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On the dynamics of the damped extensible beam 1D-equation

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A B S T R A C T

We show that the solutions of the linearized damped extensible beam equation exhibit a chaotic or stable behavior that depends on the distribution of the physical parameters of the equation. Such dynamical behavior is achieved in Herzog-like spaces. Our results provide new insights into the damped extensible beam equation by finding a critical parameter whose sign determines such qualitative properties.

1. Introduction

Evolution equations have long attracted the attention of researchers because they model a wide range of dynamical processes ranging from the natural to the social sciences. Starting with PDEs and integral equations, there is now a broad variety of them. In order to mathematically understand its qualitative behavior, the original evolution equation is usually simplified in such a way that, sometimes, the physical parameters of the equation are absent in the analysis, see e.g., [18,24]. However, this simplification most of the time reduces the possibility of having a more precise understanding of the dynamics of the model.

How should these parameters be chosen to have Devaney chaos or stability?. This article explores a plausible answer to this general problem for a specific evolution equation. The proposed methodology provides a mechanism to examine the chaos (see Definition 3.2) and stability of a wide range of evolutionary processes.

We will analyze the dynamics of the linearized damped extensible beam (DEB) equation [26] which can be stated as follows:

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$$\partial_t^2 u(t,x) + \delta\partial_t u(t,x) + \kappa\partial_x^4 u(t,x) + \eta\partial_t\partial_x^4 u(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1}$$

where κ, η are positive real numbers, and $\delta \in \mathbb{R}$. In this model, η represents the effective viscosity, $\kappa := EI$ where E is the Young's modulus and I is the cross sectional second-moment of area, and δ is the coefficient of external damping, which can be negative [1]. We suppose that the coefficient of mass for unit length (in front of the term ∂_t^2) is equal to 1. Our analysis will be carried out within the framework of Herzog spaces.

Equation (1) arises as the linearization of a model of an extensible beam proposed by Ball [1], where he assumes that the beam has linear structural (Kelvin-Voigt) and external (frictional) damping. Ball studies asymptotic properties and shows that when time t tends to infinity, provided that δ is not large and *possibly negative*, any solution of the nonlinear model converges in a suitable topology to an equilibrium position. The singular perturbation problem for (1) was studied by Fitzgibbon [19], by Racke and Yoshikawa [26], and abstractly using semigroup theory by Fitzgibbon [18] and by Massatt [24] who uses the theory of sectorial operators. Exponential attractors were studied by Eden and Milani [16, Section 8] and periodic solutions by Cwiszewski [14], among others.

Herzog-type spaces were introduced by Herzog in [21] when studying the universality of the solutions of the classical heat equation. These function spaces of analytic functions regulated by a parameter allow to control their growth in infinity. See for instance [6] where the dynamics of the translation operator on spaces of analytic functions of slow growth were considered. Due to this advantage on their growth control, researchers have analyzed the presence of Devaney chaos for the C_0-semigroups associated to PDEs in such spaces.

From a dynamical point of view, the existence of Devaney chaos implies that small changes in the initial states of a system may lead, after some time, to large discrepancies in the orbits of its solutions [20]. Note however that definitions of linear chaos may vary [13,22,23].

In the reference [9], Conejero et al. studied the Devaney chaos for the hyperbolic heat transfer equation in absence of internal heat sources and the wave equation. In [11] the authors studied the telegraph equation

$$\partial_t^2 u(t,x) + \beta\partial_t u(t,x) + \theta u(t,x) - \alpha\partial_x^2 u(t,x) = 0, \quad \beta, \theta, \alpha \geq 0,$$

which includes the hyperbolic heat transfer equation ($\theta = 0$), and the wave equation ($\theta = \beta = 0$), and obtained conditions on the sign of the critical parameter $\gamma := \beta^2 - 4\theta$ to ensure chaos or stability.

Following a similar approach, the analysis of the chaotic behavior of the Moore-Gibson-Thompson equation

$$\tau\partial_t^3 u(t,x) + \alpha\partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) - b\partial_t\partial_x^2 u(t,x) = 0, \quad \tau, \alpha \geq 0, \quad c, b > 0,$$

describing propagation process in acoustics or vibrations in elastic structures, were performed in [7]. In this case, it was shown that when the critical parameter $\gamma := \alpha - \frac{\tau c^2}{b}$ is negative, it describes chaos, while the positive sign describes stability.

