Nonlinear Analysis 232 (2023) 113277

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Periodic solutions for the Blackstock–Crighton–Westervelt equation

Carlos Lizama[a,*], Marina Murillo-Arcila[b]

[a] *Universidad de Santiago de Chile, Facultad de Ciencias, Departamento de Matemática y Ciencia de la Computación, Casilla 307, Correo 2, Santiago, Chile*
[b] *Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022, València, Spain*

A R T I C L E I N F O

Article history:
Received 27 November 2020
Accepted 7 March 2023
Communicated by Francesco Maggi

MSC:
primary 35B65
74F05
secondary 35B10
35L80

Keywords:
Well-posedness
Blackstock–Crighton–Westervelt equation
R-boundedness

A B S T R A C T

We investigate the Blackstock–Crighton–Westervelt equation which models non-linear acoustic wave propagation in thermally relaxing viscous fluids. We prove existence and regularity, in a $L^p - L^q$ setting, of time-periodic solutions for a given sufficiently small time-periodic forcing data, and homogeneous Dirichlet boundary conditions over a cylindrical domain. We show maximal L^p-regularity for the abstract linearized model. We use techniques of operator-valued Fourier multiplier theorems combined with a generalized version of the implicit function theorem.

1. Introduction

The Westervelt equation

$$u_{tt} - b\Delta u_t - c^2\Delta u = \left(\frac{1}{c^2}(1+\frac{B}{2A})(u_t)^2\right)_t$$

where u represents the potential of the velocity field, c is the speed of sound, b denotes the diffusivity of sound and B/A is an acoustic parameter of nonlinearity, is a classical, and widely used model describing the propagation of sound in fluids and is characterized by the presence of a viscoelastic damping [19]. Taking into account the heat conductivity of the fluid, denoted by a, we obtain the Blackstock–Crighton–Westervelt (BCW) equation

$$(a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = \left(\frac{1}{c^2}(1+\frac{B}{2A})(u_t)^2\right)_{tt} \tag{1.1}$$

* Corresponding author.
E-mail addresses: carlos.lizama@usach.cl (C. Lizama), mamuar1@upv.es (M. Murillo-Arcila).

https://doi.org/10.1016/j.na.2023.113277

which is a higher-order model in nonlinear acoustics that describes the propagation of sound waves in monatomic gases such as Helium, Xenon or Argon. In such model, $a := \nu Pr^{-1}$, where ν is the kinematic viscosity and Pr denotes the Prandtl number. For a detailed derivation of Eq. (1.1), we refer to [3, Chapter 1] and [5].

The BCW equation has been recently intensively investigated [3,4,6,7,14,17]. R. Brunnhuber and B. Kaltenbacher [3,6] studied the homogeneous Dirichlet boundary problem (1.1) and showed local existence of a unique (weak) solution in the Hilbert space $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open, bounded and connected, by means of a combination between regularity results for the heat equation and the linearized Westervelt equation together with Banach's fixed point theorem. Then, global well-posedness and exponential decay of solutions were proved by using energy estimates for the linearized problem and a fixed point argument [6].

However, in [6] a restriction on the dimension of Ω is considered ($n = 1, 2, 3$) in order to use various embedding theorems. This restriction was lifted in [7] by Brunnhuber and Meyer using techniques based on maximal regularity and the implicit function theorem. Using such technique, they are able to study the linearized version of (1.1) in $L^q(\Omega)$ when $1 < q < \infty$ and the BCW equation for $q > \max\{n/4 + 1/2, n/3\}$. Notably, they proved the existence of a unique global solution of the inhomogeneous Dirichlet boundary problem (1.1) within a certain regularity of class $L^1(\mathbb{R}_+; L^q(\Omega))$ and which depends continuously on sufficiently small initial and boundary data. We note that the results from [6] were also extended by Brunnhuber in [4] where existence of strong solutions for (1.1) in the vector-valued Lebesgue space $L^2(0, T; L^2(\Omega))$ was proved.

The main objective of this paper is to study well-posedness of the BCW equation with *periodic* initial conditions and a time-periodic forcing term belonging to the vector-valued Lebesgue space $L^p(\mathbb{T}, L^q(\Omega))$.

The mathematical analysis of the BCW equation with a time-periodic forcing term and inhomogeneous Dirichlet boundary conditions was recently initiated by Celik and Kyed [14]. These authors showed that for time-periodic data f sufficiently restricted in size and belonging to the vector-valued Lebesgue space $L^q(\mathbb{T}, L^q(\Omega))$, a time-periodic solution $u \in W^{3,q}_{per}(\mathbb{T}, L^q(\Omega)) \cap W^{1,q}_{per}(\mathbb{T}, W^{4,q}(\Omega))$ always exists whenever $\max\{2, n/2\} < q < \infty$ [14, Theorem 1.1]. Physically, this infers that the dissipative effects given by the term $b\Delta u_t$ are enough to avoid the occurrence of an unbounded solution when the system is excited by a periodic force within the Blackstock–Crighton–Westervelt model [14, Section 1]. The proofs in [14] are based on a priori estimates for the linear problem and an application of the contraction mapping principle.

In this paper, we are able to obtain solvability of the initial–boundary value problem in the vector-valued Lebesgue spaces $L^p(\mathbb{T}; L^q(\Omega))$ for the full range $1 < p, q < \infty$. This will be achieved employing a technique that uses *operator-valued* Fourier multipliers (or symbols) associated to the linearized equation in order to obtain L^p-maximal regularity. This technique has been used by several authors [8–11,13,22] and allows to obtain in a simpler way a priori estimates for the linearized equation, to the cost of a previous checking of a so-called R-boundedness condition on the symbols. Then a generalized version the implicit function theorem can be employed. Our method relies on very recent abstract results regarding the linearized non homogeneous version for the Moore–Gibson–Thompson equation proved in the general setting of UMD spaces [15], and abstract results for cylindrical boundary value problems due to Nau et al. [25,26], combined with a criteria of R-boundedness due to Denk, Hieber and Prüss [16, Proposition 4.10]. Another advantage is that our method is sufficiently general to admit Ω as a *cylindrical domain*.

It should be noted that a similar approach has been taken for the authors in recent works for other two models of interest, namely, the Moore– Gibson–Thompson equation with two temperatures [24], and the Van Wijngaarden–Eringen equation [23]. However, the present paper differs from these in several aspects. First, the Blackstock–Crighton–Westervelt equation has a different dynamics characterized by a operator-valued Fourier multiplier that has their own geometry and, consequently, need different conditions for well-posedness than those considered in [23,24]. Second, and most importantly, in the present paper we are interested in the *nonlinear equation* (1.1) instead of only the linear one, as in the above cited references. This new challenge is settled by a procedure using the implicit function theorem, that takes into account the very particular form of the nonlinear term given by the BCW-equation.

