A KALLMAN-ROTA INEQUALITY FOR GENERATORS OF FAMILIES OF BOUNDED OPERATORS

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Abstract. A Kallman-Rota type inequality for generators of a wide class of strongly continuous families of bounded and linear operators defined on a Banach space is shown. Our approach allows us to recover (in a unified way) known results about uniformly bounded C_0-semigroups and cosine functions as well as to prove new results for other families of operators. In particular, if A is the generator of an α-times integrated family of bounded and linear operators arising from the well posedness of fractional differential equations of order $\beta+1$ then, we prove that the inequality

$$||Ax||^2 \leq 8M^2 \frac{\Gamma(\alpha+\beta+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2\beta+3)} ||x|| \, ||A^2x||,$$

holds for all $x \in D(A^2)$.

1. introduction

A well-known result established by Hardy, Littlewood and Pólya asserts that $||f'||_2^2 \leq 2||f||_2||f''||_2$ for any function f on $\mathbb{R}_+$ such that $f, f', f'' \in L^2(\mathbb{R}_+)$, ([9, p.187]). Kallman and Rota have proved that

$$||Ax||^2 \leq 4||x|| \, ||A^2x|| \tag{1.1}$$

whenever A is the infinitesimal generator of a strongly continuous contraction semigroup on a Banach space $(X, \|\cdot\|)$, and x, and Ax are in the domain of A ([12]). This result was extended by Kraljević and Kurepa in [15] to bounded strongly continuous semigroups with the bound constant $M>0$, as $||Ax||^2 \leq 4M^2||x|| \, ||A^2x||$, for $x \in D(A^2)$. Moreover, Kraljević and Kurepa proved that in case that A generates a strongly continuous contraction cosine function, the constant 4 in the inequality (1.1) can be replaced by 4/3 ([15]).

There exists a vast and interesting literature about this amazing topic. In [4], it is shown that the Hardy-Littlewood-Pólya inequality implies the Kallman-Rota inequality for C_0-semigroups. A special interest has been taken to improve (and in some case to obtain the optimal) constant in the inequality (1.1). In Hilbert spaces, the optimal constant in (1.1) for a C_0-contraction semigroup is 2 ([7]); in C-Euclidean spaces it is treated in [11] and in the case that A generates an analytic semigroup in [22]; see [7] and [21] and references therein for more details.

Key words and phrases. Kallman-Rota inequality; Integrated semigroups; Integrated cosine functions; Abstract differential equations.

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Our main result of this paper is to show a Kallman-Rota inequality when A is the generator of certain (a,k)-regularized resolvents defined on a Banach space X (Theorem 3.1). The concept of (a,k)-regularized resolvent allows to treat different families of bounded operators in a unified way. Let $k \in C(\mathbb{R}_+)$ be an scalar kernel and let $a \in L^1_{loc}(\mathbb{R}_+)$. Assume that A is a closed linear operator. Following [17], a strongly continuous operator family $(R(t))_{t\geq 0} \subset \mathcal{B}(X)$ is called an (a,k)-regularized resolvent with generator A (or A generates a (a,k)-regularized resolvent) if and only if the following holds:

(i) $R(t)Ax = AR(t)x$ for all $x \in D(A), t \geq 0$ and $R(0) = k(0)I$;

(ii) $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)xds, x \in D(A), t > 0.$

The theory of (a,k)-regularized resolvent was developed in recent years starting with the work [17]. This concept unifies several notions of strongly continuous operators families, as for example k-convoluted semigroups ($a(t) \equiv 1$) and k-convoluted cosine functions ($a(t) \equiv t$). Note that in case $k(t) \equiv 1$ we include the theory of resolvent families associated to Volterra equation with kernel a of scalar type, and in this case for $a(t) \equiv 1$, C_0-semigroups and for $a(t) \equiv t$, cosine functions (see more details in [20]). As a proper example, we note that for every $n \in \mathbb{N}$, there exists an exponentially bounded kernel k_n such that the polyharmonic operator Δ^{2^n} generates an exponentially bounded (g_β, k_n)-regularized resolvent, where $g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, and $\beta \in [1/2, 1)$, see [14] and references therein.

