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ON SEMIDISCRETE MODELS DOMINATED BY THE HEAT, WAVE AND LAPLACE EQUATIONS

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Dedicated to the memory of my beloved sister Olga Lizama (1959-2024)

(Communicated by Masaharu Taniguchi)

Abstract. We introduce a method to solve linear semidiscrete equations that provides for the first time explicit solutions for some well-known models such as the semidiscrete advection-diffusion equation and the semidiscrete Lighthill-Whitham-Richards equation, among others. We find conditions in the parameters of the model under which it can be dominated by the heat, wave or Laplace equations. We illustrate the fundamental solutions for all the cases based on the newly developed solution formulas. We also study spatial and time regularity in Lebesgue spaces for these models.

1. **Introduction.** It is well-known that the one-dimensional advection-diffusion equation

$$u_t(t,x) - \alpha u_{xx}(t,x) + \beta u_x(t,x) = 0, \quad t \geq 0, \ x \in \mathbb{R}, \tag{1}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$, can be solved by the substitution

$$u(t,x) = w(t,x) e^{\frac{\beta}{2\alpha}x} e^{-\frac{\beta^2}{4\alpha}t}, \quad t \geq 0, \ x \in \mathbb{R}, \tag{2}$$

where $w(t,x)$ is the solution of the pure diffusion equation

$$w_t(t,x) = \alpha w_{xx}(t,x), \quad t \geq 0, \ x \in \mathbb{R}, \tag{3}$$

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so that we can completely reduce the mathematical analysis of the advection-diffusion equation (1) to the well-known wave equation (3). In particular, the substitution (2) shows how the behavior of $u(t,x)$ depends on the relative magnitude of the parameters α and β in equation (1). Unfortunately, we rapidly note that attempts to apply a similar transformation as (2) to other known PDEs fails.

Having this in mind, it may seem surprising that when we try to apply this substitution for semidiscrete linear evolution equations, a rich diversity of situations appear.

Although the theory of semidiscrete linear equations is well established, few literature sources provide complete details on how to solve them and the influence of parameters on their evolution. Bateman [3] presents several methods to solve some general-type semidiscrete equations. Slavík and Stehlík [25] studied semidiscrete linear equations in the more general context of partial dynamic equations focusing on the existence and uniqueness of solutions for initial value problems. Interesting results on the asymptotic behavior of the semidiscrete diffusion equation were recently discovered by Slavík [23]. Variable coefficients appearing in semidiscrete equations have been studied in the references [3, 24]. Other important contributions in the area can be found in [14, 27, 28, 29, 30] and in the references therein.

This article aims to develop a theory around the method suggested by (2). We will show that this type of substitution allows obtaining explicit solutions for a wide range of semidiscrete equations in terms of the solutions of the standard models given by the semidiscrete heat, wave and Laplace equations. We will provide a detailed procedure that shows the influence of the parameters on the evolution of the solution and provides a kind of classification of them according to their behavior.

More precisely, for each semidiscrete equation, we will obtain a set of parameters where this procedure gives information which is in some cases complete. This method allows to rapidly obtain new insights on the full behavior of a given semidiscrete equation by means of a completely new approach.

As methods, we will make extensive use of properties of special functions (Bessel), the fact that Δ_d, given by $\Delta_d f(n) = f(n-1) - 2f(n) + f(n+1)$, is a bounded operator in Lebesgue spaces of sequences where it has a well-known spectrum [17], as well as tools and methods of functional analysis for the study of ℓ^p regularity.

We expect that the ideas introduced in this article can serve as motivation for future studies that allow us to better understand the intrinsic structure of semidiscrete equations, seeing them as a starting point for the study of the dynamics of PDEs as a function of their physical parameters.

This article is organized as follows: In Section 2, we review and complete the theory on what is known about the explicit solution for the following three standard models:

(i) Semidiscrete heat (or diffusion) equation: $u_t(t,n) = \Delta_d u(t,n)$;
(ii) Semidiscrete wave equation: $u_{tt}(t,n) = \Delta_d u(t,n)$;
(iii) Semidiscrete Laplace equation: $u_{tt}(t,n) + \Delta_d u(t,n) = 0$.

While it is well-known that the explicit solution of (i) is given in terms of the modified Bessel function [3, 7, 26], this seems not to be widely familiar for (ii) even though there are some references that provide an explicit solution, see [3, 17]. In the case of (iii), and as far as we know, the explicit solution presented below seems to be completely new, see Theorem 2.3.

In all cases, we take an operator theoretical approach. It means that we express the solution of each equation in terms of a uniformly continuous one-parameter family of operators (semigroups and cosine families) [1].

In summary, and observing the fundamental solution in each case, we demonstrate that the following Bessel functions determine the behavior of the solution for each model:

$$(i)\ e^{-2t}I_n(2t), \qquad (ii)\ J_{2n}(2t), \qquad (iii)\ J_{2n}(2it) = (-1)^n I_{2n}(2t),$$

where I_n denotes the modified Bessel function and J_n is the Bessel function of first kind.

Section 3 is dedicated to determining for which set of parameters we can show an explicit solution in terms of (i), (ii) or (iii) for the following semidiscrete equations:

(a) The semidiscrete advection-diffusion equation

$$u_t(t,n) - \alpha\Delta_d u(t,n) + \beta[u(t,n) - u(t,n-1)] = 0.$$

(b) The semidiscrete model

$$u_t(t,n) = \alpha\Delta_d u(t,n) + \beta u(t,n) + \gamma u(t,n-1),$$

that generalizes (a) and appears as a birth-and-death type population model with proliferation [2]. It also models chemical turbulence and some biological processes [19].

(c) The semidiscrete Lighthill-Whitham-Richards equation given by

$$u_t(t,n) + c[u(t,n) - u(t,n-1)] + \beta u_{tt}(t,n) - \alpha\Delta_d u(t,n) = 0,$$

which arises in traffic modelling for describing traffic at macroscopic level [5, 8] and contains, as a special case, the semidiscrete hyperbolic heat transfer equation.

We emphasize that the listed models are not exhaustive and that the ideas and methods presented in this article could be extended to many other models of interest.

We highlight our main findings for the listed models: In the case of (b) (and (a)) the model is dominated by the semidiscrete heat equation as long as the parameters of the model satisfy the condition $(\alpha,\beta,\gamma) \in \Omega := (0,\infty)\times\mathbb{R}\times[-\alpha,\infty)$. In the case of (c) we have to distinguish two interesting situations:

(c-1) If $c\beta > 1/4$ then the model is dominated by the semidiscrete wave equation, and

(c-2) If $c\beta < 1/4$ then the model is dominated by the semidiscrete Laplace equation.

Furthermore, in both cases, the parameters must belong to the set

$$\Omega := \{(\beta,c,\alpha) \in (0,\infty)\times(0,\infty)\times(0,\infty) : (4c\beta-1)^2 - 16\alpha\beta = 0\}.$$

We also illustrate the fundamental solutions obtained with the newly discovered solution formulas.

Finally, in Section 4, we prove the spatial regularity of the solutions for models (a), (b) and (c), and the temporal regularity for models (a) and (b) in Lebesgue sequence spaces $\ell^q(\mathbb{Z})$. Although the spatial regularity can be well established under minimal assumptions about the parameters to have time decay of the solution in the case of models (a) and (b), (see Theorems 4.1, 4.2 and 4.3), the case of temporal regularity is more subtle. First, this type of regularity, also called ℓ^p-maximal regularity, is only available for those models dominated by the semidiscrete heat equation, that is, our model (b) (which includes (a)). Secondly, the method of proof requires using more sophisticated tools from functional analysis and operator theory, which we briefly recall at the beginning of our main result in this section (Theorem 4.4). With these tools we show that, in addition to the condition on the parameters obtained for case (b), we must add the condition $\beta+\gamma<-4\alpha$ to obtain the desired regularity.

