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L^p-L^q-Well Posedness for the Moore–Gibson–Thompson Equation with Two Temperatures on Cylindrical Domains

Carlos Lizama [1] and Marina Murillo-Arcila [2,*]

[1] Departamento de Matemática y Ciencia de la Computación, Facultad de Ciencias, Universidad de Santiago de Chile, Las Sophoras 173, Estación Central, Santiago 9160000, Chile; carlos.lizama@usach.cl

[2] Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 València, Spain

* Correspondence: mamuar1@upv.es

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Abstract: We examine the Cauchy problem for a model of linear acoustics, called the Moore–Gibson–Thompson equation, describing a sound propagation in thermo-viscous elastic media with two temperatures on cylindrical domains. For an adequate combination of the parameters of the model we prove L^p-L^q-well-posedness, and we provide maximal regularity estimates which are optimal thanks to the theory of operator-valued Fourier multipliers.

Keywords: well-posedness; Moore–Gibson–Thompson equation; degenerate evolution equations; R-boundedness

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1. Introduction

In this paper, we consider the Moore–Gibson–Thompson equation with two temperatures in a cylindrical domain $\Omega = U \times V \subset \mathcal{R}^{n+d}$ endowed with Dirichlet boundary conditions:

$$\begin{cases} \tau c(I - a\Delta)\partial_{ttt}u(x,y,t) + c(I - a\Delta)\partial_{tt}u(x,y,t) \\ \qquad = r^*\Delta u(x,y,t) + r\Delta\partial_t u(x,y,t) + f(x,y,t), \text{ for } (x,y,t) \in \Omega \times (0,2\pi); \\ \mathcal{B}_U u(x,y,t) = 0, \text{ for } (x,y,t) \in \partial U \times V \times (0,2\pi); \\ \mathcal{B}_V u(x,y,t) = 0, \text{ for } (x,y,t) \in U \times \partial V \times (0,2\pi); \\ u(x,y,0) = u(x,y,2\pi), \, \partial_t u(x,y,0) = \partial_t u(x,y,2\pi), \, \partial_{tt} u(x,y,0) = \partial_{tt} u(x,y,2\pi). \end{cases} \tag{1}$$

The model (1) has been recently proposed by Quintanilla [1]. This equation considers two distinct temperatures acting on the heat conduction by means of the Moore–Gibson–Thompson equation: The conductive temperature and thermodynamic temperature. The two-temperatures theory of generalized thermoelasticity was introduced by Chen et al. [2–4]. They proved that the variation between the two temperatures is proportional to the heat supply and both temperatures become identical for time-independent situations where heat supply does not appear. On the other hand, when there exists time-dependence both temperatures are distinct, in spite of the heat supply.

The existence, stability, convergence properties, and spatial behavior of two-temperatures thermoelasticity were analyzed by Quintanilla [5] and Youssef [6]. Magaña and Quintanilla [7] studied the uniqueness and growth of solutions based on the theory provided by Youssef [6]. Very recently, Quintanilla [8] considered a thermoelastic theory where the heat conduction is expressed in terms of the Moore–Gibson–Thompson equation. See also [9,10] and references therein.

Whereas Quintanilla in [1] found sufficient conditions on the parameters to ensure the well-posedness and stability/instability of the solutions of (1) in the context of the Hilbert space $L^2(B)$, where B is a three-dimensional domain whose boundary is smooth enough, the present paper analyzes the time/space regularity, i.e., L^p-L^q-well posedness, as well as optimal (or maximal) regularity estimates in the context of the Banach space $L^q(\Omega)$, $1 < q < \infty$, where Ω is a cylindrical domain. More precisely, Ω will denote the Cartesian product of half spaces and a standard domain V having compact boundary. It is well known that many situations in applied sciences are naturally modeled in cylindrical domains Ω. We refer, e.g., to the textbook [11] and [12] and the references [13–15] where a wide variety of problems on such Ω are considered.