In [8] Devaney chaos was obtained for the Lighthill-Whitham-Richards equation

$$\partial_t u(t,x) + c\partial_x u(t,x) + T\partial_t^2 u(t,x) - D\partial_x^2 u(t,x) = 0, \quad c, T, D \geq 0,$$

that describes traffic at the macroscopic level. It is interesting to observe that, unlike the previous cases, for this model chaos is always present in some Herzog-type spaces and there is no critical parameter. The corresponding study for the van Wijgaarden–Eringen equation

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) - (Re_d)^{-1}\partial_t\partial_x^2 u(t,x) - a_0^2\partial_t^2\partial_x^2 u(t,x) = 0, \quad Re_d, a_0 > 0,$$

that models the acoustic planar propagation in bubbly liquids can be found in [10]. For such an equation it was shown that assuming $a_0 < 1$ and $\frac{\sqrt{5}}{6} < a_0 Re_b < \frac{1}{2}$ chaos is obtained. However, the stability problem was left open. See [12] for a review on this topic.

This article is organized as follows. Section 2 is devoted to collect basic definitions and results about the dynamics of C_0-semigroups. In Section 3 we present the novel results of this article. More concretely, for the critical parameter $\gamma := \eta\delta - \kappa$ we show that $\gamma \leq 0$ is sufficient to ensure Devaney chaos. As a corollary, the dynamics of the (DEB) equation are proven to be distributionally chaotic under the same assumption. We also succeed in proving that the complementary condition $\gamma > 0$ provides stability which displays the dependence of the dynamics behavior for the (DEB) equation through a complete analysis of the critical parameter found. This critical parameter provides new insights and motivation for a complete future discussion on the dynamics of the solutions of the nonlinear model. Indeed, assuming a *possibly negative value of* δ, as shown in the reference [1, Conditions (7.1) and (8.5)], our result proves that $\gamma < 0$ implying Devaney chaos. However, if that value of δ is positive, then the condition $\gamma > 0$, i.e. $\delta > \eta/\kappa$, is necessary for stability.

2. Preliminaries

We first need to recall the notion of C_0-semigroup.

Definition 2.1. Let X be a Banach space. A one-parameter family $\{T_t\}_{t\geq 0} \subset \mathcal{B}(X)$, is a C_0*-semigroup* if $T_0 = I$, $T_{t+s} = T_t \circ T_s$ and $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$. The operator

$$Ax := \lim_{t\to 0} \frac{1}{t}(T_t x - x), \tag{2}$$

exists on a dense subspace of X; the set of these x, the domain of A, is denoted by $D(A)$. Then A, or rather $(A, D(A))$, is called the *infinitesimal generator of the semigroup*.

As a consequence of the Hille-Yosida theorem [5, Theorem 7.4], it follows that the solution of the abstract Cauchy problem on X given by:

$$\begin{cases} \partial_t u(t) = Au(t), \\ u(0) = \varphi, \end{cases} \tag{3}$$

can be provided by means of a C_0-semigroup $\{T_t\}_{t\geq 0}$ on X whose infinitesimal generator is A. If $A \in \mathcal{B}(X)$, then the semigroup is uniformly continuous and can be represented as $T_t = e^{tA} = \sum_{k=0}^{\infty}(tA)^n/n!$ for all $t \geq 0$ (see [17, Ch. I, Prop. 3.5]). We can now provide the definition of Devaney chaotic C_0-semigroup.

Definition 2.2. An element $x \in X$ is called a *periodic point* for $\{T_t\}_{t\geq 0}$ if there exists some $t > 0$ such that $T_t x = x$. A C_0-semigroup $\{T_t\}_{t\geq 0}$ is called *Devaney chaotic* if there exists $x \in X$ such that the set $\{T_t x : t \geq 0\}$ is dense in X and the set of periodic points is dense in X.

We point out that these two conditions in the definition of Devaney chaos imply the sensitive dependence on the initial conditions, as it was stated by Banks et al. [2,20].

There exists another notion of chaos, namely distributional chaos, introduced for the first time by Schweizer & Smítal [27].