2. **Semidiscrete heat, wave and Laplace equations.** The solution of the semidiscrete heat equation

$$\begin{cases} u_t(t,n)=\Delta_d u(t,n), \quad t>0,\, n\in\mathbb{Z}, \\ u(0,n)=\varphi(n), \quad n\in\mathbb{Z}, \end{cases} \tag{4}$$

is known to be *the semidiscrete heat semigroup*, which is represented by the series

$$\mathcal{T}_t\varphi(n)=\sum_{m\in\mathbb{Z}} T_t(n-m)\varphi(m), \quad t\geq 0, \quad n\in\mathbb{Z}$$

where the kernel is given by

$$T_t(n)=e^{-2t}I_n(2t), \quad t\geq 0, \quad n\in\mathbb{Z}, \tag{5}$$

and I_n denotes the modified Bessel function. This result is well-established within the framework of probability theory, particularly in the context of birth-and-death processes and random walks, as outlined in [12, Ch. XVII.5]. Additionally, it has been independently discovered through statistical methods, [32] and via Fourier series techniques, as demonstrated in [11]. This result finds applications in various fields, including the examination of physical models [13], mutation models in population biology [6, Section 4], and in the development of harmonic analysis associated with the discrete Laplacian [7]. Furthermore, an alternative approach to derive the kernel (5) as the fundamental solution of the semidiscrete heat equation was done in [15]. A comprehensive overview from the perspective of operator theory is available in [17, Theorem 1.1].

Less known but equally important for us is the explicit solution of the semidiscrete wave equation

$$\begin{cases} u_{tt}(t,n)=\Delta_d u(t,n), \quad t>0,\, n\in\mathbb{Z}, \\ u(0,n)=\varphi(n), \quad n\in\mathbb{Z}, \\ u_t(0,n)=\psi(n), \quad n\in\mathbb{Z}, \end{cases} \tag{6}$$

which is given by [17, Theorem 1.2 and Theorem 1.5]

$$u(t,n)=\mathcal{C}_t\varphi(n)+\mathcal{S}_t\psi(n), \tag{7}$$

where $\mathcal{C}_t$ denotes *the semidiscrete cosine family*,

$$\mathcal{C}_t\xi(n)=\sum_{m\in\mathbb{Z}} C_t(n-m)\xi(m), \quad t\in\mathbb{R},\ n\in\mathbb{Z}, \tag{8}$$

and $\mathcal{S}_t\xi=\int_0^t \mathcal{C}_s\xi ds$ denotes the associated *sine family.* The associated kernels are given by

$$C_t(n):=J_{2n}(2t), \quad t\in\mathbb{R}, \quad n\in\mathbb{Z}, \tag{9}$$

and

$$S_t(n):=\int_0^t J_{2n}(2s)ds, \quad t\geq 0, \quad n\in\mathbb{Z}, \tag{10}$$

with J_k the Bessel function of first kind.

Remark 2.1. Using [22, Formulae 1.8.1 (12) and (14)] we obtain the explicit representation

$$S_t(n) = t(1-2t\pi)J_0(2t) + \frac{\pi}{2}tJ_1(2t) - \sum_{k=0}^{n-1} J_{2k+1}(2t).$$

Equations (4) and (6) are at the basis of parabolic and hyperbolic lattice models, respectively. To complete the above results with the elliptic model, in this section we will show an explicit solution for the Laplace equation in the half-lattice $\mathbb{R}_+ \times \mathbb{Z}$

$$u_{tt}(t,n) + \Delta_d u(t,n) = 0. \tag{11}$$

In order to do that, we need to rotate the semidiscrete cosine family defined above from $\mathbb{R}$ to $i\mathbb{R}$. We observe that the right hand side of (9) still makes sense for values in the complex plane. Then, using some tools from [17] we obtain the following result.

Theorem 2.2. *Given $t \in \mathbb{R}$, let $\mathcal{L}_t$ be the operators defined by*

$$\mathcal{L}_t\xi(n) = \sum_{m\in\mathbb{Z}} L_t(n-m)\xi(m), \quad n \in \mathbb{Z}, \tag{12}$$

where

$$L_t(n) := J_{2n}(2it). \tag{13}$$

Then,

(i) We have $\{\mathcal{L}_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\ell^p)$ and it is an uniformly continuous cosine family, with bounded generator $-\Delta_d$ on ℓ^p, $1 \le p \le \infty$.

(ii) We have

1. *$\|L_t\|_{\ell^p} \le (2\cosh|t|^p - 1)^{1/p}$, for $1 \le p < \infty$.*
2. *The spectrum of $\mathcal{L}_t$ is $\sigma(\mathcal{L}_t) = \{\cosh(2t|\sin\theta/2|)\}_{\theta\in(-\pi,\pi]} = [1, \cosh(2t)]$.*

Proof. We begin with part (i). We recall that the Bessel function $J_k(z)$ satisfies (see [33, Ch. II, 2.1, p. 15])

$$J_{-k}(z) = (-1)^k J_k(z), \quad z \in \mathbb{C}, \tag{14}$$

for each $k \in \mathbb{Z}$, and the following upper bound for ν real and greater than $-1/2$, see [33, Ch. III, 3.31, p. 49]

$$|J_\nu(z)| \le \frac{|\frac{1}{2}z|^\nu e^{|\operatorname{Im} z|}}{\Gamma(\nu+1)}, \quad z \in \mathbb{C}. \tag{15}$$

Observe that (15) remains valid for all $\nu \in \mathbb{Z}$, due to (14).

Then we have, for each t fixed,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |J_{2m}(2it)\varphi(n-m)|^p &\le \|\varphi\|_{\ell^p}^p \sum_{m\in\mathbb{Z}} |J_{2m}(2it)| \\ &\le \|\varphi\|_{\ell^p}^p \Big(2\sum_{m=1}^{\infty} \frac{|t|^{2m}}{\Gamma(2m+1)} + |J_0(2it)|\Big) \\ &\le \|\varphi\|_{\ell^p}^p \big(2(\cosh|t| - 1) + |J_0(2it)|\big) \\ &\le (2\cosh|t| - 1)\|\varphi\|_{\ell^p}^p,\end{aligned}$$

so $\mathcal{L}_t$ is a linear and bounded operator on ℓ^p. It is clear that

$$J_0(0) = 1 \quad \text{and} \quad J_k(0) = 0 \quad \text{for} \quad k \neq 0. \tag{16}$$

Therefore, we obtain

$$\mathcal{L}_0\varphi(n) = \sum_{m\in\mathbb{Z}} L_0(n-m)\varphi(m) = \sum_{m\in\mathbb{Z}} J_{2n-2m}(0)\varphi(m) = \varphi(n).$$

In [22, Formula 7, p.660] (with $n = 0$) we find the following formula for Bessel functions

$$\sum_{k\in\mathbb{Z}} J_{2k}(z)(-1)^k t^{2k} = \cos(\Big(\frac{t^2+1}{2t}\Big)z), \quad z \in \mathbb{C}. \tag{17}$$

Then, for $z, w \in \mathbb{C}$ we find

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \Big(J_{2n}(z+w) + J_{2n}(z-w)\Big)(-1)^n t^{2n} \\
&= 2\cos(\Big(\frac{t^2+1}{2t}\Big)z)\cos(\Big(\frac{t^2+1}{2t}\Big)w) \\
&= 2\Big(\sum_{k\in\mathbb{Z}} J_{2k}(z)(-1)^k t^{2k}\Big)\Big(\sum_{m\in\mathbb{Z}} J_{2m}(w)(-1)^m t^{2m}\Big) \\
&= 2\sum_{n\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} J_{2k}(z)J_{2n-2k}(w)\Big)(-1)^n t^{2n},
\end{aligned}$$

and equating coefficients we obtain the identity

$$J_{2n}(z+w) + J_{2n}(z-w) = 2\sum_{k\in\mathbb{Z}} J_{2k}(z)J_{2n-2k}(w), \quad z, w \in \mathbb{C}. \tag{18}$$

