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On the existence of chaos for the fourth-order Moore–Gibson–Thompson equation

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ABSTRACT

We analyze the existence of chaos for the fourth-order Moore–Gibson–Thompson equation. We obtain sufficient conditions on the parameters of the equation so that it exhibits a chaotic behavior in the Devaney sense. Such dynamic behavior is achieved in Herzog-like spaces revealing the structure of critical parameters.

1. Introduction

Our concern in this article is to answer an open question about the existence of sufficient conditions on the parameters of the so-named fourth-order Moore–Gibson–Thompson equation

$$\tau\frac{\partial^4 u}{\partial t^4}(t,x)+\alpha\frac{\partial^3 u}{\partial t^3}(t,x)+\beta\frac{\partial^2 u}{\partial t^2}u(t,x) -\gamma\frac{\partial^4 u}{\partial t^2\partial x^2}(t,x)-\delta\frac{\partial^3 u}{\partial t\partial x^2}(t,x)-\rho\frac{\partial^2 u}{\partial x^2}u(t,x)=0, \tag{1}$$

where $t \geq 0$, $x \in \mathbb{R}$, so that we may have chaotic behavior of the associated semigroup. This model has its origin in the scope of acoustics. A prototypical model consists of the linearized part of the Westervelt equation [1] given by

$$\frac{\delta}{c_0^4}\frac{\partial^3 u}{\partial t^3}(t,x)+\Delta u(t,x)-\frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2}u(t,x)=0, \quad t\geq 0,$$

where u represents the sound pressure, c_0 denotes the small signal sound speed, δ symbolizes the sound diffusivity and Δ is the Laplacian operator. The Moore–Gibson–Thompson (MGT) equation given by

$$\tau\frac{\partial^3 u}{\partial t^3}(t,x)+\frac{\partial^2 u}{\partial t^2}u(t,x)-c^2\Delta\frac{\partial u}{\partial t}u(t,x)-b_0\Delta u(t,x)=0, \quad t\geq 0, \tag{2}$$

where $b_0 = \delta + \tau c^2$, is an expansion of the Westervelt equation, which considers second sound effects and the corresponding thermal relaxation in viscous fluids, see [2–5]. The MGT equation with memory

$$\frac{\partial^3 u}{\partial t^3}(t,x)+a\frac{\partial^2 u}{\partial t^2}u(t,x)-b\Delta\frac{\partial u}{\partial t}u(t,x)-c\Delta u(t,x) +\int_0^t g(t-s)\Delta u(s,x)ds=0, \quad t\geq 0, \tag{3}$$

was analyzed in [6–8]. If $g \neq 0$, the memory term introduces additional dissipation. From a physical perspective, the most significant scenario related to (3) is:

$$g(s)=de^{-\ell s}, \qquad d,\ell>0.$$

Considering the above kernel, model (1) arises from (3) adding $\frac{\partial(3)}{\partial t}+\ell(3)$, i.e., adding the derivative with respect to time of Eq. (3) with ℓ times Eq. (3).

It is worth noting that third and fourth order time derivatives find applications in various research domains. In physics and engineering, it is crucial to consider them when dealing with vibrations, especially when such excitation leads to multi-resonant modes of vibration (see [9]). Additionally, they should be taken into account during transitional phases such as startup and shutdown, take-off and landing, and acceleration and deceleration as indicated in [10]. Fourth order time derivatives also come into play in diverse contexts, such as the evaluation of the kinematic performance of long-dwell mechanisms of linkage type, which are utilized in automated machinery to generate intermittent motions as shown in [11] or in the Taylor series expansion of the Hubble law [12].

The dynamic behavior of (1) began to be studied by Dell'Oro and Pata [6] in Hilbert spaces. They stated that in case $\tau = 1$ the conditions

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$$\chi := \gamma - \frac{\delta}{\alpha} > 0 \quad \text{and} \quad \varpi := \beta - \frac{\rho\alpha}{\delta} > -\lambda_1 \chi \tag{4}$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian, are necessary and sufficient to exhibit exponential stability of the linked semigroup. More recently, the sufficiency of the conditions $\chi > 0$ and $\varpi > 0$, among others, for decay of solutions of Eq. (1) with memory, was recently established in [13].

After the seminal work of Dell'Oro and Pata, Eq. (1) has been studied by Abouelregal et al. [14] as a new model of magneto-thermo-viscoelasticity in which heat waves can travel at limited speeds. In the same line of work, the article [15] used (1) as a theoretical framework to establish a new photothermal model that reveals the thermo-magneto-mechanical properties of semiconductor materials. Well-posedness and general decay of solutions for (1) with a memory term, in the context of a Hilbert space, and using the Faedo–Galerkin method, was analyzed by Liu, Chen and Tu in the Ref. [13]. With an analogous technique, Mesloub et al. [16] studied solvability of the non-local mixed boundary value problem. In the Ref. [17], Murillo studied the well-posedness of (1) in the scale of vector-valued Hölder continuous functions. The method used in such a paper was operator-valued Fourier multipliers. Using the same methodology, well-posedness for the nonhomogeneous equation in the scales of Lebesgue, Besov and Triebel–Lizorkin spaces were characterized in the Ref. [18].

In accordance with a widely accepted viewpoint, chaos and nonlinearity are closely interconnected and it is generally assumed that a linear system exhibits predictable behavior. However, in 1929, Birkhoff [19] provided a linear operator that possessed a crucial aspect of chaos: the presence of a dense orbit, that is, the concept of hypercyclicity. Later, MacLane [20] and Rolewicz [21] observed the same phenomenon in the differentiation operator and linear shifts respectively. It was not until 1991 when G. Godefroy and J.H. Shapiro [22] proposed adopting Devaney's definition of chaos (typically associated with nonlinear systems) as the appropriate definition for linear chaos. The relatively recent discovery of chaos in linear systems can be attributed to the requirement for an infinite-dimensional framework, as Rolewicz highlighted, where hypercyclicity, and consequently linear chaos, can manifest, see also [23]. Consequently, we will consider the notion of Devaney chaos throughout this paper.

