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Maximum and Minimum Results for the Green's Functions in Delta Fractional Difference Settings

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Abstract: The present paper is dedicated to the examination of maximum and minimum results based on Green's functions via delta fractional differences for a class of fractional boundary problems. For such a purpose, we built the corresponding Green's functions based on the falling factorial functions. In addition, using the constructed Green's function, the positivity of the function and its corresponding delta function are presented. We also verified the occurrence of two distinct functions with the same Green's function. The maximality and minimality of the Green's function show a good qualitative agreement. Finally, we considered some special examples to explain the obtained results.

Keywords: Riemann–Liouville operators; Green's functions; positivity results; max and min results

MSC: 26A48; 26A51; 39A12; 39B62

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1. Introduction

In the last two decades, a large number of fractional differential and difference equations have been studied with their application in fractional calculus, telecommunication, mathematical modeling, biological modeling, and so on; see e.g., [1–4]. One of the fundamental problems in science is the development of suitable fractional operators to extract useful information from fractional and discrete fractional calculus; see e.g., [5–7]. These operators are important in many fields of science, including physics, applied science, mathematics, and scientific computing, as well as some related research fields, such as engineering sciences, fluid dynamics, number theory, mathematical physics, and quantum mechanics; see for example [8–11].

The study of the fractional boundary value problems (FBVPs) has attracted the attention of researchers through the world, and these models have rarely been investigated in the context of discrete fractional calculus [12,13]. The problem of finding the existence and uniqueness of discrete FBVPs in relation to the homogeneous and inhomogeneous boundary conditions is critical for mathematical and physical applications. For this reason, different models have been proposed in the literature aiming to calculate the existence and uniqueness of discrete FBVP models, using analytical and numerical or experimental approaches; see for example [14,15] to be familiar with these operators.

On the other hands, the FBVPs have been extensively modelled since its beginning, and with the flourishing development of discrete fractional calculus, the qualitative analysis of FBVPs for fractional difference equations has become an active research field. There are several books and papers devoted to fractional difference modelling in both the commonly used fractional differences: Riemann–Liouville and Liouville–Caputo settings; see, e.g., [16–22], for instance. Moreover, the existence and uniqueness of the solutions of FBVPs have been investigated via other types of fractional differences, including the Attangana–Baleanu and Caputo–Fabrizio fractional operators; see for example [23–26].

We have previously introduced several models of FBVPs to better understand their interactions; for example in [27,28]. Motivated by the FBVP used in [27], we aim to examine the following FBVP:

$$\begin{cases} -\left({}^{\mathrm{RL}}_{b_0+1}\Delta^{\ell} w\right)(t) = g(t+\ell), \quad t \in \mathbb{J}_{(b_0+2,b)},\ \ell \in I_2, \\ \alpha_1 w(b_0) - \alpha_2 (\nabla w)(b_0+1) = 0, \\ \delta_2 w(b) + \delta_1 (\nabla w)(b) = 0. \end{cases} \tag{1}$$

where $g : \mathbb{J}_{(b_0+2,b)} \to \mathbb{R}$, and $\alpha_1^2 + \alpha_2^2 > 0$, $\delta_2^2 + \delta_1^2 > 0$, for $\alpha_1, \alpha_2, \delta_2, \delta_1 \in \mathbb{R}$ and $\mathbb{J}_{(b_0+2,b)} = \{b_0+2, b_0+3, \ldots, b\}$. With these motivations and considerations in mind, in this article, we will establish bounded results to the Green's functions obtained from the above delta fractional operator.

The remaining part of this paper has structured in the sequence: In the next section, we have briefly presented fundamental structures and basic theorems related to Green's functions and FBVPs. Then, in Section 3, we have studied the conduction of Green's functions associated with the proposed FBVP. In addition, we have two parts for the main results in this section: In Section 3.1, the essential positivity results of the operators have been deducted along with their existence results, and we have presented the bounded results in Section 3.2 regarding the maximality and minimality. Then, in Section 4, we have presented a related numerical example. Concluding remarks, together with the future directions, are detailed in Section 5.

2. Preliminaries

Let $I_n = (n-1, n)$, $\mathbb{J}_{(b_0)} = \{b_0, b_0+1, \ldots\}$, $n \in \mathbb{J}_{(1)}$, and $\Upsilon(t) = t+1$. We refer to Definition 2.25 in [2]; the delta-fractional sum is given as follows:

$$\left({}_{b_0}\Delta^{-\ell} w\right)(t) = \sum_{t_2=b_0}^{t-\ell} \mathcal{W}_{\ell-1}(t, \Upsilon(t_2)) w(t_2), \quad \text{for } t \in \mathbb{J}_{(b_0+\ell)}, \tag{2}$$

and Theorem 2.2 in [29]; the delta-fractional difference is given as follows:

$$\left({}^{\mathrm{RL}}_{b_0}\Delta^{\ell} w\right)(t) = \sum_{t_2=b_0}^{t+\ell} \mathcal{W}_{-\ell-1}(t, \Upsilon(t_2)) w(t_2), \quad \text{for } t \in \mathbb{J}_{(b_0+n-\ell)}, \tag{3}$$

for $\ell \in I_n$ and w is defined by $\mathbb{J}_{(b_0)}$. Also, we have

$$\mathcal{W}_{\ell}(t, t_2) := \frac{(t-t_2)^{\underline{\ell}}}{\Gamma(\ell+1)} = \frac{\Gamma(t-t_2+1)}{\Gamma(\ell+1)\,\Gamma(t-t_2+1-\ell)}. \tag{4}$$

Next, we recall some properties of $\mathcal{W}_{\ell}(t, t_2)$.