Therefore it follows that

$$J_{2n}(2it+2is) + J_{2n}(2it-2is) = 2\sum_{m\in\mathbb{Z}} J_{2n-2m}(2is)J_{2m}(2it).$$

Since $L_t(n) = J_{2n}(2it)$ we arrive at

$$\begin{aligned}
\mathcal{L}_t(\mathcal{L}_s\varphi)(n) &= \sum_{k\in\mathbb{Z}} L_t(n-k)\Big(\sum_{m\in\mathbb{Z}} L_s(k-m)\varphi(m)\Big) \\
&= \sum_{m\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} L_t(n-k)L_s(k-m)\Big)\varphi(m) \\
&= \frac{1}{2}\sum_{m\in\mathbb{Z}}\big(\mathcal{L}_{t+s}(n-m) + \mathcal{L}_{t-s}(n-m)\big)\varphi(m) \\
&= \frac{1}{2}(\mathcal{L}_{t+s} + \mathcal{L}_{t-s})\varphi(n).
\end{aligned}$$

Now we will perform the necessary analysis to show the family of cosine operators $\{\mathcal{C}_t\}_{t\in\mathbb{R}}$ has the bounded generator $-\Delta_d$ and, in particular, is uniformly continuous. Indeed, we will check that $-\Delta_d\varphi = \frac{\partial^2}{\partial t^2}\mathcal{L}_t|_{t=0}\varphi$, for any $\varphi \in \ell^p$. Using the fact that the Bessel function $J_k(z)$ satisfies $J_k'(z) = -\frac{1}{2}(J_{k+1}(z) - J_{k-1}(z))$ we get $\frac{d}{dt}J_k(2it) = -i(J_{k+1}(2it) - J_{k-1}(2it))$ and hence

$$\frac{d^2}{dt^2}J_k(2it) = -\Big(J_{k+2}(2it) - 2J_k(2it) + J_{k-2}(2it)\Big).$$

Then, we have

$$\frac{\partial^2}{\partial t^2}\mathcal{C}_t\varphi(n) = -\sum_{m\in\mathbb{Z}}\big(J_{2(n-m+1)}(2t) - 2J_{2(n-m)}(2t) + J_{2(n-m-1)}(2t)\big)\varphi(m).$$

Consequently, from (16), we arrive at the intended conclusion. Now we prove (ii). To show (ii)-(1) we reason as in part (i) obtaining

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |J_{2n}(2it)|^p &\leq 2\sum_{n=1}^{\infty}\Big(\frac{|t|^{2n}}{\Gamma(2n+1)}\Big)^p + |J_0(2it)|^p \\ &\leq 2\sum_{n=1}^{\infty}\frac{|t|^{2np}}{\Gamma(2n+1)} + |J_0(2it)|^p \\ &\leq (2\cosh|t|^p - 1).\end{aligned}$$

Finally, part (2) is a consequence of $\sigma(\Delta_d) = \{-4\sin^2(\theta/2)\}_{\theta\in(-\pi,\pi]} = [-4,0]$ (see [17, Theorem 1.1, (iii)]) and the spectral mapping theorem for cosine operator functions, i.e., the identity

$$\sigma(\mathcal{L}_t) = \cosh(t\sqrt{\sigma(-\Delta_d)}) = [1, \cosh(2t)].$$

The theorem is proved. □

In what follows we define

$$M_t(n) := \int_0^t L_s(n)ds \text{ and } \mathcal{M}_t\psi(n) := \int_0^t \mathcal{L}_s\psi(n)ds. \tag{19}$$

The following result is a direct consequence of the above theorem and operator cosine family theory [1, Section 3.14, Corollary 3.14.12].

Theorem 2.3. *The unique solution of the semidiscrete Laplace equation*

$$\begin{cases} u_{tt}(t,n) + \Delta_d u(t,n) = 0, & t>0,\ n\in\mathbb{Z}, \\ u(0,n) = \varphi(n), & n\in\mathbb{Z}, \\ u_t(0,n) = \psi(n), & n\in\mathbb{Z}, \end{cases} \tag{20}$$

is given by

$$u(t,n) = \mathcal{L}_t\varphi(n) + \mathcal{M}_t\psi(n). \tag{21}$$

Remark 2.4. For further use, we observe that as $u(t,n) = \mathcal{T}_t\varphi(n)$ solves the Cauchy problem

$$u_t(t,n) = \Delta_d u(t,n)$$

with initial condition $u(0,n) = \varphi(n)$ we have that $v(t,n) = \mathcal{T}_{at}\varphi(n)$ where $a>0$, solves the problem

$$u_t(t,n) = a\Delta_d u(t,n)$$

with the same initial condition. In the same way, for the second order Cauchy problem

$$u_{tt}(t,n) = \Delta_d u(t,n)$$

with initial conditions $u(0,n) = \varphi(n)$ and $u_t(0,n) = \psi(n)$ we have that $v(t,n) = \mathcal{C}_{at}\varphi(n) + \mathcal{S}_{at}\psi(n)$, where $a>0$, solves the problem

$$u_{tt}(t,n) = a\Delta_d u(t,n)$$

with identical initial conditions. Finally, the same occurs when dealing with the semidiscrete Laplace equation.

3. **Semidiscrete dominating equations.** In this section, we use the theoretical results obtained in Section 2 to obtain explicit solutions for a variety of semidiscrete equations. We graphically display the behavior of each analysed system to show how different constants affect the evolution.

We introduce the following definition.

Definition 3.1. We say that the solution $u : [0, \infty) \times \mathbb{Z} \to \mathbb{R}$ of a semidiscrete evolution equation is dominated if there exists a nonempty subset $\Omega \subset \mathbb{R}^N$ consisting of parameters of the equation such that for each choice of parameters from Ω there exist functions $T : [0, \infty) \to \mathbb{R}$ and $X : \mathbb{Z} \to \mathbb{R}$ depending on such parameters so that we can write $u(t, n) = w(t, n)T(t)X(n)$ where $w(t, n)$ is the solution of either the semidiscrete heat, wave or Laplace equations.

In the following semidiscrete version of the advection-diffusion equation we assume the one-step backward Euler discretization for the spatial derivative operator given by

$$\frac{\partial u(t,x)}{\partial x} \sim u(t,n) - u(t,n-1).$$

Theorem 3.2. *The solution of the semidiscrete advection-diffusion equation*

$$u_t(t,n) - \alpha\Delta_d u(t,n) + \beta[u(t,n) - u(t,n-1)] = 0 \tag{22}$$

with initial condition $u(0, n) = \varphi(n)$ *is dominated by the semidiscrete heat equation*

$$w_t(t,n) = \sqrt{\alpha(\alpha+\beta)}\Delta_d w(t,n), \quad w(0,n) = \Big(\frac{\alpha+\beta}{\alpha}\Big)^{-n/2}\varphi(n) \tag{23}$$

whenever $(\alpha, \beta) \in \Omega := (0, \infty) \times (0, \infty)$.