We note that in the special case $\tau = 0$ and $\gamma = 0$, model (1) reduces to the classical Moore–Gibson–Thompson equation, which arises in acoustics. For that equation, it is well known that the condition $\varpi > 0$ is sufficient for stability [3,5], while the opposite condition $\varpi < 0$ is sufficient for chaos in the Devaney sense [24]. This provides a complete picture of its dynamic behavior. Another special case is $\tau = \alpha = 0$ which corresponds to the viscous van Wijngaarden–Eringen equation, and which models the acoustic planar propagation in bubbly liquids. The existence of chaos for such an equation has been studied in the Ref. [25].

As intimated in [18], chaos for higher order time derivatives appears in the study of jerk and hyper jerk systems [26,27], among other appealing areas of research. However, aside from analyzing the stability and decay properties for (1) carried out so far, the study of chaos for the linear system (1) remains a wide open problem.

In this article, we give an important step towards solving this question, providing a first answer and new insights into this interesting problem. In passing we solve an open problem proposed in [28, Problem 6.5]. Our main result, Theorem 3.1, ensures that the conditions

$$\chi \leq 0, \qquad \varpi \geq 0, \qquad \alpha \geq 2\sqrt{\beta}, \tag{5}$$

are sufficient to exhibit Devaney chaos of the associated semigroup. Our results confirm that χ is the critical parameter that determines the dynamic behavior of model (1).

It is worth noting that there is a large gap between the sufficient and necessary criteria for chaos; at present any 'if and only if' result about the occurrence of chaos in a general linear dynamical system seems to be far beyond our understanding of this phenomenon.

Our method to solve our problem is based on the Desch–Schappacher–Webb criterion [23] that provides sufficient conditions for Devaney chaos of a C_0-semigroup in a separable Banach space. This criterion will be used in the context of Herzog spaces, which are isometrically isomorphic to the space of vanishing sequences c_0, such that the underlying partial differential operator $\frac{\partial^2}{\partial x^2}$ in (1) remains bounded in such space. Therefore the additional problems of working with an unbounded operator can be avoided. The main difficulty then lies in establishing delicate estimates of the characteristic symbol

$$R_\lambda := \frac{\lambda^4 + \beta\lambda^2 + \alpha\lambda^3}{\gamma\lambda^2 + \delta\lambda + \rho},$$

and providing an exhaustive analysis on sufficient conditions for the location of its zeros within a convenient subset of the complex plane.

This article is organized as follows. In the next section, we give a brief review on the necessary preliminaries. In Section 3, we prove our main general result, Theorem 3.1, and its complementary result, namely Theorem 3.2 (case $\gamma = 0$). We then turn our attention to the situation $\alpha < 2\sqrt{\beta}$ where instead of $\varpi \geq 0$ we discover that the condition

$$|\rho - \beta(2+\gamma)| > \sqrt{\beta}(2\alpha + \delta),$$

together with $\chi \leq 0$ are sufficient to ensure Devaney chaos, see Theorem 3.3. We end this article with a complementary theorem in case $\gamma = 0$ (Theorem 3.4).

2. Preliminaries

We start with the notion of a C_0-semigroup.

Definition 2.1. Let X be a Banach space. The family of operators $\{T_t\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be a C_0-semigroup if the following assertions hold:

- $T_0 = I$,
- $T_{t+s} = T_t \circ T_s$,
- $\lim_{s\to t} T_s x = T_t x$ for all $x \in X$ and $t \geq 0$.

We also recall the operator

$$Ax := \lim_{t\to 0} \frac{1}{t}(T_t x - x),$$

exists on a dense subspace of X; the so-called domain of A which is denoted by $D(A)$. Then $(A, D(A))$ is called the infinitesimal generator of the semigroup.

Accordingly to the Hille–Yosida theorem [29, Theorem 7.4], the solution of the abstract Cauchy problem on X stated as:

$$\begin{cases} \partial_t u(t) = Au(t), \\ u(0) = \varphi \in D(A), \end{cases} \tag{6}$$

can be expressed on terms of the C_0-semigroup $\{T_t\}_{t\geq 0}$ on X generated by the operator A. Furthermore, if $A \in \mathcal{B}(X)$, then the semigroup is *uniformly continuous* and it is given by $T_t = e^{tA} = \sum_{k=0}^{\infty} (tA)^n/n!$ for all $t \geq 0$ (see [30, Ch. I, Prop. 3.5]).

We now provide the notion of Devaney chaos for C_0-semigroups. See also [23] for more information and related notions.

Definition 2.2. A C_0-semigroup $\{T_t\}_{t\geq 0}$ is Devaney chaotic if there exists $x \in X$ such that the set $\{T_t x : t \geq 0\}$ is dense in X and the set of periodic points, that is, those $x \in X$ such that $T_t x = x$ for some $t > 0$, is dense in X.

As it was stated by Banks et al. [23,31], the existence of a vector with dense orbit and the density of the set of periodic points imply the sensitive dependence on the initial conditions.

The following criterion stated in [23, Theorem 7.30] provides sufficient conditions in terms of the generator A and its eigenvectors of a C_0-semigroup to be Devaney chaotic. We also refer the reader to [32] for the original version of it, the so-called Desch–Schappacher–Webb criterion.

Theorem 2.3. *Assume that X is a complex separable Banach space, $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on X whose infinitesimal generator is $(A, D(A))$ and there exists an open connected subset $U \subset \mathbb{C}$ and a weakly holomorphic function $g : V \to X$ such that*

(i) $V \cap i\mathbb{R} \neq \emptyset$,
(ii) $g(\mu) \in \ker(\mu I - A)$ *for every* $\mu \in V$,
(iii) for any $x^* \in X^*$, *if* $\langle g(\mu), x^*\rangle = 0$ *for all* $\mu \in V$, *then* $x^* = 0$.