In the next section we prove some technical lemmata which we are used in the third section. We also introduce a new condition, the "CP-condition", which is related with the solution of certain integral equation. The main results of the paper appear in the third section. Our approach is applied to a large number of operators which include generators of C_0-semigroups and cosine functions. Moreover, this approach fits and explains perfectly why constants 4 and 4/3 appear in both cases. To conclude the paper, we apply our results to several concrete examples in the last section.

2. Some technical conditions and results

In what follows, we assume that $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ and we denote by $a * k$ the usual convolution product,

$$a * k(t) := \int_0^t a(t-s)k(s)ds, \qquad t > 0.$$

In the case that a is a positive function $a.e.$, then

$$\int_0^\infty a * a(t)dt = \left(\int_0^\infty a(t)dt\right)^2$$

and $\int_0^\infty a * a(t)dt = \infty$ if and only if $\int_0^\infty a(t)dt = \infty.$

Definition 2.1. *We say that the pair (a,k) satisfies the CP-condition if for any $\lambda > 0$ there exists $t_\lambda > 0$ such that*

$$\lambda k(t_\lambda) = (a * a * k)(t_\lambda). \tag{2.2}$$

We denote by $\chi_{[0,+\infty)}$ the characteristic function in the interval $[0,\infty)$; $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha > 0$ and $e_\lambda(t) := e^{\lambda t}$ for $\lambda \in \mathbb{R}$. It is easy to check that (g_ν, g_μ) with $\nu, \mu > 0$, (e_1, e_1),

(e_{-1}, e_{-1}) and (e_1, e_{-1}) satisfy the CP-condition; however the pair (e_{-1}, e_1) does not satisfy it:

$$e_{-1} * e_{-1} * e_1(s) = \frac{e^s - e^{-s}}{4} - \frac{se^{-s}}{2}, \qquad s > 0.$$

In the next Lemma, we show some necessary conditions to get pairs (a, k) which verify the CP-condition.

Lemma 2.2. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be a positive function and $k \in C(\mathbb{R}_+)$.*

(i) *If $k(0) > 0$, k is decreasing and $\int_0^\infty a(t)dt = \infty$, then the pair (a, k) satisfies the CP-condition.*

(ii) *The pair $(a, \chi_{[0,\infty)})$ satisfies the CP-condition if and only if $\int_0^\infty a(t)dt = \infty$*

Proof. (i) Fixed $\lambda > 0$, we apply Bolzano's theorem to the function $g := \lambda k - a * a * k$. Note that $g(0) > 0$ and

$$\lim_{t\to\infty} \frac{a * a * k(t)}{k(t)} \geq \lim_{t\to\infty} \int_0^t a * a(s)ds = +\infty,$$

and then $\lim_{t\to\infty} g(t) = -\infty$. We conclude that there exists $t_\lambda > 0$ such that $\lambda k(t_\lambda) = (a * a * k)(t_\lambda)$.

(ii) If the pair $(a, \chi_{[0,\infty)})$ satisfies the CP-condition, then there exists t_λ such that

$$\int_0^{t_\lambda} a * a(s)ds = \lambda$$

for all $\lambda > 0$. We may conclude that $\int_0^\infty a * a(s)ds = \infty$ and then $\int_0^\infty a(s)ds = \infty$. The converse statement is proven in a similar way. □

Examples. The pair $(g_\alpha \cdot e_\lambda, g_\beta)$ with $\alpha > 0$, $\lambda > 0$ and $0 < \beta \leq 1$ satisfies the CP-condition using a similar proof as part (i) in Lemma 2.2. The pair $(g_\alpha \cdot e_\lambda, \chi_{[0,\infty)})$ for $\alpha > 0$ satisfies the CP-condition if and only if $\lambda > 0$ by Lemma 2.2 (ii).