By an appropriate combination of these described tools, we are able to prove the existence of a unique *time* periodic solution of the nonlinear problem

$$\begin{cases} (a\Delta - \partial_t)(u_{tt} - b\Delta u_t - c^2\Delta u) = \left(\frac{1}{c^2}(1+\frac{B}{2A})(u_t)^2\right)_{tt} + f(t), & t \in \mathbb{T} := [0,2\pi]; \\ u(0) = u(2\pi) \quad u'(0) = u'(2\pi) \quad u''(0) = u''(2\pi), \end{cases} \tag{1.2}$$

on the cylindrical domain $\Omega = \mathbb{R}^n_+ \times V$, $V \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions. Moreover, this solution depends continuously on a sufficiently small time-periodic forcing term $f \in L^p(\mathbb{T}; L^q(\Omega))$ where $1 < p, q < \infty$.

This paper is organized as follows: Section 2 is devoted to the necessary tools that we will use throughout the paper. They include the vector-valued Banach spaces of periodic functions, and the notion of R-boundedness, needed to recall the main abstract result of [15]. We also recall in this section the notion of R-sectorial operator and how this property is connected with R-boundedness of certain sets by means of a functional calculus (Theorem 2.9). Section 3 states the main abstract result of this paper (Theorem 3.1). It says that for certain classes of closed and linear operators A, the linearized BCW equation

$$(-aA - \partial_t)(u''(t) + c^2 Au(t) + bAu'(t)) = f(t), \quad t \in \mathbb{T} := [0,2\pi], \tag{1.3}$$

admits maximal L^p-regularity. The section finishes with an application to the linearized Blackstock–Crighton–Westervelt equation with periodic initial conditions and subject to homogeneous Dirichlet boundary conditions on a cylindrical domain (see Theorem 3.4).

Finally, in Section 4 we study the existence and regularity of solutions for the Blackstock–Crighton–Westervelt Eq. (1.1) adding an external forcing term $f \in L^p(\mathbb{T}, L^q(\Omega))$ where $\Omega = U \times V$ is a cylindrical domain. Our main result in this section (Theorem 4.5) asserts the existence of a unique solution that depends continuously on f and belongs to the following maximal regularity space

$$\begin{aligned} S_p(L^q(\Omega)) := \{u \in W^{1,p}_{per}(\mathbb{T}, W^{4,q}(\Omega)) \cap L^p(\mathbb{T}, W^{4,q}(\Omega)) : \\ u' \in L^p(\mathbb{T}, W^{4,q}(\Omega)) \cap W^{2,p}_{per}(\mathbb{T}, L^q(\Omega)), \Delta_q u' \in W^{1,p}_{per}(\mathbb{T}, L^q(\Omega))\}. \end{aligned}$$

2. Preliminaries

In what follows, X will denote a Banach space and $1 < p < \infty$. We will first recall some preliminary results about well-posedness in vector-valued L^p-spaces for third order degenerate equations that can be found in [12,15]. In the mentioned papers, the authors succeed in characterizing L^p-well-posedness for the model:

$$\alpha(Mu')''(t) + (Nu')'(t) - \beta Eu(t) - \gamma Hu'(t) = f(t), \quad t \in \mathbb{T} := [0,2\pi], \tag{2.1}$$

with initial conditions $u(0) = u(2\pi), u'(0) = u'(2\pi), u''(0) = u''(2\pi)$ and α, β, γ are real numbers, M, N, E and H are closed linear operators defined on a Banach space X with domains $D(M), D(N), D(E)$ and $D(H)$, respectively.

We define the vector-valued function spaces:

$$W^{n,p}_{per}(\mathbb{T}, X) := \{u \in L^p(\mathbb{T}, X) : \text{there exists } v \in L^p(\mathbb{T}, X),\ \hat{v}(k) = (ik)^n \hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

where

$$\hat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)dt$$

for all $k \in \mathbb{Z}$ denotes the kth Fourier coefficient of a function $f \in L^1(\mathbb{T}, X)$.

We recall the following important properties related to the spaces $W^{n,p}_{per}(\mathbb{T},X)$:

(i) Let $m,n\in\mathbb{N}$. If $n\le m$, then $W^{m,p}_{per}(\mathbb{T},X)\subseteq W^{n,p}_{per}(\mathbb{T},X)$.

(ii) If $u\in W^{n,p}_{per}(\mathbb{T},X)$, then for any $0\le k\le n-1$, we obtain $u^{(k)}(0)=u^{(k)}(2\pi)$.

(iii) Let $u\in L^p(\mathbb{T},X)$, then $u\in W^{1,p}_{per}(\mathbb{T},X)$ if and only if u is differentiable a.e. on $\mathbb{T}$ and $u'\in L^p(\mathbb{T},X)$, in this case u is actually continuous and $u(0)=u(2\pi)$ [2, Lemma 2.1].

Analogously, given $u\in L^p(\mathbb{T},X)$, then $u\in W^{2,p}_{per}(\mathbb{T},X)$ if and only if u is twicely differentiable a.e., $u',u''\in L^p(\mathbb{T},X)$, and $u(0)=u(2\pi)$, $u'(0)=u'(2\pi)$.

In [15] the following space of maximal regularity was introduced:

Definition 2.1. Let us define the following space of maximal regularity:

$$\begin{aligned}S_p(E,H,M,N)&:=\{u\in W^{1,p}_{per}(\mathbb{T},[D(E)\cap D(H)])\cap L^p(\mathbb{T},[D(E)\cap D(H)]):\\&u'\in L^p(\mathbb{T},[D(E)\cap D(H)]),Mu'\in W^{2,p}_{per}(\mathbb{T},X),Nu'\in W^{1,p}_{per}(\mathbb{T},X)\}.\end{aligned}$$

The space $S_p(E,H,M,N)$ is a Banach space endowed with the norm

$$\begin{aligned}\|u\|_{S_p(E,H,M,N)}:=&\|u\|_{L^p(\mathbb{T},X)}+\|u'\|_{L^p(\mathbb{T},X)}+\|Hu'\|_{L^p(\mathbb{T},X)}+\|Eu\|_{L^p(\mathbb{T},X)}+\|(Nu')'\|_{L^p(\mathbb{T},X)}\\&+\|(Mu')''\|_{L^p(\mathbb{T},X)}+\|Nu'\|_{L^p(\mathbb{T},X)}+\|Mu'\|_{L^p(\mathbb{T},X)}.\end{aligned}$$

Eq. (2.1) is said to be *strongly $L^p(X)$-well-posed* if for each $f\in L^p(\mathbb{T},X)$, there exists a unique solution $u\in S_p(E,H,M,N)$ that satisfies (2.1) for almost all $t\in\mathbb{T}$. We also need the following definition provided in [15]:

Definition 2.2. Let M,N,E and H be closed linear operators defined on a Banach space X with domains $D(M),D(N),D(E)$ and $D(H)$ such that $D(E)\cap D(H)\subset D(M)\cap D(N)$, then the (M,N)-resolvent of E and H is defined as the set:

$$\begin{aligned}\rho_{M,N}(E,H):=\{s\in\mathbb{R}:\alpha is^3M+s^2N+\beta E+\gamma isH:[D(E)\cap D(H)]\to X\\ \text{is invertible and } [\alpha is^3M+s^2N+\beta E+\gamma isH]^{-1}\in\mathcal{B}(X)\}.\end{aligned}\tag{2.2}$$

Here, $[D(E)\cap D(H)]$ is a Banach space with the standard norm $\|x\|_{[D(E)\cap D(H)]}:=\|x\|+\|Ex\|+\|Hx\|$.