Proof. Let assume a solution u for equation (22) has the form

$$u(t,n) := w(t,n)T(t)X(n)$$

for a suitable choice of w, T and X. First, a computation shows that:

$$\begin{aligned}\Delta_d u(t,n) = [&\Delta_d w(t,n)X(n+1) + 2(w(t,n) - w(t,n-1))(X(n+1) - X(n)) \\ &+ w(t,n-1)(\Delta_d X)(n)]T(t)\end{aligned} \tag{24}$$

and

$$u_t(t,n) = w_t(t,n)T(t)X(n) + w(t,n)T'(t)X(n). \tag{25}$$

Now, substituting $u(t, n)$ in equation (22) we get

$$\begin{aligned} 0 &= u_t(t,n) - \alpha\Delta_d u(t,n) + \beta[u(t,n) - u(t,n-1)] \\ &= w_t(t,n)T(t)X(n) + w(t,n)T'(t)X(n) - \alpha\Delta_d w(t,n)X(n+1)T(t) \\ &\quad -2\alpha(w(t,n) - w(t,n-1))(X(n+1) - X(n))T(t) \\ &\quad -\alpha w(t,n-1)(\Delta_d X)(n)T(t) \\ &\quad +\beta w(t,n)X(n)T(t) - \beta w(t,n-1)X(n-1)T(t). \end{aligned} \tag{26}$$

A reordering in (26) leads to

$$\begin{aligned} 0 &= u_t(t,n) - \alpha\Delta_d u(t,n) + \beta[u(t,n) - u(t,n-1)] \\ &= [w_t(t,n)X(n) - \alpha\Delta_d w(t,n)X(n+1)]T(t) \\ &\quad + w(t,n)[-2\alpha(X(n+1) - X(n))T(t) + T'(t)X(n) + \beta X(n)T(t)] \\ &\quad + w(t,n-1)[2\alpha(X(n+1) - X(n)) - \alpha(X(n+1) \end{aligned} \tag{27}$$

$$- 2X(n) + X(n-1))$$
$$- \beta X(n-1)]T(t).$$

We now look for a function $X : \mathbb{Z} \to \mathbb{R}$ that solves the difference equation:

$$2\alpha(X(n+1) - X(n)) - \alpha(X(n+1) - 2X(n) + X(n-1)) - \beta X(n-1) = 0$$

where we obtain

$$X(n) = \Big(\frac{\alpha+\beta}{\alpha}\Big)^{n/2}. \tag{28}$$

In what follows, denote $\mu := \Big(\frac{\alpha+\beta}{\alpha}\Big)^{1/2}$. We now find a function $T : [0,\infty) \to \mathbb{R}$ that satisfies the equation

$$- 2\alpha(X(n+1) - X(n))T(t) + T'(t)X(n) + \beta X(n)T(t) = 0. \tag{29}$$

Taking into account (28), equation (29) reduces to

$$T'(t) = (2\alpha\mu - 2\alpha - \beta)T(t),$$

whose solution is given by $T(t) = e^{-(2\alpha+\beta-2\alpha\mu)t}$. Finally, if we select $w(t,n)$ as the solution of the semidiscrete heat equation given by

$$w_t(t,n) - \alpha\mu\Delta_d w(t,n) = w_t(t,n) - \sqrt{\alpha(\alpha+\beta)}\Delta_d w(t,n) = 0, \tag{30}$$

it is immediate that $u(t,n) := w(t,n)T(t)X(n)$ solves equation (22). Observe that given the definition of u it follows directly from $w(0,n) = \Big(\frac{\alpha+\beta}{\alpha}\Big)^{n/2}\varphi(n)$ that $u(0,n) = \varphi(n)$ and the conclusion holds. □

Remark 3.3. From the proof, we have $T(t) = e^{-\gamma t}$ and $X(n) = \mu^n$ where $\gamma := 2\alpha + \beta - 2\alpha\mu$ obtaining the explicit solution

$$u(t,n) = \sum_{m\in\mathbb{Z}} e^{-\gamma t} T_{at}(n-m)\mu^{n-m}\varphi(m), \quad n \in \mathbb{Z},\ t \geq 0, \tag{31}$$

with $a := \sqrt{\alpha(\alpha+\beta)}$. In case $\varphi(n) = \delta_0(n)$ we obtain the fundamental solution

$$u(t,n) = e^{-(2\alpha+\beta)t} I_n(2\sqrt{\alpha(\alpha+\beta)}t)\Big(\frac{\alpha+\beta}{\alpha}\Big)^{n/2}, \quad n \in \mathbb{Z},\ t \geq 0, \tag{32}$$

see Figure 1. We point out that (32) is a special case of the fundamental solution obtained in Example 3.1 of [26].

Remark 3.4. From [16] we know that the fundamental solution of the advection-diffusion equation

$$u_t(t,x) - \alpha u_{xx}(t,x) + \beta u_x(t,x) = 0, \tag{33}$$

is given by

$$u(t,x) = \frac{1}{\sqrt{4\pi\alpha t}} e^{-\frac{(x-\beta t)^2}{4\alpha t}}.$$

To compare with (32) we observe that we can rewrite $u(t,x)$ as

$$u(t,x) = \Big(\frac{1}{\sqrt{4\pi\alpha t}} e^{-\frac{x^2}{4\alpha t}}\Big)\Big(e^{\frac{x}{2}\frac{\beta}{\alpha}}\Big)\Big(e^{-\frac{\beta^2}{4\alpha}t}\Big). \tag{34}$$

Then, we note that the first term is the fundamental solution of the heat equation which plays the same role as the fundamental solution of the semidiscrete heat equation (23) given by $e^{-2t\sqrt{\alpha(\alpha+\beta)}} I_n(2\sqrt{\alpha(\alpha+\beta)}t)$. The second term is the

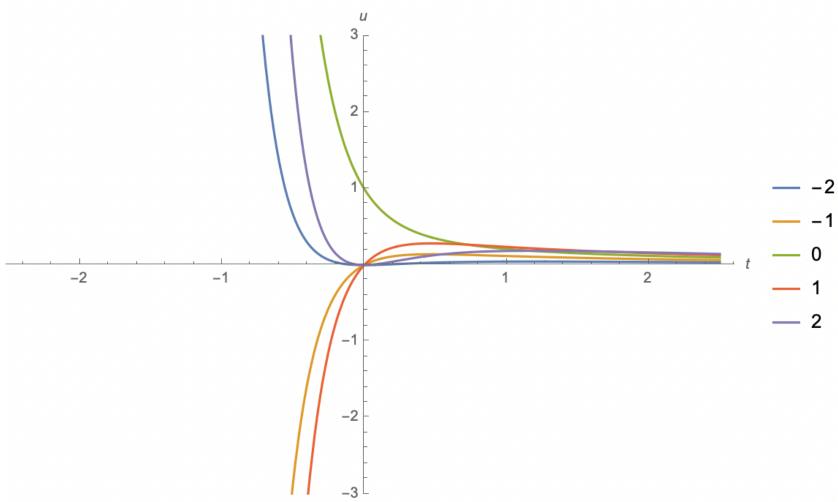

FIGURE 1. The fundamental solution of (22) for $\alpha = \beta = 1$ and $n = -2, ..., 2$

continuous analogue of $(1 + \frac{\beta}{\alpha})^{n/2}$. Finally, the third term corresponds to $e^{-(2\alpha+\beta-2\sqrt{\alpha(\alpha+\beta)})t}$ in the discrete fundamental solution.

Remark 3.5. We point out that if we apply either the forward Euler or the centered discretization for the spatial derivative operator to the advection-diffusion equation we obtain the solution of the corresponding semidiscrete advection-diffusion equation is never dominated, i.e., $\Omega = \emptyset$.

In our next example we consider the semidiscrete linear model

$$u_t(t,n) = du(t,n+1) + au(t,n) + bu(t,n-1), \quad n \in \mathbb{Z}, \quad t \geq 0, \tag{35}$$

$a, b, d \in \mathbb{R}$, which appears when modelling chemical turbulence [19] and biological processes [2, 20]. An explicit solution for this model was previously obtained under different techniques in [26].

A computation shows the above model can be rewritten as a diffusion equation with finite delay:

$$u_t(t,n) = d\Delta_d u(t,n) + (2d+a)u(t,n) + (b-d)u(t,n-1) \tag{36}$$

Our method provides an explicit solution for this model as we prove in the following result.