In the next section the function f_α defined by

$$f_\alpha(t) := \frac{\Gamma(\alpha + t + 2)^2}{\Gamma(\alpha + 1)\Gamma(\alpha + 2t + 3)}, \qquad t \in [-1, +\infty), \tag{2.3}$$

for $\alpha > -1$ will play a crucial role in several estimates, see Theorem 3.2 below. In the following Proposition, we collect some interesting properties of f_α which will be used.

Proposition 2.3. *Let $\alpha > -1$ and f_α defined by (2.3). Then $f_\alpha(-1) = 1$, $0 < f_\alpha(t) \leq 1$, f_α is a decreasing function in $(-1, +\infty)$ for any $\alpha > -1$ and $\lim_{\alpha\to\infty} f_\alpha(t) = 1$ for any $t > -1$.*

Proof. We directly check that $f_\alpha(-1) = 1$. To show that f_α is decreasing, we prove that $f'_\alpha(t) < 0$ for $t > -1$. Note that

$$f'_\alpha(t) = \frac{2\Gamma(\alpha + t + 2)}{\Gamma(\alpha + 1)} \big(\Gamma'(\alpha + t + 2)\Gamma(\alpha + 2t + 3) - \Gamma(\alpha + t + 2)\Gamma'(\alpha + 2t + 3)\big) < 0$$

if and only if

$$\frac{\Gamma'(\alpha + t + 2)}{\Gamma(a + t + 2)} < \frac{\Gamma'(\alpha + 2t + 3)}{\Gamma(a + 2t + 3)}, \qquad t > -1.$$

We consider the function ψ given by $\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)}$. As $\psi'(s) > 0$ (see for example [18, p.14]) we conclude that ψ is increasing and f_α decreasing for any $\alpha > -1$. Then $1 = f_\alpha(-1) \geq f_\alpha(t)$ for any $t > -1$.

It is known that $\lim_{s\to\infty} \frac{\Gamma(s+c)}{\Gamma(s)} e^{-c\log s} = 1$ for $c > 0$, see [8, 8.328 (2)]. Then

$$\lim_{\alpha\to\infty} f_\alpha(t) = \lim_{\alpha\to\infty} \frac{\Gamma(\alpha+t+2)}{\Gamma(\alpha+1)} e^{-(t+1)\log\alpha} \frac{\Gamma(\alpha+t+2)}{\Gamma(\alpha+2t+3)e^{-(t+1)\log\alpha}} = 1,$$

and we conclude the proof. □

Remark. There exists an alternative way to show that $f_\alpha(t) \leq 1$ for any $t > -1$. We write

$$f_\alpha(t) = \frac{\beta(\alpha+t+2, \alpha+t+2)}{\beta(\alpha+1, \alpha+2t+3)}, \qquad t \in [-1, +\infty),$$

where β denotes the beta function defined by $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. We claim that $\beta(\mu-\delta, \mu+\delta) \geq \beta(\mu,\mu)$ for each $\delta > 0$ and $\mu > \delta$. Indeed, we have that

$$\begin{aligned}
\beta(\mu-\delta, \mu+\delta) &= \tfrac{1}{2}\beta(\mu+\delta, \mu-\delta) + \tfrac{1}{2}\beta(\mu-\delta, \mu+\delta) \\
&= \frac{1}{2}\int_0^1 x^{\mu+\delta-1}(1-x)^{\mu-\delta-1}dx + \frac{1}{2}\int_0^1 x^{\mu-\delta-1}(1-x)^{\mu+\delta-1}dx \\
&= \int_0^1 x^{\mu-1}(1-x)^{\mu-1}\frac{1}{2}((\frac{x}{1-x})^\delta + (\frac{1-x}{x})^\delta)dx \\
&\geq \int_0^1 x^{\mu-1}(1-x)^{\mu-1}dx = \beta(\mu,\mu).
\end{aligned}$$

Finally we choose $\mu = \alpha+\beta+2$ and $\delta = \beta+1$ to show the desired inequality.

3. Norm inequalities

In what follows, we assume that $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ are both positive functions. Now, we give the main result in this paper.