Lemma 1 (see [2,14]). *If $\ell \in \mathbb{R}^+$, then*

(i) $\nabla \mathcal{W}_{\ell}(t, b_0) = \mathcal{W}_{\ell-1}(t-1, b_0)$.

(ii) *For $t \in \mathbb{J}_{(b_0)}$, we have*

$$\mathcal{W}_{\ell}(t+\ell-1, b_0) - \mathcal{W}_{\ell-1}(t+\ell-2, b_0) = \mathcal{W}_{\ell}(t+\ell-1, \Upsilon(b_0)) = \mathcal{W}_{\ell}(t+\ell-2, b_0).$$

(iii) *For $t \in \mathbb{J}_{(b_0+n)}$ as $\ell \in I_n$, we have*

$$\left({}_{b_0+n-\alpha_1}^{\text{RL}}\Delta^{-\ell}\ {}_{b_0}^{\text{RL}}\Delta^{\ell} w\right)(t) = w(t). \tag{5}$$

(iv) *$t \in \mathbb{J}_{(b_0+1)}$, we have*

$$\sum_{t_2=b_0+1}^{t} \mathcal{W}_\ell(t_2+\ell-1, b_0) = \mathcal{W}_{\ell+1}(t+\ell, b_0),$$
$$\sum_{t_2=b_0+1}^{t} \mathcal{W}_\ell(t_2+\ell+1, \Upsilon(t_2)) = \mathcal{W}_{\ell+1}(t+\ell, b_0).$$

Lemma 2 (see [28])**.** *Let $t_2 \in \mathbb{J}_{(b_0)}$. Then, one can have*

(i) *If $\ell > 0$, then*
- *$\mathcal{W}_\ell(t+\ell+1, \Upsilon(t_2))$ is decreasing with reference to t_2, for $t \in \mathbb{J}_{(t_2-1)}$.*
- *$\mathcal{W}_\ell(t+\ell+1, \Upsilon(t_2))$ is increasing with reference to t, for $t \in \mathbb{J}_{(t_2)}$.*

(ii) *If $\ell > -1$, then*
- *$\mathcal{W}_\ell(t+\ell+1, \Upsilon(t_2)) \geq 0$, for $t \in \mathbb{J}_{(t_2-1)}$.*
- *$\mathcal{W}_\ell(t+\ell+1, \Upsilon(t_2)) > 0$, for $t \in \mathbb{J}_{(t_2)}$.*

(iii) *If $0 > \ell > -1$, then*
- *$\mathcal{W}_\ell(t+\ell+1, \Upsilon(t_2))$ is increasing with reference to t_2, for $t \in \mathbb{J}_{(t_2)}$.*
- *$\mathcal{W}_\ell(t+\ell+1, \Upsilon(t_2))$ is increasing with reference to t, for $t \in \mathbb{J}_{(t_2+1)}$.*

(iv) *If $\ell \geq 0$, then $\mathcal{W}_\ell(t+\ell+1, \Upsilon(t_2))$ is non-decreasing with reference to t, for $t \in \mathbb{J}_{(t_2-1)}$.*

Lemma 3 (see [28])**.** *For $t_2 \in \mathbb{J}_{(b_0+1)}$, $t \in \mathbb{J}_{(t_2)}$ and $\ell > -1$, we define*

$$T_\ell(t, t_2) = \frac{\mathcal{W}_\ell(t+\ell+1, \Upsilon(t_2))}{\mathcal{W}_\ell(t+\ell-1, b_0)}. \tag{6}$$

Then, we have

i- *$T_\ell(t, t_2) > 0$.*
ii- *$T_\ell(t, t_2) \leq 1$, where $\ell > 0$, and $T_\ell(t, t_2) \geq 1$, where $-1 < \ell < 0$, specifically, $T_0(t, t_2) = 1$.*

iii- *The function $T_\ell(t, t_2)$ is non-increasing with reference to t, where $\ell > 0$.*
iv- *The function $T_\ell(t, t_2)$ is non-increasing with reference to t, where $-1 < \ell < 0$.*

Lemma 4 (see [28])**.** *The general solution of*

$$\left({}_{b_0}^{\text{RL}}\Delta^{\ell} w\right)(t) = -g(t+\ell), \quad t \in \mathbb{J}_{(b_0+2)},$$

is given as follows

$$w(t) = c_1\, \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(b_0)\big) + c_2\, \mathcal{W}_{\ell-2}\big(t+\ell-1, \Upsilon(b_0)\big) - \left({}_{b_0+2}\Delta^{-\alpha_1} g\right)(t+\ell), \quad t \in \mathbb{J}_{(b_0)}, \tag{7}$$

where $\ell \in I_2$, c_1 and c_2 are arbitrary constants.