Then the semigroup $\{T_t\}_{t\geq 0}$ is Devaney chaotic.

Finally, we introduce the Herzog space of analytic functions X_p defined in [33], which is isometrically isomorphic to the Banach space of sequences $c_0(\mathbb{N}_0) := \{s : \mathbb{N}_0 \to \mathbb{C} : |s_n| \to 0 \text{ as } n \to \infty\}$ endowed with the natural norm.

Definition 2.4. Given $p > 0$, the space of analytic functions given by

$$X_p = \left\{ f : \mathbb{R} \to \mathbb{C} \ / \ f(x) = \sum_{n=0}^{\infty} \frac{a_n p^n}{n!} x^n, \quad (a_n)_{n\geq 0} \in c_0(\mathbb{N}_0) \right\}$$

and endowed with the norm $\|f\|_p := \sup_{n\geq 0} |a_n|$ is called the Herzog space.

3. Devaney chaos

In this section, we will study the chaotic behavior of the fourth-order Moore–Gibson–Thompson equation

$$\frac{\partial^4 u}{\partial t^4}(t,x) + \alpha\frac{\partial^3 u}{\partial t^3}(t,x) + \beta\frac{\partial^2 u}{\partial t^2}u(t,x) - \gamma\frac{\partial^4 u}{\partial t^2\partial x^2}(t,x) - \delta\frac{\partial^3 u}{\partial t\partial x^2}(t,x) - \rho\frac{\partial^2 u}{\partial x^2}u(t,x) = 0, \tag{7}$$

in the space of analytic functions of Herzog type. Since the second order differential operator $\frac{\partial}{\partial x^2}$ is a bounded operator on X_p, we can express (7) as a first-order equation on the product space $X := X_p \oplus X_p \oplus X_p \oplus X_p$. Setting $u_1 = u$, $u_2 = \frac{\partial u_1}{\partial t}$, $u_3 = \frac{\partial u_2}{\partial t}$ and $u_4 = \frac{\partial u_3}{\partial t}$ we can pose the following abstract Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \rho\partial_{xx} & \delta\partial_{xx} & \gamma\partial_{xx} - \beta & -\alpha \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}; \\ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \\ u_3(0,x) \\ u_4(0,x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \\ \varphi_4(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{cases} \tag{8}$$

Then, the operator-valued matrix

$$A := \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \rho\partial_{xx} & \delta\partial_{xx} & \gamma\partial_{xx} - \beta & -\alpha \end{pmatrix} \tag{9}$$

defines a bounded operator on X and, consequently, we have that $\{e^{tA}\}_{t\geq 0}$ is the solution C_0-semigroup of (8). See [24, Section 3] for a similar representation for the case of the classical Moore–Gibson–Thompson equation.

We begin with the following general result:

Theorem 3.1. *Let $\alpha, \beta, \delta, \rho, \gamma > 0$. Suppose that*

$$\gamma - \frac{\delta}{\alpha} \leq 0, \qquad \alpha \geq 2\sqrt{\beta}, \qquad \beta\delta - \rho\alpha \geq 0. \tag{10}$$

Then for each p satisfying

$$p^2 > \max\{\beta, r_0^2\}, \quad r_0 := \frac{-(2\alpha+\delta)}{2(2+\gamma)} + \sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}}, \tag{11}$$

the operator A generates a uniformly continuous semigroup which is Devaney chaotic on $X_p \oplus X_p \oplus X_p \oplus X_p$.

Proof. We set $U := \{z \in \mathbb{C} : |z| < r_0\}$. It is clear that condition (i) in Theorem 2.3 is satisfied. For each $\lambda \in U$ we define

$$R_\lambda := \frac{\lambda^4 + \beta\lambda^2 + \alpha\lambda^3}{\gamma\lambda^2 + \delta\lambda + \rho},$$

and weakly analytic functions $f_{z_0,z_1} : U \to X_p \oplus X_p \oplus X_p \oplus X_p$ by

$$f_{z_0,z_1}(\lambda) := \begin{pmatrix} \varphi_\lambda \\ \lambda\varphi_\lambda \\ \lambda^2\varphi_\lambda \\ \lambda^3\varphi_\lambda \end{pmatrix}, \tag{12}$$

where $\varphi_\lambda(x) := z_0\cosh(\sqrt{R_\lambda}x) + z_1\sinh(\sqrt{R_\lambda}x)$, with $z_0, z_1 \in \mathbb{C}$ and $x \in \mathbb{R}$. It is easy to verify that

$$\rho\varphi''_\lambda(x) + \delta\lambda\varphi''_\lambda(x) + \gamma\lambda^2\varphi''_\lambda(x) - \beta\lambda^2\varphi_\lambda(x) - \alpha\lambda^3\varphi_\lambda(x) = \lambda^4\varphi_\lambda(x), \quad x \in \mathbb{R},$$

and therefore $Af_{z_0,z_1}(\lambda) = \lambda f_{z_0,z_1}(\lambda)$.

We claim that $f_{z_0,z_1}(\lambda) \in X_p \oplus X_p \oplus X_p \oplus X_p$ for all $\lambda \in U$. Indeed, first note that we can rewrite φ_λ as follows:

$$\varphi_\lambda(x) = \cosh\left(px\sqrt{\frac{R_\lambda}{p^2}}\right)z_0 + \sinh\left(px\sqrt{\frac{R_\lambda}{p^2}}\right)z_1 = \sum_{n=0}^{\infty} a_n(\lambda)\frac{(px)^n}{n!}, \ x \in \mathbb{R},$$

where $a_n(\lambda) = z_0\frac{R_\lambda^{n/2}}{p^n}, n = 0, 2, 4, \ldots$ and $a_n(\lambda) = z_1\sqrt{R_\lambda}\frac{R_\lambda^{(n-1)/2}}{p^n}, n = 1, 3, 5, \ldots$ Therefore, to prove the claim, it is enough to show that $\left|\frac{R_\lambda}{p^2}\right| < 1$ for each $\lambda \in U$.