Theorem 3.1. *Let (a,k) be a pair satisfying the CP-condition and*

$$C_{a,k} := \sup_{t>0} \frac{a*a*k(t)k(t)}{(k*a)^2(t)} < \infty. \tag{3.1}$$

Now suppose that A is the generator of an (a,k)-regularized resolvent $\{R(t)\}_{t\geq 0}$, such that

$$||R(t)|| \leq Mk(t), \quad t \geq 0, \tag{3.2}$$

with $M \geq 1$. Then the Kallman-Rota inequality,

$$||Ax||^2 \leq 8M^2 C_{a,k}||x||\,||A^2x||, \tag{3.3}$$

holds for all $x \in D(A^2)$.

Proof. For all $x \in D(A^2)$ and $t \geq 0$ we have $R(t)x \in D(A)$ and $AR(t)x \in D(A)$, hence

$$\begin{aligned}
R(t)x &= (a*AR)(t)x + k(t)x \\
&= a*A[(a*AR)(t)x + k(t)x] + k(t)x \\
&= (a*a*A^2R)(t)x + (a*k)(t)Ax + k(t)x.
\end{aligned}$$

Therefore,

$$||(a*k)(t)Ax|| \leq ||R(t)x|| + ||(a*a*A^2R)(t)x|| + ||k(t)x||. \tag{3.4}$$

Note that by (3.2), we have

$$\begin{aligned} ||(a*a*A^2R)(t)x|| &= ||\int_0^t (a*a)(t-s)A^2R(s)xds|| \\ &\leq \int_0^t (a*a)(t-s)||R(s)A^2x||ds \\ &\leq M\int_0^t (a*a)(t-s)k(s)||A^2x||ds \\ &= M(a*a*k)(t)||A^2x||. \end{aligned}$$

Hence, again by (3.2), we conclude from (3.4) that

$$||(a*k)(t)Ax|| \leq Mk(t)||x|| + M(a*a*k)(t)||A^2x|| + k(t)||x||,$$

or, equivalently,

$$||Ax|| \leq 2M\frac{k(t)}{(a*k)(t)}||x|| + M\frac{(a*a*k)(t)}{(a*k)(t)}||A^2x||, \quad t>0, \tag{3.5}$$

Define $f(t) = p\frac{k(t)}{(a*k)(t)} + q\frac{(a*a*k)(t)}{(a*k)(t)}$, where $p = 2M||x||$ and $q = M||A^2x||$. From $(\sqrt{q}\sqrt{(a*a*k)(t)} - \sqrt{p}\sqrt{k(t)})^2 \geq 0$, we obtain

$$f(t) \geq 2\sqrt{pq}\sqrt{\frac{(a*a*k)(t)k(t)}{(a*k)^2(t)}}, \quad \text{for all } t>0, \tag{3.6}$$

and the equality is given for those $t>0$ such that $\sqrt{q}\sqrt{(a*a*k)(t)} - \sqrt{p}\sqrt{k(t)} = 0$. Since the pair (a,k) satisfies the CP-condition, we conclude that there exists $t_0>0$, depending on p and q, such that

$$\frac{q}{p}k(t_0) = (a*a*k)(t_0). \tag{3.7}$$

Hence,

$$f(t_0) = 2\sqrt{pq}\sqrt{\frac{(a*a*k)(t_0)k(t_0)}{(a*k)^2(t_0)}} = 2q\frac{k(t_0)}{(a*k)(t_0)}. \tag{3.8}$$

From (3.5) we deduce that for all $x \in D(A^2)$,

$$||Ax|| \leq \min_{t>0} f(t) \leq f(t_0) = 2M\sqrt{2||x||||A^2x||}\sqrt{\frac{(a*a*k)(t_0)k(t_0)}{(a*k)^2(t_0)}}. \tag{3.9}$$

Hence,

$$||Ax||^2 \leq 8M^2C_{a,k}||x||\,||A^2x||,$$

concluding the proof. □

In what follows we will deduce from our main result examples concerning different types of strongly continuous families arising in applications to abstract evolution equations. We begin with norm inequalities for generators of α-times integrated β-regularized resolvents