3. Main Results

First, we study our essential results on Green's functions. By considering (1), we define the following notations:

$$\begin{aligned} B_1 &= \alpha_1 + \alpha_2\,(1-\ell), \\ B_2 &= B_1 + \alpha_2 = \alpha_1 + \alpha_2\,(2-\ell), \\ f_1(r) &= \delta_2\, \mathcal{W}_{\ell-1}\big(b+\ell, \Upsilon(r)\big) + \delta_1\, \mathcal{W}_{\ell-2}\big(b+\ell-1, \Upsilon(r)\big), \quad r \in \mathbb{J}_{(b_0,b)}, \\ f_2(r) &= B_2\, \mathcal{W}_{\ell-1}\big(r+\ell, \Upsilon(b_0)\big) - B_1\, \mathcal{W}_{\ell-2}\big(r+\ell-1, \Upsilon(b_0)\big), \quad r \in \mathbb{J}_{(b_0,b)}, \\ A &= \delta_2\, \mathcal{W}_{\ell-2}\big(b+\ell-1, \Upsilon(b_0)\big) + \delta_1\, \mathcal{W}_{\ell-3}\big(b+\ell-2, \Upsilon(b_0)\big), \\ \lambda &= B_2\, f_1(b_0) - B_1\, A. \end{aligned}$$

Theorem 1. *There is a unique solution for the* FBVP (1), *which is given by*

$$w(t) = \sum_{t_2=b_0+2}^{b} \mathscr{G}(t,t_2) g(t_2), \quad t \in \mathbb{J}_{(b_0,b)}, \tag{8}$$

where

$$\mathscr{G}(t,t_2) = \begin{cases} \mathscr{G}_1(t,t_2) := \frac{f_2(t)}{\lambda} f_1(t_2), & t \in \mathbb{J}_{(b_0,t_2-1)}; \\ \mathscr{G}_2(t,t_2) := \mathscr{G}_1(t,t_2) - \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big), & t \in \mathbb{J}_{(t_2,b)}. \end{cases} \tag{9}$$

Proof. The general solution of (1) is given by (7). It follows from this and Lemma 1 that

$$\begin{aligned} \big(\nabla\, w\big)(t) &= c_1\, \mathcal{W}_{\ell-2}\big(t+\ell-1, \Upsilon(b_0)\big) + c_2\, \mathcal{W}_{\ell-3}\big(t+\ell-2, \Upsilon(b_0)\big) \\ &\quad - \nabla\Big({}_{b_0+2}\Delta^{-\alpha_1} g\Big)(t+\ell) \\ &= c_1\, \mathcal{W}_{\ell-2}\big(t+\ell-1, \Upsilon(b_0)\big) + c_2\, \mathcal{W}_{\ell-3}\big(t+\ell-2, \Upsilon(b_0)\big) \\ &\quad - \Big({}_{b_0+2}\Delta^{1-\alpha_1} g\Big)(t+\ell-1), \end{aligned} \tag{10}$$

for $t \in \mathbb{J}_{(b_0,b)}$. By using the BCs of (1) in (7) and (10), respectively, we obtain

$$c_1\, B_1 + c_2\, B_2 = 0,$$

and

$$c_1\, f_1(b_0) + c_2\, A = \sum_{t_2=b_0+2}^{b} f_1(t_2) g(t_2).$$

From these equations, it follows that

$$c_1 = \frac{B_2}{\lambda} \sum_{t_2=b_0+2}^{b} f_1(t_2) g(t_2),$$

and

$$c_2 = \frac{-B_1}{\lambda} \sum_{t_2=b_0+2}^{b} f_1(t_2) g(t_2).$$

By substituting the values of c_1 and c_2 in (7), we obtain the desired result. Therefore, the proof is complete. □

We have divided the main results of this section into two parts.

3.1. Positivity Results

This subsection is dedicated to prove some necessary lemmas for the positivity of operators.

Lemma 5. *Let $\alpha_1, \alpha_2, \delta_2, \delta_1 > 0$ s.t. $\alpha_1 \geq \alpha_2$. Then, we have*

(a) $B_1, B_1, f_1(r) > 0$, *for* $r \in \mathbb{J}_{(b_0,b)}$;
(b) $f_1(b_0) - A > 0$;
(c) $\lambda > 0$;
(d) $f_2(r) \geq 0$, *for* $r \in \mathbb{J}_{(b_0,b)}$;
(e) $(\nabla f_2)(r) > 0$, *for* $r \in \mathbb{J}_{(b_0+1,b)}$.

Proof. By considering Lemma 2 (ii), we obtain (a).
Next, by using the hypothesis and Lemmas 1–2 (ii), we see that

$$\begin{aligned} f_1(b_0) - A &= \delta_2\Big[\mathcal{W}_{\ell-1}(b+\ell, \Upsilon(b_0)) - \mathcal{W}_{\ell-2}(b+\ell-1, \Upsilon(b_0))\Big] \\ &\quad + \delta_1\Big[\mathcal{W}_{\ell-2}(b+\ell-1, \Upsilon(b_0)) - \delta_1\, \mathcal{W}_{\ell-3}(b+\ell-2, \Upsilon(b_0))\Big] \\ &= \delta_2\, \mathcal{W}_{\ell-1}(b+\ell-2, b_0) + \delta_1\, \mathcal{W}_{\ell-2}(b+\ell-3, b_0) > 0, \end{aligned}$$