Before we recall the main theorem obtained in [15] that will serve us a tool for providing one of our main results we need to define the concepts of R-boundedness and UMD-spaces.

Definition 2.3. Let X and Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X,Y)$ is called R-bounded if there is a constant $c\ge0$ such that

$$\|(T_1y_1,\dots,T_ny_n)\|_R\le c\|(y_1,\dots,y_n)\|_R,\tag{2.3}$$

for all $T_1,\dots,T_n\in\mathcal{T},\ y_1,\dots,y_n\in X,\ n\in\mathbb{N}$ where

$$\|(y_1,\dots,y_n)\|_R:=\frac{1}{2^n}\sum_{\epsilon_j\in\{-1,1\}^n}\Big\|\sum_{j=1}^n\epsilon_jy_j\Big\|,\quad y_1,\dots,y_n\in X.$$

We refer the reader to [16] for the properties preserved under R-boundedness. We also recall the notion of UMD spaces. For more details [1, Section III.4.3-III.4.5].

Definition 2.4. A Banach space X is said to have the Unconditional Martingale Difference property (UMD) if for each $p\in(1,\infty)$ there exists a constant $C_p>0$ such that for any martingale $(f_n)_{n\ge0}\subset L^p(\Omega,\Sigma,\mu;X)$ and any choice of signs $(\xi_n)_{n\ge0}\subset\{-1,1\}$ and any $N\in\mathbb{Z}_+$ the following estimate holds

$$\Big\|f_0+\sum_{n=1}^N\xi_n(f_n-f_{n-1})\Big\|_{L^p(\Omega,\Sigma,\mu;X)}\le C_p\|f_N\|_{L^p(\Omega,\Sigma,\mu;X)}.$$

The following characterization obtained in [15] will serve us as a useful tool for providing L^p-well-posedness for the Blackstock–Crighton–Westervelt equation. In this characterization well-posedness is equivalent to the R-boundedness of certain sets of operators related to the model.

Theorem 2.5 (*[15]*)**.** *Let $1<p<\infty$ and $\alpha,\beta,\gamma\in\mathbb{R}$. Let E,H,M,N be closed linear operators on a UMD space X satisfying $D(E)\cap D(H)\subset D(M)\cap D(N)$. Then, the next two assertions are equivalent:*

(i) (2.1) is strongly $L^p(X)$-well posed;
(ii) $\mathbb{Z}\subset\rho_{M,N}(E,H)$ and the sets $\{ik^3\alpha MN_k:k\in\mathbb{Z}\}$, $\{k^2NN_k:k\in\mathbb{Z}\}$, $\{k\gamma HN_k:k\in\mathbb{Z}\}$ $\{kN_k:k\in\mathbb{Z}\}$ are R-bounded where

$$N_k:=-[i\alpha k^3M+k^2N+\beta E+ik\gamma H]^{-1},\quad k\in\mathbb{Z}. \tag{2.4}$$

Moreover, for each $f\in L^p(\mathbb{T},X)$ there exists a constant $C>0$, independent of f, such that

$$\|u\|_{S_p(E,H,M,N)}\le C\|f\|_{L^p}.$$

Given $\Sigma_\phi\subset\mathbb{C}$, we define the open sector $\Sigma_\phi:=\{\lambda\in\mathbb{C}\setminus\{0\}:|\arg\lambda|<\phi\}$. In what follows we denote as $\mathcal{H}(\Sigma_\phi)$ the set of holomorphic functions $f:\Sigma_\phi\to\mathbb{C}$. If a function $f\in\mathcal{H}(\Sigma_\phi)$ is bounded too then we say that $f\in\mathcal{H}^\infty(\Sigma_\phi)$. This set can be endowed with the norm $\|f\|_\infty^\phi:=\sup_{|\arg\lambda|<\phi}|f(\lambda)|$.

We define the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows: $\mathcal{H}_0(\Sigma_\phi):=\bigcup_{\alpha,\beta<0}\{f\in\mathcal{H}(\Sigma_\phi):\|f\|_{\alpha,\beta}^\phi<\infty\}$, with $\|f\|_{\alpha,\beta}^\phi:=\sup_{|\lambda|\le1}|\lambda^\alpha f(\lambda)|+\sup_{|\lambda|\ge1}|\lambda^{-\beta}f(\lambda)|$.

We now recall the notion of sectorial operators. Most of the properties concerning this class of operators can be found in [18].

Definition 2.6 (*[21]*)**.** Given a closed linear operator A defined on a complex Banach space X, A is said to be a sectorial operator if:

(i) $\overline{D(A)}=X,\overline{R(A)}=X,(-\infty,0)\subset\rho(A)$;
(ii) $\|t(t+A)^{-1}\|\le M$ for all $t>0$ and some $M>0$.

Remark 2.7. If condition (ii) is replaced by the R-boundedness of the set $\{t(t+A)^{-1}\}_{t>0}$ then A is said to be R-sectorial.