$(S_{\alpha,\beta}(t))_{t\geq 0}$. That means, according to the definition given in the introduction, that they satisfy:

$$S_{\alpha,\beta}(t)x = \frac{t^\alpha}{\Gamma(\alpha+1)}x + \int_0^t \frac{(t-s)^\beta}{\Gamma(\beta+1)} AS_{\alpha,\beta}(s)xds, \quad t>0, \quad x\in X,$$

i.e., a $(g_{\beta+1}, g_{\alpha+1})$-regularized resolvent for some $\alpha, \beta > -1$. Recall that for $\alpha = 0$, the existence of $(S_{0,\beta}(t))_{t\geq 0}$ is equivalent to the well-posedness of the abstract fractional differential equation

$$D_t^{\beta+1}u(t) = Au(t), \quad t>0, \quad \beta > -1, \tag{3.10}$$

with some initial conditions, where $D_t^{\beta+1}$ denotes the Caputo's fractional derivative , see [3]. In case $\alpha > 0$, these families corresponds to α-times integrated solutions of the above equation.

Theorem 3.2. *Let A be generator of an α-times integrated β-regularized resolvent $(S_{\alpha,\beta}(t))_{t\geq 0}$ for some $\alpha, \beta > -1$ and suppose that there is $M \geq 1$ such that*

$$||S_{\alpha,\beta}(t)|| \leq M\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad t\geq 0.$$

Then for all $x \in D(A^2)$ we have

$$||Ax||^2 \leq 8M^2\frac{\Gamma(\alpha+\beta+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2\beta+3)}||x||\,||A^2x||. \tag{3.11}$$

Proof. The pair $(g_{\beta+1}, g_{\alpha+1})$ satisfies the CP-condition and using the formula $t^a * t^b = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}t^{a+b+1}(a, b > -1)$, the following equality

$$\frac{(a*a*k)(t)k(t)}{(a*k)^2(t)} = \frac{\Gamma(\alpha+\beta+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2\beta+3)}$$

holds for any $t > 0$. Hence, the conclusion follows from Theorem 3.1. □

Remarks. In the case of the well-posed fractional equation (3.10), we first analyze the qualitative behavior of $f_\alpha(\beta)$, given in Proposition 2.3, for different values of α. Then we apply it in the study of the Kallman-Rota inequality (3.11).

(I) When $\alpha \to -1$ then $f_\alpha(\beta) \to 0$ for any $\beta > -1$. As a consequence, we can always choose $\epsilon_{\alpha,\beta}$ as smaller as we want, such that

$$||Ax||^2 \leq \epsilon_{\alpha,\beta}||x||\,||A^2x||, \qquad x\in D(A^2). \tag{3.12}$$

In particular for $0 < \beta < 1$, the inequality (3.12) holds for operators A governing fractional differential equations that are between parabolic and hyperbolic type. Moreover, the constant $\epsilon_{\alpha,\beta}$ near to $\beta = 1$ (parabolic) is smaller that near to $\beta = 0$ (hyperbolic).

(II) When $\alpha = 0$ then $f_0(\beta) = \frac{\Gamma(\beta+2)^2}{\Gamma(2\beta+3)}$ for $\beta > -1$. Note that such function is decreasing so that the constant in case of the second order abstract differential equation ($f_0(1) = \frac{1}{6}$), will be always smaller that the constant in case of the first order equation ($f_0(0) = \frac{1}{2}$). As a consequence, inequality (3.11) shows in full generality the continuous transition between the constants 4 and 4/3 appearing in the Kallman-Rota inequality (1.1) in case that A generates a C_0-semigroup and cosine function, respectively.

(III) When $\alpha \to \infty$ then $f_\alpha(\beta) \to 1$. The situation is somewhat the inverse as in case (I): If we regularize by $k(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ with α going to ∞, then the constant $\epsilon_{\alpha,\beta}$ in (3.12) goes to $8M^2$. Moreover, again the constant near to the abstract differential equation of order 2 is smaller than the constant near to the abstract differential equation of order 1, for the same value of α.