which proves (b).
For the next one, we use (a) and (b) to obtain

$$\begin{aligned} \lambda &= B_2\, f_1(b_0) - B_1\, A \\ &= (B_1 + \alpha_2) f_1(b_0) - B_1\, A \\ &= B_1(f_1(b_0) - A) + \alpha_2\, f_1(b_0) > 0, \end{aligned}$$

which gives the proof of (c).
By considering (a), Lemma 1 (ii), Lemma 2 (i,ii), we have

$$\begin{aligned} f_2(r) &= B_2\, \mathcal{W}_{\ell-1}(r+\ell, \Upsilon(b_0)) - B_1\, \mathcal{W}_{\ell-2}(r+\ell-1, \Upsilon(b_0)) \\ &= (B_1+\alpha_2)\mathcal{W}_{\ell-1}(r+\ell, \Upsilon(b_0)) - B_1\, \mathcal{W}_{\ell-2}(r+\ell-1, \Upsilon(b_0)) \\ &= B_1[\mathcal{W}_{\ell-1}(r+\ell, \Upsilon(b_0)) - \mathcal{W}_{\ell-2}(r+\ell, \Upsilon(b_0))] + \alpha_2\, \mathcal{W}_{\ell-1}(r+\ell, \Upsilon(b_0)) \\ &= B_1\, \mathcal{W}_{\ell-1}(r+\ell-2, b_0) + \alpha_2\, \mathcal{W}_{\ell-1}(r+\ell, \Upsilon(b_0)) \geq 0, \end{aligned}$$

for $r \in \mathbb{J}_{(b_0,b)}$. This has proved (d).
The final item can be proved by using (d) and Lemma 1 (i) as follows:

$$\begin{aligned} (\nabla f_2)(r) &= \nabla\Big[B_1\, \mathcal{W}_{\ell-1}(r+\ell-2, b_0) + \alpha_2\, \mathcal{W}_{\ell-1}(r+\ell, \Upsilon(b_0))\Big] \\ &= B_1\, \mathcal{W}_{\ell-2}(r+\ell-3, b_0) + \alpha_2\, \mathcal{W}_{\ell-2}(r+\ell-1, \Upsilon(b_0)) > 0, \end{aligned}$$

for $r \in \mathbb{J}_{(b_0+1,b)}$. Hence, the proof is complete. □

Lemma 6. *With the same assumptions as the above lemma, we have*

$$\mathscr{G}(t, t_2) \geq 0, \quad (t, t_2) \in \mathbb{J}_{(b_0,b)} \times \mathbb{J}_{(b_0+2,b)}.$$

Proof. Considering Theorem 1, we have

$$\mathscr{G}_1(t, t_2) = \frac{f_2(t)}{\lambda} f_1(t_2) \geq 0, \tag{11}$$

as $f_2(t) \geq 0, \lambda > 0, f_1(t_2) > 0$ for $t \in \mathbb{J}_{(b_0,b)}$ and $t_2 \in \mathbb{J}_{(b_0+2,b)}$ according to Lemma 5.

Moreover, we have

$$\begin{aligned}\mathscr{G}_2(t,t_2) &= \frac{f_2(t)}{\lambda} f_1(t_2) - \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big) \\ &= \frac{1}{\lambda}\Big[f_2(t)\, f_1(t_2) - \lambda\, \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\Big] \\ &= \frac{1}{\lambda}\Big[B_1\delta_2\, E_1 + B_1\delta_1\, E_2 + \alpha_2\delta_2\, E_3 + \alpha_2\delta_1\, E_4\Big],\end{aligned} \tag{12}$$

where

$$\begin{aligned}E_1 = {} & \mathcal{W}_{\ell-1}\big(b+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(t+\ell-2, b_0\big) \\ & - \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(b+\ell-2, b_0\big);\end{aligned}$$

$$\begin{aligned}E_2 = {} & \mathcal{W}_{\ell-2}\big(b+\ell-1, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(t+\ell-2, b_0\big) \\ & - \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-2}\big(b+\ell-3, b_0\big);\end{aligned}$$

$$\begin{aligned}E_3 = {} & \mathcal{W}_{\ell-1}\big(b+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(b_0)\big) \\ & - \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(b+\ell, \Upsilon(b_0)\big);\end{aligned}$$

$$\begin{aligned}E_4 = {} & \mathcal{W}_{\ell-2}\big(b+\ell-1, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(b_0)\big) \\ & - \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-2}\big(b+\ell-1, \Upsilon(b_0)\big).\end{aligned}$$

Computing these values in turn, we have

$$\begin{aligned}E_1 &= \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(b+\ell-2, b_0\big) \\ &\quad \times \left[\frac{\mathcal{W}_{\ell-1}\big(b+\ell, \Upsilon(t_2)\big)}{\mathcal{W}_{\ell-1}\big(b+\ell-2, b_0\big)}\, \frac{\mathcal{W}_{\ell-1}\big(t+\ell-2, b_0\big)}{\mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)} - 1\right] \\ &= \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(b+\ell-2, b_0\big)\left[\frac{T_{\ell-1}(b,t_2)}{T_{\ell-1}(t,t_2)} - 1\right] \geq 0,\end{aligned}$$

where we used $\mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(b+\ell-2, b_0\big) > 0$ according to Lemma 2 (ii), and $T_{\ell-1}(b,t_2) \geq T_{\ell-1}(t,t_2)$ according to Lemma 3. Also,