If A is a sectorial operator then we have $\Sigma_\theta\subset\rho(-A)$ for some $\theta>0$ and moreover $\sup_{|\arg\lambda|<\theta}\|\lambda(\lambda+A)^{-1}\|<\infty$. We denote the *spectral angle* of a sectorial operator A by

$$\theta_A=\inf\{\theta:\Sigma_{\pi-\theta}\subset\rho(-A),\quad\sup_{\lambda\in\Sigma_{\pi-\theta}}\|\lambda(\lambda+A)^{-1}\|<\infty\}.$$

Definition 2.8 (*[21]*)**.** Given a sectorial operator A defined on a complex Banach space X, we denote as $\mathcal{H}^\infty(X)$ the space of all operators that admit a bounded $\mathcal{H}^\infty-$ calculus, that is, if there exist $\theta>\theta_A$ and a constant $K_\theta>0$ such that $\|f(A)\|\le K_\theta\|f\|_\infty^\theta$ for all $f\in\mathcal{H}_0(\Sigma_\theta)$. We also define the $\mathcal{H}^\infty(X)$-angle as

$$\theta_A^\infty:=\inf\{\theta>\theta_A:\|f(A)\|\le K_\theta\|f\|_\infty^\theta\text{ for all }f\in\mathcal{H}_0(\Sigma_\theta)\text{ holds }\}.$$

In the setting of R-boundedness, there are an analogous concept for operators admitting an R-bounded $\mathcal{H}^\infty-$calculus (or simply $\mathcal{RH}^\infty-$calculus), and this class of operators is denoted as $\mathcal{RH}^\infty(X)$. Specifically, if $A\in\mathcal{H}^\infty(X)$ satisfies that the set

$$\{g(A):g\in\mathcal{H}^\infty(\Sigma_\theta),\|g\|_\infty^\theta\le1\}$$

is R-bounded for some $\theta > 0$ then it is said that A *admits an R-bounded* $\mathcal{H}^\infty-$*calculus* and that A belongs to the class $\mathcal{RH}^\infty(X)$. The corresponding angle will be noted as $\theta_A^{R_\infty}$. See [21].

For examples of operators that admit a $\mathcal{RH}^\infty-$calculus and bounded $\mathcal{H}^\infty$-calculus we refer to [20,21,24] or [23] and references therein.

Finally, we need the following theorem given in [16, Proposition 4.10] which provides an easier condition to ensure the R-boundedness of a certain set of operators.

Theorem 2.9. *Let $A \in \mathcal{RH}^\infty(X)$ and suppose that $\{g_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta_A^{R_\infty}$, where Λ is an arbitrary index set. Then the set $\{g_\lambda(A)\}_{\lambda\in\Lambda}$ is R-bounded.*

We recall the general implicit function theorem. In such theorem D will denote the Fréchet derivative where, if needed, a subindex indicates the variable under which such derivative acts.

Theorem 2.10. *Let X, Y, and Z be Banach spaces and let Ω be an open subset of $X \times Y$. Let C be a continuously differentiable map from Ω to Z. If $(\hat{x}, \hat{y}) \in \Omega$ is a point such that $D_yC(\hat{x}, \hat{y})$ is a bounded, invertible, linear map from Y to Z, then there is an open neighborhood G of $\hat{x}$, and a unique function $\psi : G \to Y$ such that*

$$C(x, \psi(x)) = C(\hat{x}, \hat{y}), \quad \forall x \in G.$$

Moreover, ψ is continuously differentiable and $\psi'(x) = -[D_yC(x, \psi(x))]^{-1}D_xC(x, \psi(x))$.

3. Strongly $L^p(X)$-well-posedness for the linearized Blackstock–Crighton–Westervelt equation

The linearization of the BCW equation in a general abstract form on a Hilbert space was first considered by Brunnhuber and Kaltenbacher in [6]. They apply the theory of operator semigroups and prove that the underlying semigroup is analytic on two different phase spaces which leads, together with certain spectral properties of the generator, to two exponentially decaying energy functionals. Moreover, they provide existence and uniqueness results for the solutions of the linear model. From a different perspective, in the Ref. [17] the authors are able to give an explicit representation of the solution for the linear model by means of two classes of related strongly continuous families of bounded and linear operators.

In this section we will take a direct approach to the linearized BCW equation, in the setting of a Banach space X satisfying a geometrical condition (UMD), avoiding reduction to a first-order system as done in [6].

We state the main abstract result of this paper.

Theorem 3.1. *Let X be a UMD-space, $1 < p < \infty$, $a, b, c > 0$. Suppose that $A \in \mathcal{RH}^\infty(X)$ has angle $\theta_A^{R_\infty} \in (0, \arctan(b/c^2))$ and $0 \in \rho(A)$. Then for each $f \in L^p(\mathbb{T}, X)$ the linearized Blackstock–Crighton–Westervelt equation:*

$$(-aA - \partial_t)(u''(t) + c^2Au(t) + bAu'(t)) = f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{3.1}$$

admits a unique solution $u \in W^{1,p}_{per}(\mathbb{T}, D(A^2)) \cap L^p(\mathbb{T}, X)$ with $u' \in W^{2,p}_{per}(\mathbb{T}, X) \cap L^p(\mathbb{T}, D(A^2))$ and $Au \in W^{1,p}_{per}(\mathbb{T}, X)$ that satisfies (3.3) for a.a. $t \in \mathbb{T}$. (i.e. is strongly $L^p(X)$-well posed).

Proof. We first note that Eq. (3.3) can be expressed as:

$$-u'''(t) - (a + b)Au''(t) - (abA + c^2)Au'(t) - ac^2A^2u(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{3.2}$$

and then it labels into (2.1) for $M = I$, $N = -(a + b)A$ $E = A^2$, $H = (abA^2 + c^2A)$, $\alpha = -1$ and $\beta = ac^2$, $\gamma = 1$.

Note that the operators M, N and E are closed. Moreover, since $0 \in \rho(A)$ and A is closed, we can easily deduce that H is closed, too. Also, we have that $D(E) \cap D(H) = D(A^2) \subset D(A) = D(M) \cap D(N)$.

We point out that for proving L^p-well posedness of our equation we have to check assertion (ii) in Theorem 2.5, where

$$N_k := -[i\alpha k^3 M + k^2 N + \beta E + ik\gamma H]^{-1} = -[-ik^3 - k^2(a+b)A + ac^2A^2 + ik(abA^2 + c^2A)]^{-1},$$

for $k \in \mathbb{Z}$. An easy computation shows that:

$$N_k = [-aA - ik]^{-1}[-k^2 + c^2A + bAik]^{-1}.$$

It follows that

$$\begin{aligned} N_k &= \frac{1}{a(c^2+bik)}\left[\frac{ik}{a} + A\right]^{-1}\left[\frac{-k^2}{(c^2+bik)} + A\right]^{-1} \\ &= \frac{1}{ik^3} c_k(c_k + A)^{-1} d_k(d_k + A)^{-1}, \end{aligned}$$

where $c_k = \frac{ik}{a}$ and $d_k = \frac{-k^2}{(c^2+bik)}$.