The main tool to address the problem of regularity for (1) is the theory of discrete operator-valued Fourier multipliers. Applying Fourier series, we asked under which conditions an operator-valued symbol defines a bounded operator in $L^p((0, 2\pi), X)$ where X is a Banach space. The answer to this question was provided by Arendt and Bu in [16], where they obtain a discrete operator-valued Fourier multiplier result in UMD spaces and they also obtain applications to Cauchy problems of the first and second order in Lebesgue spaces. An extension of the results obtained in [16] to certain evolution equations in Lebesgue-, Besov-, and Hölder-spaces was obtained in [17]. In [18], a suitable treatment for second order differential equations in Lebesgue- and Hölder-spaces is provided. In particular, the special case of the linearized Kuznetsov equation, i.e., $\tau = 0, a = 0, c = 1$ is investigated. Recently, the references [19,20] began to treat the case of the Moore–Gibson–Thompson equation using these methods.

The usage of operator-valued Fourier multipliers to treat cylindrical domains was first carried out in [21] in a Besov-space setting. In that paper the author obtains semiclassical fundamental solutions for a wide variety of elliptic operators on infinite cylindrical domains $\mathcal{R}^n \times V$. As a result, they succeed obtaining the key for solving related elliptic and parabolic, as well as hyperbolic problems. Operators defined on a cylindrical domain with the same splitting property as in the present paper were also examined by Nau et.al. in [13–15,22,23].

In this paper, we directly apply general results of [20] and [14] to the Moore–Gibson–Thompson equation with two temperatures (1), and we obtain a L^p-L^q well-posedness result with maximal regularity estimates. The main difficulty relies in the verification of the so-called R-boundedness property that must be satisfied by certain sets of operators. To overcome this difficulty, we will employ a criteria established by Denk, Hieber, and Prüss in the reference [24] that reduces the problem to the localization of the spectrum of the Laplacian. We highlight that our method is sufficiently general to admit a wider class of operators than the Laplacian in (1) allowing also the possibility of the fractional Laplacian, the bi-Laplacian Δ^2, or other operators of practical interest. Therefore, we first establish our main result in an abstract setting, that roughly says that under certain conditions of sectoriality of the operator A and for all $\eta > 0$ the equation:

$$\tau c(I + aA^\eta)u'''(t) + c(I + aA^\eta)u''(t) + r^* A^\eta u(t) + rA^\eta u'(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{2}$$

is strongly L^p-L^q-well posed. Then, using the results of [14], we will establish our main findings concerning (1). It is important to observe that this result covers not only (1) taking $A = -\Delta$ and $\eta = 1$, but also the cases of the fractional Laplacian: $(-\Delta_q)^\eta, 0 < \eta < 1$ which intersects a recent work by Bezerra and Santos [25] that analyzes fractional powers for evolution equations of third order in time and the case of the bi-Laplacian operator: Δ^2 that appears in case $a = 0$ (i.e. the Moore–Gibson–Thompson equation) as a possible model for the vertical displacement in viscoelastic plates [26].

2. Preliminaries

Let $1 \leq p < \infty$ and X be a Banach space. In this section, we would like to provide optimal conditions to ensure the well-posedness of the following problem:

$$\tau c(I - a\Delta)u'''(x,t) + c(I - a\Delta)u''(x,t) - r^*\Delta u(x,t) - r\Delta u'(x,t) = f(x,t),\ t \in [0, 2\pi] \tag{3}$$

for each $x \in U \times V$ in the scale of vector valued L^p-spaces: $L^p((0, 2\pi), X), 1 < p < \infty$, and where Δ denotes de Laplacian operator subject to appropriate boundary conditions, and $a, c, \tau, r, r^* > 0$. For this purpose, we need to recall the preliminary results obtained in [20] where, in particular, the authors obtained a full characterization of L^p-well-posedness for the following abstract third order degenerate equation:

$$\alpha(Mu')''(t) + (Nu')'(t) - \beta Bu(t) - \gamma Cu'(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{4}$$

where α, β, γ are real numbers, M, N, B and C are closed linear operators defined on a complex Banach space X with respective domains $D(M), D(N), D(B)$, and $D(C)$. Assuming that $D(B) \cap D(C) \subset D(M) \cap D(N)$, the M-resolvent of B and C is defined as:

$$\rho_{M,N}(B,C) := \{s \in \mathbb{R} : \alpha i s^3 M + s^2 N + \beta B + \gamma i s C : [D(B) \cap D(C)] \to X$$
$$\text{is invertible and } [\alpha i s^3 M + s^2 N + \beta B + \gamma i s C]^{-1} \in \mathcal{B}(X)\}. \tag{5}$$