$$\begin{aligned}E_2 &= \mathcal{W}_{\ell-2}\big(b+\ell-1, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-1}\big(t+\ell-2, b_0\big) \\ &\qquad - \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-2}\big(b+\ell-3, b_0\big) \\ &> \mathcal{W}_{\ell-2}\big(b+\ell-1, \Upsilon(b_0)+1\big)\, \mathcal{W}_{\ell-1}\big(t+\ell-2, t_2-1\big) \\ &\qquad - \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big)\, \mathcal{W}_{\ell-2}\big(b+\ell-3, b_0\big) = 0,\end{aligned}$$

where we have used Lemma 2 (i,iii) and

$$\begin{aligned}\mathcal{W}_{\ell-2}\big(b+\ell-1, \Upsilon(b_0)+1\big) &= \mathcal{W}_{\ell-2}\big(b+\ell-3, b_0\big), \\ \mathcal{W}_{\ell-1}\big(t+\ell-2, t_2-1\big) &= \mathcal{W}_{\ell-1}\big(t+\ell, \Upsilon(t_2)\big),\end{aligned}$$

for $t_2 \in \mathbb{J}_{(b_0+2,b)}$ and $\ell \in I_2$.

Again, by using Lemma 2 (ii) and Lemma 3, we have

$$\begin{aligned}
E_3 &= \mathcal{W}_{\ell-1}(b+\ell, \Upsilon(t_2))\, \mathcal{W}_{\ell-1}(t+\ell, \Upsilon(b_0)) \\
&\quad - \mathcal{W}_{\ell-1}(t+\ell, \Upsilon(t_2))\, \mathcal{W}_{\ell-1}(b+\ell, \Upsilon(b_0)) \\
&= \mathcal{W}_{\ell-1}(t+\ell, \Upsilon(t_2))\, \mathcal{W}_{\ell-1}(b+\ell, \Upsilon(b_0)) \\
&\quad \times \left[\frac{\mathcal{W}_{\ell-1}(b+\ell, \Upsilon(t_2))}{\mathcal{W}_{\ell-1}(b+\ell, \Upsilon(b_0))} \frac{\mathcal{W}_{\ell-1}(t+\ell, \Upsilon(b_0))}{\mathcal{W}_{\ell-1}(t+\ell, \Upsilon(t_2))} - 1 \right] \\
&= \mathcal{W}_{\ell-1}(t+\ell, \Upsilon(t_2))\, \mathcal{W}_{\ell-1}(b+\ell, \Upsilon(b_0)) \left[\frac{T_{\ell-1}(b, t_2)}{T_{\ell-1}(t, t_2)} - 1 \right] \geq 0.
\end{aligned}$$

Finally, by using Lemma 2 (i,iii), we have

$$\begin{aligned}
E_4 &= \mathcal{W}_{\ell-2}(b+\ell-1, \Upsilon(t_2))\, \mathcal{W}_{\ell-1}(t+\ell, \Upsilon(b_0)) \\
&\quad - \mathcal{W}_{\ell-1}(t+\ell, \Upsilon(t_2))\, \mathcal{W}_{\ell-2}(b+\ell-1, \Upsilon(b_0)) \\
&\quad > \mathcal{W}_{\ell-2}(b+\ell-1, \Upsilon(b_0))\, \mathcal{W}_{\ell-1}(t+\ell, \Upsilon(s)) \\
&\quad - \mathcal{W}_{\ell-1}(t+\ell, \Upsilon(t_2))\, \mathcal{W}_{\ell-2}(b+\ell-1, \Upsilon(b_0)) = 0.
\end{aligned}$$

Therefore, by considering the E_ℓ's values in (12) and the hypotheses, we have

$$\mathscr{G}_2(t, t_2) = \frac{f_2(t)}{\lambda} f_1(t_2) \geq 0, \tag{13}$$

for $t \in \mathbb{J}_{(b_0,b)}$ and $t_2 \in \mathbb{J}_{(b_0+2,b)}$. Hence, $\mathscr{G}(t, t_2) \geq 0$, for $(t, t_2) \in \mathbb{J}_{(b_0,b)} \times \mathbb{J}_{(b_0+2,b)}$, according to (11) and (13). This completes the proof. □

3.2. Max and Min Results

The maximality and minimality of the proposed Green's function will be stated in the following theorems.

Theorem 2. *With the same assumptions as the Lemma 5, we have*

$$\max_{t \in \mathbb{J}_{(b_0,b)}} \mathscr{G}(t, t_2) = \mathscr{G}(t_2 - 1, t_2), \quad t_2 \in \mathbb{J}_{(b_0+2,b)}.$$