Theorem 3.6. *The solution of the semidiscrete model*

$$u_t(t,n) = \alpha\Delta_d u(t,n) + \beta u(t,n) + \gamma u(t,n-1) \tag{37}$$

with initial condition $u(0,n) = \varphi(n)$ *is dominated by the semidiscrete heat equation*

$$w_t(t,n) = \sqrt{\alpha(\alpha+\gamma)}\Delta_d w(t,n), \quad w(0,n) = \Big(\frac{\alpha+\gamma}{\alpha}\Big)^{-n/2}\varphi(n) \tag{38}$$

whenever $(\alpha,\beta,\gamma)\in\Omega:=(0,\infty)\times\mathbb{R}\times[-\alpha,\infty)$*. Moreover the explicit solution of* (37) *is given by*

$$u(t,n)=\sum_{m\in\mathbb{Z}}e^{-(2\alpha-\beta-2\sqrt{\alpha(\alpha+\gamma)})t}T_{t\sqrt{\alpha(\alpha+\gamma)}}(n-m)\Big(\frac{\alpha+\gamma}{\alpha}\Big)^{\frac{n-m}{2}}\varphi(m),n\in\mathbb{Z},\ t\geq 0. \tag{39}$$

Proof. Observe that except for the different parameters γ and β, the structure of equation (37) is similar to that of (22) and hence the proof follows the same steps. Therefore, we omit it. □

As an immediate consequence of Theorem 3.6 and (36) we can provide an explicit solution for the standard birth-and-death type population model with proliferation (35) where a,b and d represent the growth rates. It is immediate this model labels into (37) with $\alpha=d,\ \beta=+(2d+a),\gamma=b-d$.

Corollary 3.7. *The fundamental solution for* (35) *in case* $b,d>0$ *is given by*

$$u(t,n):=e^{at}I_n(2t\sqrt{db})\Big(\frac{b}{d}\Big)^{n/2},\quad n\in\mathbb{Z},\quad t\geq 0. \tag{40}$$

We note that the behavior of the solution in this case is similar to that of Figure 1.

Remark 3.8. Since the fundamental solution of the semidiscrete heat equation given by (38) is known to decay exponentially, it is sufficient in order to determine the exponential decay in t of the function (40) that the condition

$$4d+a-2\sqrt{bd}>0$$

holds.

Model (35) has been previously studied in the literature. In [31] it was found in the so-called sub-critical case, i.e., when $0<b<d$ that $\sum_{n=1}^{\infty}u(t,n)$ decays exponentially in t to 0 if and only if $a+2\sqrt{bd}\leq 0$. In [2, Theorem 1] the authors analysed this model on a weighted $\ell^p_s(\mathbb{N})$ space and they showed the semigroup solution was exponentially stable if and only if $s\in(\sigma_-,\sigma_+)$ where $\sigma_\pm=\frac{-a\pm\sqrt{a^2-4bd}}{2d}$ and the condition $a+2\sqrt{bd}<0$ holds.

If we take $d=b=\frac{r}{2}$ and $a=1-r$ with $0<r<1$ in (35), this model was previously considered in [19] for describing cyclic chemical oscillation where r represents the strength of diffusion. From Theorem 3.6, we can obtain the following corollary.

Corollary 3.9. *For any* $0<r<1$*, the fundamental solution for the equation*

$$u_t(t,n)=\frac{r}{2}u(t,n+1)+(1-r)u(t,n)+\frac{r}{2}u(t,n-1),\quad n\in\mathbb{Z},\quad t\geq 0,$$

is given by

$$u(t,n):=e^{(1-r)t}I_n(rt),\quad n\in\mathbb{Z},\quad t\geq 0.$$

We will now consider a semidiscrete version for the Lighthill-Whitham-Richards equation

$$u_t(t,x)+cu_x(t,x)+\beta u_{tt}(t,x)-\alpha u_{xx}(t,x)=0 \tag{41}$$

which arises in traffic modelling for describing traffic at macroscopic level [5, 8]. Here, $\beta>0$ represents the inertial time constant for speed variation, $c>0$ is the wave speed, and $\alpha > 0$ is the diffusion coefficient that shows how drivers respond to changes far away from their position.

Theorem 3.10. *The solution of the semidiscrete Lighthill-Whitham-Richards equation given by*

$$u_t(t,n) + c[u(t,n) - u(t,n-1)] + \beta u_{tt}(t,n) - \alpha\Delta_d u(t,n) = 0 \tag{42}$$

with initial conditions $u(0,n) = \varphi(n)$ *and* $u_t(0,n) = \psi(n)$ *is dominated by the semidiscrete equation*

$$w_{tt}(t,n) = \frac{\mu\alpha}{\beta}\Delta_d w(t,n), \quad w(0,n) = \mu^{-n}\varphi(n) \quad \text{and} \quad w_t(0,n) = \mu^{-n}\psi(n) \tag{43}$$

with $\mu := \frac{c}{2\alpha} + 1 - \frac{1}{8\alpha\beta}$ *whenever*

$$(\beta, c, \alpha) \in \Omega := \{(x,y,z) \in (0,\infty)\times(0,\infty)\times(0,\infty) : (4xy-1)^2 - 16xz = 0\}.$$

Proof. Let assume that $u(t,n) := w(t,n)T(t)X(n)$ for a suitable choice of w, T and X solves equation (42). Taking into account (24), (25) and the fact that:

$$u_{tt}(t,n) = w_{tt}(t,n)T(t)X(n) + 2w_t(t,n)T'(t)X(n) + w(t,n)T''(t)X(n) \tag{44}$$

we replace $u(t,n)$ in equation (42) obtaining:

$$\begin{aligned} 0 &= u_t(t,n) + c[u(t,n) - u(t,n-1)] + \beta u_{tt}(t,n) - \alpha\Delta_d u(t,n) \\ &= w_t(t,n)T(t)X(n) + w(t,n)T'(t)X(n) + cw(t,n)T(t)X(n) \\ &\quad - cw(t,n-1)T(t)X(n-1) + \beta w_{tt}(t,n)T(t)X(n) + 2\beta w_t(t,n)T'(t)X(n) \\ &\quad + \beta w(t,n)T''(t)X(n) - \alpha\Delta_d w(t,n)X(n+1)T(t) \\ &\quad - 2\alpha(w(t,n) - w(t,n-1))(X(n+1) - X(n))T(t) \\ &\quad - \alpha w(t,n-1)(\Delta_d X)(n)T(t). \end{aligned} \tag{45}$$

An appropriate regrouping of the terms in (45) leads to

$$\begin{aligned} 0 &= u_t(t,n) + c[u(t,n) - u(t,n-1)] + \beta u_{tt}(t,n) - \alpha\Delta_d u(t,n) \\ &= [\beta w_{tt}(t,n)X(n) - \alpha\Delta_d w(t,n)X(n+1)]T(t) \\ &\quad + w(t,n)[-2\alpha(X(n+1) - X(n))T(t) + T'(t)X(n) + cX(n)T(t) \\ &\quad + \beta T''(t)X(n)] \\ &\quad + w_t(t,n)[T(t) + 2\beta T'(t)] \\ &\quad + w(t,n-1)[2\alpha(X(n+1) - X(n)) - \alpha(X(n+1) \\ &\quad - 2X(n) + X(n-1)) \\ &\quad - cX(n-1)]T(t). \end{aligned}$$

We now look for a function $T : [0,\infty) \to \mathbb{R}$ that satisfies the differential equation

$$T(t) + 2\beta T'(t) = 0.$$

Then we get $T(t) := e^{-\frac{1}{2\beta}t}$. We now obtain a function $X : \mathbb{Z} \to \mathbb{R}$ that solves the system of difference equations given by:

$$\begin{aligned} &-2\alpha(X(n+1) - X(n))T(t) + T'(t)X(n) + cX(n)T(t) + \beta T''(t)X(n) \\ &= 0, \\ &2\alpha(X(n+1) - X(n)) - \alpha(X(n+1) - 2X(n) + X(n-1)) - cX(n-1) \\ &= 0 \end{aligned} \tag{46}$$

