrast, when $\alpha<2$, we obtain that the difference equation (38) is equivalent to $\frac{1}{\tau^\alpha}(b_n*u)_n=Au^n+f^n$, $n\in\mathbb{N}$ where $b_n=\frac{1}{\tau^\alpha}(g_{3-\alpha})^{(1)}_{\nabla}(n+1)$. Consequently, we are in the conditions of our abstract main result with the sequence $b_n=\frac{1}{\tau^\alpha\Gamma(3-\alpha)}[(n+1)^{2-\alpha}-n^{2-\alpha}]$, $n\in\mathbb{N}_0$, and $\widehat{b}(t)=\frac{2}{\tau^\alpha}\frac{1-e^{-it}}{1+e^{-it}}\frac{(1-e^{-it})^2}{e^{-it}\Gamma(3-\alpha)}Li_{\alpha-2}(e^{-it})$. The corresponding theorem follows similarly as the case $0<\alpha<1$ as a consequence of the 1-regularity of the kernel sequence b_n.

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REFERENCES

[1] R. P. Agarwal, C. Cuevas and C. Lizama, *Regularity of Difference Equations on Banach Spaces*, Springer-Verlag, Cham, 2014.

[2] G. Akrivis and E. Katsoprinakis, Maximum angles of $A(\theta)$-stability of backward difference formulae, *BIT Numer. Math.*, **60** (2020), 93–99.

[3] G. Akrivis, B. Li and C. Lubich, Combining maximal regularity and energy estimates for time discretizations of quasilinear parabolic equations, *Math. Comp.*, **86** (2017), 1527–1552.

[4] H. Amann, *Linear and Quasilinear Parabolic Problems. Vol. I. Abstract Linear Theory*, Monographs in Mathematics, 89. Birkhäuser-Verlag, Basel, 1995.

[5] F. M. Atici and P. W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Amer. Math. Soc.*, **137** (2009), 981–989.

[6] S. Blunck, Analyticity and discrete maximal regularity on Lp-spaces, *J. Funct. Anal.*, **183** (2001), 211–230.

[7] S. Blunck, Maximal regularity of discrete and continuous time evolution equations, *Studia Math.*, **146** (2001), 157–176.

[8] M. Caputo, Diffusion of fluids in porous media with memory, *Geothermics*, **28** (1999), 113–130.

[9] E. Cuesta, C. Lubich and C. Palencia, Convolution quadrature time discretization of fractional diffusion-wave equations, *Math. Comp.*, **75** (2006), 673–696.

[10] R. Denk, M. Hieber and J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, *Mem. Amer. Math. Soc.*, **166** (2003).

[11] P. Flajolet, Singularity analysis and asymptotics of Bernoulli sums, *Theoret. Comput. Sci.*, **215** (1999), 371–381.

[12] C. S. Goodrich and C. Lizama, A transference principle for nonlocal operators using a convolutional approach: Fractional monotonicity and convexity, *Israel J. Math.*, **236** (2020), 533–589.

[13] B. Jin, R. Lazarov and Z. Zhou, An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data, *IMA Journal of Numerical Analysis*, **36** (2016), 197–221.

[14] B. Jin, B. Li and Z. Zhou, Discrete maximal regularity of time-stepping schemes for fractional evolution equations, *Numer. Math.*, **138** (2018), 101–131.

[15] T. Kemmochi, Discrete maximal regularity for abstract Cauchy problems, *Studia Math.*, **234** (2016), 241–263.

[16] P. Klouček and F. S. Rys, Stability of the fractional step θ-scheme for the nonstationary Navier–Stokes equations, *SIAM J. Numer. Anal.*, **31** (1994), 1312–1335.

[17] A. N. Kochubei, General fractional calculus, evolution equations, and renewal processes, *Integr. Equ. Oper. Theory*, **71** (2011), 583–600.

[18] B. Kovács, B. Li and C. Lubich, A-stable time discretizations preserve maximal parabolic regularity, *SIAM J. Numer. Anal.*, **54** (2016), 3600–3624.

[19] D. Leykekhman, B. Vexler and D. Walter, Numerical analysis of sparse initial data identification for parabolic problems, *ESAIM Math. Model. Numer. Anal.*, **54** (2020), 1139–1180.

[20] B. Li and W. Sun, Regularity of the diffusion-dispersion tensor and error analysis of FEMs for a porous media flow, *SIAM J. Numer. Anal.*, **53** (2015), 1418–1437.

[21] C. Lizama, The Poisson distribution, abstract fractional difference equations, and stability, *Proc. Amer. Math. Soc.*, **145** (2017), 3809–3827.

[22] C. Lizama, ℓ_p-maximal regularity for fractional difference equations on UMD spaces, *Math. Nach.*, **288** (2015), 2079–2092.

[23] C. Lizama and M. Murillo-Arcila, Discrete maximal regularity for Volterra equations and nonlocal time-stepping schemes, *Discrete Contin. Dyn. Syst. Series A*, **40** (2020), 509–528.

[24] C. Lubich, Convolution quadrature and discretized operational calculus. I., *Numer. Math.*, **52** (1988), 129–145.

[25] M. D. Ortigueira, F. J. V. Coito and J. J. Trujillo, Discrete-time differential systems, *Signal Processing*, **107** (2015), 198–217.

[26] P. Portal, Discrete time analytic semigroups and the geometry of Banach spaces, *Semigroup Forum*, **67** (2003), 125–144.

[27] J. Prüss, *Evolutionary Integral Equations and Applications*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 1993.

[28] S. G. Samko and R. P. Cardoso, Integral equations of the first kind of Sonine type, *Int. J. Math. and Math. Sci.*, **57** (2003), 3609–3632.

[29] E. Süli, *Numerical Solution of Ordinary Differential Equations*, Mathematical Institute, Oxford University, 2014.

[30] Y. Sun and P. C. B. Phillips, Understanding the Fisher equation, *J. Appl. Econ.*, **19** (2004), 869–886.

[31] Z.-Z. Sun and X. Wu, A fully discrete scheme for a diffusion-wave system, *Appl. Numer. Math.*, **56** (2006), 193–209.

[32] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers. Volume I. Background and Theory*, Nonlinear Physical Science, Higher Education Press, Beijing; Springer, Heidelberg, 2013.

[33] V. Vergara and R. Zacher, Optimal decay estimates for time-fractional and other non-local subdiffusion equations via energy methods, *SIAM J. Math. Anal.*, **47** (2013), 210–239.

[34] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, *Math. Ann.*, **319** (2001), 735–758.

[35] D. Wood, *The Computation of Polylogarithms*, Technical Report 15-92, University of Kent, Computing Laboratory, University of Kent, Canterbury, UK, June 1992.

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