Here $[D(B) \cap D(C)]$ is a Banach space endowed with the norm $\|x\|_{[D(B)\cap D(C)]} := \|x\| + \|Bx\| + \|Cx\|$. The notion of L^p-well-posedness is given as follows:

Definition 1. *Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{T}, X)$ be given. We say that Equation (4) is strongly L^p-well-posed if for each $f \in L^p(\mathbb{T}, X)$, there exists a unique solution u that belongs to a maximal regularity space and satisfies (4) for almost all $t \in \mathbb{T}$.*

The following result obtained in [20] gives a computable criterion that completely characterizes the well-posedness of Equation (4) in terms of some R-boundedness conditions on certain sets of operators associated with the equation. We refer the reader to [27] where a precise definition and all the properties preserved under R-boundedness are summarized.

Theorem 1. *Let $1 < p < \infty$ and $\alpha, \beta, \gamma \in \mathcal{R}$. Assume A, B and M, N are closed linear operators defined on a UMD space X such that $D(B) \cap D(C) \subset D(M) \cap D(N)$. The following assertions are equivalent:*

(i) Equation (4) is strongly L^p-well posed;
(ii) $\mathbb{Z} \subset \rho_{M,N}(B,C)$ and the sets $\{ik^3\alpha M N_k : k \in \mathbb{Z}\}$, $\{k^2 N N_k : k \in \mathbb{Z}\}$, $\{k\gamma C N_k : k \in \mathbb{Z}\}$ $\{k N_k : k \in \mathbb{Z}\}$ are R-bounded where:

$$N_k := -[i\alpha k^3 M + k^2 N + \beta B + ik\gamma C]^{-1}, \quad k \in \mathbb{Z}. \tag{6}$$

Before we show our main abstract result we need to recall some preliminaries on sectorial operators. All these notions and more information about this class of operators can be found in [28].

Let $\Sigma_\phi \subset \mathbb{C}$ be the open sector $\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$. We define the following spaces of functions: $\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$ and $\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}$ which is endowed with the norm $||f||_\infty^\phi = \sup_{|\arg \lambda| < \phi} |f(\lambda)|$.

We also introduce the subspace $\mathcal{H}_0(\Sigma_\phi)$ of $\mathcal{H}(\Sigma_\phi)$ as follows: $\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha,\beta<0} \{f \in \mathcal{H}(\Sigma_\phi) : ||f||_{\alpha,\beta}^\phi < \infty\}$, with $||f||_{\alpha,\beta}^\phi = \sup_{|\lambda|\leq 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\geq 1} |\lambda^{-\beta} f(\lambda)|$.

Definition 2 ([28]). *Given a closed linear operator A in X, we say that A is sectorial if A satisfies the following conditions (i) $\overline{D(A)} = X, \overline{R(A)} = X, (-\infty, 0) \subset \rho(A)$; (ii) $||t(t + A)^{-1}|| \leq M$ for all $t > 0$ and some $M > 0$. The operator A is called R-sectorial if the set $\{t(t + A)^{-1}\}_{t>0}$ is R-bounded.*

If A is sectorial then $\Sigma_\phi \subset \rho(-A)$ for some $\phi > 0$ and $\sup_{|\arg\lambda|<\phi} ||\lambda(\lambda + A)^{-1}|| < \infty$. We denote the *spectral angle* of a sectorial operator A by:

$$\phi_A = \inf\{\phi : \Sigma_{\pi-\phi} \subset \rho(-A), \quad \sup_{\lambda\in\Sigma_{\pi-\phi}} ||\lambda(\lambda + A)^{-1}|| < \infty\}.$$

Definition 3 ([28]). *Given a sectorial operator A, we say that it admits a bounded $\mathcal{H}^\infty$- calculus if there exist $\phi > \phi_A$ and a constant $K_\phi > 0$ such that $||f(A)|| \leq K_\phi ||f||^\phi_\infty$ for all $f \in \mathcal{H}_0(\Sigma_\phi)$. This class of operators will be denoted as $\mathcal{H}^\infty(X)$ and $\phi^\infty_A = \inf\{\phi > \phi_A : ||f(A)|| \leq K_\phi ||f||^\phi_\infty$ for all $f \in \mathcal{H}_0(\Sigma_\phi)$ holds $\}$ denotes the $\mathcal{H}^\infty(X)$ angle.*