Taking now into consideration the choice of T, system (46) reduces to:

$$\begin{aligned}\Big(c+2\alpha-\frac{1}{4\beta}\Big)X(n)-2\alpha X(n+1) &= 0, \\ \alpha X(n+1)-(c+\alpha)X(n-1) &= 0,\end{aligned} \tag{47}$$

whose unique solution is given by

$$X(n)=\Big(\frac{c}{2\alpha}+1-\frac{1}{8\alpha\beta}\Big)^{n}=\Big(\frac{c}{\alpha}+1\Big)^{n/2} \tag{48}$$

where the last equality holds due to the fact that $(\beta, c, \alpha)\in\Omega$. Finally, if we select $w(t,n)$ as the solution of the semidiscrete equation given by

$$\beta w_{tt}(t,n)-\alpha\Big(\frac{c}{2\alpha}+1-\frac{1}{8\alpha\beta}\Big)\Delta_d w(t,n)=0, \tag{49}$$

or equivalenty,

$$w_{tt}(t,n)=\frac{\alpha}{\beta}\Big(\frac{c}{2\alpha}+1-\frac{1}{8\alpha\beta}\Big)\Delta_d w(t,n), \tag{50}$$

it is immediate that $u(t,n):=w(t,n)T(t)X(n)$ is a solution of equation (42). Also, a computation shows that $w(0,n)=\mu^{-n}\varphi(n)$ and $w_t(0,n)=\mu^{-n}\psi(n)$ implies $u(0,n)=\varphi(n)$ and $u_t(0,n)=\psi(n)$ and then the conclusion holds. □

Corollary 3.11. *Assume $(\beta, c, \alpha)\in\Omega$. If $c\beta>\frac{1}{4}$ equation (50) corresponds to the semidiscrete wave equation and the solution of (42) is given by*

$$u(t,n)=\sum_{m\in\mathbb{Z}}e^{-\gamma t}C_{at}(n-m)\mu^{n-m}\varphi(m)+\sum_{m\in\mathbb{Z}}e^{-\gamma t}S_{at}(n-m)\mu^{n-m}\psi(m). \tag{51}$$

If $c\beta<\frac{1}{4}$ equation (50) corresponds to the semidiscrete Laplace equation and the solution of (42) is given by

$$u(t,n)=\sum_{m\in\mathbb{Z}}e^{-\gamma t}L_{at}(n-m)\mu^{n-m}\varphi(m)+\sum_{m\in\mathbb{Z}}e^{-\gamma t}M_{at}(n-m)\mu^{n-m}\psi(m) \tag{52}$$

where $\gamma:=\frac{1}{2\beta}$, $a:=\frac{\alpha}{\beta}\Big(\frac{c}{2\alpha}+1-\frac{1}{8\alpha\beta}\Big)$ and $\mu:=\Big(\frac{c}{\alpha}+1\Big)^{1/2}$.

Proof. Since $(\beta, c, \alpha)\in\Omega$ it follows that $(4c\beta-1)^2-16\alpha\beta=0$. Then, the coefficient of equation (50) given by

$$\frac{1}{8\beta^2}(4c\beta+8\alpha\beta-1)=\frac{1}{8\beta^2}(4c\beta-1)(\frac{1}{2}+2c\beta).$$

Consequently, this coefficient will be positive if $c\beta>\frac{1}{4}$ and negative if $c\beta<\frac{1}{4}$ and the conclusion holds. □

Remark 3.12. Setting $\varphi=\delta_0$ and $\psi\equiv 0$ in Theorem 3.10 with α, β, c satisfying

$$(4\beta c-1)^2-16\beta\alpha=0,$$

we obtain if $c\beta>1/4$ the fundamental solution

$$u(t,n)=e^{-\frac{1}{2\beta}t}J_{2n}\Big(t\Big(\frac{4c\beta+8\alpha\beta-1}{8\beta^2}\Big)\Big)\Big(\frac{c}{\alpha}+1\Big)^{n/2} \tag{53}$$

and, if $c\beta<1/4$ the fundamental solution

$$u(t,n)=e^{-\frac{1}{2\beta}t}(-1)^n I_{2n}\Big(t\Big(\frac{4c\beta+8\alpha\beta-1}{8\beta^2}\Big)\Big)\Big(\frac{c}{\alpha}+1\Big)^{n/2} \tag{54}$$

respectively.

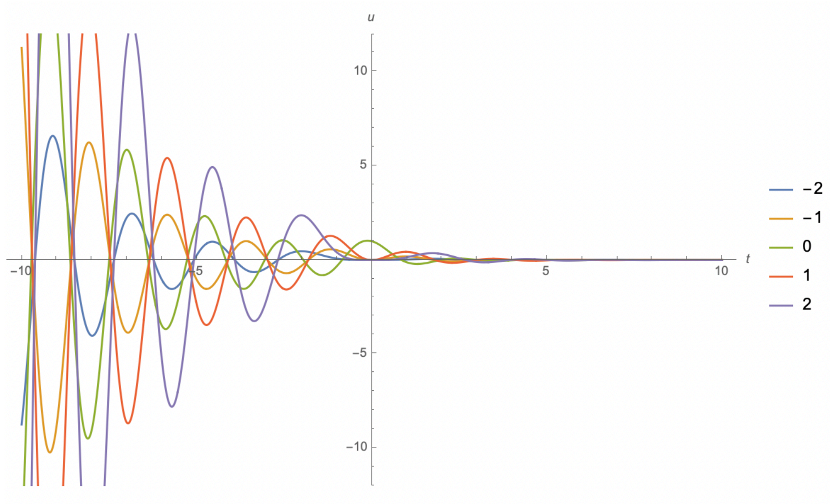

FIGURE 2. The fundamental solution of (42) for $\alpha = \beta = 1$ and $c = 5/4$ with $n = -2, ..., 2$

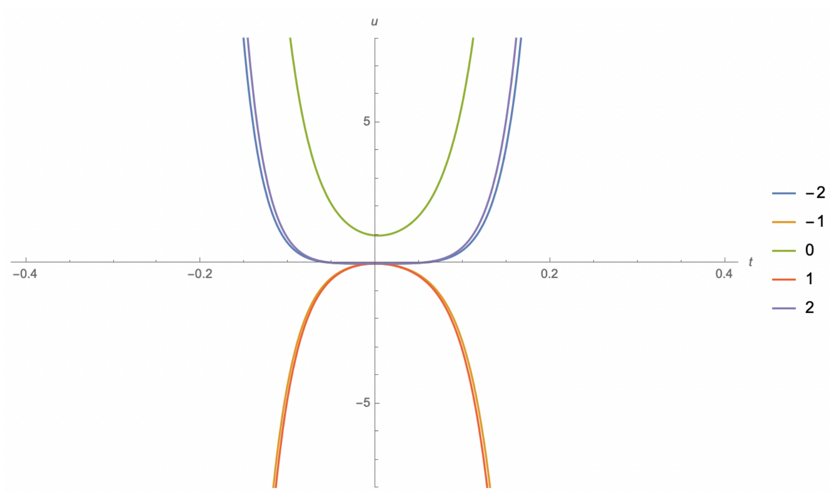

FIGURE 3. The fundamental solution of (42) for $\alpha = 9/4$, $\beta = 1/4$ and $c = 1/4$ with $n = -2, ..., 2$

We now pay special attention to the case $c = 0$. In this case, equation (41) reduces to the hyperbolic heat transfer equation given by

$$u_t(t, x) + \beta u_{tt}(t, x) - \alpha u_{xx}(t, x) = 0 \tag{55}$$

which has been extensively studied in the literature [9]. Here $\beta > 0$ denotes the thermal relaxation time and $\alpha > 0$ represents the thermal diffusivity. As a corollary of Theorem 3.10 we immediately obtain the following result.