For each $\lambda \in U$ we have

$$\left|\frac{R_\lambda}{p^2}\right| = \frac{|\lambda|^2(|\lambda|^2 + \alpha|\lambda| + \beta)}{p^2|\gamma\lambda^2 + \delta\lambda + \rho|} < \frac{r_0^2(r_0^2 + \alpha r_0 + \beta)}{p^2|\rho - \gamma r_0^2 - \delta r_0|}. \tag{13}$$

For $\gamma > 0$ we define

$$f(x) := \rho - \gamma x^2 - \delta x, \quad x \in \mathbb{R}.$$

This is a quadratic polynomial whose two roots are given by

$$r_1 := -\frac{\delta}{2\gamma} - \sqrt{\frac{\delta^2}{4\gamma^2} + \frac{\rho}{\gamma}} \quad \text{and} \quad r_2 := -\frac{\delta}{2\gamma} + \sqrt{\frac{\delta^2}{4\gamma^2} + \frac{\rho}{\gamma}}. \tag{14}$$

Note that $f(x) > 0$ for every $r_1 < x < r_2$. We will check that $r_1 < r_0 < r_2$. Condition $r_1 < r_0$ holds trivially. On the other hand, since by hypothesis (10) we have $\delta - \alpha\gamma \geq 0$ we obtain that the following identity

$$\delta^2(2+\gamma) + \gamma(2\alpha+\delta)^2 + 4\rho\gamma(2+\gamma) < 4\rho(1+\gamma)^2 + 2\delta(1+\gamma)(2\alpha+\delta)$$

holds. Multiplying by $\frac{\rho}{\gamma^2(2+\gamma)^2}$ we obtain

$$\frac{\delta^2\rho}{\gamma^2(2+\gamma)} + \frac{\rho(2\alpha+\delta)^2}{\gamma(2+\gamma)^2} + \frac{4\rho^2}{\gamma(2+\gamma)} < \frac{4\rho^2(1+\gamma)^2}{\gamma^2(2+\gamma)^2} + \frac{2\rho(1+\gamma)\delta(2\alpha+\delta)}{\gamma^2(2+\gamma)^2}$$

which turns out to be equivalent to

$$4\Big(\frac{\delta^2}{4\gamma^2} + \frac{\rho}{\gamma}\Big)\Big(\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}\Big) < \Big(\frac{2\rho(1+\gamma)}{\gamma(2+\gamma)} + \frac{\delta(2\alpha+\delta)}{2\gamma(2+\gamma)}\Big)^2 = \Big(\frac{\rho}{\gamma} + \frac{\rho}{2+\gamma} + \frac{\delta(2\alpha+\delta)}{2\gamma(2+\gamma)}\Big)^2.$$

Therefore, taking square root, we get the following identity

$$\frac{\delta^2}{4\gamma^2} - \frac{\delta(2\alpha+\delta)}{2\gamma(2+\gamma)} + \frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} < \frac{\delta^2}{4\gamma^2} + \frac{\rho}{\gamma} - 2\sqrt{\frac{\delta^2}{4\gamma^2} + \frac{\rho}{\gamma}}\sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}} + \frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\rho}$$

which is easy to see is equivalent to

$$0 \le \frac{\delta - \alpha\gamma}{\gamma(2+\gamma)} = \frac{\delta}{2\gamma} - \frac{2\alpha+\delta}{2(2+\gamma)} < \sqrt{\frac{\delta^2}{4\gamma^2} + \frac{\rho}{\gamma}} - \sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}}.$$

As a consequence, the inequality $r_0 < r_2$ holds. We conclude that $f(r_0) > 0$ and hence from (13) we have

$$\left|\frac{R_\lambda}{p^2}\right| < \frac{r_0^2(r_0^2 + \alpha r_0 + \beta)}{p^2(\rho - \gamma r_0^2 - \delta r_0)} = \frac{r_0^2}{p^2}\left(\frac{r_0^2 + \alpha r_0}{\rho - \gamma r_0^2 - \delta r_0} + \frac{\beta}{\rho - \gamma r_0^2 - \delta r_0}\right). \quad (15)$$

Due to the choice of r_0 we have $(2+\gamma)r_0^2 + (2\alpha+\delta)r_0 - \rho = 0$. Thus,

$$\frac{r_0^2 + \alpha r_0}{\rho - \gamma r_0^2 - \delta r_0} = \frac{1}{2} \quad \text{and} \quad \frac{\beta}{\rho - \gamma r_0^2 - \delta r_0} = \frac{\beta}{2(r_0^2 + \alpha r_0)}.$$

Therefore, from (15) and using hypothesis (11) we obtain

$$\left|\frac{R_\lambda}{p^2}\right| < \frac{r_0^2}{2p^2} + \frac{\beta r_0^2}{2p^2(r_0^2 + \alpha r_0)} < \frac{1}{2} + \frac{1}{2} = 1.$$

This proves the claim, and condition (ii) in Theorem 2.3.

It only remains to show that for any $x^* \in X_p^* \oplus X_p^* \oplus X_p^* \oplus X_p^*$ the functions $\lambda \to \langle f_{z_0,z_1}(\lambda), x^*\rangle$, $z_0, z_1 \in \mathbb{C}$, are holomorphic on U, and if they all vanish on U, then $x^* = 0$. Since X_p is isometrically isomorphic to c_0, in what follows, we identify the dual space X_p^* with ℓ^1.