Since $0 < \theta_A^{R_\infty} < \arctan(b/c^2)$ there exists $s > \theta_A^{R_\infty}$ such that $s < \arctan(b/c^2)$. For each $z \in \Sigma_s$ and $k \in \Lambda := \mathbb{Z} \setminus \{0\}$, define

$$\begin{aligned} F_1(k,z) &= c_k(c_k + z)^{-1} = \left(1 + \frac{az}{ik}\right)^{-1} \\ F_2(k,z) &= d_k(d_k + z)^{-1} = \left(1 + \frac{-(c^2+ibk)z}{k^2}\right)^{-1}. \end{aligned}$$

Observe that the fraction $\frac{az}{ik}$ belongs to the sector $\Sigma_{s+\frac{\pi}{2}}$ and that if $s < \pi/2$ then such a sector does not contain the semi-axis $(-\infty, 0]$. On the other hand, for all $k \in \Lambda$ we note that the fraction $\frac{1}{d_k} = -\frac{c^2}{k^2} - i\frac{b}{k}$ belongs to the sector $\Sigma_{\pi-\arctan(b/c^2)}$. Thus, the fraction $\frac{-(c^2+ibk)z}{k^2}$ belongs to the sector $\Sigma_{s+\pi-\arctan(b/c^2)}$ and, therefore, if $s < \arctan(b/c^2) \leq \pi/2$ then such a sector does not contain the semi-axis $(-\infty, 0]$.

We conclude that both $\frac{z}{c_k}$ and $\frac{z}{d_k}$ belong to the sector $\Sigma_{s+\pi-\arctan(b/c^2)}$ where $s < \arctan(b/c^2)$ and we can assert that the distance from the sector $\Sigma_{s+\pi-\arctan(b/c^2)}$ to -1 is always positive. As a consequence, there exist positive constants $C_1, C_2 > 0$ independent of $k \in \mathbb{Z}$ and $z \in \Sigma_s$ such that:

$$|F_1(k,z)| = \left|\frac{1}{1+\frac{z}{c_k}}\right| \leq C_1, \quad |F_2(k,z)| = \left|\frac{1}{1+\frac{z}{d_k}}\right| \leq C_2.$$

Theorem 2.9 shows that the sets $\{F_1(k,A)\}_{k\in\Lambda}$ and $\{F_2(k,A)\}_{k\in\Lambda}$ are R-bounded. Moreover, since A is invertible, and $c_0 = d_0 = 0$, the operators $H_1(k) := (c_k + A)^{-1}$ and $H_2(k) := (d_k + A)^{-1}$ exist for all $k \in \mathbb{Z}$. Consequently, $H_1(k), H_2(k)$ belong to $\mathcal{B}(X)$ for all $k \in \mathbb{Z}$ and the sequences $\{c_k(c_k + A)^{-1}\}_{k\in\mathbb{Z}}$ and $\{d_k(d_k + A)^{-1}\}_{k\in\mathbb{Z}}$ are R-bounded.

As a consequence of the above and the permanence properties of R-bounded sets [16], we obtain that $\{N_k\}_{k\in\mathbb{Z}}$ is R-bounded. Now, we proceed to prove the R-boundedness of the sets $\{ik^3\alpha MN_k : k \in \mathbb{Z}\}$, $\{k^2NN_k : k \in \mathbb{Z}\}$, $\{k\gamma HN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$. Indeed,

$$ik^3MN_k = c_k(c_k + A)^{-1}d_k(d_k + A)^{-1}$$

and then the conclusion follows immediately. Considering the identity $A(c_k + A)^{-1} = I - c_k(c_k + A)^{-1}$ we have:

$$k^2NN_k = -k^2(a+b)AN_k = -\frac{(a+b)}{ik}d_k(d_k + A)^{-1} + \frac{(a+b)}{ik}c_k(c_k + A)^{-1}d_k(d_k + A)^{-1}.$$

Since the sets $\{\frac{(a+b)}{ik}\}_{k\in\Lambda}$, $\{d_k(d_k+A)^{-1}\}$ and $\{c_k(c_k+A)^{-1}d_k(d_k+A)^{-1}\}_{k\in\mathbb{Z}}$ are R-bounded, we obtain using the previous identity that the set $\{k^2NN_k\}_{k\in\mathbb{Z}}$ is R-bounded, too. Moreover, using again the identities $A(d_k+A)^{-1}=I-d_k(d_k+A)^{-1}$ and $A(c_k+A)^{-1}=I-c_k(c_k+A)^{-1}$ we obtain

$$\begin{aligned}
k\gamma CN_k &= kabA^2N_k + kc^2ANN_k = kabA^2N_k \\
&+ \frac{c^2}{ik^2}[d_k(d_k+A)^{-1} - c_k(c_k+A)^{-1}d_k(d_k+A)^{-1}] \\
&= -\frac{b}{ik^2} + \frac{b}{ik^2}d_k(d_k+A)^{-1} + \frac{b}{ik^2}c_k(c_k+A)^{-1} - \frac{b}{ik^2}c_k(c_k+A)^{-1}d_k(d_k+A)^{-1} \\
&+ \frac{c^2}{ik^2}d_k(d_k+A)^{-1} - \frac{c^2}{ik^2}c_k(c_k+A)^{-1}d_k(d_k+A)^{-1}
\end{aligned}$$

where the sets $\{\frac{c^2}{aik^2}\}_{k\in\Lambda}$ and $\{\frac{b}{ik^2}\}_{k\in\Lambda}$ are bounded and the sets $\{c_k(c_k+A)^{-1}\}$ and $\{d_k(d_k+A)^{-1}\}$ are R-bounded. We conclude that the set $\{k\gamma HN_k\}_{k\in\mathbb{Z}}$ is R-bounded. To end, we have

$$kN_k = \frac{1}{ik^2}(ik^3MN_k),$$

and then $\{kN_k\}_{k\in\mathbb{Z}}$ is also R-bounded. By Theorem 2.5 we conclude that Eq. (3.3) is strongly $L^p(X)$-well posed. $\square$

Remark 3.2. Observe that if A is a sectorial operator that admits a bounded $\mathcal{H}^\infty-$calculus of angle $\theta_A^{\mathcal{R}\infty}\in(0,\arctan(b/c^2))$ and $0\in\rho(A)$, then the conclusion of Theorem 3.1 also holds.

In case of $A=-\Delta$ with Dirichlet boundary conditions on $X=L^q(\Omega)$ we immediately get the following result that complements and extends recent results in [14].

Corollary 3.3. *Given $1<p,q<\infty$ and $a,b,c>0$. For each $f\in L^p(\mathbb{T},L^q(\Omega))$ the linearized Blackstock–Crighton–Westervelt equation:*

$$(a\Delta-\partial_t)(u''(t)-c^2\Delta u(t)-b\Delta u'(t)) = f(t), \quad t\in\mathbb{T}:=[0,2\pi], \tag{3.3}$$

admits a unique solution $u\in W^{1,p}_{per}(\mathbb{T},W^{4,q}(\Omega))\cap L^p(\mathbb{T},L^q(\Omega))$ with $u'\in W^{2,p}_{per}(\mathbb{T},L^q(\Omega))\cap L^p(\mathbb{T},W^{4,q}(\Omega))$ and $Au\in W^{1,p}_{per}(\mathbb{T},L^q(\Omega))$ that satisfies (3.3) for a.a. $t\in\mathbb{T}$. (i.e. is strongly $L^p(L^q(\Omega))$-well posed).