In the context of R-boundedness, we have the analogous notion of operators admitting an R-bounded $\mathcal{H}^\infty$-calculus. More concretely, if $A \in \mathcal{H}^\infty(X)$ verifies that $\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\theta), ||h||^\theta_\infty \leq 1\}$ is R-bounded for some $\theta > 0$ then it is said that A *admit an R-bounded $\mathcal{H}^\infty$-calculus* and that A belongs to the class $\mathcal{RH}^\infty(X)$. The corresponding angle will be noted as $\theta^{R_\infty}_A$. See [28] for more information about these concepts.

Remark 1. *Given A a sectorial operator on a Hilbert space, Lebesgue spaces $L^p(\Omega)$, $1 < p < \infty$, Sobolev spaces $W^{s,p}(\Omega)$, $1 < p < \infty$, $s \in \mathbb{R}$, or Besov spaces $B^s_{p,q}(\Omega)$, $1 < p, q < \infty$, $s \in \mathbb{R}$ that admit a bounded $\mathcal{H}^\infty$-calculus of angle β, then it follows that A also admit a $\mathcal{RH}^\infty$ calculus on the same angle β on the corresponding space above mentioned (see Kalton and Weis [29]). Moreover, this property holds whenever X is a UMD space endowed with the so called property (α) (see [28,29]).*

Some operators that admit a bounded $\mathcal{H}^\infty$-calculus are: M-accretive and normal sectorial operators defined on Hilbert spaces, generators of C_0-groups, whenever bounded, and negative generators contraction semigroups, whenever positive, on L_p-spaces. See [28] for more information about well-known operators in the literature that admit a bounded $\mathcal{H}^\infty$-calculus. The following remark recalls that the Dirichlet Laplacian operator defined in a suitable space also verifies this property.

Remark 2. *Let $1 < q < \infty$ and denote by Δ the Laplacian operator in $\mathcal{R}^n$. By [24] [Theorem 7.2] we obtain that the $L^q(\mathcal{R}^n)$ realization Δ_q of the Laplacian operator admits an R-bounded $\mathcal{H}^\infty$–calculus for each $0 < \theta^{\mathcal{R}_\infty}_A < \pi$. Moreover, by [24] [Corollary 7.3] the same is true for $-\Delta_D$, the negative Dirichlet Laplacian in $\mathcal{R}^{n+1}_+$.*

The following result will be the key in our method for establishing L^p-L^q well posedness for Equation (3) because, under certain conditions on the operator A, the hypothesis of R-boundedness in Theorem 1 could be restricted to simply checking uniform boundedness of a set of one-parameter functions on the complex plane. It is obtained in [27] [Proposition 4.10].

Proposition 1. *Let $A \in \mathcal{RH}^\infty(X)$ be given and assume that the set $\{h_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta^{R_\infty}_A$, where Λ is an arbitrary index set. Then the set $\{h_\lambda(A)\}_{\lambda\in\Lambda}$ is R-bounded.*

3. Main Results

Let $a, c, \tau, r, r^* > 0$ be given. We start defining a sequence of complex numbers that depends on the parameters of the general equation:

$$\tau c(I + aA^\eta)u'''(t) + c(I + aA^\eta)u''(t) + r^* A^\eta u(t) + rA^\eta u'(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi],$$

and given by:

$$d_k := \frac{-ck^2(1 + i\tau k)}{(r^* - cak^2) + i(rk - \tau ack^3)} \qquad k \in \mathbb{Z}.$$

We also define $\Phi(k) := \arg(d_k)$. Let $\theta^* := \sup_{k\in\mathbb{Z}} \Phi(k)$. Since $\Phi(0) = 0$ we have $0 \leq \theta^* \leq \pi$. Moreover, we have:

$$\Re(d_k) = \frac{ck^2[\tau^2ack^4 + (ac - \tau r)k^2 - r^*]}{(r^* - ack^2)^2 + (ac\tau k^3 - rk)^2}$$

and

$$\Im(d_k) = \frac{ck^3(r - \tau r^*)}{(ack^2 - r^*)^2 + (ac\tau k^3 - rk)^2}.$$

It is clear that $\Phi(k) = \pi$ if and only if $\Re(d_k) \leq 0$ and $\Im(d_k) = 0$. We can then conclude that conditions $r^* < ca - \tau r$ or $\tau r^* \neq r$ are sufficient to ensure that $\Phi(k) < \pi$. After these preliminaries, we can state the main abstract result of this paper.