Definition 2.3. A C_0-semigroup $\{T_t\}_{t\geq 0}$ on X is *distributionally chaotic* if there is an uncountable subset $K \subset X$ and $\delta > 0$ such that, for all points $x, y \in K$ with $x \neq y$ and for every $\varepsilon > 0$, we have $\overline{\text{Dens}}(\{s \geq$

$0; ||T_s x - T_s y|| > \delta\}) = 1$ and $\overline{\mathrm{Dens}}(\{s \geq 0; ||T_s x - T_s y|| < \varepsilon\}) = 1$, where $\overline{\mathrm{Dens}}$ corresponds to the upper density of a set $I \subset \mathbb{R}_+$, that is,

$$\overline{\mathrm{Dens}}(I) := \limsup_{N\to\infty} \frac{\mu(I \cap [0, N])}{N},$$

where μ is the Lebesgue measure on $\mathbb{R}_+$.

We recall the following criterion [20, Th. 7.30] which gives the necessary conditions to obtain a Devaney chaotic C_0-semigroup. The original version of it, the so-called Desch-Schappacher-Webb (DSW) criterion can be found in [15].

Theorem 2.4. *Let X be a complex separable Banach space, and $\{T_t\}_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator $(A, D(A))$. Assume that there exists an open connected subset $U \subset \mathbb{C}$ and a weakly holomorphic function $f : U \to X$ (for all $x^* \in X^*$, $\lambda \to \langle f(\lambda), x^*\rangle$ is holomorphic, where $\langle\cdot,\cdot\rangle$ is the dual product between X and X^*) such that*

(i) $U \cap i\mathbb{R} \neq \emptyset$,
(ii) $f(\lambda) \in \ker(\lambda I - A)$ *for every* $\lambda \in U$,
(iii) *for any* $x^* \in X^*$, *if* $\langle f(\lambda), x^*\rangle = 0$ *for all* $\lambda \in U$, *then* $x^* = 0$.

Then the semigroup $\{T_t\}_{t\geq 0}$ is Devaney chaotic.

Now, we define the Herzog space of analytic functions introduced in [21], which is isometrically isomorphic to the Banach space of sequences $c_0(\mathbb{N}_0) := \{s : \mathbb{N}_0 \to \mathbb{C} \ : \ |s_n| \to 0 \text{ as } n \to \infty\}$ endowed with the natural norm.

Definition 2.5. Given $\rho > 0$, the Herzog space of analytic functions is defined as:

$$X_\rho = \left\{ f : \mathbb{R} \to \mathbb{C}; f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, (a_n)_{n\geq 0} \in c_0(\mathbb{N}_0) \right\} \tag{4}$$

endowed with the norm $\|f\|_\rho := \sup_{n\geq 0} |a_n|$.

For example, the functions $f(x) = \cosh(bx)$ and $g(x) = e^{bx}$ where $b \in \mathbb{C}$, belong to X_ρ if $\rho > |b|$. It is worth noting that classical partial differential operators, e.g. ∂_x^n, $n \in \mathbb{N}$, become *bounded* operators on Herzog spaces and hence they are generators of uniformly continuous semigroups (see for instance [25, Section 1.1]).

3. Chaos and stability

In this section we will examine the damped extensible beam equation on $(0, \infty) \times \mathbb{R}$ with prescribed initial conditions on Herzog spaces

$$\begin{cases} \partial_t^2 u(t,x) + \delta\partial_t u(t,x) + \kappa\partial_x^4 u(t,x) + \eta\partial_t\partial_x^4 u(t,x) = 0 & (t,x) \in (0,\infty)\times\mathbb{R}; \\ u(0,x) = u^0(x), \quad \partial_t u(0,x) = u^1(x), & x \in \mathbb{R}. \end{cases} \tag{5}$$

In order to study the dynamics for the equation (5), we first reduce it into a first order system by setting $u_1 = u$ and $u_2 = \partial_t u$. It reads as follows:

$$\begin{cases} \partial_t \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\kappa\partial_x^4 & -(\delta I + \eta\partial_x^4) \end{pmatrix} \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}; \\ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix} = \begin{pmatrix} u^0(x) \\ u^1(x) \end{pmatrix}. \end{cases} \tag{6}$$

Since the operator ∂_x^4 is bounded on any Herzog space X_ρ, it follows that the operator-valued matrix

$$A := \begin{pmatrix} 0 & I \\ -\kappa\partial_x^4 & -(\delta I + \eta\partial_x^4) \end{pmatrix}, \tag{7}$$

is bounded on $X := X_\rho \oplus X_\rho$ for every $\rho > 0$. By [25, Theorem 1.2 and Corollary 1.4] we immediately get the following result on well-posedness.