Proof. According to Theorem 1 one can have

$$(\nabla_t \mathscr{G}_1)(t, t_2) = \frac{(\nabla_t f_2)(t)}{\lambda} f_1(t_2) > 0,$$

according to Lemma 5, for $(t, t_2) \in \mathbb{J}_{(b_0+1,t_2-1)} \times \mathbb{J}_{(b_0+2,b)}$.
Next, from Lemma 6, we have

$$\begin{aligned}
(\nabla_t \mathscr{G}_2)(t, t_2) &= \frac{1}{\lambda} \Big[(\nabla_t f_2)(t)\, f_1(t_2) - \lambda\, \mathcal{W}_{\ell-2}(t+\ell-1, \Upsilon(t_2)) \Big] \\
&= \frac{1}{\lambda} \Big[B_1 \delta_2\, F_1 + B_1 \delta_1\, F_2 + \alpha_2 \delta_2\, F_3 + \alpha_2 \delta_1\, F_4 \Big],
\end{aligned}$$

where

$$
\begin{aligned}
F_1 = \left(\nabla_t E_1\right) &= \mathcal{W}_{\ell-1}\left(b+\ell, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(t+\ell-3, b_0\right) \\
&\quad - \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-1}\left(b+\ell-2, b_0\right); \\
F_2 = \left(\nabla_t E_2\right) &= \mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(t+\ell-3, b_0\right) \\
&\quad - \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(b+\ell-3, b_0\right); \\
F_3 = \left(\nabla_t E_3\right) &= \mathcal{W}_{\ell-1}\left(b+\ell, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(b_0)\right) \\
&\quad - \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-1}\left(b+\ell, \Upsilon(b_0)\right); \\
F_4 = \left(\nabla_t E_4\right) &= \mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(b_0)\right) \\
&\quad - \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(b_0)\right).
\end{aligned}
$$

By using the same techniques used in the previous lemma, we can calculate each F_ℓs as follows:

$$
\begin{aligned}
F_1 = \mathcal{W}_{\ell-1}\left(b+\ell, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(t+\ell-3, b_0\right) & \\
- \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-1}\left(b+\ell-2, b_0\right) & \\
< \mathcal{W}_{\ell-1}\left(b+\ell-2, b_0\right) \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) & \\
- \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-1}\left(b+\ell-2, b_0\right) &= 0,
\end{aligned}
$$

$$
\begin{aligned}
F_2 &= \mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(t+\ell-3, b_0\right) \\
&\quad - \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(b+\ell-3, b_0\right) \\
&= \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(b+\ell-3, b_0\right) \\
&\quad \times \left[\frac{\mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(t_2)\right)}{\mathcal{W}_{\ell-2}\left(b+\ell-3, b_0\right)} \frac{\mathcal{W}_{\ell-2}\left(t+\ell-3, b_0\right)}{\mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right)} - 1\right] \\
&= \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(b+\ell-3, b_0\right) \left[\frac{T_{\ell-2}(b, t_2)}{T_{\ell-2}(t, t_2)} - 1\right] \\
&\leq 0,
\end{aligned}
$$

$$
\begin{aligned}
F_3 = \mathcal{W}_{\ell-1}\left(b+\ell, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(b_0)\right) & \\
- \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-1}\left(b+\ell, \Upsilon(b_0)\right) & \\
< \mathcal{W}_{\ell-1}\left(b+\ell, \Upsilon(b_0)\right) \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(s)\right) & \\
- \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-1}\left(b+\ell, \Upsilon(b_0)\right) &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
F_4 &= \mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(b_0)\right) \\
&\quad - \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(b_0)\right) \\
&= \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(b_0)\right) \\
&\quad \times \left[\frac{\mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(t_2)\right)}{\mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(b_0)\right)} \frac{\mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(b_0)\right)}{\mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right)} - 1\right] \\
&= \mathcal{W}_{\ell-2}\left(t+\ell-1, \Upsilon(t_2)\right) \mathcal{W}_{\ell-2}\left(b+\ell-1, \Upsilon(b_0)\right) \left[\frac{T_{\ell-2}(b, t_2)}{\mathcal{W}_{\ell-2}(t, t_2)} - 1\right] \\
&\leq 0.
\end{aligned}
$$

Therefore, we conclude from the hypotheses and these values

$$\big(\nabla_t \mathscr{G}_2\big)(t,t_2) < 0,$$

for $(t,t_2) \in \mathbb{J}_{(t_2,b)} \times \mathbb{J}_{(b_0+2,b)}$.
As a consequence, for $t_2 \in \mathbb{J}_{(b_0+2,b)}$, we obtain the result

$$\begin{aligned}\max_{t\in\mathbb{J}_{(b_0,b)}} \mathscr{G}(t,t_2) &= \max_{t\in\mathbb{J}_{(b_0,b)}} \Big\{\mathscr{G}_1(t_2-1,t_2), \mathscr{G}_2(t_2,t_2)\Big\} \\ &= \mathscr{G}(t_2-1,t_2),\end{aligned}$$

where we have used

$$\begin{aligned}\mathscr{G}_2(t_2,t_2) - \mathscr{G}_1(t_2-1,t_2) &= \frac{f_2(t_2)}{\lambda} f_1(t_2) - \mathcal{W}_{\ell-1}\big(t_2+\ell, \Upsilon(t_2)\big) - \frac{f_2(t_2-1)}{\lambda} f_1(t_2) \\ &= \frac{f_1(t_2)}{\lambda}\Big[f_2(t_2) - f_2(t_2-1)\Big] - \mathcal{W}_{\ell-1}\big(t_2+\ell, \Upsilon(t_2)\big) \\ &= \frac{\big(\nabla_{t_2} f_2\big)(t_2)}{\lambda} f_1(t_2) - \mathcal{W}_{\ell-2}\big(t_2+\ell-1, \Upsilon(t_2)\big) \\ &= \big(\nabla_{t_2}\mathscr{G}_2\big)(t_2,t_2) < 0.\end{aligned}$$