Let $x^* \in X_p^* \oplus X_p^* \oplus X_p^* \oplus X_p^*$. It can be represented in a canonical way by $x^* = (x_1^*, x_2^*, x_3^*, x_4^*) = ((x_{1,n}^*)_{n\ge0}, (x_{2,n}^*)_{n\ge0}, (x_{3,n}^*)_{n\ge0}, (x_{4,n}^*)_{n\ge0}) \in \ell^1 \oplus \ell^1 \oplus \ell^1 \oplus \ell^1$. Then, we have

$$0 = \langle f_{z_0,z_1}(\lambda), x^*\rangle = \langle \varphi_\lambda, x_1^*\rangle + \langle \lambda\varphi_\lambda, x_2^*\rangle + \langle \lambda^2\varphi_\lambda, x_3^*\rangle + \langle \lambda^3\varphi_\lambda, x_4^*\rangle,$$

for all $\lambda \in U, z_0, z_1 \in \mathbb{C}$. This last equation can be reformulated in the following way:

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} a_n(\lambda)x_{1,n}^* + \lambda\sum_{n=0}^{\infty} a_n(\lambda)x_{2,n}^* + \lambda^2\sum_{n=0}^{\infty} a_n(\lambda)x_{3,n}^* + \lambda^3\sum_{n=0}^{\infty} a_n(\lambda)x_{4,n}^* \\ &= z_0x_{1,0}^* + \lambda z_0 x_{2,0}^* + \lambda^2 z_0 x_{3,0}^* + \lambda^3 z_0 x_{4,0}^* \\ &+ \frac{z_1}{\rho}\sqrt{R_\lambda}x_{1,1}^* + \frac{z_1}{\rho}\lambda\sqrt{R_\lambda}x_{2,1}^* + \frac{z_1}{\rho}\lambda^2\sqrt{R_\lambda}x_{3,1}^* + \frac{z_1}{\rho}\lambda^3\sqrt{R_\lambda}x_{4,1}^* \\ &+ \frac{z_0}{\rho^2}R_\lambda x_{1,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda x_{2,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda^2 x_{3,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda^3 x_{4,2}^* \dots \end{aligned} \quad (16)$$

Define now

$$\lambda_0 = 0, \quad \lambda_1 := -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta} \quad \text{and} \quad \lambda_2 := -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \beta}. \quad (17)$$

It is clear that $R_\lambda = 0$ if $\lambda \in \{\lambda_0, \lambda_1, \lambda_2\}$. We claim that $\lambda_0, \lambda_1, \lambda_2 \in U$.

In fact, clearly $\lambda_0 \in U$. By hypothesis (10) we have $\lambda_2 < \lambda_1 < 0$. Therefore, it is enough to show that $\lambda_2 \in U$.

By hypothesis (10) we obtain that the following inequality

$$\begin{aligned} 0 &< \rho^2 + 4\rho\beta + 2\rho\beta\gamma + 4\beta^2 + 4\gamma\beta^2 + \gamma^2\beta^2 + 2\alpha^2\beta\gamma \\ &\quad + (6\alpha + \delta + \alpha\gamma)(\beta\delta - \alpha\rho) + 4\beta\gamma^2 + 4\beta\alpha^2 + 2\alpha^2\rho, \end{aligned}$$

holds. A computation shows that the above inequality is equivalent to

$$0 < \left(\frac{\rho}{2+\gamma} + \beta\right)^2 + \frac{\beta(2\alpha+\delta)^2 - \alpha\rho(2\alpha+\delta)}{(2+\gamma)^2} + \frac{\beta\alpha(2\alpha+\delta) - \rho\alpha^2}{2+\gamma}.$$

which is the same as:

$$\begin{aligned} &4\left(\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}\right)\left(\frac{\alpha^2}{4} - \beta\right) < \frac{\rho^2}{(2+\gamma)^2} + \beta^2 \\ &\quad - \frac{2\rho\beta}{(2+\gamma)} + \frac{\beta\alpha(2\alpha+\delta) - \rho\alpha(2\alpha+\delta)}{(2+\gamma)^2} \\ &+ \frac{\alpha^2(2\alpha+\delta)^2}{4(2+\gamma)^2} = \left(\frac{\rho}{(2+\gamma)} - \beta - \frac{\alpha(2\alpha+\delta)}{2(2+\gamma)}\right)^2. \end{aligned}$$

Taking now square roots we arrive to:

$$2\sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}}\sqrt{\frac{\alpha^2}{4} - \beta} < \left(\frac{\rho}{(2+\gamma)} - \beta - \frac{\alpha(2\alpha+\delta)}{2(2+\gamma)}\right).$$

Adding $\frac{\alpha^2}{4} + \frac{(2\alpha+\delta)^2}{4(2+\gamma)^2}$ in both sides of the previous inequality we get:

$$\begin{aligned} \frac{\alpha(2\alpha+\delta)}{2(2+\gamma)} + \frac{\alpha^2}{4} + \frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} &< \frac{\alpha^2}{4} + \frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} \\ &+ \frac{\rho}{(2+\gamma)} - \beta - 2\sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}}\sqrt{\frac{\alpha^2}{4} - \beta}. \end{aligned}$$

This inequality is equivalent to:

$$\left(\frac{\alpha}{2} + \frac{(2\alpha+\delta)}{2(2+\gamma)}\right)^2 < \left(\sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}} - \sqrt{\frac{\alpha^2}{4} - \beta}\right)^2.$$

Note that the hypothesis $\delta - \alpha\gamma > 0$ implies $\sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}} > \sqrt{\frac{\alpha^2}{4} - \beta}$. Therefore, taking square roots in the previous inequality leads to:

$$|\lambda_2| = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta} < -\frac{(2\alpha+\delta)}{2(2+\gamma)} + \sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}} = r_0,$$

which proves the claim.

Finally, we will check condition (iii) in Theorem 2.3. Indeed, evaluating (16) in λ_0, we have the following equation:

$$z_0 x_{1,0}^* = 0 \quad (18)$$

for all $z_0 \in \mathbb{C}$. Therefore, $x_{1,0}^* = 0$.