Theorem 3.1. *The operator A is the generator of a uniformly continuous semigroup $(e^{tA})_{t\geq 0}$ on $X := X_\rho \oplus X_\rho$ for every $\rho > 0$, which determines a unique solution of the Cauchy problem (5).*

Adopting the notion of Devaney chaos for C_0-semigroups, we introduce the following definition that directly appeal to the equation (5).

Definition 3.2. Let $\rho > 0$ be given. The dynamic of the equation (5) is said to be Devaney chaotic on $X := X_\rho \oplus X_\rho$ if

(a) There exists $(u^0, u^1) \in X$ such that the unique solution of (5) satisfies that the set of trajectories $\{(u(t,\cdot), \partial_t u(t,\cdot)) : t \geq 0\}$ is dense in X.
(b) The set of periodic points $\{(\varphi_1, \varphi_2) \in X : \exists\, t > 0,\ u(t,x) = \varphi_1(x) \text{ and } \partial_t u(t,x) = \varphi_2(x) \text{ for all } x \in \mathbb{R}\}$ is dense in X.

Concerning distributional chaos, we introduce the following definition.

Definition 3.3. Let $\rho > 0$ be given. The dynamic of the equation (5) is said to be distributionally chaotic on $X := X_\rho \oplus X_\rho$ if there is an uncountable subset $K \subset X$ and $\delta > 0$ such that, for all points $(\varphi_1, \varphi_2), (\psi_1, \psi_2) \in K$ with $(\varphi_1, \varphi_2) \neq (\psi_1, \psi_2)$, and for every $\varepsilon > 0$, we have $\overline{\text{Dens}}(\{s \geq 0; ||(u(s,\cdot) - v(s,\cdot), u'(s,\cdot) - v'(s,\cdot)||_{\rho\times\rho} > \delta\}) = 1$, $\overline{\text{Dens}}(\{s \geq 0; ||(u(s,\cdot) - v(s,\cdot), u'(s,\cdot) - v'(s,\cdot)||_{\rho\times\rho} < \varepsilon\}) = 1$ where $u(0,\cdot) = \varphi_1, u'(0,\cdot) = \varphi_2, v(0,\cdot) = \psi_1$ and $v'(0,\cdot) = \psi_2$.

We can now state the first main result of this section where we prove Devaney chaos for the damped extensible beam equation (5) under a condition on the parameters of the equation.

Theorem 3.4. *Suppose that*

$$\eta\delta < \kappa.$$

Then the dynamic of the equation (5) is Devaney chaotic on $X := X_\rho \oplus X_\rho$ whenever $\rho^4 > \frac{3}{2}\frac{\delta\eta+\kappa}{\eta^2}$.

Proof. We define $U := \{z \in \mathbb{C} : |z| < r_0\}$, where

$$r_0 := \frac{\delta\eta + \kappa}{2\eta} > 0.$$

It is clear that $U \cap i\mathbb{R} \neq \emptyset$ and then hypothesis (i) in Theorem 2.4 is verified. For each $\lambda \in \mathbb{C} \setminus \{-\kappa/\eta\}$ we define the analytic function $R_\lambda \equiv R(\lambda) := -\frac{\lambda^2+\delta\lambda}{\kappa+\eta\lambda}$.

Let now prove (ii) in Theorem 2.4. We start solving $Ag_{z_0,z_1,z_2,z_3}(\lambda) = \lambda g_{z_0,z_1,z_2,z_3}(\lambda)$, $\lambda \in U$, $z_0, z_1, z_2, z_3 \in \mathbb{C}$. By the structure in the first line of the matrix A, we find that the eigenvectors of A must have the form:

$$g_{z_0,z_1,z_2,z_3}(\lambda) = \begin{pmatrix} \psi_\lambda \\ \lambda\psi_\lambda \end{pmatrix}, \tag{8}$$

and a simple calculus shows that

$$\psi_\lambda(x) := \exp\left(\sqrt[4]{R_\lambda}x\right) z_0 + \exp\left(-\sqrt[4]{R_\lambda}x\right) z_1 + \sin\left(\sqrt[4]{R_\lambda}x\right) z_2 + \cos\left(\sqrt[4]{R_\lambda}x\right) z_3. \tag{9}$$