We now consider the linearized Blackstock–Crighton–Westervelt equation subject to homogeneous Dirichlet boundary conditions on a cylindrical domain Ω:

$$\begin{cases}
(-a(-\Delta)-\partial_t)(\partial_t^2u(x,y,t)+c^2(-\Delta)u(x,y,t)+b(-\Delta)\partial_tu(x,y,t)) \\
\qquad = f(x,y,t), \text{ for } (x,y,t)\in\Omega\times(0,2\pi); \\
\mathcal{B}_Uu(x,y,t)=0, \text{ for } (x,y,t)\in\partial U\times V\times(0,2\pi); \\
\mathcal{B}_Vu(x,y,t)=0, \text{ for } (x,y,t)\in U\times\partial V\times(0,2\pi); \\
u(x,y,0)=u(x,y,2\pi),\ \partial_tu(x,y,0)=\partial_tu(x,y,2\pi), \\
\partial_t^2u(x,y,0)=\partial_t^2u(x,y,2\pi),\ \partial_t^3u(x,y,0)=\partial_t^3u(x,y,2\pi),
\end{cases} \tag{3.4}$$

for each $(x,y)\in\Omega$, with $\Omega=U\times V\subset\mathbb{R}^{n+d}$ where $U=\mathbb{R}^n_+, n\in\mathbb{N}$ and $V\subset\mathbb{R}^d, d\in\mathbb{N}_0$ has a compact boundary (i.e. is a C^2-standard domain, see [26, Definition 4.5]).

Moreover, in (3.4) Δ stands for a cylindrical decomposition of the Dirichlet Laplacian on the space $L^q(\Omega)$ with reference to the two cross-sections i.e. $\Delta = \Delta_1 + \Delta_2$ where Δ_i behaves on the corresponding part of Ω. According to [25] we introduce L^q-realizations $\Delta_{q,i} = \Delta_i$ in the following way:

$$\begin{aligned} D(\Delta_{q,1}) &:= \{u \in W^{2,q}(\mathbb{R}^n_+, L^q(V)) \,:\, \mathcal{B}_U = 0\}; \\ D(\Delta_{q,2}) &:= W^{2,q}(V) \cap W^{1,q}_0(V). \end{aligned}$$

More details about $\Delta_{q,2}$ can be found in [27].

We introduce the Laplacian Δ_q in $L^q(\Omega)$ subject to the Dirichlet boundary conditions $\mathcal{B}_U$ and $\mathcal{B}_V$ to be

$$\begin{aligned} D(\Delta_q) &:= D(\Delta_{q,1}) \cap D(\Delta_{q,2}) \\ \Delta_q u &:= \Delta_{q,1}u + \Delta_{q,1}u = \Delta u, \quad u \in D(\Delta_q). \end{aligned}$$

As a consequence of [25, Theorem 4.2] we can assert that $-\Delta_q \in \mathcal{RH}^\infty(L^q(\Omega))$ and Δ_q is invertible. Also, by [25, Remark 4.7] we obtain that $\theta^{\mathcal{R}_\infty}_{-\Delta_q} = 0$. As an application of Theorem 3.1 with $A = -\Delta_q$ we obtain the following main result on the strongly $L^p(L^q(\Omega))$-well-posedness for the linearized Blackstock–Crighton–Westervelt equation.

Theorem 3.4. *Let $a, b, c > 0$ and $1 < p, q < \infty$. For any given $f \in L^p(\mathbb{T}, L^q(\Omega))$ the problem (3.4) has a unique solution u that belongs to the maximal regularity space:*

$$\begin{aligned} &S_p(L^q(\Omega)) := \{u \in W^{1,p}_{per}(\mathbb{T}, W^{4,q}(\Omega)) \cap L^p(\mathbb{T}, W^{4,q}(\Omega)) : \\ &u' \in L^p(\mathbb{T}, W^{4,q}(\Omega)) \cap W^{2,p}_{per}(\mathbb{T}, L^q(\Omega)),\ \Delta_q u' \in W^{1,p}_{per}(\mathbb{T}, L^q(\Omega))\}. \end{aligned} \tag{3.5}$$

Moreover, for any $1 < p, q < \infty$ there exists a constant $C > 0$ such that the following estimate

$$\begin{aligned} &\|u'''\|_{L^p(\mathbb{T},L^q(\Omega))} + \|u'\|_{L^p(\mathbb{T},L^q(\Omega))} + \|u\|_{L^p(\mathbb{T},L^q(\Omega))}|| + \|\Delta u'||_{L^p(\mathbb{T},L^q(\Omega))} + \|\Delta u''||_{L^p(\mathbb{T},L^q(\Omega))} \\ &\quad + \|(-\Delta)^2 u||_{L^p(\mathbb{T},L^q(\Omega))} + \|\Delta u||_{L^p(\mathbb{T},L^q(\Omega))} \le C\|f\|_{L^p(\mathbb{T},L^q(\Omega))}, \end{aligned}$$

holds.

4. $L^p(L^q)$-well-posedness for the Blackstock–Crighton–Westervelt equation with time-periodic forcing term

We now analyze $L^p(L^q)$-well-posedness of periodic solutions for the Blackstock–Crighton–Westervelt equation on a cylindrical domain with an external forcing term and homogeneous Dirichlet boundary conditions:

$$\begin{cases} (-a(-\Delta) - \partial_t)(\partial_t^2 u(x,y,t) + c^2(-\Delta)u(x,y,t) + b(-\Delta)\partial_t u(x,y,t)) \\ \qquad = \partial_t^2 \left(\dfrac{1}{c^2} \left(1 + \dfrac{B}{2A} \right) (\partial_t u(x,y,t))^2 \right) + \rho f(x,y,t), \text{ for } (x,y,t) \in \Omega \times \mathbb{T}; \\ \mathcal{B}_U u(x,y,t) = 0, \text{ for } (x,y,t) \in \partial U \times V \times \mathbb{T}; \\ \mathcal{B}_V u(x,y,t) = 0, \text{ for } (x,y,t) \in U \times \partial V \times \mathbb{T}; \\ u(x,y,0) = u(x,y,2\pi),\ \partial_t u(x,y,0) = \partial_t u(x,y,2\pi), \\ \partial_t^2 u(x,y,0) = \partial_t^2 u(x,y,2\pi),\ \partial_t^3 u(x,y,0) = \partial_t^3 u(x,y,2\pi), \end{cases} \tag{4.6}$$

where $\Omega := U \times V$, $\rho \in \mathbb{R}$ and $f \in L^p(\mathbb{T}, L^q(\Omega))$. We consider the Dirichlet Laplacian Δ_q in $L^q(\Omega)$ with boundary conditions $\mathcal{B}_U$ and $\mathcal{B}_V$ as in the previous section. We now state the main result of this section that provides the existence of solutions for the nonlinear Eq. (4.6). Our method of proof is based on the general implicit function theorem and the maximal regularity estimate established in the previous section for the linearized Blackstock–Crighton–Westervelt equation.

