UNIFORM STABILITY OF $(a,k)-$REGULARIZED FAMILIES

CARLOS LIZAMA, PEDRO J. MIANA, AND FELIPE POBLETE

Abstract. In this article we study the uniform stability of an (a,k)-regularized family $\{S(t)\}_{t\geq 0}$ generated by a closed operator A. We give sufficient conditions, on the scalar kernels a, k and the operator A, to ensure the uniform stability of the family $\{S(t)\}_{t\geq 0}$ in Hilbert spaces. Our main result is a generalization of [5, Theorem 1], concerning the stability of resolvent families, and can be seen as a substantial generalization of the Gearhart-Greiner-Prüss characterization of exponential stability for strongly continuous semigroups.

1. Introduction

Let A be a closed operator with domain $D(A)$ defined on a complex Banach space X; $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$. Recall that an (a,k)-regularized family generated by A is a strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ (the set of bounded and linear operators defined in X) which satisfies the following conditions:

(i) $S(0) = k(0)I$;

(ii) $S(t)x \in D(A)$ and $S(t)Ax = AS(t)x$ for all $x \in D(A)$ and $t \geq 0$;

(iii) $S(t)x = k(t)x + \int_0^t a(t-s)AS(s)x ds,\ t \geq 0,\ x \in D(A)$.

This notion generalizes the theories of C_0-semigroups, r-times-integrated semigroups, k-convoluted semigroups, r-times integrated cosine families, and k-times resolvent families, among others. We observe that existence as well as structural properties of (a,k)-regularized families have been studied by several authors in recent years (see [4, 6] and references therein).

Existence, uniqueness and qualitative properties of solutions for wide classes of linear evolution equations are associated to (a,k)-regularized families. For example, the abstract Cauchy problem of first and second order, Volterra equations of convolution type like

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t),$$

and fractional order differential equations, among others, see for example [3, 4] .

We note that a large number of results concerning C_0-semigroups, resolvent families, convoluted semigroups and cosine functions, can be presented in a new and unified look on the theory of general (a,k)-regularized families. However, the study of stability for this general structure remains an untreated topic in the literature.

In this paper, we study uniform stability of (a,k)-regularized families. Note that the theory of stability is important since stable (a,k)-regularized families correspond one-to-one to stable well-posed abstract linear equations. It is also important since stability plays a central role in the structural theory of operators.

2010 *Mathematics Subject Classification.* 45M10; 47D06; 47D09; 47D62.

Key words and phrases. Stability; strongly continuous semigroups; strongly continuous cosine families; Hilbert spaces .

The first author is partially supported by Proyecto FONDECYT 1100485. The second author has been partly supported by the Ministry Education and Science DGI-FEDER (MTM 2010-16679) and the DGA project (E-64).

We give sufficient conditions for the uniform stability of the (a,k)-regularized family in Hilbert spaces. Our main result can be seen as substantial generalization of the Gearhart-Greiner-Prüss characterization of exponential stability for strongly continuous semigroups, see for example [2, Theorem V.1.11]. Our results also allow to study the identification of the kernels a and k. More precisely, we prove the following main result in the third section.

Suppose a, k are 1-regular kernels and that k satisfies the **(H)**-condition (see below). Assume that A generates an (a,k)-resolvent family $\{S(t)\}_{t\geq 0}$ with finite growth bound in a Hilbert space $\mathcal{H}$, and the following conditions hold.

(H1) $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\Re(\lambda) \geq 0$, $\lambda \neq 0$.

(H2) For all $x \in \mathcal{H}$, $\lim\limits_{\lambda\to 0} \frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1}x = Bx$ exists.

(H3) $\sup\limits_{\Re\lambda>0} \|\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1}\| < \infty$.

Then $\{S(t)\}_{t\geq 0}$ is uniformly stable (Theorem 3.1).

In the second section we present some technical results about Laplace transforms and its estimates. We finish this paper with some examples and comments concerning stability of strongly continuous cosine families and α-resolvent families associated to fractional differential equations, see Section 4. Finally, an example concerning stability of the solutions of the Basset equation is also considered.

2. Estimates of Laplace transform

We say that $k \in L^1_{loc}(\mathbb{R}_+)$ is of subexponential growth if for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|k(t)| \leq C_\varepsilon e^{\varepsilon t}$ *a.e.* $t \geq 0$. In this case the Laplace transform, $\hat{k}(\lambda)$, given by

$$\hat{k}(\lambda) := \int_0^\infty e^{-\lambda t} k(t)dt,$$

exists for all $\Re\lambda > 0$.

Definition 2.1. Let $k \in L^1_{loc}(\mathbb{R}_+)$ be of subexponential growth such that there exists $\lim\limits_{\lambda\to i\rho} \hat{k}(\lambda) = \hat{k}(i\rho)$ for all $|\rho| \geq 1$. We say that k satisfies the **(H)**-condition if there exists a constant $M > 0$ such that

$$\frac{1}{|\rho\hat{k}(i\rho)|} \leq M$$

for all $|\rho| \geq 1$.

In what follows, we denote:

$$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} \text{ for } \alpha > 0 \text{ and } e_{-b}(t) := e^{-bt} \text{ for } b \geq 0.$$

Example. The functions $e_{-b}\cos$, for $b > 0$; g_α, for $0 < \alpha \leq 1$; $\chi_{(1,\infty)}g_{n+1}$, for $n \in \mathbb{N}$ and e_{-b} for $b \geq 0$ satisfy the **(H)**-condition. However the characteristic function $\chi_{(0,1)}$, sin, cos and g_α for $\alpha > 1$ do not satisfy this **(H)**-condition.