Let us now prove that the functions $g_{z_0,z_1,z_2,z_3}(\lambda)$ belong to $X_\rho \oplus X_\rho$ for all $\lambda \in U$. We first point out that the functions $\psi_\lambda(x)$ can be rewritten as:

$$\begin{aligned}\psi_\lambda(x) &= \exp\left(\sqrt[4]{\frac{R_\lambda}{\rho^4}}\rho x\right) z_0 + \exp\left(-\sqrt[4]{\frac{R_\lambda}{\rho^4}}\rho x\right) z_1 + \sin\left(\sqrt[4]{\frac{R_\lambda}{\rho^4}}\rho x\right) z_2 + \cos\left(\sqrt[4]{\frac{R_\lambda}{\rho^4}}\rho x\right) z_3 \\ &= \sum_{n=0}^{\infty}(a_n(\lambda) + b_n(\lambda) + c_n(\lambda))\frac{(\rho x)^n}{n!}, \quad x \in \mathbb{R}, \quad \lambda \in U,\end{aligned}$$

where $a_n(\lambda) := \frac{R_\lambda^{n/4}}{\rho^n} z_0$, $b_n(\lambda) := \frac{(-1)^n R_\lambda^{n/4}}{\rho^n} z_1$ and

$$c_n(\lambda) = \begin{cases} \frac{(-1)^k R_\lambda^{n/4}}{\rho^n} z_2 & \text{if} \quad n = 2k+1,\ k \in \mathbb{N}_0, \\ \frac{(-1)^k R_\lambda^{n/4}}{\rho^n} z_3 & \text{if} \quad n = 2k,\ k \in \mathbb{N}_0. \end{cases}$$

Consequently, from the definition of X_ρ, it is enough to prove that $\left|\frac{R_\lambda}{\rho^4}\right| < 1$ for all $\lambda \in U$ in order to obtain that $\psi_\lambda, \lambda\psi_\lambda \in X_\rho$. Indeed, let $\lambda \in U$ that satisfies $|\lambda| < r_0 = \frac{\kappa+\delta\eta}{2\eta}$. Using the identity $\frac{az+b}{cz+d} = \frac{a}{c} + \frac{1}{c}\frac{bc-ad}{cz+d}$, we can write

$$\frac{\lambda^2+\delta\lambda}{\kappa+\eta\lambda} = \frac{\lambda}{\eta} + \frac{\lambda}{\eta}\frac{\delta\eta-\kappa}{\eta\lambda+\kappa}. \tag{10}$$

Since $\eta\delta < \kappa$ we have that $|\lambda| < r_0 = \frac{\kappa+\delta\eta}{2\eta} < \frac{\kappa}{\eta}$, and we obtain

$$|\eta\lambda+\kappa| \geq |\eta|\lambda| - \kappa| = \kappa - \eta|\lambda| > \kappa - \eta r_0.$$

Using again $\eta\delta < \kappa$ and the definition of r_0, we have

$$\left|\frac{\lambda^2+\delta\lambda}{\eta\lambda+\kappa}\right| \leq \frac{r_0}{\eta} + \frac{r_0}{\eta}\frac{|\delta\eta-\kappa|}{\kappa-\eta r_0} = \frac{r_0}{\eta} + \frac{r_0}{\eta}\frac{\kappa-\delta\eta}{\kappa-\eta\frac{\kappa+\delta\eta}{2\eta}} = 3\frac{r_0}{\eta} < \rho^4,$$

where the last inequality follows from the hypothesis. This proves the claim.