Thus, we have completed our proof. □

Theorem 3. *With the same assumptions as the Lemma 5, we have*

$$\max_{t\in\mathbb{J}_{(b_0,b)}} \mathscr{G}(t,t_2) \ge \chi\,\mathscr{G}(t_2-1,t_2), \quad t_2 \in \mathbb{J}_{(b_0+2,b)},$$

where

$$\chi = \frac{1}{f_2(b-1)} \min\left\{ f_2(b_0), f_2(b) - \frac{\lambda}{\delta_2 + \delta_1\left(\frac{\ell-1}{b-b_0+\ell-3}\right)} \right\}.$$

Proof. Theorem 2 implies that

$$\mathscr{G}_1(b_0,t_2) \le \mathscr{G}_1(t,t_2) \le \mathscr{G}_1(t_2-1,t_2), \tag{14}$$

for $(t,t_2) \in \mathbb{J}_{(b_0,t_2-1)} \times \mathbb{J}_{(b_0+2,b)}$

$$\mathscr{G}_2(b,t_2) \le \mathscr{G}_2(t,t_2) \le \mathscr{G}_2(t_2,t_2), \tag{15}$$

for $(t,t_2) \in \mathbb{J}_{(t_2,b)} \times \mathbb{J}_{(b_0+2,b)}$.
We consider

$$\frac{\mathscr{G}(t,t_2)}{\mathscr{G}(t_2-1,t_2)} = \begin{cases} \frac{\mathscr{G}_1(t,t_2)}{\mathscr{G}(t_2-1,t_2)}, & t \in \mathbb{J}_{(b_0,t_2-1)}; \\ \frac{\mathscr{G}_2(t,t_2)}{\mathscr{G}(t_2-1,t_2)}, & t \in \mathbb{J}_{(t_2,b)}. \end{cases}$$

It follows from (14) and (15) that

$$\frac{\mathscr{G}(t,t_2)}{\mathscr{G}(t_2-1,t_2)} \geq \begin{cases} \frac{\mathscr{G}_1(b_0,t_2)}{\mathscr{G}(t_2-1,t_2)}, & t \in \mathbb{J}_{(b_0,t_2-1)}; \\ \frac{\mathscr{G}_2(b,t_2)}{\mathscr{G}(t_2-1,t_2)}, & t \in \mathbb{J}_{(t_2,b)}, \end{cases}$$

$$= \begin{cases} \frac{f_2(b_0)}{f_2(t_2-1)}, & t \in \mathbb{J}_{(b_0,t_2-1)}; \\ \frac{f_2(b)}{f_2(t_2-1)} - \frac{\lambda\, \mathcal{W}_{\ell-1}\big(b+\ell,\Upsilon(t_2)\big)}{f_2(t_2-1)f_1(t_2)}, & t \in \mathbb{J}_{(t_2,b)}, \end{cases}$$

$$= \frac{1}{f_2(t_2-1)} \begin{cases} f_2(b_0), & t \in \mathbb{J}_{(b_0,t_2-1)}; \\ f_2(b) - \lambda\, \frac{\mathcal{W}_{\ell-1}\big(b+\ell,\Upsilon(t_2)\big)}{f_1(t_2)}, & t \in \mathbb{J}_{(t_2,b)}. \end{cases} \tag{16}$$

Calculating the second term, we have

$$\begin{aligned} \frac{\mathcal{W}_{\ell-1}\big(b+\ell,\Upsilon(t_2)\big)}{f_1(t_2)} &= \frac{\mathcal{W}_{\ell-1}\big(b+\ell,\Upsilon(t_2)\big)}{\delta_2\, \mathcal{W}_{\ell-1}\big(b+\ell,\Upsilon(t_2)\big) + \delta_1\, \mathcal{W}_{\ell-2}\big(b+\ell-1,\Upsilon(t_2)\big)} \\ &= \frac{1}{\delta_2 + \delta_1 \frac{\mathcal{W}_{\ell-2}\big(b+\ell-1,\Upsilon(t_2)\big)}{\mathcal{W}_{\ell-1}\big(b+\ell,\Upsilon(t_2)\big)}} \\ &= \frac{1}{\delta_2 + \delta_1\left(\frac{\ell-1}{b-t_2+\ell-1}\right)} \\ &\leq \frac{1}{\delta_2 + \delta_1\left(\frac{\ell-1}{b-b_0+\ell-3}\right)}, \end{aligned} \tag{17}$$

for $t_2 \in \mathbb{J}_{(b_0+2,b)}$. From (16) and (17), we can conclude that

$$\frac{\mathscr{G}(t,t_2)}{\mathscr{G}(t_2-1,t_2)} \geq \frac{1}{f_2(t_2-1)} \begin{cases} f_2(b_0), & t \in \mathbb{J}_{(b_0,t_2-1)}; \\ f_2(b) - \frac{\lambda}{\delta_2+\delta_1\left(\frac{\ell-1}{b-b_0+\ell-3}\right)}, & t \in \mathbb{J}_{(t_2,b)}. \end{cases} \tag{18}$$

Since $\big(\nabla f_2\big)(t_2) > 0$, for $t_2 \in \mathbb{J}_{(b_0+1,b)}$, according to Lemma 5 (e), we can say that

$$f_2(b-1) \geq f_2(t_2-1) \geq f_2(b_0+1), \quad t_2 \in \mathbb{J}_{(b_0+2,b)}. \tag{19}$$

By making the use of (19) in (18), we obtain the desired result. Hence, the proof is complete. □

4. An Application

The following example is dedicated to understand the applicability of the above main results.