Corollary 3.13. *The solution of the semidiscrete hyperbolic heat transfer equation*

$$u_t(t, n) + \beta u_{tt}(t, n) - \alpha \Delta_d u(t, n) = 0 \tag{56}$$

with initial conditions $u(0,n) = \varphi(n)$ *and* $u_t(0,n) = \psi(n)$ *is dominated by the semidiscrete Laplace equation*

$$w_{tt}(t,n) = -\frac{\alpha}{\beta}\Delta_d w(t,n), \quad w(0,n) = \mu^{-n}\varphi(n) \quad \textit{and} \quad w_t(0,n) = \mu^{-n}\psi(n) \tag{57}$$

whenever

$$(\alpha,\beta) \in \Omega := \{(x,y) \in (0,\infty)\times(0,\infty) : 16xy = 1\}$$

and the explicit solution of (56) *is given by*

$$u(t,n) = \sum_{m\in\mathbb{Z}} e^{-\frac{t}{2\beta}} L_{\frac{\alpha}{\beta}t}(n-m)(-1)^{n-m}\varphi(m) + \sum_{m\in\mathbb{Z}} e^{-\frac{t}{2\beta}} M_{\frac{\alpha}{\beta}t}(n-m)(-1)^{n-m}\psi(m). \tag{58}$$

Proof. Observe the proof follows the same steps as the one of Theorem 3.10 until Formula (48). We take $T(t) = e^{\frac{-1}{2\beta}t}$ and we arrive to

$$X(n) = \Big(1 - \frac{1}{8\alpha\beta}\Big)^n = (-1)^n \tag{59}$$

where the last equality holds due to the fact that $(\alpha,\beta) \in \Omega$. Finally, if we select $w(t,n)$ as the solution of the semidiscrete Laplace equation given by

$$\beta w_{tt}(t,n) - \alpha\Big(1 - \frac{1}{8\alpha\beta}\Big)\Delta_d w(t,n) = \beta w_{tt}(t,n) + \alpha\Delta_d w(t,n) = 0, \tag{60}$$

it is immediate that (52) is a solution of equation (56). $\square$

Remark 3.14. It is interesting to observe that there exists a set Ω, described in Corollary 3.13, where the behavior of the solution of equation (56) can be described by means of the solution of the semidiscrete Laplace equation instead of the semidiscrete heat equation as one could expect. In Figure 4 we note that the profile of the solution, for $t > 0$, is consistent with the model, i.e., the solution decays in time as $t \to \infty$.

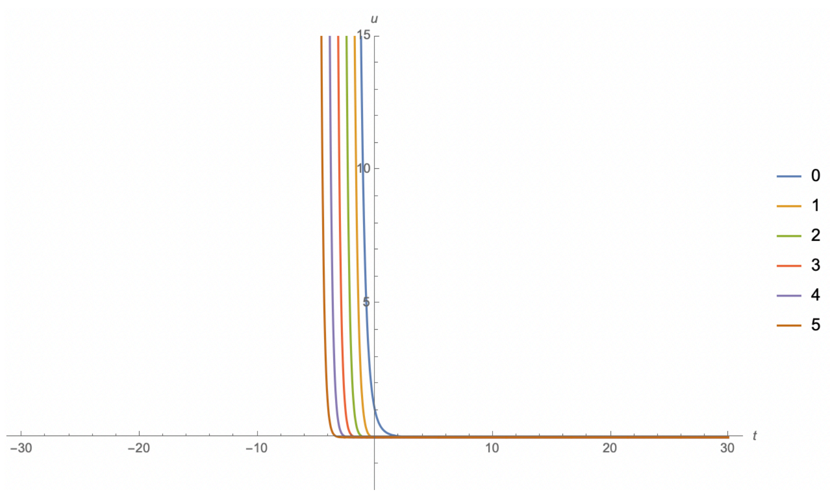

FIGURE 4. The fundamental solution of (56) for $\alpha = \beta = 1/4$ and $c = 0$ with $n = 0, ..., 5$

4. **ℓ^p-regularity.** We start this section studying spatial regularity for all the models analysed in the previous Section.

Theorem 4.1. *Let* $(\alpha, \beta, \gamma) \in \Omega := (0, \infty) \times \mathbb{R} \times [-\alpha, \infty)$ *and suppose* $2\alpha - \beta - 2\sqrt{\alpha(\alpha+\gamma)} > 0$. *For any* $\varphi \in \ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$, *the solution of* (37) *with initial condition* $u(0, n) = \varphi(n)$ *satisfies* $u(t, \cdot) \in \ell^p(\mathbb{Z})$ *for all* $t \geq 0$.

Proof. We observe that the structure of the solution of (37) has the form

$$u(t, n) = \sum_{m\in\mathbb{Z}} K_t(n-m)\mu^{n-m}\varphi(m), \quad n \in \mathbb{Z},\ t \geq 0 \tag{61}$$

where $K_t(n) := e^{-\omega t} T_{at}(n)$, $\omega := 2\alpha - \beta - 2\sqrt{\alpha(\alpha+\gamma)}, a := \sqrt{\alpha(\alpha+\gamma)}$ and $\mu := \left(\frac{\alpha+\gamma}{\alpha}\right)^{1/2}$.

Let $t \geq 0$ be fixed. In order to show that $u(t, \cdot) \in \ell^p(\mathbb{Z})$ it is sufficient to check that the sequence $\mu^n K_t(n)$ belongs to $\ell^1(\mathbb{Z})$ and then the result follows from an application of Young's inequality for the convolution.

Since $\omega > 0$ by hypothesis, we only need to verify that $\sum_{n\in\mathbb{Z}} |\mu^n T_{at}(n)| < \infty$.

Indeed, by (5), (15), using the properties $I_k(z) = I_{-k}(z)$, $J_n(iz) = (i)^n I_n(z)$ and the fact that $\mu > 1$ we obtain the estimates

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |\mu^n T_{at}(n)| &= \sum_{n\in\mathbb{Z}} \mu^n e^{-2at} |I_n(2at)| \\
&= \sum_{n=0}^{\infty} \mu^n e^{-2at} |I_n(2at)| + \sum_{n=1}^{\infty} \mu^{-n} e^{-2at} |I_n(2at)| \\
&\leq \sum_{n=0}^{\infty} \mu^n e^{-2at} |J_n(2ait)| + \sum_{n=1}^{\infty} e^{-2at} |J_n(2ait)| \\
&\leq \sum_{n=0}^{\infty} \mu^n \frac{(at)^n}{n!} + \sum_{n=1}^{\infty} \frac{(at)^n}{n!} = e^{\mu at} + e^{at} - 1,
\end{aligned}$$

proving the claim. □

With an analogous proof we obtain the corresponding result for model (22).

Theorem 4.2. *Let* $(\alpha, \beta) \in \Omega := (0, \infty) \times (0, \infty)$ *and suppose* $2\alpha - \beta - 2\sqrt{\alpha(\alpha+\beta)} > 0$. *For any* $\varphi \in \ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$, *the solution of* (22) *with initial condition* $u(0, n) = \varphi(n)$ *satisfies* $u(t, \cdot) \in \ell^p(\mathbb{Z})$ *for all* $t \geq 0$.

We next prove time regularity for the semidiscrete Lighthill-Whitham-Richards equation.

Theorem 4.3. *Let* $(\beta, c, \alpha) \in \Omega := \{(x, y, z) \in (0, \infty) \times (0, \infty) \times (0, \infty) : (4xy - 1)^2 - 16xz = 0\}$. *For any* $\varphi, \psi \in \ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$, *the solution of* (42) *with initial conditions* $u(0, n) = \varphi(n)$, $u_t(0, n) = \psi(n)$ *satisfies* $u(t, \cdot) \in \ell^p(\mathbb{Z})$ *for all* $t \geq 0$.