Finally, let us show condition (iii) in Theorem 2.4 is fulfilled, that is, for any $x^* \in X_\rho^* \oplus X_\rho^*$ the functions $\lambda \to \langle g_{z_0,z_1,z_2,z_3}(\lambda), x^*\rangle$, $z_0, z_1, z_2, z_3 \in \mathbb{C}$, are holomorphic on U, and if they all vanish on U, then $x^* = 0$. Indeed, we first observe that from Definition 2.5, the dual space X_ρ^* is isomorphic to ℓ^1. Now, given $x^* \in X_\rho^* \oplus X_\rho^*$ it follows that $x^* = (x_1^*, x_2^*) = ((x_{1,n}^*)_{n\geq 0}, (x_{2,n}^*)_{n\geq 0}) \in \ell^1 \oplus \ell^1$. Then, we have

$$0 = \langle g_{z_0,z_1,z_2,z_3}(\lambda), x^* \rangle = \langle \psi_\lambda, x_1^* \rangle + \langle \lambda\psi_\lambda, x_2^* \rangle, \tag{11}$$

for all $\lambda \in U, z_0, z_1, z_2, z_3 \in \mathbb{C}$. This last equation can be rewritten as follows:

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty}(a_n(\lambda) + b_n(\lambda) + c_n(\lambda))x_{1,n}^* + \lambda\sum_{n=0}^{\infty}(a_n(\lambda) + b_n(\lambda) + c_n(\lambda))x_{2,n}^* \\
&= (z_0 + z_1 + z_3)x_{1,0}^* + \lambda(z_0 + z_1 + z_3)x_{2,0}^* \\
&+ \frac{\sqrt[4]{R_\lambda}}{\rho}(z_0 - z_1 + z_2)x_{1,1}^* + \lambda\frac{\sqrt[4]{R_\lambda}}{\rho}(z_0 - z_1 + z_2)x_{2,1}^* + \\
&+ \frac{\sqrt{R_\lambda}}{\rho^2}(z_0 + z_1 - z_3)x_{1,2}^* + \lambda\frac{\sqrt{R_\lambda}}{\rho^2}(z_0 + z_1 - z_3)x_{2,2}^* + \dots
\end{aligned} \tag{12}$$

Let $\lambda_0 = 0$. It is clear that $\lambda_0 \in U$ and $R_{\lambda_0} = 0$. Replacing in (12) the value of $\lambda_0 = 0$, we get the following equation:

$$(z_0 + z_1 + z_3)x_{1,0}^* = 0 \tag{13}$$

for all $z_0, z_1, z_3 \in \mathbb{C}$. Consequently, $x_{1,0}^* = 0$. Now, observe that from the hypothesis $\eta\delta < \kappa$ we obtain $\delta < \frac{\kappa+\delta\eta}{2\eta} = r_0$ and therefore $-\delta \in U$. Hence, we can evaluate (12) in $\lambda_1 = -\delta$ obtaining $R_{\lambda_1} = 0$. It immediately follows that $x_{2,0}^* = 0$. We now divide (12) by $\sqrt[4]{R_\lambda}$ obtaining:

$$\begin{aligned}
0 &= \frac{1}{\sqrt[4]{R_\lambda}}\left(\sum_{n=0}^{\infty}(a_n(\lambda) + b_n(\lambda) + c_n(\lambda))x_{1,n}^* + \lambda\sum_{n=0}^{\infty}(a_n(\lambda) + b_n(\lambda) + c_n(\lambda))x_{2,n}^*\right) \\
&= \frac{1}{\rho}(z_0 - z_1 + z_2)x_{1,1}^* + \frac{\lambda}{\rho}(z_0 - z_1 + z_2)x_{2,1}^* + \frac{\sqrt[4]{R_\lambda}}{\rho^2}(z_0 + z_1 - z_3)x_{1,2}^* \\
&+ \lambda\frac{\sqrt[4]{R_\lambda}}{\rho^2}(z_0 + z_1 - z_3)x_{2,2}^* + \dots
\end{aligned} \tag{14}$$

Evaluating equation (14) in $\lambda = \lambda_0$ we get:

$$\frac{1}{\rho}(z_0 - z_1 + z_2)x_{1,1}^* = 0 \tag{15}$$

for all $z_0, z_1, z_3 \in \mathbb{C}$. As a consequence, $x_{1,1}^* = 0$. Now, replacing λ by λ_1 in equation (14) we obtain:

$$\frac{\lambda_1}{\rho}(z_0 - z_1 + z_2)x_{2,1}^* = 0 \tag{16}$$

and then $x_{2,1}^* = 0$. Proceeding by induction we finally arrive to $x_{i,n}^* = 0$ for all $i = 1, 2$ and $n \in \mathbb{N}$ and then $x^* = 0$. As an application of Theorem 2.4, the conclusion holds. □

Theorem 3.5. *Suppose that*

$$\eta\delta = \kappa.$$

Then the dynamic of the equation (5) is Devaney chaotic on $X := X_\rho \oplus X_\rho$ whenever $\rho^4 > \frac{3\kappa}{\eta^2}$.