Now, we divide (16) by λ and we get:

$$\begin{aligned} 0 &= \frac{1}{\lambda}\left(\sum_{n=0}^{\infty} a_n(\lambda)x_{1,n}^* + \lambda\sum_{n=0}^{\infty} a_n(\lambda)x_{2,n}^* + \lambda^2\sum_{n=0}^{\infty} a_n(\lambda)x_{3,n}^* + \lambda^3\sum_{n=0}^{\infty} a_n(\lambda)x_{4,n}^*\right) \\ &= z_0x_{2,0}^* + \lambda z_0 x_{3,0}^* + \lambda^2 z_0 x_{4,0}^* + \frac{z_1}{\rho}\frac{\sqrt{R_\lambda}}{\lambda}x_{1,1}^* \\ &\quad + \frac{z_1}{\rho}\sqrt{R_\lambda}x_{2,1}^* + \frac{z_1}{\rho}\lambda\sqrt{R_\lambda}x_{3,1}^* + \frac{z_1}{\rho}\lambda^2\sqrt{R_\lambda}x_{4,1}^* \\ &+ \frac{z_0}{\rho^2}\frac{R_\lambda}{\lambda}x_{1,2}^* + \frac{z_0}{\rho^2}R_\lambda x_{2,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda x_{3,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda^2 x_{4,2}^* \dots \end{aligned} \quad (19)$$

As $R_{\lambda_0} = 0$, evaluating (19) in $\lambda_0 = 0$ we obtain:

$$z_0x_{2,0}^* + \frac{z_1}{\rho}\sqrt{\frac{\beta}{\rho}}x_{1,1}^* = 0 \quad (20)$$

for all $z_0, z_1 \in \mathbb{C}$. Then, $x_{2,0}^* = 0$ and $x_{1,1}^* = 0$.

We then obtain

$$\begin{aligned} 0 &= \frac{1}{\lambda}\left(\sum_{n=0}^{\infty} a_n(\lambda)x_{1,n}^* + \lambda\sum_{n=0}^{\infty} a_n(\lambda)x_{2,n}^* + \lambda^2\sum_{n=0}^{\infty} a_n(\lambda)x_{3,n}^* + \lambda^3\sum_{n=0}^{\infty} a_n(\lambda)x_{4,n}^*\right) \\ &= \lambda z_0x_{3,0}^* + \lambda^2 z_0 x_{4,0}^* + \frac{z_1}{\rho}\sqrt{R_\lambda}x_{2,1}^* + \frac{z_1}{\rho}\lambda\sqrt{R_\lambda}x_{3,1}^* + \frac{z_1}{\rho}\lambda^2\sqrt{R_\lambda}x_{4,1}^* \\ &+ \frac{z_0}{\rho^2}\frac{R_\lambda}{\lambda}x_{1,2}^* + \frac{z_0}{\rho^2}R_\lambda x_{2,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda x_{3,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda^2 x_{4,2}^* \dots \end{aligned} \quad (21)$$

Since $\lambda_1, \lambda_2 \in U$ we now evaluate (21) in λ_1 and λ_2 getting:

$$\lambda_1 z_0 x_{3,0}^* + \lambda_1^2 z_0 x_{4,0}^* = 0 \quad \text{and} \quad \lambda_2 z_0 x_{3,0}^* + \lambda_2^2 z_0 x_{4,0}^* = 0 \quad (22)$$

for all $z_0, z_1 \in \mathbb{C}$. Since $\lambda_1 \neq \lambda_2$ then $x_{3,0}^* = 0$ and $x_{4,0}^* = 0$.

Now, we divide (21) by λ again getting:

$$\begin{aligned} 0 &= \frac{z_1}{\rho}\frac{\sqrt{R_\lambda}}{\lambda}x_{2,1}^* + \frac{z_1}{\rho}\sqrt{R_\lambda}x_{3,1}^* + \frac{z_1}{\rho}\lambda\sqrt{R_\lambda}x_{4,1}^* \\ &+ \frac{z_0}{\rho^2}\frac{R_\lambda}{\lambda^2}x_{1,2}^* + \frac{z_0}{\rho^2}\frac{R_\lambda}{\lambda}x_{2,2}^* + \frac{z_0}{\rho^2}R_\lambda x_{3,2}^* + \frac{z_0}{\rho^2}R_\lambda\lambda x_{4,2}^* \dots \end{aligned} \quad (23)$$

Evaluating (23) in λ_0 we get:

$$\frac{z_1}{\rho}\sqrt{\frac{\beta}{\rho}}x_{2,1}^* + \frac{z_0}{\rho^2}\frac{\beta}{\rho}x_{1,2}^* = 0 \quad (24)$$

for all $z_0, z_1 \in \mathbb{C}$. Therefore $x^*_{2,1} = 0$ and $x^*_{1,2} = 0$. Dividing again (23) by λ and evaluating in λ_0 we arrive to $x^*_{3,1} = 0$ and $x^*_{2,2} = 0$.

Proceeding inductively, we will get that $x^*_{i,n} = 0$ for $i = 1,2,3,4$ and $n \in \mathbb{N}$. We finally have $x^* = 0$ and we conclude the result by applying Theorem 2.3. □

Special attention deserves the case $\gamma = 0$, that is, the model

$$\frac{\partial^4 u}{\partial t^4}(t,x) + \alpha\frac{\partial^3 u}{\partial t^3}(t,x) + \beta\frac{\partial^2 u}{\partial t^2}u(t,x) - \delta\frac{\partial^3 u}{\partial t\partial x^2}(t,x) - \rho\frac{\partial^2 u}{\partial x^2}u(t,x) = 0, \quad t \geq 0,\ x \in \mathbb{R}. \tag{25}$$

The model (25) appears in the Ref. [15, formula (12)] and represents the so-called modified fourth-order Moore–Gibson–Thompson heat equation. In such case u represents the thermodynamical temperature of a semiconductor. It explains the interaction of thermal-plasma-elastic waves. For the meaning of the constitutive parameters α, β, δ and ρ we refer to [15].