Theorem 2. *Assume that X is a UMD-space, $1 < p < \infty$, $\theta^* < \pi$ and suppose that $A \in \mathcal{RH}^\infty(X)$ with angle $\theta_A^{R_\infty} \in (0, \frac{\pi-\theta^*}{\eta})$ and $0 \in \rho(A)$. Then for all $\eta > 0$ the equation:*

$$\tau c(I + aA^\eta)u'''(t) + c(I + aA^\eta)u''(t) + r^*A^\eta u(t) + rA^\eta u'(t) = f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{7}$$

is strongly L^p-well posed.

Proof. First observe that, our Equation (7) labels into (4) for $M = N = c(I + aA^\eta)$, $\beta = -r^*$, $\alpha = \tau$,$\gamma = -r$ and $B = C = A^\eta$. As a consequence, in order to prove well-posedness for (3) we only need to show that condition (ii) in Theorem 1 holds. Now, it suffices to show that the sets $\{ik^3\alpha MN_k : k \in \mathbb{Z}\}$,$\{k^2NN_k : k \in \mathbb{Z}\}$, $\{k\gamma CN_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded. Indeed, we have:

$$N_k = [-c(k^2 + i\tau k^3) + ((r^* - ack^2) + i(rk - ac\tau k^3))A^\eta]^{-1}.$$

It follows that:

$$\begin{aligned} N_k &= \frac{1}{(r^* - ack^2) + ik(r - ac\tau k^2)}[\frac{-ck^2(1 + i\tau k)}{(r^* - ack^2) + i(rk - ac\tau k^3)} + A^\eta]^{-1} \\ &= \frac{-1}{ck^2(1 + i\tau k)}d_k(d_k + A^\eta)^{-1}, \end{aligned}$$

where $d_k = \frac{-ck^2(1+i\tau k)}{(r^*-cak^2)+i(rk-ac\tau k^3)}$.

Due to the fact that $0 < \theta_A^{R_\infty} < \frac{\pi-\theta^*}{\eta}$ there exists $s > \theta_A^{R_\infty}$ such that $s < \frac{\pi-\theta^*}{\eta}$. For each $z \in \Sigma_s$ and $k \in \mathbb{Z}, k \neq 0$, define:

$$F(k, z) = d_k(d_k + z^\eta)^{-1}.$$

Note that $\frac{z^\eta}{d_k}$ belongs to the sector $\Sigma_{s\eta+\theta^*}$ where $s\eta + \theta^* < \pi$ and then the distance from the sector $\Sigma_{s\eta+\theta^*}$ to -1 is always positive. As a result, there exists $M > 0$ independent of $k \in \mathbb{Z}$ and $z \in \Sigma_\tau$ that satisfies the following:

$$|F(k, z)| = \left|\frac{1}{1 + \frac{z^\eta}{d_k}}\right| \leq M.$$

Now, from Proposition 1 we can conlude that the set $\{F(k, A)\}_{k\in\mathbb{Z}\setminus\{0\}}$ is R-bounded. Moreover, due to the fact that A is invertible, the operators $H(k) := (d_k + A^\eta)^{-1}$ exist for all $k \in \mathbb{Z}$. As a consequence, $H(k)$ belongs to $\mathcal{B}(X)$ for all $k \in \mathbb{Z}$ and the sequence $\{d_k(d_k + A^\eta)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded. Taking into account the identity $A^\eta(d_k + A^\eta)^{-1} = I - d_k(d_k + A^\eta)^{-1}$ we have:

$$\begin{aligned} ik^3\tau MN_k &= \frac{-i\tau k^3}{c(i\tau k^3 + k^2)}c(I + aA^\eta)d_k(d_k + A^\eta)^{-1} \\ &= \frac{ia\tau k^3}{(i\tau k^3 + k^2)}d_kd_k(d_k + A^\eta)^{-1} - \frac{ia\tau k^3}{(i\tau k^3 + k^2)}d_kI - \frac{i\tau k^3}{(i\tau k^3 + k^2)}d_k(d_k + A^\eta)^{-1}. \end{aligned}$$