Proof. The proof follows the same steps as the one of Theorem 3.4. We consider $U := \{z \in \mathbb{C} : |z| < \frac{\kappa}{\eta} \}$, which clearly verifies $U \cap i\mathbb{R} \neq \emptyset$ and $0 \in U$. For each $\lambda \in \mathbb{C}$ we consider the analytic function $R_\lambda := -\frac{\lambda}{\eta}$. Proceeding as in Theorem 3.4, it can be shown the eigenvectors of A have the form provided in (8) and (9). In order to prove (ii) in Theorem 2.4 it only remains to show that $\left|\frac{R_\lambda}{\rho^4}\right| < 1$ for all $\lambda \in U$ to obtain $\psi_\lambda, \lambda\psi_\lambda \in X_\rho$. Indeed, given $\lambda \in U$ we have

$$\left|\frac{\lambda}{\eta}\right| < \frac{\kappa}{\eta^2} < \frac{3\kappa}{\eta^2} < \rho^4,$$

and the claim is proven.

Finally, it only remains to show that hypothesis (iii) of Theorem 2.4 holds. Indeed, evaluating equation (12) in $\lambda_0 = 0$, we get $x^*_{1,0} = 0$. We now divide (12) by $\sqrt[4]{R_\lambda}$ obtaining:

$$\begin{aligned} 0 &= \frac{1}{\sqrt[4]{R_\lambda}} \left(\sum_{n=0}^{\infty} (a_n(\lambda) + b_n(\lambda) + c_n(\lambda)) x^*_{1,n} + \lambda \sum_{n=0}^{\infty} (a_n(\lambda) + b_n(\lambda) + c_n(\lambda)) x^*_{2,n} \right) \\ &= \frac{1}{\sqrt[4]{\eta}} \left(\sqrt[4]{\lambda^3}(z_0 + z_1 + z_3) x^*_{2,0} + \frac{1}{\rho}(z_0 - z_1 + z_2) x^*_{1,1} + \frac{\lambda}{\rho}(z_0 - z_1 + z_2) x^*_{2,1} \right. \\ &\quad \left. + \frac{\sqrt[4]{R_\lambda}}{\rho^2}(z_0 + z_1 - z_3) x^*_{1,2} + \lambda \frac{\sqrt[4]{R_\lambda}}{\rho^2}(z_0 + z_1 - z_3) x^*_{2,2} + \dots \right) \end{aligned} \tag{17}$$

and evaluating again in $\lambda_0 = 0$ we get $x^*_{1,1} = 0$.

Dividing now (17) by $\sqrt[4]{R_\lambda}$ we get:

$$\begin{aligned} 0 &= \frac{1}{\sqrt[4]{R_\lambda}} \left(\sum_{n=0}^{\infty} (a_n(\lambda) + b_n(\lambda) + c_n(\lambda)) x^*_{1,n} + \lambda \sum_{n=0}^{\infty} (a_n(\lambda) + b_n(\lambda) + c_n(\lambda)) x^*_{2,n} \right) \\ &= \frac{1}{\sqrt{\eta}} \left(\sqrt{\lambda}(z_0 + z_1 + z_3) x^*_{2,0} + \frac{\sqrt[4]{\lambda^3}}{\rho}(z_0 - z_1 + z_2) x^*_{2,1} + \frac{1}{\rho^2}(z_0 + z_1 - z_3) x^*_{1,2} \right. \\ &\quad \left. + \frac{\lambda}{\rho^2}(z_0 + z_1 - z_3) x^*_{2,2} + \dots \right) \end{aligned} \tag{18}$$

We substitute $\lambda = 0$ in (18) which leads to $x^*_{1,2} = 0$. In the next step, we divide (18) by $\sqrt{\lambda}$ obtaining $x^*_{2,0} = 0$. Afterwards, we divide again by $\sqrt[4]{R_\lambda}$ getting $x^*_{2,1} = 0$. Proceeding by induction we finally arrive to $x^*_{i,n} = 0$ for all $i = 1, 2$ and $n \in \mathbb{N}$ and the proof is concluded. □

Since it is well known that there is distributional chaos if the DSW criterion holds [3,4,28], we immediately get the following corollary.