The cases $\beta = 0$ and $\beta = 1$ in Theorem 3.2 give, respectively, the following corollaries. We notice that they were first proved by Cioranescu in the unpublished note [5].

Corollary 3.3. [5, Theorem 1] *Let A be the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\geq 0}$ for some $\alpha \geq 0$ and suppose that there is $M \geq 1$ such that*

$$||S_\alpha(t)|| \leq M\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad t \geq 0. \tag{3.13}$$

Then for all $x \in D(A^2)$ we have

$$||Ax||^2 \leq 8M^2\left(\frac{\alpha+1}{\alpha+2}\right)||x||\,||A^2x||. \tag{3.14}$$

As a simple consequence of the next corollary, we recover a main result of [15]: Let A be the generator of a bounded strongly continuous cosine function with bound constant one; then for all $x \in D(A^2)$ we have

$$||Ax||^2 \leq \frac{4}{3}||x||\,||A^2x||. \tag{3.15}$$

Corollary 3.4. [5, Theorem 2] *Let A be the generator of an α-times integrated cosine function $(C_\alpha(t))_{t\geq 0}$ for some $\alpha \geq 0$ and suppose that there is $M \geq 1$ such that*

$$||C_\alpha(t)|| \leq M\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad t \geq 0. \tag{3.16}$$

Then for all $x \in D(A^2)$ we have

$$||Ax||^2 \leq 8M^2\frac{(\alpha+1)(\alpha+2)}{(\alpha+3)(\alpha+4)}||x||\,||A^2x||. \tag{3.17}$$

The case $k = \chi_{[0,\infty)}$ gives an explicit bound and the below result, which is new for the theory of Volterra equations of convolution type. Remind that the existence of a resolvent family $\{S(t)\}_{t\geq 0}$ (which corresponds to a $(a, \chi_{[0,\infty)})$-regularized resolvent as observed in the Introduction) is equivalent of the well-posedness of the Volterra equation

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t), \quad t > 0,$$

where $a \in L^1_{loc}(\mathbb{R}_+)$ and $f \in L^1_{loc}(\mathbb{R}_+; X)$. See more details in [20].

Theorem 3.5. *Suppose that A is the generator of a resolvent family $\{S(t)\}_{t\geq 0}$ with $a \in L^1_{loc}(\mathbb{R}_+)$, positive and $\int_0^\infty a(t)dt = +\infty$. Moreover, assume that there is $M \geq 1$ such that*

$$||S(t)|| \leq M, \quad t \geq 0.$$

Then for all $x \in D(A^2)$ we have

$$||Ax||^2 \leq 8M^2||A^2x||\,||x||.$$

Proof. By Lemma 2.2 (ii), we have that the pair $(a, \chi_{[0,\infty)})$ satisfies the CP-condition. We follow the proof of Theorem 3.1 and apply formulae (3.9) and (3.8) to conclude that

$$||Ax|| \leq \frac{2M}{\displaystyle\int_0^{(1*a*a)^{-1}(\frac{1}{2}\frac{||A^2x||}{||x||})} a(s)ds} ||A^2x|| = 2M\sqrt{2||x||\,||A^2x||}\sqrt{\frac{(a*a*\chi_{[0,\infty)})(t_0)}{(a*\chi_{[0,\infty)})^2(t_0)}} \tag{3.18}$$

for some $t_0 > 0$. Since $a * a * \chi_{[0,\infty)}(t) \leq (a * \chi_{[0,\infty)})^2(t)$ for all $t \geq 0$, we conclude the result. □

Remarks. Note that the first inequality in the formula (3.18) is sharp in the sense that for $a = \chi_{[0,\infty)}$ and $a(t) = t$, we obtain the known inequality for uniformly bounded C_0-semigroups and cosine functions (respectively) commented in the Introduction.

4. Examples, applications and comments

In this last section, we apply our results in concrete examples including some extensions of the classical Kallman-Rota inequalities for C_0-semigroups and cosine functions.