Associated to (25) we have the operator-valued matrix

$$B := \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \rho\partial_{xx} & \delta\partial_{xx} & -\beta & -\alpha \end{pmatrix}$$

as the generator of an uniformly continuous semigroup on the space c_0. Note that the proof of Theorem 3.1 breaks down in step (14) for $\gamma = 0$. The analysis of this case is contained in the following result.

Theorem 3.2. *Let $\alpha, \beta, \delta, \rho > 0$ Suppose that*

$$\beta\delta - \rho\alpha \geq 0 \qquad \text{and} \qquad \alpha \geq 2\sqrt{\beta}. \tag{26}$$

Then for each p satisfying

$$p^2 > \max\{\beta, r_0^2\}, \quad r_0 := -\frac{2\alpha+\delta}{4} + \sqrt{\frac{(2\alpha+\delta)^2}{16} + \frac{\rho}{2}} \tag{27}$$

the operator B generates a uniformly continuous semigroup which is Devaney chaotic on $X_p \oplus X_p \oplus X_p \oplus X_p$.

Proof. We will only include the main steps of the proof which are different with respect to Theorem 3.1. First, we define $U := \{\lambda \in \mathbb{C} : |\lambda| < r_0\}$. For each $\lambda \in U$ we define

$$R_\lambda := \frac{\lambda^4 + \beta\lambda^2 + \alpha\lambda^3}{\delta\lambda + \rho},$$

and weakly analytic functions f_{z_0,z_1} on U as in (14). We will show that $f_{z_0,z_1}(\lambda) \in X_p \oplus X_p \oplus X_p \oplus X_p$ for all $\lambda \in U$ by proving that $|\frac{R_\lambda}{p^2}| < 1$. For each $\lambda \in U$ we have the inequality

$$|\delta\lambda + \rho| \geq |\rho - \delta r_0|.$$

We claim that $\rho - \delta r_0 > 0$. Indeed, we have

$$0 < \frac{2\rho^2 + 2\delta\alpha\rho}{2\delta^2} = \frac{2\rho^2 + \delta\rho(2\alpha+\delta) - \delta^2\rho}{2\delta^2}. \tag{28}$$

Observe that inequality (28) is equivalent to

$$\frac{(2\alpha+\delta)^2}{16} + \frac{\rho}{2} < \frac{\rho^2}{\delta^2} + \frac{(2\alpha+\delta)^2}{16} + \frac{\rho(2\alpha+\delta)}{2\delta} = \left(\frac{\rho}{\delta} + \frac{(2\alpha+\delta)}{4}\right)^2. \tag{29}$$

Taking square roots to (29) we get

$$\sqrt{\frac{(2\alpha+\delta)^2}{16} + \frac{\rho}{2}} < \frac{\rho}{\delta} + \frac{(2\alpha+\delta)}{4} \tag{30}$$

which is equivalent to

$$r_0 := -\frac{2\alpha+\delta}{4} + \sqrt{\frac{(2\alpha+\delta)^2}{16} + \frac{\rho}{2}} < \frac{\rho}{\delta}.$$

This proves the claim.

For each $\lambda \in U$ we have

$$\left|\frac{R_\lambda}{p^2}\right| = \frac{|\lambda|^2(|\lambda|^2 + \alpha|\lambda| + \beta)}{p^2|\delta\lambda + \rho|} < \frac{r_0^2(r_0^2 + \alpha r_0 + \beta)}{p^2(\rho - \delta r_0)} = \frac{r_0^2}{p^2}\left(\frac{r_0^2 + \alpha r_0}{\rho - \delta r_0} + \frac{\beta}{\rho - \delta r_0}\right).$$

Note that r_0 satisfies the equation $2r_0^2 + (2\alpha+\delta)r_0 - \rho = 0$, therefore we get:

$$\frac{r_0^2 + \alpha r_0}{\rho - \delta r_0} = \frac{1}{2} \quad \text{and} \quad \frac{\beta}{\rho - \delta r_0} = \frac{\beta}{2(r_0^2 + \alpha r_0)}.$$

Hence,

$$\left|\frac{R_\lambda}{p^2}\right| < \frac{r_0^2}{2p^2} + \frac{\beta r_0^2}{2p^2(r_0^2 + \alpha r_0)} < \frac{1}{2} + \frac{1}{2} = 1, \tag{31}$$

since by hypothesis we have $p^2 > \max\{\beta, r_0^2\}$. The remaining proof follows the same lines as Theorem 3.1 just making $\gamma = 0$. □

It is interesting to note that there exists a different combination of parameters that still produces a chaotic behavior for the fourth-order model (7). This combination is, in a certain sense, less natural than the one given in Theorem 3.1.

Theorem 3.3. *Let $\alpha, \beta, \delta, \rho, \gamma > 0$. Suppose that*

$$\gamma - \frac{\delta}{\alpha} \leq 0, \qquad \alpha < 2\sqrt{\beta}, \qquad |\rho - \beta(2+\gamma)| > \sqrt{\beta}(2\alpha+\delta). \tag{32}$$

Then for each p satisfying (11) the operator A generates a uniformly continuous semigroup which is Devaney chaotic on $X_p \oplus X_p \oplus X_p \oplus X_p$.

Proof. We set r_0 as in (11) and then follow the steps of Theorem 3.1 until the expression in (16). We define λ_0, λ_1 and λ_2 as in (17). Recall that $R_\lambda = 0$ if $\lambda \in \{\lambda_0, \lambda_1, \lambda_2\}$. We claim that $\lambda_1, \lambda_2 \in U$.

In fact, by hypothesis (32) we have $\lambda_1 = \frac{\alpha}{2} + i\sqrt{\beta - \frac{\alpha^2}{4}}$ and $\lambda_2 = \frac{\alpha}{2} - i\sqrt{\beta - \frac{\alpha^2}{4}}$. Consequently, it is sufficient to prove that $|\lambda_1| = |\lambda_2| = \sqrt{\beta} < r_0$.