We recall that $b \in L^1_{loc}(\mathbb{R}_+)$ of subexponential growth is called n-regular ($n \in \mathbb{N}$) if there exist a constant $c > 0$ such that

$$|\lambda^k \frac{d^k\hat{b}}{d\lambda^k}(\lambda)| \leq c|\hat{b}(\lambda)|$$

for all $\Re\lambda > 0$ and $0 \le k \le n$, see [9, Definition 3.3]. In this paper, we will use only the definition of 1-regular.

Example 2.2. We give some examples of 1-regular functions.

- Let $a = e_{-b}g_\alpha$ where $\Re b \ge 0$, $\alpha > 0$. We note that
$$\sup_{\Re\lambda>0}\left|\frac{\lambda\hat{a}'(\lambda)}{\hat{a}(\lambda)}\right| = \sup_{\Re\lambda>0}\frac{\alpha|\lambda|}{|\lambda+b|} < \infty.$$
Then a is an 1-regular function.
- Let $0<\alpha<1$ and $a = g_{1-\alpha} - (g_{1-\alpha} * e_{-1})$. We have that $\hat{a}(\lambda) = \frac{\lambda^\alpha}{\lambda+1}$, then
$$\left|\frac{\lambda\hat{a}'(\lambda)}{\hat{a}(\lambda)}\right| = \left|\frac{\alpha+(\alpha-1)\lambda}{\lambda+1}\right|,$$
which is bounded for all $\Re\lambda > 0$.

Remark 2.3. Note that if $b \in L^1_{loc}(\mathbb{R}_+)$ is 1-regular, then

(i) $\hat{b}(i\rho) = \lim_{\lambda\to i\rho}\hat{b}(\lambda)$ exist for each $\rho \neq 0$.

(ii) $\hat{b}(\lambda) \neq 0$ for all $\Re\lambda \ge 0$, $\lambda \neq 0$.

(iii) $|\rho\hat{b}'(i\rho)| \le c|\hat{b}(i\rho)|$ for a.a. $\rho \in \mathbb{R}$.

see [9, Lemma 8.1].

Lemma 2.4. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ be of subexponential growth and 1-regular; Assume that k satisfies the* **(H)**-*condition and let $\omega > 0$ be fixed. Then*

(i)
$$\sup_{|\rho|\ge 1}\left|\frac{1}{\hat{k}(i\rho)}\left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)}-1\right)\right| < \infty.$$

(ii) If $\hat{k}(\omega+i(\cdot)) \in L^2(\mathbb{R}\setminus[-1,1])$ then $\left(1-\frac{\hat{k}(\omega+i(\cdot))}{\hat{k}(i(\cdot))}\right) \in L^2(\mathbb{R}\setminus[-1,1])$.

(iii) If $\lim_{\lambda\to\infty}\lambda\hat{k}(\lambda)$ exists, then $\hat{k}(\omega+i(\cdot)) \in L^2(\mathbb{R}\setminus[-1,1])$.

Proof. We note that 1-regularity of a implies that $i\rho\left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)}-1\right)$ is bounded for $|\rho| \ge 1$. Hence, from the identity
$$\frac{1}{\hat{k}(i\rho)}\left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)}-1\right) = \frac{i\rho}{i\rho\hat{k}(i\rho)}\left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)}-1\right)$$
and the **(H)**-condition, the conclusion (i) follows. To show (ii), note that
$$\left(1-\frac{\hat{k}(\omega+i\rho)}{\hat{k}(i\rho)}\right) = i\rho\left(\frac{\hat{k}(i\rho)}{\hat{k}(\omega+i\rho)}-1\right)\frac{1}{i\rho\hat{k}(i\rho)}\hat{k}(\omega+i\rho)$$
where $i\rho\left(\frac{\hat{k}(i\rho)}{\hat{k}(\omega+i\rho)}-1\right)$ and $\frac{1}{i\rho\hat{k}(i\rho)}$ are bounded on $\mathbb{R}\setminus[-1,1]$ by 1-regularity of k and **(H)**-condition, respectively.

To show (iii) note that by hypothesis there exists $M > 0$ such that
$$|\hat{k}(\omega+i\rho)| = \frac{1}{|\omega+i\rho|}|(\omega+i\rho)\hat{k}(\omega+i\rho)| \le \frac{M}{|\omega+i\rho|},$$
which yields the claim. □

Remark 2.5. Recall that in case that $\lim_{t\to 0} k(t)$ exists, then

$$\lim_{t\to 0} k(t) = \lim_{\lambda\to\infty} \lambda\hat{k}(\lambda),$$

(see for example [1, Proposition 4.1.3]) and hence we may replace the condition $\lim_{\lambda\to\infty} \lambda\hat{k}(\lambda)$ by $\lim_{t\to 0} k(t)$ in *(iii)* of Lemma 2.4.

3. Main result

Recall that a family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ satisfying

$$\lim_{t\to\infty} \|S(t)\| = 0,$$

is called *uniformly stable.* The next theorem gives sufficient conditions about uniformly stability of (a, k)-regularized family $\{S(t)\}_{t\geq 0}$. The case of C_0-semigroups is known as Gearhart-Greiner-Prüss theorem and may be found in [2, 8]. For resolvent families of operators, see [5, Theorem 1]. In this section, we modify the proof of [5] to consider the much more general case of (a, k)-regularized families. We write

$$H(\lambda) := \hat{k}(\lambda)(I - \hat{a}(\lambda)A)^{-1},$$

whenever is well defined. In view of [6, Proposition 3.1], if A generates an (a, k)-regularized family $\{S(t)\}_{t\geq 0}$, exponentially bounded of type (M, ω) (i.e. $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ with $M > 0$ and $\omega \in \mathbb{R}$) then $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\Re\lambda > \omega$ and

$$\hat{S}(\lambda) = H(\lambda), \text{ for all } \Re\lambda > \omega.$$

Theorem 3.1. *Suppose $a \in L^1_{loc}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$ are 1-regular and k satisfies the **(H)**-condition. Assume that A generates an (a, k)-resolvent family $\{S(t)\}_{t\geq 0}$ with finite growth bound in a Hilbert space $\mathcal{H}$, and the following conditions:*

- **(H1)** $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ *for all* $\Re(\lambda) \geq 0$, $\lambda \neq 0$.
- **(H2)** *For all* $x \in \mathcal{H}$, $\lim_{\lambda\to 0} H(\lambda)x = Bx$ *defines a bounded operator.*
- **(H3)** $\sup_{\lambda\in\mathbb{C}_+} \|H(\lambda)\| < \infty$ *where* $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$.