4.1. **The Laplacian operator.** Let X be one of the Banach spaces $L^p(\mathbb{R}^N), 1 \leq p < \infty$, with the usual Lebesgue norm $\|\ \|_p$, or $C_0(\mathbb{R}^N), BUC(\mathbb{R}^N), C_b(\mathbb{R}^N)$ with the sup-norm $\|\ \|_\infty$. We consider the Laplace operator Δ in all these spaces with its maximal distributional domain. It is well known that in X the operator Δ generates the Gaussian semigroup (which is, in fact, a contraction C_0-semigroup) and then

$$||\Delta f||_p^2 \leq 4||f||_p\,||\Delta^2 f||_p, \quad f \in D(\Delta^2), \qquad 1 \leq p \leq \infty. \tag{4.1}$$

In the case that $N = 1$ or $p = 2$, the Laplacian generates a contractive cosine function in X and

$$||\Delta f||_p^2 \leq \frac{6}{5}||f||_p\,||\Delta^2 f||_p, \quad f \in D(\Delta^2), \qquad 1 \leq p \leq \infty,$$

as consequence of [22, Theorem 3]. Note that this constant is sharp in $L^1(\mathbb{R})$, see [22, p. 52].

In the case that $N \geq 2$ and $p \neq 2$, the Laplacian generates an α-times integrated cosine function $(C_\alpha(t))_{t>0}$ in X such that

$$\|C_\alpha(t)\| \leq M\frac{t^\alpha}{\Gamma(\alpha+1)}, \qquad t \geq 0,$$

for all $\alpha > (N-1)|\frac{1}{2} - \frac{1}{p}|$, ([6, Proposition 3.2], [10, Theorem 4.3]). By Corollary 3.4, we conclude that

$$||\Delta f||_p^2 \leq 8M^2\frac{(\alpha+1)(\alpha+2)}{(\alpha+3)(\alpha+4)}||f||_p\,||\Delta^2 f||_p, \quad f \in D(\Delta^2), \qquad 1 \leq p \leq \infty.$$

Under the natural condition that $M = 1$ (we have not found a suitable reference of this fact), we may conclude that there are p closer to 2 such that the constant 4 in the inequality (4.1) can be improved in those Lebesgue spaces $L^p(\mathbb{R}^N)$.

4.2. **Continuous-time Markov chains on $\mathbb{Z}_+$.** A matrix $Q = \{q_{ij};\ i, j \in \mathbb{Z}_+\}$, is called a q-matrix (stable) if it satisfies the following conditions

$$0 \leq q_{ij} < \infty, \quad i \neq j, \tag{4.2}$$

$$\sum_{i \neq j} q_{i,j} \leq q_{i,i} \equiv q_i < \infty, \quad \forall i \in \mathbb{Z}_+. \tag{4.3}$$

For a q-matrix Q, there always exists a transition function $F(t) = \{f_{ij}(t);\ i, j \in \mathbb{Z}_+\}$; see [1]. That is, $F(t)$ satisfy the conditions:
(i) $f_{ij}(t) \geq 0$ and $f_{ij}(0) = \delta_{i,j}$ for all $t \geq 0, \quad i, j \in \mathbb{Z}_+$;
(ii) $\sum_{j \in \mathbb{Z}_+} f_{ij}(t) \leq 1$ for all $t \geq 0, \quad i \in \mathbb{Z}_+$;
(iii) $f_{ij}(s + t) = \sum_{k \in \mathbb{Z}_+} f_{ik}(s) f_{kj}(t)$ for all $t \geq 0, \quad i, j \in \mathbb{Z}_+$;
(iv) $\lim_{t \to 0} f_{ii}(t) = 1$ for all $t \geq 0, \quad i \in \mathbb{Z}_+$.
Moreover,