From the hypothesis (32) we obtain the inequality

$$0 < (\rho - \beta(2+\gamma))^2 - \beta(2\alpha+\delta)^2. \tag{33}$$

Dividing (33) by $(2\alpha+\delta)^2$ we obtain the equivalent form:

$$0 < \left(\frac{\rho}{2\alpha+\delta} - \frac{\beta(2+\gamma)}{2\alpha+\delta}\right)^2 - \beta = \frac{\rho^2}{(2\alpha+\delta)^2} + \frac{\beta^2(2+\gamma)^2}{(2\alpha+\delta)^2} - \frac{2\rho\beta(2+\gamma)}{(2\alpha+\delta)^2} - \beta. \tag{34}$$

Adding $\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{(2+\gamma)}$ in both sides of inequality (34) we have:

$$\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{(2+\gamma)} < \left(\frac{(2\alpha+\delta)}{2(2+\gamma)} + \frac{\rho}{(2\alpha+\delta)} - \frac{\beta(2+\gamma)}{(2\alpha+\delta)}\right)^2$$

and then root squaring, we arrive to:

$$\sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{(2+\gamma)}} < \frac{(2\alpha+\delta)}{2(2+\gamma)} + \frac{\rho}{(2\alpha+\delta)} - \frac{\beta(2+\gamma)}{(2\alpha+\delta)}. \tag{35}$$

We now multiply (35) by $\frac{(2\alpha+\delta)}{(2+\gamma)}$ which leads to:

$$\beta < \frac{(2\alpha+\delta)^2}{2(2+\gamma)^2} + \frac{\rho}{(2+\gamma)} - \frac{(2\alpha+\delta)}{(2+\gamma)}\sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{(2+\gamma)}} \tag{36}$$
$$= \left(\sqrt{\frac{(2\alpha+\delta)^2}{4(2+\gamma)^2} + \frac{\rho}{2+\gamma}} - \frac{(2\alpha+\delta)}{2(2+\gamma)}\right)^2.$$

Finally, taking square root to (36) we get $\sqrt{\beta} < r_0$. This proves the claim. The remaining proof follows the same steps as Theorem 3.1. □

Parallel to Theorem 3.2, we can state the corresponding result for the model (25).

Theorem 3.4. *Let $\alpha, \beta, \delta, \rho > 0$ Suppose that*

$$\alpha < 2\sqrt{\beta} \qquad \text{and} \qquad |\rho - 2\beta| > \sqrt{\beta}(2\alpha+\delta). \tag{37}$$

Then for each p satisfying (27) the operator B generates a uniformly continuous semigroup which is Devaney chaotic on $X_p \oplus X_p \oplus X_p \oplus X_p$.

Proof. We set r_0 as in (27) and define $U := \{\lambda \in \mathbb{C} : |\lambda| < r_0\}$. We also define weakly analytic functions f_{z_0,z_1} on U as in (14). Our task is to show that $f_{z_0,z_1}(\lambda) \in X_p \oplus X_p \oplus X_p \oplus X_p$ for all $\lambda \in U$, and this is achieved following the same lines of Theorem 3.2 by proving that $|\frac{R_\lambda}{p^2}| < 1$ for all $\lambda \in U$ where $R_\lambda = \frac{\lambda^4+\beta\lambda^2+\alpha\lambda^3}{\delta\lambda+\rho}$. Next, we note that the hypothesis $\alpha < 2\sqrt{\beta}$ in (37) allows to define λ_1 and λ_2 as $\lambda_1 = \frac{\alpha}{2} + i\sqrt{\beta - \frac{\alpha^2}{4}}$ and $\lambda_2 = \frac{\alpha}{2} - i\sqrt{\beta - \frac{\alpha^2}{4}}$. Then, thanks to the hypothesis $|\rho - 2\beta| > \sqrt{\beta}(2\alpha + \delta)$, we can follow the same proof of Theorem 3.3 to obtain that $\lambda_1, \lambda_2 \in U$. From this point, we finish the proof following the same steps of Theorem 3.1. □

We end this article with the following example.

Example 1. Let $a, b, c, d, \ell \geq 0$ and consider the following form of the fourth-order Moore–Gibson–Thompson equation

$$\frac{\partial^4 u}{\partial t^4}(t,x) + (a+\ell)\frac{\partial^3 u}{\partial t^3}(t,x) + a\ell\frac{\partial^2 u}{\partial t^2}u(t,x) - b\frac{\partial^4 u}{\partial t^2 \partial x^2}(t,x) - (c+b\ell)\frac{\partial^3 u}{\partial t \partial x^2}(t,x) - (c\ell - d)\frac{\partial^2 u}{\partial x^2}u(t,x) = 0, \tag{38}$$

It was considered in [6] and labels into the model (7) with $\alpha = a+\ell, \beta = a\ell, \gamma = b, \delta = c+b\ell$ and $\rho = c\ell - d$.

We will apply Theorem 3.1 to (38). Note that

$$\gamma - \frac{\delta}{\alpha} \leq 0 \text{ if and only if } b - \frac{c}{a} \leq 0.$$

We now observe that

$$\alpha \geq 2\sqrt{\beta} \text{ if and only if } (a-\ell)^2 > 0. \tag{39}$$

and

$$\beta\delta - \rho\alpha > 0 \quad \text{if and only if} \quad (c-ab)\ell^2 - dl - ad < 0$$

where the last conditions hold if

$$b - \frac{c}{a} < 0 \quad \text{and} \quad 0 \leq \ell < \frac{\sqrt{d^2 + 4ad(c-ab)}}{2(c-ab)} + \frac{d}{2(c-ab)},$$

and therefore they are sufficient conditions to exhibit chaos for model (38).

CRediT authorship contribution statement

Carlos Lizama: Conceptualization, Methodology, Writing – original draft, Investigation, Supervision, Writing – review & editing. **Marina Murillo-Arcila:** Conceptualization, Methodology, Investigation, Validation, Writing – review & editing.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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