Corollary 3.6. *Suppose that*

$$\eta\delta \leq \kappa.$$

Then the dynamic of the equation (5) *is distributionally chaotic on* $X := X_\rho \oplus X_\rho$ *whenever* $\rho^4 > \frac{3}{2}\frac{\kappa + \delta\eta}{\eta^2}$.

Our second main result shows stability under a complementary condition on the parameters of the equation. Recall that a uniformly continuous semigroup is said to be stable, if $\lim_{t\to\infty} \|e^{At}\| = 0$. According to [17, Theorem 3.14] this is equivalent to prove that $\Re(\lambda) < 0$ for all $\lambda \in \sigma(A)$.

Theorem 3.7. *Suppose that*

$$\eta\delta > \kappa.$$

Then the dynamic of the equation (5) *is stable on* $X := X_\rho \oplus X_\rho$ *whenever* $\rho^4 < \frac{1}{2}\frac{\delta\eta-\kappa}{\eta^2}$.

Proof. Since A is bounded, it is enough to show that the spectrum of A is contained in the open left halfplane. To this end, we consider the problem

$$Af = \lambda f, \quad \lambda \in \mathbb{C}, \text{ for every } f \in X. \tag{19}$$

As in the proof of the above theorem, this leads to the equation

$$\varphi''''(x) = R(\lambda)\varphi(x), \quad x \in \mathbb{R}, \quad \varphi \in X_\rho, \tag{20}$$

where

$$R(\lambda) := -\frac{\lambda^2 + \delta\lambda}{\kappa + \eta\lambda}, \quad \lambda \in \mathbb{C} \setminus \{-\kappa/\eta\}. \tag{21}$$

Since $\varphi \in X_\rho$ we have $\varphi(x) = \sum_{n=0}^{\infty} \frac{a_n\rho^n}{n!}x^n$, where $a_n \to 0$. From (20) we deduce that the sequence $(a_n)_n$ must satisfy the identity $\rho^4 a_{n+4} = R(\lambda)a_n$, $n \in \mathbb{N}_0$. It follows that the sequence $(a_n)_n$ is defined as powers of $\frac{R(\lambda)}{\rho^4}$. This, together with the condition $a_n \to 0$, implies that

$$|R(\lambda)| < \rho^4, \quad \lambda \in \mathbb{C} \setminus \{-\kappa/\eta\}. \tag{22}$$

Let $\lambda = a + ib$, $a, b \in \mathbb{R}$. It is not difficult to prove that for $a \in \mathbb{R}$ arbitrary but fixed, we have

$$\left|\frac{R(\lambda)}{\lambda} - \frac{1}{\eta}\right| \to 0 \quad \text{as } b \to \infty. \tag{23}$$

Therefore, there exists $N > 0$ such that for all $|b| > N$ we have

$$\left|\frac{R(\lambda)}{\lambda} - \frac{1}{\eta}\right| < \frac{\delta\eta - \kappa}{2\kappa\eta}. \tag{24}$$

From (21) and using (22) and (24) we obtain

$$|\lambda + \delta| = \left|\kappa\frac{R(\lambda)}{\lambda} + \eta R(\lambda)\right| < \kappa\left(\frac{\delta\eta - \kappa}{2\kappa\eta} + \frac{1}{\eta}\right) + \eta\rho^4. \tag{25}$$

Since $\rho^4 < \frac{1}{2}\frac{\delta\eta-\kappa}{\eta^2}$, we obtain

$$|\lambda + \delta| < \frac{\delta\eta - \kappa}{2\eta} + \frac{\kappa}{\eta} + \eta\left(\frac{1}{2}\frac{\delta\eta - \kappa}{\eta^2}\right) = \delta. \tag{26}$$

Therefore $\Re(\lambda) = a < 0$. This proves the claim and the theorem. □

Remark 3.8. In terms of the model equation (5) the conclusion of stability expresses that

$$\lim_{s\to\infty} \|(u(s,\cdot), u'(s,\cdot))\|_{\rho\times\rho} = 0,$$

uniformly for all initial conditions $u(0,\cdot) = \varphi$, $u'(0,\cdot) = \psi$ satisfying $\|(\varphi,\psi)\|_{\rho\times\rho} \leq 1$.

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