Then $\{S(t)\}_{t\geq 0}$ is uniformly stable.

Proof. By hypothesis there are constants $M > 0$ and $\omega_0 \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{t\omega_0}$ for all $t \geq 0$. We may suppose that $\omega_0 \geq 0$. Let $\omega > \omega_0 + 1$ be given and define $R(t) := e^{-t\omega}S(t)$ and $\|R(t)\| \leq Me^{-(\omega-\omega_0)t}$ for $t \geq 0$. Let $x \in \mathcal{H}$ be fixed, and observe that $\chi_{[0,\infty)}(\cdot)R(\cdot)x$ is in $L^2(\mathbb{R}, \mathcal{H})$ where $\chi_{[0,\infty)}$ denotes the characteristic function. In fact,

$$\begin{aligned} \|\chi_{[0,\infty)}(\cdot)R(\cdot)x\|_2^2 &\leq \int_0^\infty \|R(t)x\|^2 dt \leq M^2 \int_0^\infty \left(e^{-(\omega-\omega_0)t}\|x\|\right)^2 dt \\ &\leq \frac{M^2\|x\|^2}{2(\omega-\omega_0)}; \end{aligned}$$

Hence

$$\|\chi_{[0,\infty)}(\cdot)R(\cdot)x\|_2 \leq \frac{M\|x\|}{\sqrt{2(\omega-\omega_0)}}.$$

Because $\mathcal{H}$ is a Hilbert space, the Plancherel theorem shows us that the Fourier transform $\mathcal{F}$ satisfies $\|\mathcal{F}f\|_2 = \sqrt{2\pi}\|f\|_2$ for all $f \in L^2(\mathbb{R}, \mathcal{H})$. On the other hand, because $\{S(t)\}_{t\geq 0}$ is an exponentially bounded family, its Laplace transform $\hat{S}(\lambda)$ is well-defined, holomorphic and satisfies $H(\lambda)=\hat{S}(\lambda)$ for all $\Re\lambda>0$. Hence, we have for all $x\in\mathcal{H}$ and $s\in\mathbb{R}$,

$$\begin{aligned} H(\omega+is)x &= \hat{S}(\omega+is)x=\int_0^\infty e^{-(\omega+is)t}S(t)xdt=\int_0^\infty e^{-\omega t}e^{-ist}S(t)xdt \\ &= \int_0^\infty e^{-ist}R(t)xdt=\int_{-\infty}^\infty e^{-ist}\chi_{[0,\infty)}(t)R(t)xdt \\ &= \mathcal{F}(\chi_{[0,\infty)}(\cdot)R(\cdot)x)(s) \end{aligned}$$

It follows from the Plancherel theorem that $H(\omega+i(\cdot))x\in L^2(\mathbb{R},\mathcal{H})$ and

$$\|H(\omega+i\cdot)x\|_2=\sqrt{2\pi}\|\chi_{[0,\infty)}(\cdot)R(\cdot)x\|_2\le M\sqrt{\frac{\pi}{\omega-\omega_0}}\|x\|. \tag{3.1}$$

We observe that by (**H2**) $\lim_{\lambda\to 0}H(\lambda)x=Bx$ exist for all $x\in\mathcal{H}$ and $B\in B(\mathcal{H})$. Also, from 1-regularity of a,k, Remark 2.3 and (**H1**), we obtain that $H(i\rho)x:=\lim_{\lambda\to i\rho}H(\lambda)x$ is well defined for all $x\in\mathcal{H}$ and, by (**H3**) and the Banach-Steinhaus theorem, that $H(i\rho)$ is bounded for each $\rho\in\mathbb{R}$. It follows from the uniform boundedness principle that H is also uniformly bounded in the imaginary axis $i\mathbb{R}$.

On the other hand, the identity

$$H(\lambda)x-\hat{a}(\lambda)AH(\lambda)x=k(\lambda)x$$

is valid for all $x\in\mathcal{H}$ and $\Re\lambda\ge 0$, $\lambda\ne 0$. It follows, replacing x by $H(\lambda_2)x$ and λ by λ_1, that

$$H(\lambda_1)H(\lambda_2)x-\hat{a}(\lambda_1)AH(\lambda_1)H(\lambda_2)x=\hat{k}(\lambda_1)H(\lambda_2)x$$

for all $x\in\mathcal{H}$ and $\Re\lambda_1,\Re\lambda_2\ge 0$, $\lambda_1,\lambda_2\ne 0$. Now, taking into account the above equation for $\lambda_1=\omega+i\rho$ and $\lambda_2=i\rho$ we have,