Since the sets $\{d_k\}_{k\in\mathbb{Z}}$, $\{\frac{ia\tau k^3}{(i\tau k^3+k^2)}\}_{k\in\mathbb{Z}}$ are bounded and the set $\{d_k d_k(d_k+A^\eta)^{-1}\}$ is R-bounded, it follows from the above identity that the set $\{ik^3\tau MN_k\}_{k\in\mathbb{Z}}$ is R-bounded, too. Since $M=N$ we also have:

$$k^2NN_k = \frac{1}{k}k^3MN_k$$

for all $k\in\mathbb{Z}\setminus\{0\}$. Therefore, the set $\{k^2NN_k\}_{k\in\mathbb{Z}}$ is R-bounded. On the other hand, using again the identity $A^\eta(d_k+A^\eta)^{-1} = I - d_k(d_k+A^\eta)^{-1}$ we obtain:

$$krCN_k = \frac{-kr}{c(i\tau k^3+k^2)}A^\eta d_k(d_k+A^\eta)^{-1} = \frac{-kr}{c(i\tau k^3+k^2)}d_k d_k(d_k+A)^{-1} + \frac{kr}{c(i\tau k^3+k^2)}d_k,$$

where the set $\{\frac{kr}{c(i\tau k^3+k^2)}\}_{k\in\mathbb{Z}}$ is bounded. We deduce that the set $\{krCN_k\}_{k\in\mathbb{Z}}$ is R-bounded. Finally, we have:

$$kN_k = \frac{-k}{c(i\tau k^3+k^2)}d_k(d_k+A^\eta)^{-1},$$

where the set $\{\frac{-k}{c(i\tau k^3+k^2)}\}_{k\in\mathbb{Z}}$ is bounded. This, together with the hypothesis, implies that the set $\{kN_k\}_{k\in\mathbb{Z}}$ is R-bounded. We conclude that Equation (7) is L^p-well posed. □

Immediately from Remark 1 we obtain the following useful corollaries:

Corollary 1. *Let $1<p,q<\infty$ be given. Suppose that $\tau r^* \neq r$ and that A is a sectorial operator that admits a bounded $\mathcal{H}^\infty$-calculus of angle $\theta_A^{\mathcal{R}_\infty} \in (0, \frac{\pi-\theta^*}{\eta})$ and $0\in\rho(A)$. Then for all $\eta>0$ the Equation (7) is strongly L^p-L^q-well posed.*

Taking into account that $\Re(b_k)\geq 0$ for all $k\in\mathbb{Z}$ under the condition $r^*<ca-\tau r$ we obtain the following result.

Corollary 2. *Let $1<p,q<\infty$ be given. Suppose that $r^*<ca-\tau r$ and that A is a sectorial operator that admits a bounded $\mathcal{H}^\infty$-calculus of angle $\theta_A^{\mathcal{R}_\infty}\in(0,\frac{\pi}{2\eta})$ and $0\in\rho(A)$. Then for all $\eta>0$ the Equation (7) is strongly L^p-L^q-well posed.*

Finally, we consider the Moore–Gibson–Thompson Equation (1) with two temperatures in a cylindrical domain $\Omega = U\times V\subset\mathbb{R}^{n+d}$ where $U=\mathbb{R}^n_+$, $n\in\mathbb{N}$ and $V\subset\mathbb{R}^d$, $d\in\mathbb{N}_0$ is bounded, open, and connected. Moreover, in (1) Δ denotes a cylindrical decomposition of the Dirichlet Laplacian operator on $L^q(\Omega)$ with respect to the two cross-sections i.e., $\Delta=\Delta_1+\Delta_2$ where Δ_i acts on the according component of Ω. Following [14] we introduce L^q-realizations $\Delta_{q,i}=\Delta_i$ as follows:

$$\begin{aligned} D(\Delta_{q,1}) &:= \{u\in W^{2,q}(\mathcal{R}^n_+, L^q(V)) : \mathcal{B}_U = 0\};\\ D(\Delta_{q,2}) &:= W^{2,q}(V)\cap W_0^{1,q}(V), \end{aligned}$$