$$\lim_{t \to 0} \frac{F(t) - I}{t} = Q \tag{4.4}$$

holds componentwise. Every transition function $F(t)$ is a positive continuous contraction semigroup on $l^1(\mathbb{R}_+)$, but $F(t)$ is generally not a continuous semigroup on $l^\infty(\mathbb{Z}_+)$ except for the trivial case that the q-matrix Q is uniformly bounded. For further information on this topic, see e.g. [1]. However, it was proved in [16, Theorem 5.2] that a q-matrix Q generates a positive once integrated semigroup of contractions in $l^\infty(\mathbb{Z}_+)$ if and only if $\lambda - Q$ is injective on $l^\infty(\mathbb{Z}_+)$ for some $\lambda > 0$. Hence, Corollary 3.3 applies with $\alpha = 1$ and $M = 1$, obtaining the following: For each q-matrix Q for which there exists $\lambda > 0$ such that $(\lambda - Q)y = 0$ implies $y = 0$, we have

$$||Qx||_\infty^2 \leq \frac{16}{3} ||x||_\infty ||Q^2 x||_\infty,$$

for all $x = (x_n) \in D(Q^2) \subset l^\infty(\mathbb{Z}_+)$.

4.3. **Integrated semigroups in ordered spaces.** Recall that Arendt proved in [2] that a resolvent positive operator A generates a once integrated semigroup (satisfying the condition (3.13) for $\alpha = 1$) if the underlying Banach lattice has order continuous norm. We conclude that there is $M \geq 1$ such that

$$||Ax||^2 \leq \frac{16}{3} M^2 ||x||\, ||A^2 x||, \qquad x \in D(A^2).$$

See other examples of α-times integrated semigroups generated by differential operators in Euclidean spaces in [10].

4.4. **Fractional relaxation equation.** Let X be a Banach space and for $0 < \alpha < 1$ consider the fractional relaxation equation

$$u'(t) - AD_t^\alpha u(t) + u(t) = f(t), \quad t > 0, \tag{4.5}$$

with initial condition $u(0) = 0$ and f an appropriate X-valued function. Equation (4.5) corresponds to the abstract version of the Basset problem (see [13]). We recall that the Basset equation arises in fluid dynamics concerning the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, see e.g. [19]. Recall that $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha > 0$, and $e_{-1}(t) = e^{-t}$ for $t > 0$. As stated in [13, Section 3], well posedness of equation

(4.5) is equivalent to the existence of an (a, k)-regularized family $(R(t))_{t\geq 0}$ generated by A, with

$$a = g_{1-\alpha} - g_{1-\alpha} * e_{-1}, \text{ and } k = e_{-1}.$$

By the Lemma 2.2 (i), we get that (a, k) satisfies the CP-condition. Note that

$$k * a = g_{1-\alpha} * (e_1 - te_1),$$

and

$$k * a * a \quad = \quad g_{2-2\alpha} * (e_1 - 2te_1 + \tfrac{1}{2}t^2 e_1).$$

We define the function g by

$$g(t) := \frac{k * a * a(t)k(t)}{(k * a)^2(t)} = \frac{\Gamma(1-\alpha)^2}{\Gamma(2-2\alpha)} \frac{\int_0^t s^{1-2\alpha} e^s (1 - 2(t-s) + \frac{1}{2}(t-s)^2) ds}{\left(\int_0^t s^{-\alpha} e^s (1-(t-s)) ds\right)^2}, \qquad t > 0.$$

Note that $\lim_{t\to 0} g(t) = \frac{\Gamma(2-\alpha)\Gamma(1-\alpha)}{2\Gamma(2-2\alpha)}$. Now we use the asymptotic expansion as $t \to \infty$ of

$${}_1F_1(a; c; t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{ts} s^{(a-1)} (1-s)^{c-a-1} ds = \frac{\Gamma(c)}{\Gamma(a)} e^t t^{a-c} [1 + O(|t|^{-1})]$$

for $c > a > 0$ (see [18, p. 289]) to conclude that $\lim_{t\to\infty} g(t) < \infty$ and $\|g\|_\infty < \infty$. In the case that $\|R(t)\| \leq Me^{-t}$ for $t > 0$, we may apply Theorem 3.1 to obtain that

$$||Ax||^2 \leq 8M^2 \|g\|_\infty ||x|| \, ||A^2 x||, \qquad x \in D(A^2).$$

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