$$\begin{aligned} H(i\rho)x &= H(\omega+i\rho)x+H(i\rho)x-\frac{1}{\hat{k}(i\rho)}\left[H(\omega+i\rho)H(i\rho)\right]x \\ &\quad +\frac{1}{\hat{k}(i\rho)}\Big[H(i\rho)H(\omega+i\rho)-\hat{k}(i\rho)H(\omega+i\rho)\Big]x \\ &= H(\omega+i\rho)x+H(i\rho)x-\frac{1}{\hat{k}(i\rho)}\Big[\hat{k}(\omega+i\rho)H(i\rho)+\hat{a}(\omega+i\rho)AH(\omega+i\rho)H(i\rho)\Big]x \\ &\quad +\frac{1}{\hat{k}(i\rho)}[\hat{a}(i\rho)AH(i\rho)H(\omega+i\rho)]x \\ &= H(\omega+i\rho)x+\left(1-\frac{\hat{k}(\omega+i\rho)}{\hat{k}(i\rho)}\right)H(i\rho)x \\ &\quad +\frac{1}{\hat{k}(i\rho)}\left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)}-1\right)\hat{a}(\omega+i\rho)[AH(\omega+i\rho)H(i\rho)]x, \end{aligned}$$

and finally we obtain that

$$\begin{aligned} H(i\rho)x &= H(\omega+i\rho)x+\left(1-\frac{\hat{k}(\omega+i\rho)}{\hat{k}(i\rho)}\right)H(i\rho)x \\ &+ \frac{1}{\hat{k}(i\rho)}\left(\frac{\hat{a}(i\rho)}{\hat{a}(\omega+i\rho)}-1\right)H(i\rho)[H(\omega+i\rho)-\hat{k}(\omega+i\rho)]x. \end{aligned} \tag{3.2}$$

Choose a function ϕ in $C_0^\infty(\mathbb{R})$, defined by $\phi(\rho) = 1$ for $|\rho| < 1$ and $\phi(\rho) = 0$ for $|\rho| \geq 2$. Define $\psi(\rho) = 1 - \phi(\rho)$, $\rho \in \mathbb{R}$. Then using the uniform boundedness of $H(i\cdot)$ in $\mathbb{R}$ and (3.1) in (3.2) together with Lemma 2.4 implies that $\psi(\cdot)H(i\cdot)x \in L^2(\mathbb{R}, \mathcal{H})$ and

$$\begin{aligned}
\|\psi(\cdot)H(i\cdot)x\|_2^2 &= \int_\infty^\infty \|\psi(\rho)H(i\rho)x\|^2 d\rho \\
&\leq \int_{|\rho|\geq 2} \|H(i\rho)x\|^2 d\rho + \int_{1<|\rho|<2} \psi(\rho)\|H(i\rho)x\|^2 d\rho \\
&\leq M_0\|x\|^2.
\end{aligned}$$

Analogously, we can prove that $H(\omega + i(\cdot))^* x \in L^2(\mathbb{R}, \mathcal{H})$ and following the same argument as above we conclude that $\psi(\cdot)H(i(\cdot))^* x \in L^2(\mathbb{R}, \mathcal{H})$. By Parseval's theorem, there exists a function $u \in L^2(\mathbb{R}, \mathcal{H})$ such that

$$\mathcal{F}(u(\cdot)x)(\rho) = \psi(\rho)H(i\rho)x \text{ for a.a } \rho \in \mathbb{R}.$$

It follows that

$$\mathcal{F}(u(\cdot)x)'(\rho) = \psi'(\rho)H(i\rho)x + i\psi(\rho)H'(i\rho)x. \tag{3.3}$$

Note that for all $\Re(\lambda) \geq 0,\ \lambda \neq 0$ we have from the definition of $H(\lambda)$

$$\begin{aligned}
H'(\lambda) &= \left(\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}\right)' (\frac{1}{\hat{a}(\lambda)} - A)^{-1} - \frac{\hat{k}(\lambda)\hat{a}(\lambda)'}{\hat{a}(\lambda)^3}(\frac{1}{\hat{a}(\lambda)} - A)^{-2} \\
&= \frac{\hat{k}(\lambda)'\hat{a}(\lambda) - \hat{k}(\lambda)\hat{a}(\lambda)'}{\hat{a}(\lambda)^2}(\frac{1}{\hat{a}(\lambda)} - A)^{-1} - \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)\hat{k}(\lambda)}\left(\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}\right)^2 (\frac{1}{\hat{a}(\lambda)} - A)^{-2} \\
&= \frac{\hat{k}(\lambda)'}{\hat{k}(\lambda)}\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1} - \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)}\frac{\hat{k}(\lambda)}{\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)} - A)^{-1} - \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)\hat{k}(\lambda)}H(\lambda)^2 \\
&= \frac{\hat{k}(\lambda)'}{\hat{k}(\lambda)}H(\lambda) - \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)}H(\lambda) - \frac{\hat{a}(\lambda)'}{\hat{a}(\lambda)\hat{k}(\lambda)}H(\lambda)^2.
\end{aligned}$$

Replacing $\lambda = i\rho,\ \rho \neq 0$ we get from (3.3) that

$$\mathcal{F}(u(\cdot)x)'(\rho) = \psi'(\rho)H(i\rho)x + i\psi(\rho)\left(\frac{\hat{k}'(i\rho)}{\hat{k}(i\rho)}H(i\rho)x - \frac{\hat{a}(i\rho)'}{\hat{a}(i\rho)}H(i\rho)x - \frac{\hat{a}'(i\rho)}{\hat{k}(i\rho)\hat{a}(i\rho)}H(i\rho)^2 x\right).$$