Proof. Let $t \geq 0$ be fixed. We note that the structure of the solutions of this model carries out expressions of the form (61) where the kernel K_t can be either $e^{\frac{-t}{2\beta}} C_{at}, e^{\frac{-t}{2\beta}} S_{at}, e^{\frac{-t}{2\beta}} L_{at}$ or $e^{\frac{-t}{2\beta}} M_{at}$, where $a := \frac{4c\beta+8\alpha\beta-1}{8\beta^2}$. Therefore, it is enough to prove that the sequences $\mu^n K_t(n)$ belong to $\ell^1(\mathbb{Z})$ in each case.

Since $\mu = (\frac{c}{\alpha}+1)^{1/2} > 1$ we can proceed as in the proof of the above theorem to obtain

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\mu^n C_{at}(n)| &= \sum_{n\in\mathbb{Z}} \mu^n |J_{2n}(2at)| = \sum_{n=0}^{\infty} \mu^n |J_{2n}(2at)| + \sum_{n=1}^{\infty} \mu^{-n} |J_{2n}(2at)| \\ &\le \sum_{n=0}^{\infty} \mu^n |J_{2n}(2at)| + \sum_{n=1}^{\infty} |J_{2n}(2at)| \\ &\le \sum_{n=0}^{\infty} \mu^n \frac{(|a|t)^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{(|a|t)^{2n}}{(2n)!} \\ &\le \sum_{n=0}^{\infty} \frac{(\mu^{1/2}|a|t)^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{(|a|t)^{2n}}{(2n)!} \\ &= \cosh(\mu|a|^2t^2) + \cosh(|a|^2t^2) - 1.\end{aligned}$$

The same proof applies to $S_{at}(n)$ by definition (10). For $L_{at}(n) := J_{2n}(2it)$ and M_{at} as in (19) the proof is analogous and we omit it. □

We next study time regularity. More precisely, we analyse ℓ^p-maximal regularity of the non-homogeneous version for the general model (37). Our methods are based on the Dore-Venni theorem [18, Theorem 8.4.4].

We first recall that an operator A is said to be non-negative if $(-\infty,0) \subset \rho(A)$ and $\sup_{\lambda>0} \|\lambda(\lambda+A)^{-1}\| < \infty$. If A is non-negative and $0 \in \rho(A)$ the operator A is said to be positive, see [18, Definition 1.1.2].

Let X be a complex Banach space. Given $\theta_A > 0$, we say that an injective and sectorial operator A belongs to the class $BIP(X,\theta_A)$ if $\{A^{it}\}_{t\in\mathbb{R}} \subset \mathcal{B}(X)$ and there is a positive constant C such that

$$\|A^{it}\| \le Ce^{\theta|t|}, \tag{62}$$

see [18, Definition 8.1.1]. The number θ_A is the infimum of all θ such that (62) holds.

Our result is the following.

Theorem 4.4. *Let $(\alpha,\beta,\gamma) \in \Omega := (0,\infty)\times\mathbb{R}\times[-\alpha,\infty)$. Assume*

$$\beta+\gamma < -4\alpha. \tag{63}$$

Then, for any $1<p,q<\infty$, and $f \in L^p(\mathbb{R}_+,\ell^q(\mathbb{Z}))$, the solution of the semidiscrete model

$$u_t(t,n) = \alpha\Delta_d u(t,n) + \beta u(t,n) + \gamma u(t,n-1) + f(t,n) \tag{64}$$

with initial condition $u(0,n)=0$ belongs to

$$L^p(\mathbb{R}_+,\ell^q(\mathbb{Z})) := \{v:\mathbb{R}_+ \to \ell^q(\mathbb{Z}) : \left(\int (\sum_{n\in\mathbb{Z}} |v(t,n)|^q)^{p/q}\right)^{1/p} < \infty\}$$

and satisfies the maximal regularity estimate

$$\|u\|_{L^p(\mathbb{R}_+,\ell^q(\mathbb{Z}))} + \|u_t\|_{L^p(\mathbb{R}_+,\ell^q(\mathbb{Z}))} + \|Bu\|_{L^p(\mathbb{R}_+,\ell^q(\mathbb{Z}))} \le C\|f\|_{L^p(\mathbb{R}_+,\ell^q(\mathbb{Z}))} \tag{65}$$

where $B := \alpha\Delta_d + \beta I + \gamma F$ and F is the forward shift operator.

Proof. We will use [18, Theorem 8.5.2, p.215]. Since $X = \ell^q(\mathbb{Z})$ is a UMD space for $1<q<\infty$, we have to verify two hypothesis: (i) $-B$ is a positive operator and (ii) $-B \in BIP(\ell^q;\theta_B)$ with $0<\theta_B<\pi/2$.

We prove (i): Since $\sigma(\Delta_d) = [-4, 0]$ (see [17, Theorem 1.1 (iii)]) and $\sigma(F) = \mathbb{T}$, the torus (see [10]), we obtain that

$$\sigma(B) \subset \alpha\sigma(\Delta_d) + \beta\sigma(I) + \gamma\sigma(F) \subset [-4\alpha + \beta - \gamma, \beta + \gamma] \times [-1, 1], \tag{66}$$

where the first inclusion follows form the fact that $\ell^q(\mathbb{Z})$ is a Banach algebra with unity [4, Chapter 7] and the operators Δ_d and F commute.

By hypothesis (63) we have that $\beta + \gamma < 0$. In particular, (66) implies that $[0, \infty) \subset \rho(B)$, i.e. $(-\infty, 0] \subset \rho(-B)$ and hence the first condition to be $-B$ positive is satisfied [18, Definition 1.1.1]. It remains to show that

$$\sup_{\lambda>0} \|\lambda(\lambda - B)^{-1}\| < \infty. \tag{67}$$

Since B is a bounded operator on $\ell^q(\mathbb{Z})$, the identity $\lambda(\lambda - B)^{-1} = I + B(\lambda - B)^{-1}$ and $0 \in \rho(B)$ imply that it is enough to prove

$$\sup_{\lambda>0} \|(\lambda - B)^{-1}\| < \infty, \tag{68}$$

which follows from the identity $(\lambda - B)^{-1} = \int_0^\infty e^{Bt}e^{-\lambda t}dt$ (see e.g. [1, Theorem 3.1.7]), the continuity of $\lambda \to (\lambda - B)^{-1}$, and the estimate

$$\|(\lambda - B)^{-1}\| \leq \frac{1}{\lambda - \|B\|} < \frac{1}{\|B\|}, \quad \text{for} \quad \lambda > 2\|B\|. \tag{69}$$

We prove (ii): Note that for any bounded and non-negative operator D with $0 \in \rho(D)$ we have that $D^{i\tau}$ is bounded and there exists a constant $C_D > 0$ such that $\|D^{i\tau}\| \leq C_D e^{\pi|\tau|}$, $\tau \in \mathbb{R}$, see [18, p.172]. In particular, $\theta_D < \pi$.

Since by hypothesis (63) we have $i\mathbb{R}_+ \subset \rho(B)$ it follows that $(-\infty, 0] \subset \rho(B^2)$. Also, the fact that $\sup_{\lambda>0} \|\lambda(\lambda + B^2)^{-1}\| < \infty$ can be proved analogously as (67).

It implies that B^2 is non-negative with $0 \in \rho(B^2)$ and consequently with $D = B^2$ we obtain $\|(B^2)^{i\tau}\| \leq C_{B^2}e^{\pi|\tau|}, \tau \in \mathbb{R}$, that is $\|B^{is}\| \leq C_{B^2}e^{\frac{\pi}{2}|s|}$, $s \in \mathbb{R}$. Hence $\theta_B < \pi/2$. □

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