Example 1. *Let us suppose that*

$$b_0 = 0,\ b = 2,\ \delta_1 = \alpha_2 = 0,\ \alpha_1 = \delta_2 = 1.$$

This implies that $B_1 = B_2 = \alpha_1 = 1$*. Therefore,*

$$
\begin{aligned}
A &= \mathcal{W}_{\ell-2}(\ell+1,1) = \frac{1}{2}(\ell-1)\ell, \\
f_1(t) &= \mathcal{W}_{\ell-1}(\ell+2,t+1) = \frac{\Gamma(\ell+2-t)}{\Gamma(\ell)\Gamma(3-t)} \quad \Longrightarrow \quad f_1(b_0) = \frac{1}{2}\ell(\ell+1), \\
\lambda &= f_1(b_0) - A = \ell, \\
f_2(t) &= \mathcal{W}_{\ell-1}\big(t+\ell,\Upsilon(b_0)\big) - \mathcal{W}_{\ell-2}\big(t+\ell-1,\Upsilon(b_0)\big) \\
&= \mathcal{W}_{\ell-1}\big(t+\ell-1,\Upsilon(b_0)\big) = \frac{\Gamma(t+\ell-1)}{\Gamma(t)\Gamma(\ell)} \\
&\Longrightarrow f_2(b_0) \to 0, \quad f_2(b) = \ell, \quad f_2(b-1) = 1.
\end{aligned}
$$

Also, we know that

$$
\mathcal{W}_{\ell-1}\big(t+\ell,\Upsilon(t_2)\big) = \frac{\Gamma(t+\ell-t_2)}{\Gamma(t-t_2+1)\Gamma(\ell)}.
$$

Since $t_2 = 2$ *and*

$$
\mathcal{W}_{\ell-1}\big([t_2-1]+\ell,\Upsilon(t_2)\big) = \frac{\Gamma(\ell-1)}{\Gamma(0)\Gamma(\ell)} \to 0,
$$

we conclude that

$$
\mathscr{G}_1(t_2-1,t_2) = \mathscr{G}_2(t_2-1,t_2) = \frac{1}{\Gamma(\ell)\Gamma(\ell+1)} \frac{\Gamma(t_2+\ell-2)\Gamma(\ell-t_2+2)}{\Gamma(t_2-1)\Gamma(3-t_2)} = \frac{1}{\ell}.
$$

Moreover, in view of (9)*, we have*

$$
\mathscr{G}(t,t_2) = \begin{cases} \frac{1}{\Gamma(\ell+1)} \frac{\Gamma(t+\ell-1)}{\Gamma(t)}, & t \in \{0,1\}; \\ \\ \frac{1}{\Gamma(\ell+1)} \frac{\Gamma(t+\ell-1)}{\Gamma(t)} - \frac{1}{\Gamma(\ell)} \frac{\Gamma(t+\ell-2)}{\Gamma(t-1),} & t \in \{2\}. \end{cases}
$$

$$
= \begin{cases} 0, & t = 0,2; \\ \\ \frac{1}{\ell} & t = 1. \end{cases}
$$

Thus, we can deduce that

$$
\max_{t \in \mathbb{J}_{(0,2)}} \mathscr{G}(t,t_2) = \frac{1}{\ell} = \mathscr{G}(t_2-1,t_2), \quad t_2 = 2,
$$

which confirms the validity of Theorem 2*.*

On the other hands, we observe that

$$
\begin{aligned}
\chi &= \frac{1}{f_2(b-1)} \min\left\{ f_2(b_0), f_2(b) - \frac{\lambda}{\delta_2 + \delta_1\left(\frac{\ell-1}{b-b_0+\ell-3}\right)} \right\} \\
&= \min\{0,0\} = 0.
\end{aligned}
$$

Therefore, for $t_2 = 2$*, we have*

$$
\frac{1}{\ell} = \max_{t \in \mathbb{J}_{(0,2)}} \mathscr{G}(t,t_2) \geq 0 = \chi \mathscr{G}(t_2-1,t_2),
$$

for each $\ell \in I_2$. This confirms the validity of Theorem 3.

5. Conclusions

In this paper, we have considered the delta FBVP (1). For this, a new Green's function in the domain $\mathbb{J}_{(b_0,b)}$ has been constructed, together with some essential properties. The proposed Green's function is formulated via some functions and the positivity of these functions has been derived. In fact, it is proven that the maximality of this Green's function is equal to $\mathscr{G}(t_2 - 1, t_2), \quad t_2 \in \mathbb{J}_{(b_0+2,b)}$; however, when χ is defined in Theorem 3, it is greater and equal to $\chi\mathscr{G}(t_2 - 1, t_2)$. Finally, Theorem 3 has been verified by using an example of a special FBVP.

This research direction can be extended to other types of fractional difference operators, such as Liouville–Caputo operators, and other types with Mittag-Leffler and exponential in kernels; for example, see [14,15] to find these operators.

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