But, by (**H**) and 1-regularity of $a(\cdot)$ there exists M_0 such that

$$\left|\frac{\hat{a}'(i\rho)}{\hat{k}(i\rho)\hat{a}(i\rho)}\right| = \left|\frac{1}{i\rho\hat{k}(i\rho)}\right|\left|\frac{i\rho\hat{a}'(i\rho)}{\hat{a}(i\rho)}\right| \leq M_0$$

for all $\rho \in \mathbb{R}$, $|\rho| > 1$. Moreover by the 1-regularity of $k(\cdot)$ and $a(\cdot)$, we have that $\frac{\hat{k}'(i(\cdot))}{\hat{k}(i(\cdot))}$ and $\frac{\hat{a}'(i(\cdot))}{\hat{a}(i(\cdot))}$ are bounded in $\mathbb{R}\backslash[-1,1]$. This and the fact that $\psi(\cdot)H(i(\cdot))x,\ \psi(\cdot)H(i(\cdot))^* x^*$ are in $L^2(\mathbb{R}, \mathcal{H})$ for each $x, x^* \in \mathcal{H}$ gives,

$$\int_\infty^\infty |\langle \mathcal{F}(u(\cdot)x')(\rho), x^*\rangle| d\rho = \int_{\mathbb{R}\backslash[-1,1]} |\langle \mathcal{F}(u(\cdot)x')(\rho), x^*\rangle| d\rho \leq M_0\|x\|\|x^*\|. \tag{3.4}$$

On the other hand, again from the uniform boundedness of $H(i\cdot)$ in $\mathbb{R}$ we have that for each $t>0$

$$S_0(t) := \int_{-\infty}^{\infty} \phi(\rho)H(i\rho)e^{i\rho t}d\rho = \int_{-2}^{2} \phi(\rho)H(i\rho)e^{i\rho t}d\rho.$$

Hence, by the Riemann-Lebesgue lemma it follows that $S_0(t)\to 0$ in $B(\mathcal{H})$ as $t\to+\infty$.
Finally, for $x,x^*\in\mathcal{H}$ we have that

$$\begin{aligned}\langle S(t)x,x^*\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle H(i\rho)x,x^*\rangle e^{i\rho t}d\rho\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle \phi(\rho)H(i\rho)x,x^*\rangle e^{i\rho t}d\rho + \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle \psi(\rho)H(i\rho)x,x^*\rangle e^{i\rho t}d\rho.\end{aligned}$$

Integrating by parts in the second integral, we get

$$\begin{aligned}\langle S(t)x,x^*\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle \phi(\rho)H(i\rho)x,x^*\rangle e^{i\rho t}d\rho + \frac{1}{2\pi it}\int_{-\infty}^{\infty}\langle (\psi(\rho)H(i\rho))'x,x^*\rangle e^{i\rho t}d\rho\\ &= \frac{1}{2\pi}\langle S_0(t)x,x^*\rangle + \frac{1}{2\pi it}\int_{-\infty}^{\infty}\langle (\mathcal{F}(u(\cdot)))'(\rho)x,x^*\rangle e^{i\rho t}d\rho.\end{aligned}$$

Therefore $\|S(t)\| \le \frac{1}{2}\|S_0(t)\| + \frac{1}{2\pi t}M_0$, from which we obtain the result. □

Remark 3.2. Consider the following integral Volterra equation of scalar type:

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t), \tag{3.5}$$

where A is a closed and linear operator with domain $D(A)$ dense in X, $a\in L^1_{loc}(\mathbb{R}_+)$ is a scalar kernel and $f\in W^{1,1}(\mathbb{R}_+;X)$. It is well known that equation (3.5) is well-posed if and only if it admits a resolvent family, see for example [9]. In terms of the theory of (a,k)-regularized families, this correspond to an $(a,1)$-regularized family generated by A.
We recover the following result concerning stability of resolvent families which appeared in [5, Theorem 1].

Corollary 3.3. *Suppose $a\in L^1_{loc}(\mathbb{R}_+)$ is 1-regular; Let A be the generator of a resolvent family $\{S(t)\}_{t\ge0}$ with finite growth bound in a Hilbert space $\mathcal{H}$, and the following conditions:*

(1) $\frac{1}{\hat{a}(\lambda)}\in\rho(A)$ *for all* $\Re\lambda\ge 0$ $\lambda\neq 0$.
(2) $\lim_{\lambda\to0}\lambda\hat{a}(\lambda) =: a(\infty)\neq 0$ *and* $0\in\rho(A)$.
(3) $(\lambda-\lambda\hat{a}(\lambda)A)^{-1}$ *is uniformly bounded in* $\mathbb{C}_+ := \{\lambda\in\mathbb{C} : \Re\lambda>0\}$.

Then $\{S(t)\}_{t\ge0}$ is uniformly stable.

Proof. We observe that the scalar kernel $k=\chi_{[0,\infty)}$ is 1-regular and satisfies the (**H**)-condition. It follows from (1) and (3) that (**H1**) and (**H3**) of Theorem 3.1 are satisfied, so that we only need to verify (**H2**). For this we note that by (2)

$$\lim_{\lambda\to0}H(\lambda)x = \frac{1}{\lambda\hat{a}(\lambda)}\left(\frac{\lambda}{\lambda\hat{a}(\lambda)}-A\right)^{-1} = \frac{1}{a(\infty)}A^{-1}x,$$

and the conclusion follows. □

Recall that a strongly continuous family $S\equiv\{S(t)\}_{t\ge0}\subset\mathcal{B}(X)$ is called *uniformly integrable* if $t\mapsto\|S(t)\|$ is a measurable function and

$$\|S\|_{L^1} := \int_0^{\infty}\|S(t)\|dt < \infty.$$

Corollary 3.4. *Suppose $a \in L^1_{loc}(\mathbb{R})$ and $k \in C(\mathbb{R}_+)$ are 1-regular and k satisfies the **(H)**-condition. Assume that A generates an (a,k)-resolvent family $S \equiv \{S(t)\}_{t\geq 0}$ with finite growth bound in a Hilbert space $\mathcal{H}$. If $\{S(t)\}_{t\geq 0}$ is uniformly integrable then $\{S(t)\}_{t\geq 0}$ is uniformly stable.*