Theorem 4.5. *Let $1 < p, q < \infty$, $a, b, c > 0$. Then for each $f \in L^p(\mathbb{T}, L^q(\Omega))$ there exists $\rho^* > 0$ such that for all $0 < |\rho| < \rho^*$ Eq. (4.6) has a nontrivial solution $u_\rho \in W^{1,p}_{per}(\mathbb{T}, W^{4,q}(\Omega)) \cap L^p(\mathbb{T}, W^{4,q}(\Omega))$ with $u'_\rho \in L^p(\mathbb{T}, W^{4,q}(\Omega)) \cap W^{2,p}_{per}(\mathbb{T}, L^q(\Omega))$ and $\Delta_q u'_\rho \in W^{1,p}_{per}(\mathbb{T}, L^q(\Omega))$.*

Proof. We first recall that the space $S_p(L^q(\Omega))$ defined in (3.5) is a Banach space when endowed with the norm:

$$\|u\| := \|u'''\|_{L^p(\mathbb{T},L^q(\Omega))} + \|u'\|_{L^p(\mathbb{T},L^q(\Omega))} + \|u\|_{L^p(\mathbb{T},L^q(\Omega))}\| + ||\Delta u'\|_{L^p(\mathbb{T},L^q(\Omega))}$$

$$+||\Delta u''||_{L^p(\mathbb{T},L^q(\Omega))} + \|(-\Delta)^2 u||_{L^p(\mathbb{T},L^q(\Omega))} + \|\Delta u||_{L^p(\mathbb{T},L^q(\Omega))}.$$

We define a linear operator $\mathcal{C} : S_p(L^q(\Omega)) \to L^p(\mathbb{T}; L^q(\Omega))$ by

$$\mathcal{C}(u) := -u''' + (a+b)\Delta u'' + (-ab\Delta + c^2)\Delta u' - ac^2(-\Delta)^2 u.$$

From the definition of the operator $\mathcal{C}$ we deduce that $\|\mathcal{C}(u)\| \le M\|u\|$. As a consequence of Theorem 3.4 we can state the existence of a constant $C > 0$ such that $\|u\| \le C\|\mathcal{C}(u)\|$. Therefore $\mathcal{C}$ is an isomorphism. Moreover, as a consequence of Theorem 3.4 it follows that $\mathcal{C}$ is onto.

Let now define a mapping $G : S_p(L^q(\Omega)) \to L^p(\mathbb{T}; L^q(\Omega))$ as follows:

$$G(u) := \partial_t^2 \left(\frac{1}{c^2} \left(1 + \frac{B}{2A}\right) (\partial_t u)^2 \right) = k[(\partial_t^2 u)^2 + (\partial_t u)(\partial_t^3 u)], \quad u \in S_p(L^q(\Omega)),$$

where $k := \frac{2}{c^2}\left(1 + \frac{B}{2A}\right)$.

Note that $u \in S_p(L^q(\Omega))$ implies $u' \in W^{2,p}(\mathbb{T}; L^q(\Omega))$ and hence we have that $u'', u''' \in L^p(\mathbb{T}; L^q(\Omega))$ and u', u'' are continuous. Therefore, we have

$$\int_0^{2\pi} \|u''(t)u''(t)\|_q^p dt \le \int_0^{2\pi} \|u''(t)\|_q^p \|u''(t)\|_q^p dt \le 2\pi C_1^2$$

and

$$\int_0^{2\pi} \|u'(t)u'''(t)\|_q^p dt \le \int_0^{2\pi} \|u'(t)\|_q^p \|u'''(t)\|_q^p dt \le C_2 \int_0^{2\pi} \|u'''(t)\|_q^p dt = C_2 \|u'''\|^p_{L^p(\mathbb{T},L^q(\Omega))}$$

where $C_1 := \sup_{t\in[0,T]} \|u''(t)\|_q^p$ and $C_2 := \sup_{t\in[0,T]} \|u'(t)\|_q^p$.

Thus, $(u'')^2 \in L^p(\mathbb{T}; L^q(\Omega))$ and $u'u''' \in L^p(\mathbb{T}; L^q(\Omega))$. It implies that G is well defined as a map from $S_p(L^q(\Omega))$ to $L^p(\mathbb{T}; L^q(\Omega))$.

To prove that the map G is continuous, we first note that $0 \in \rho(A)$ implies that

$$\|w''\|_{L^p(\mathbb{T};L^q(\Omega))} = \|A^{-1}Aw''\|_{L^p(\mathbb{T};L^q(\Omega))} \le C_3\|Aw''\|_{L^p(\mathbb{T};L^q(\Omega))} \le C_3\|w\|$$

for each $w \in S_p(L^q(\Omega))$, where $C_3 := \|A^{-1}\|$. Therefore, given $(u_n) \subset S_p(L^q(\Omega))$ such that $u_n \to u \in S_p(L^q(\Omega))$ in the norm $\|\cdot\|$, we have that the sequence (u_n) is bounded and, denoting by $\|\cdot\|$ the norm

$\|\cdot\|_{L^p(\mathbb{T};L^q(\Omega))}$, we obtain

$$\begin{aligned}\|G(u_n)-G(u)\| &\le \|(\partial_t^2u_n-\partial_t^2u)(\partial_t^2u_n+\partial_t^2u)\|+\|(\partial_tu_n-\partial_tu)\partial_t^3u_n\|+\|\partial_tu(\partial_t^3u_n-\partial_t^3u)\|\\ &\le \|\partial_t^2u_n-\partial_t^2u\|(\|\partial_t^2u_n\|+\|\partial_t^2u\|)+\|\partial_tu_n-\partial_tu\|\|\partial_t^3u_n\|\\ &\quad+\|\partial_t^3u_n-\partial_t^3u\|\|\partial_tu\|\\ &\le (C_4+|||u|||)(C_3+1)|||u_n-u|||,\end{aligned}$$

where $C_4 := \sup_n |||u_n|||$. This inequality proves the continuity of G.