see also [30] for the description of $\Delta_{q,2}$. We define the Laplacian Δ_q in $L^q(\Omega)$ subject to the Dirichlet boundary conditions $\mathcal{B}_U$ and $\mathcal{B}_V$ to be:

$$\begin{aligned} D(\Delta_q) &:= D(\Delta_{q,1})\cap D(\Delta_{q,2})\\ \Delta_q u &:= \Delta_{q,1}u + \Delta_{q,1}u = \Delta u, \quad u\in D(\Delta_q). \end{aligned}$$

Suppose now that V is a C^2-standard domain (see [14] [Definition 3.1] for the precise definition). Then, applying [14] [Theorem 4.2] we have that $-\Delta_q\in\mathcal{RH}^\infty(L^q(\Omega))$ and $0\in\rho(\Delta_q)$. Moreover, by [14]

[Proposition 5.1 (i)] we have $\theta^{\mathcal{R}_\infty}_{-\Delta_q} < \frac{\pi}{2}$. Consequently, from Corollary 2 with $\eta = 1$ and $A = -\Delta_q$ we deduce the following result:

Theorem 3. *Let $1 < p, q < \infty$ and $a, c, \tau, r, r^* > 0$. Assume the condition $r^* < ca - \tau r$ then, for any given $f \in L^p(\mathbb{T}, L^q(\Omega))$ the solution u of the problem (1) exists, is unique and belongs to the space $W^{3,p}_{per}(\mathbb{T}, [D(\Delta_q)]) \cap W^{3,p}_{per}(\mathbb{T}, L^q(\Omega))$. Moreover, for any $1 < p, q < \infty$ the estimate:*

$$\begin{aligned} \|u\|_{L^p(\mathbb{T},L^q(\Omega))} + ||u'||_{W^{1,p}(\mathbb{T},L^q(\Omega))} + ||u''||_{W^{2,p}_{per}(\mathbb{T},L^q(\Omega))} + ||u'''||_{W^{3,p}_{per}(\mathbb{T},L^q(\Omega))} \\ + ||\Delta u||_{L^p(\mathbb{T},[D(\Delta_q)])} + \|\Delta u'\|_{W^{1,p}_{per}(\mathbb{T},[D(\Delta_q)])} + \|\Delta u''\|_{W^{2,p}_{per}(\mathbb{T},[D(\Delta_q)])} \\ + \|\Delta u'''\|_{W^{3,p}_{per}(\mathbb{T},[D(\Delta_q)])} \leq C||f||_{L^p(\mathbb{T},L^q(\Omega))} \end{aligned}$$

holds.

The last estimate follows from the Closed Graph Theorem. We remark that an analogous result holds when we replace the Laplacian by the fractional Laplacian $(-\Delta_q)^\eta, 0 < \eta < 1$. We observe that a fractional power approach for abstract evolution equations of third order in time has been recently proposed by Bezerra and Santos [25]. Finally, we note that also very recently it has been pointed out in [26] that in the particular case where $A = \Delta^2$ with proper boundary conditions, the Moore–Gibson–Thompson equation, i.e., (1) with $a = 0$, appears as a possible model for the vertical displacement in viscoelastic plates [31]. Of course, this last case is also included in our findings.

4. Conclusions

Taking into account a suitable combination of the parameters of the Cauchy problem for the Moore–Gibson–Thompson equation, treated in this article with two temperatures and in cylindrical domains, we could provide the existence, uniqueness, and estimates of maximal regularity for the solution in Lebesgue spaces. Our original method combined the theory of operator-valued Fourier multipliers, that reduced the problem to verify a certain property of R-boundedness of an operator-valued symbol, and a criterion established by Denk–Hieber and Prüss that gave computable conditions on the data of the problem (the closed linear operator A) and that we used to verify the mentioned property of the R-boundedness. We note, for future works on this topic, that the maximal regularity estimate provided in this article is the starting point for the analysis of the existence, uniqueness and regularity of the nonlinear problem: The Jordan–Moore–Gibson–Thomson equation, by using the implicit function theorem and fixed point arguments, see for example the reference [27] for an overview of this method.

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