Proof. The fact that S is uniformly integrable implies that $H(\lambda)$ is well defined for all $\lambda \in \overline{\mathbb{C}}_+$. On the other hand, from the definition of an (a,k)-regularized family generated by A we can see that

$$\frac{1}{\hat{k}(\lambda)} H(\lambda)(I - \hat{a}(\lambda)A)x = x \quad \text{ for all } \lambda \in \overline{\mathbb{C}}_+ \setminus \{0\},\ x \in D(A),$$

then we obtain that $(I - \hat{a}(\lambda)A)$ is an injective operator. Now using that $H(\lambda)$ commutes with $(I - \hat{a}(\lambda)A)$ for all $x \in D(A)$ and $H(\lambda)D(A) \subset D(A)$, we conclude that the operator $(I - \hat{a}(\lambda)A)$ is surjective. Moreover $(I - \hat{a}(\lambda)A)^{-1}$ is bounded i.e. $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\Re\lambda \geq 0$, $\lambda \neq 0$. Then (**H1**) holds.
It follows from the above that $\hat{S}(\lambda) = H(\lambda)$ for all $\lambda \in \overline{\mathbb{C}}_+ \setminus \{0\}$ and hence, by application of the dominated convergence theorem, we obtain

$$\lim_{\lambda\to 0} H(\lambda)x = \int_0^\infty S(t)x dt =: Bx.$$

Therefore (**H2**) is satisfied. Finally, let $\lambda \in \overline{\mathbb{C}}_+$ then we have

$$\|H(\lambda)x\| \leq \int_0^\infty \|S(t)x\| dt \leq \|S\|_{L^1}\|x\|$$

and we conclude that (**H3**) holds. Then by Theorem 3.1 $\{S(t)\}_{t\geq 0}$ is uniformly stable. □

Since the function $\chi_{[0,\infty)}$ is 1-regular and satisfies the (**H**)-condition, we recover the following result on stability of C_0-semigroups due to Gearhart, Greiner and Prüss (see [2, Theorem V. 1.11]), taking $k = a = \chi_{[0,\infty)}$ in Theorem 3.1.

Corollary 3.5. *Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ with finite growth bound defined in a Hilbert space $\mathcal{H}$. The following conditions are equivalent.*

1. $\{\lambda \in \mathbb{C} : \Re\lambda \geq 0\} \subset \rho(A)$ *and* $\sup_{\Re\lambda > 0} \|R(\lambda; A)\| < \infty$.
2. *The semigroup $\{T(t)\}_{t\geq 0}$ is uniformly stable.*

4. Examples and comments

Example 4.1. Suppose that A is the generator of an $(e_{-b}g_\alpha, e_{-b})$-regularized family $\{S(t)\}_{t\geq 0}$ for some $\alpha > 1$ and $b > 0$ satisfying the following conditions:

1. $(b+\lambda)^\alpha \in \rho(A)$ for all $\Re\lambda \geq 0$;
2. $\sup_{\Re\lambda > 0} ||(\lambda + b)^{\alpha-1}((\lambda + b)^\alpha - A)^{-1}|| < \infty$;

Then $\{S(t)\}_{t\geq 0}$ is uniformly stable.

Proof. It follows from Example 2.2 that $a = e_{-b}g_\alpha$ and $k = e_{-b}$ are 1-regular. On the other hand

$$\frac{1}{|\rho \hat{e}_{-b}(i\rho)|} = \frac{|b + i\rho|}{|\rho|} \leq M$$

for some $M > 0$ and for all $|\rho| \geq 1$, proving that the (**H**)-condition holds. By (1)

$$\lim_{\lambda\to 0} H(\lambda)x = b^{\alpha-1}(b^\alpha - A)^{-1}x$$

for all $x \in \mathcal{H}$, and therefore the above limit defines a bounded operator. Then (**H2**) holds. It also follows from (1) and (2) that (**H1**) and (**H3**) of Theorem 3.1 are satisfied. Then $\{S(t)\}_{t\geq 0}$ is uniformly stable. $\square$

Let A be the generator of a strongly continuous cosine family $\{C(t)\}_{t\geq 0}$. It is well known that $\{C(t)\}_{t\geq 0}$ cannot be stable (because of the identity $I = 2C(t)^2 - C(2t)$ for $t \geq 0$). However, our above example in case $\alpha = 2$ shows that we can give a counterpart of Corollary 3.4 for strongly continuous cosine families of operators as follows.

Corollary 4.1. *Let $b > 0$. Suppose that A is the generator of a strongly continuous cosine families of operators $\{C(t)\}_{t\geq 0}$ satisfying the following conditions:*

(1) $(b+\lambda)^2 \in \rho(A)$ *for all* $\Re\lambda \geq 0$.

(2) $\sup_{\Re\lambda>0} ||(\lambda+b)((\lambda+b)^2 - A)^{-1}|| < \infty$.

Then $\{e^{-bt}C(t)\}_{t\geq 0}$ is uniformly stable.

Example 4.2. Note that Example 4.1 includes stability for the solution of the fractional differential equation

$$D^\alpha u(t) = Au(t);$$

with initial condition $u(0) = u_0$ or, equivalently, the solution of the integral equation of convolution type:

$$u(t) = u(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s)ds, \qquad t \geq 0.$$

Indeed, take $v(t) = e^{-bt}u(t)$ where $b > 0$ is given, then the above integral equation is equivalent to

$$v(t) = e^{-bt}v(0) + \int_0^t e^{-b(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Av(s)ds, \qquad t \geq 0. \tag{4.1}$$

Hence, we conclude that if the problem (4.1) is well-posed then, under the conditions (1) and (2), we have that $v(t) \to 0$ as $t \to \infty$.