Now, we consider the uniparametric family $H : \mathbb{R}\times S_p(L^q(\Omega)) \to L^p(\mathbb{T};L^q(\Omega))$ defined by

$$H(\tau,u) = -\mathcal{C}(u)+G(u)+\tau f.$$

From its definition, it is clear that $H(0,0)=0$. On the other hand, we note that

$$DG(u) = k[2(\partial_t^2u)\partial_t^2+(\partial_t^3u)\partial_t+((\partial_tu)\partial_t^3)]$$

where we recall that D denotes the Fréchet derivative of G and therefore

$$DG(u)(h) = k[2(\partial_t^2u)(\partial_t^2h)+(\partial_t^3u)(\partial_th)+(\partial_tu)(\partial_t^3h)].$$

In particular, it implies that $DG(0)=0$. Since $D_uH(\tau,u) = -\mathcal{C}+DG(u)$ we then obtain $D_uH(0,0)=-\mathcal{C}$ which is linear, bounded and invertible. Using Theorem 2.10 we conclude that there exists a neighborhood $I\subset\mathbb{R}$ of 0 and a unique $\psi : I\to S_p(L^q(\Omega))$ such that $H(\tau,\psi(\tau))=0$ for all $\tau\in I$. Since $u\equiv 0$ is a trivial solution in case $\tau=0$, we conclude that for all $\tau\in I\setminus\{0\}$ there exists $u_\tau := \psi(\tau)\in S_p(L^q(\Omega))$ such that u_τ is a nontrivial solution of Eq. (4.6), and the proof is finished. □

Funding

The first author is partially supported by ANID under FONDECYT grant number 1180041. The second author is supported by MCIN/AEI/10.13039/501100011033, Project PID2019-105011GBI00; by Generalitat Valenciana, Project PROMETEU/2021/070 and by Junta de Andalucía, Consejería de Universidad, Investigación e Innovación, Project ProyExcel 00780: “Operator Theory: An interdisciplinary approach”.

References

[1] H. Amann, Linear and Quasilinear Parabolic Problems, in: Monographs in Mathematics, vol. 89, Birkhäuser-Verlag, Basel, 1995.
[2] W. Arendt, S. Bu, The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, Math. Z. 240 (2) (2002) 311–343.
[3] R. Brunnhuber, Well-Posedness and Long-Time Behavior of Solutions for the Blackstock-Crighton Equation (Thesis), Alpen-Adria-Universität Klagenfurt, Wien, 2015.
[4] R. Brunnhuber, Well-posedness and exponential decay of solutions for the Blackstock-Crighton-Kuznetsov equation, J. Math. Anal. Appl. 433 (2) (2016) 1037–1054.
[5] R. Brunnhuber, P.M. Jordan, On the reduction of Blackstock- model of thermoviscous compressible flow via Becker-assumption, Int. J. Non-Linear Mech. 78 (2016) 131–132.
[6] R. Brunnhuber, B. Kaltenbacher, Well-posedness and asymptotic behavior of solutions for the Blackstock-Crighton-Westervelt equation, Discrete Contin. Dyn. Sys. 34 (11) (2014) 4515–4535.
[7] R. Brunnhuber, S. Meyer, Optimal regularity and exponential stability for the Blackstock-Crighton equation in L_p-spaces with Dirichlet and Neumann boundary conditions, J. Evol. Equ. 16 (4) (2016) 945–981.
[8] S. Bu, G. Cai, Periodic solutions of second order degenerate differential equations with delay in Banach spaces, Can. Math. Bull. 61 (4) (2018) 717–737.
[9] S. Bu, G. Cai, Periodic solutions of fractional degenerate differential equations with delay in Banach spaces, Israel J. Math. 232 (2) (2019) 695–717.

[10] S. Bu, G. Cai, Periodic solutions of second order degenerate differential equations with finite delay in Banach spaces, J. Fourier Anal. Appl. 25 (1) (2019) 32–50.
[11] S. Bu, Y. Fang, Periodic solutions for second order integro-differential equations with infinite delay in Banach spaces, Studia Math. 184 (2) (2008) 103–119.
[12] G. Cai, S. Bu, Periodic solutions of third-order degenerate differential equations in vector-valued functional spaces, Israel J. Math. 212 (2016) 163–188.
[13] G. Cai, S. Bu, Periodic solutions of third-order integro-differential equations in vector-valued functional spaces, J. Evol. Equ. 17 (2) (2017) 749–780.
[14] A. Celik, M. Kyed, Nonlinear acoustics: Blackstock-Crighton equations with a periodic forcing term, J. Math. Fluid Mech. 21 (2019) 45.
[15] J.A. Conejero, C. Lizama, M. Murillo-Arcila, J.B. Seoane-Sepúlveda, Well-posedness for degenerate third order equations with delay and applications to inverse problems, Israel J. Math. 229 (2019) 219–254.
[16] R. Denk, M. Hieber, J. Prüss, R-boundedness,Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166 (2003) 788.
[17] L.R. Gambera, C. Lizama, A. Prokopczyk, Well-posedness for the abstract Blackstock Crighton Westervelt equation, J. Evol. Equ. 21 (2021) 313–337.
[18] M. Haase, The functional calculus for sectorial operators, in: Operator Theory: Advances and Applications, Vol. 169, Birkhäuser, Basel, 2006.
[19] M.F. Hamilton, D.T. Blackstock, Nonlinear Acoustics, Academic Press, 1998.
[20] N. Kalton, L. Weis, The $\mathcal{H}^{\infty}$-calculus and sums of closed operators, Math. Ann. 321 (2001) 319–345.
[21] V. Keyantuo, C. Lizama, A characterization of periodic solutions for time-fractional differential equations in UMD spaces and applications, Math. Nachr. 284 (4) (2011) 494–506.
[22] P. Kokocki, Effect of resonance on the existence of periodic solutions for strongly damped wave equation, Nonlinear Anal. 125 (2015) 167–200.
[23] C. Lizama, M. Murillo, $L^p - L^q$-Maximal regularity of the Van Wijngaarden-Eringen equation in a cylindrical domain, Adv. Diff. Equ. 2020 (1) (2020) 591.
[24] C. Lizama, M. Murillo, $L^p - L^q$-Well posedness for the Moore-Gibson-Thompson equation with two temperatures on cylindrical domains, Mathematics 8 (10) (2020) 1–8.
[25] T. Nau, The Laplacian on cylindrical domains, Integral Equations Operator Theory 75 (2013) 409–431.
[26] T. Nau, J. Saal, $\mathcal{R}$-Sectoriality of cylindrical boundary value problems, in: Parabolic Problems, in: Progr. Nonlinear Differential Equations Appl., vol. 80, Birkhäuser/Springer Basel AG, Basel, 2011, pp. 479–505.
[27] I. Wood, Maximal L_p-regularity for the Laplacian on Lipschitz domains, Math. Z. 255 (4) (2007) 855–875.