In particular, if we consider the operator $Ax = \mu x$ for all $x \in \mathcal{H}$ where:

$$0 < \mu < b^\alpha, \tag{4.2}$$

then A generates a uniformly stable $(e_{-b}g_\alpha, e_{-b})$-regularized family $\{S(t)\}_{t\geq 0}$ given by

$$S(t) := e^{-bt}E_\alpha(\mu t^\alpha), \qquad t \geq 0,$$

where E_α denotes the Mittag-Leffler function. Indeed, we have that

$$\hat{S}(\lambda) = [\widehat{e^{-bt}E_\alpha(\mu t^\alpha)}](\lambda) = \frac{(b+\lambda)^{\alpha-1}}{(b+\lambda)^\alpha - \mu} = \hat{k}(\lambda)(1-\hat{a}(\lambda)\mu)^{-1}, \qquad \Re\lambda > 0,$$

where $a = e_{-b}g_\alpha$ and $k = e_{-b}$, which shows that $\{S(t)\}_{t\geq 0}$ is an $(e_{-b}g_\alpha, e_{-b})$-regularized family. We are going now to show properties (**H1**) and (**H2**) of Theorem 3.1. In fact, we note that $(b+\lambda)^\alpha \in \rho(A)$, for all $\Re\lambda \geq 0$ if and only if $(b+\lambda)^\alpha \neq \mu$, for all $\Re\lambda \geq 0$. If $(b+\lambda)^\alpha = \mu$ for some $\Re\lambda \geq 0$ then $\Re\lambda = \mu^{1/\alpha}\cos(2k\pi/\alpha) - b$ for $k \in \mathbb{Z}$. Therefore, condition (4.2) implies $\mu^{1/\alpha}\cos(2k\pi/\alpha) < b$, for all $k \in \mathbb{Z}$, and hence $(b+\lambda)^\alpha \in \rho(A)$ for all $\Re\lambda \geq 0$. We conclude that condition (**H1**) holds. On the other hand,

$$\hat{k}(\lambda)(1-\hat{a}(\lambda)\mu)^{-1} = \frac{(b+\lambda)^{\alpha-1}}{(b+\lambda)^\alpha - \mu} \to 0$$

as $\lambda \to \infty$. In consequence, using the fact that $\hat{k}(\lambda)(1-\hat{a}(\lambda)\mu)^{-1}$ is continuous in $\overline{\mathbb{C}_+}$, we conclude that $H(\lambda)$ is uniformly bounded in $\mathbb{C}_+$, namely, the condition (**H2**) holds. The claim follows.

It is illustrative to check the result in the following particular cases:

$\alpha = 1$: $0 < \mu < b$ implies $e^{-bt}E_1(\mu t) = e^{-(b-\mu)t} \to 0$ as $t \to \infty$;

$\alpha = 1/2$: $0 < \mu < b^{1/2}$ implies $e^{-bt}E_{1/2}(\mu t^{1/2}) = e^{-(b-\mu^2)t}[1 - \mathrm{erf}(\mu t^{1/2})] \to 0$ as $t \to \infty$;

$\alpha = 2$: $0 < \mu < b^2$ implies $e^{-bt}E_2(\mu t^2) = e^{-bt}\cosh(\sqrt{\mu}t) = \frac{1}{2}[e^{-(b-\sqrt{\mu})t} + e^{-(b+\sqrt{\mu})t}] \to 0$ as $t \to \infty$;

$\alpha = 4$: $0 < \mu < b^4$ implies $e^{-bt}E_4(\mu t^4) = \frac{1}{2}e^{-bt}[\cos(\sqrt[4]{\mu}t) + \cosh(\sqrt[4]{\mu}t)] = \frac{1}{4}e^{-bt}[2\cos(\sqrt[4]{\mu}t) + e^{-(b-\sqrt[4]{\mu})t} + e^{-(b+\sqrt[4]{\mu})t}] \to 0$ as $t \to \infty$.

Example 4.3 Let $\mathcal{H}$ be a Hilbert space and for $0 < \alpha < 1$ consider the fractional relaxation equation

$$u'(t) - AD^{\alpha}u(t) + u(t) = f(t), \qquad t > 0, \tag{4.3}$$

with initial condition $u(0) = u_0$ and f an appropriate $\mathcal{H}$-valued function. Equation (4.3) corresponds to the abstract version of the Basset problem (see [3]). We recall that the Basset equation arises in fluid dynamics concerning the unsteady motion of a particle accelerating in a viscous fluid under the action of the gravity, see [7]. As stated in [3, Section 3], well-posedness of equation (4.3) is equivalent to the existence of an (a,k)-regularized family $\{S(t)\}_{t\geq 0}$ generated by A, with

$$a = g_{1-\alpha} - (g_{1-\alpha} * e_{-1}) \quad \text{and} \quad k = e_{-1}. \tag{4.4}$$

Moreover, it is easy to check that the solution of the problem in terms of $\{S(t)\}_{t\geq 0}$ is given by

$$u(t) = u(0) - \int_0^t S(s)u(0)ds + \int_0^t S(t-s)f(s)ds. \tag{4.5}$$

Corollary 4.2. *Let $0 < \alpha < 1$. Suppose that A generates an $(g_{1-\alpha} - (g_{1-\alpha} * e_{-1}), e_{-1})$-regularized family $\{S(t)\}_{t\geq 0}$ satisfying the following conditions.*

1. *$\frac{\lambda+1}{\lambda^{\alpha}} \in \rho(A)$ for all $\Re\lambda \geq 0$ and $\lambda \neq 0$.*
2. *$\sup_{\Re\lambda > 0} ||(1+\lambda-\lambda^{\alpha}A)^{-1}|| < \infty$.*

Then $\{S(t)\}_{t\geq 0}$ is uniformly stable.

Proof. It follows from the identities $\hat{a}(\lambda) = \frac{\lambda^{\alpha}}{1+\lambda}$ and $\hat{k}(\lambda) = \frac{1}{1+\lambda}$ that

$$\lim_{\lambda\to 0} H(\lambda)x = \lim_{\lambda\to 0}((1+\lambda) - \lambda^{\alpha}A)^{-1}x = x, \quad x \in \mathcal{H},$$

and the conclusion follows from Theorem 3.1. □

For instance, suppose that the operator A is scalar ($Ax = \mu x$, for some $\mu \in \mathbb{C}$ and for all $x \in \mathcal{H}$). We consider equation (4.3) with initial conditions, i.e.

$$u'(t) - \mu D^{\alpha}u(t) + u(t) = f(t), \quad u(0) = u_0.$$

We claim that if $\Re\mu < 0$ then A generates a uniformly stable (a,k)-regularized family $\{S(t)\}_{t\geq 0}$.

Firstly, we will show that there exist a bounded continuous function $S : [0,\infty) \to \mathbb{C}$ such that

$$\hat{S}(\lambda) = H(\lambda) = \frac{1}{1+\lambda-\lambda^{\alpha}\mu} \quad \text{for all } \Re\lambda > 0,$$

and we will use [6, Proposition 3.1] to show that A generates an (a,k)-regularized family, where a and k are given in (4.4). For this, we note that $\frac{1+\lambda}{\lambda^\alpha} \neq \mu$ for all $\Re\lambda > 0$. Indeed, we have the identity

$$\Re\left(\frac{1+\lambda}{\lambda^\alpha}\right) = \frac{1}{|\lambda|^\alpha}[(1+\Re\lambda)\cos(\alpha\theta_\lambda) + (\Im\lambda)\sin(\alpha\theta_\lambda)]$$

where $\theta_\lambda := \mathrm{Arg}_{[-\pi,\pi)}(\lambda)$. Then using that $0 < \alpha < 1$, we conclude that $\Re\left(\frac{1+\lambda}{\lambda^\alpha}\right) \geq 0$ for all $\Re\lambda > 0$. Since $\Re\mu < 0$ we obtain that $\frac{1+\lambda}{\lambda^\alpha} \neq \mu$ for all $\Re\lambda > 0$.

On the other hand, from the fact that $0 < \alpha < 1$ and the continuity of H and H' over $i\mathbb{R}$, we conclude that there exist $M > 0$ such that

$$|\lambda H(\lambda)| \leq M \quad \textit{and} \quad |\lambda^2 H'(\lambda)| \leq M \quad \text{for all} \quad \Re\lambda > 0.$$

Hence by [1, Theorem 2.5.2] we get that there exists a bounded function $S \in C(\mathbb{R}_+)$ such that $\hat{S}(\lambda) = H(\lambda)$, for all $\Re\lambda > 0$.

Secondly, we will apply Corollary 4.2 to conclude that the solution is uniformly stable. The fact that $\Re\mu < 0$ and $\Re(\frac{1+\lambda}{\lambda^\alpha}) \geq 0$ for all $\Re\lambda \geq 0$, implies that the condition (1) of Corollary 4.2 is satisfied. Condition (2) follows from the fact that H is analytic over $\mathbb{C}_+$ and continuous in $i\mathbb{R}$. Therefore we have

$$\lim_{|\lambda|\to\infty} H(\lambda) = \lim_{|\lambda|\to\infty} \frac{1}{1+\lambda-\lambda^\alpha\mu} = 0,$$

and hence the resolvent $\{S(t)\}_{t\geq 0}$ is uniformly stable. In particular, we conclude from (4.5) that if $\Re\mu < 0$ then the solution of the equation:

$$u'(t) - \mu D^\alpha u(t) + u(t) = 0, \quad u(0) = u_0,$$

satisfies $\mu D^\alpha u(t) - u(t) \to 0$ as $t \to \infty$.

References

[1] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems.* Second edition. Monographs in Mathematics, **96** Birkhäuser, Basel, 2011.

[2] K. J. Engel, R. Nagel. *One-parameter semigroups for linear evolution equations.* Graduate Texts in Mathematics, **194** Springer, New York, 2000.

[3] A. Karczewska, C. Lizama. *On stochastic fractional relaxation equations in Hilbert space*, Phys. Scr. **136** (2009).

[4] M. Kostic. *(a,k)-regularized C-resolvent families: regularity and local properties.* Abstr. Appl. Anal. 2009, Art. ID 858242, 27 pp.

[5] C. Lizama, V. Vergara. *Uniform stability of resolvent families.* Proc. Amer. Math. Soc. **132**(1),(2004) 175-181.

[6] C. Lizama. *Regularized solutions for abstract Volterra equations.* J. Math. Anal. Appl. **243**(2) (2000) 278-292.

[7] F. Mainardi. *Fractional calculus: some basic problems in continuum, statistical mechanics,* in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, 1997, pp. 291-348.

[8] J. van Nerveen. *The asymtotic behaviour of semigroups of linear operators.* Oper. Theory Adv. Appl. **88**, Birkhäuser, 1996.

[9] J. Prüss. *Evolutionary integral equations and applications*, Birkhäuser, Basel-Boston-Berlin, 1993.

Universidad de Santiago de Chile, Facultad de Ciencias, Departamento de Matemática y Ciencia de la Computación, Casilla 307, Correo 2, Santiago, CHILE
E-mail address: carlos.lizama@usach.cl

Universidad de Zaragoza, Departamento de Matemáticas & I.U.M.A., 50.009, Zaragoza, Spain.
E-mail address: pjmiana@unizar.es

Universidad de Chile, Facultad de Ciencias, Las Palmeras 3425, Santiago, CHILE.
E-mail address: fepobletg@ug.uchile.cl