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SEMIGROUPS ON TIME SCALES AND APPLICATIONS TO ABSTRACT CAUCHY PROBLEMS

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Abstract. In this paper, we introduce the definition of a C_0-semigroup on a time scale, which unifies the continuous, discrete and other cases which lie between them. Also, it extends the classical theory of operator semigroups to the quantum case. We study the relationship between the semigroup and its infinitesimal generator. We apply our theory to study the homogeneous and non homogeneous abstract Cauchy problem in Banach and Fréchet spaces.

1. Introduction

The theory of time scales was introduced in the literature by Stefan Hilger in 1988. Since then, this theory has been extensively investigated by several researchers, specially due to its applications in different fields of knowledge such as economics, physics, biology, engineer, chemistry, among others. See, for instance, [1], [4], [8]–[10], [14], [18], [19], [22], [26] and the references therein.

Since a time scale $\mathbb{T}$ is any closed nonempty subset of $\mathbb{R}$, this theory allows to unify several existing concepts, depending on the chosen time scale. For instance,

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choosing $\mathbb{T} = \mathbb{Z}$, we have a result for a difference equation. On the other hand, taking $\mathbb{T} = \mathbb{R}$, we obtain a result for a differential equation. Besides, choosing $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$, we get a result for a q-difference equation, which has several applications to quantum and nuclear physics, among others.

Moreover, we can choose the time scale in such way to describe continuous-discrete hybrid processes, which have many applications in several models. For instance, it can be used to investigate option-pricing and stock dynamics in finance, population models, the frequency of markets and duration of market trading in economic, large-scale models of DNA dynamics, gene mutation fixation, electric circuits, among others. We refer the reader to [8], [9], [18], [19], [26] for details.

Parallel to this, it is well known that the theory of C_0-semigroups of bounded linear operators is a very important tool to study the first order abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)) & \text{for } t \geq 0, \\ u(0) = x, \end{cases}$$

when A generates a C_0-semigroup in a Banach space X. We know that this type of equation is used to model, in abstract form, several important concrete applications described by partial differential equations (PDEs) or functional differential equations (FDEs). See, for instance, [12] and [23]. On the other hand, to the best of our knowledge, the concept of C_0-semigroup on time scales encompassing all the time scales such that $0 \in \mathbb{T}$ was not presented in the literature until now.

The great challenge to develop a theory for C_0-semigroups on time scales is that the usual way to define the concept of semigroup is based on the following algebraic property:

$$T(t+s) = T(t)T(s), \quad \text{for } s, t \geq 0. \tag{1.1}$$

However, since we are dealing with time scales, which are only nonempty and closed subsets of $\mathbb{R}$, they may not be closed for the addition operation. For instance, if we consider $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$, then clearly $\mathbb{T}$ is not closed under summation. Therefore, to overcome this difficulty, some authors have considered the following condition on the time scale $\mathbb{T}$:

(C_1) If $a, b \in \mathbb{T}$, then $a - b \in \mathbb{T}$,

(see [15]), but as we will show in this paper, this property is very strong and restricts a lot the results, since these conditions are valid only for few class of time scales. For instance, we will show that if $\mathbb{T}_0$ is a time scale such that $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = +\infty$, as well as $\mathbb{T}_0$ has the property (C_1) and 0 is a right-dense point in $\mathbb{T}_0^+ := \mathbb{T}_0 \cap [0, \infty)$, then $\mathbb{T}_0^+ = [0, \infty)$. It shows that this condition is very restrictive, and many interesting time scales do not satisfy this hypothesis.

Because of that, we present here a definition of C_0-semigroup based on the concept of Laplace transforms on time scales, which is more general and encompass all the time scales $\mathbb{T}_0$ satisfying

$$0 \in \mathbb{T}_0 \quad \text{and} \quad \sup \mathbb{T}_0 = \infty.$$

First we recall the concept of strongly continuous function.

DEFINITION 1.1. Let X be a Banach space. We say that $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$ is *strongly continuous* if $\|T(t)x - x\|_X \to 0$ as $t \to 0^+$ for each $x \in X$.

Next, we will introduce the concept of a C_0-semigroup.

DEFINITION 1.2. Let A be a closed linear operator defined in a Banach space X. We say that $T\colon \mathbb{T}_0^+ := \mathbb{T}_0 \cap [0, +\infty) \to \mathcal{L}(X)$ is a *C_0-semigroup with infinitesimal generator A* if the following conditions are satisfied:

(a) $T(0) = I$ and for every $x \in X$, the function $t \mapsto T(t)x$ is strongly continuous.

(b) There exists a λ_0 such that $(\lambda_0, \infty) \subset \rho(A)$ and

$$\widehat{T}(\lambda)x = (\lambda - A)^{-1}x, \quad \text{for } x \in X,$$

for every $\mathrm{Re}_\mu(\lambda) > \lambda_0$, for all $t \in \mathbb{T}$ and for every $\lambda \in \mathcal{D}\{T\}$, where $\mathcal{D}\{T\}$ consists of all complex numbers $\lambda \in \mathcal{R}$ for which the improper integral of Laplace transform exists. Here, the hat indicates Laplace transform.

See also Definition 4.5 below. It turns out that this definition is successful to include in the theory a wide class of time scales and, consequently, autonomous semilinear initial value problems on time scales can be analytically studied with this method. In particular, initial value problems for linear and nonlinear partial dynamic equations on time scales could be treated using this tool.

The paper is organized as follows: The second section is devoted to the basic concepts and properties of the time scales theory. In the third section, we recall the definition of Laplace transform on time scales and we prove new properties which are necessary to establish our results. For instance, given $f\colon \mathbb{T}_0^+ \to X$ an rd-continuous and Δ-differentiable function such that: $\widehat{f^\Delta}(\lambda)$ exists for $\mathrm{Re}_\mu \lambda > 0$ for every $t \in \mathbb{T}$, then $\widehat{f}(\lambda)$ exists and $\widehat{f^\Delta}(\lambda) = \lambda \widehat{f}(\lambda) - f(0)$, for those $\lambda \in \mathbb{C} \setminus \{0\} \cap \mathcal{R}$ such that

$$\lim_{t \to \infty} f(t) e_{\ominus \lambda}(t, 0) = 0.$$

In the fourth section, we develop several important properties of C_0-semigroups on time scales, most of which generalize the properties of the classical theory of semigroups. Among others, we prove that the following properties hold:

- $(\lambda - A)^{-1} T(t) = T(t)(\lambda - A)^{-1}$ for all $\lambda \in (\lambda_0, \infty) \subset \rho(A)$ and for all $t \in \mathbb{T}_0^+$.

- If $x \in D(A)$, then $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$ for all $t \in \mathbb{T}_0^+$.
- $\int_0^t T(s)x\Delta s \in D(A)$ and $A\int_0^t T(s)x\Delta s = T(t)x - x$ for all $x \in X$ and $t \in \mathbb{T}_0^+$.
- Let $x, y \in X$. Then $x \in D(A)$ and $Ax = y$, if and only if, for all $t \in \mathbb{T}_0^+$, we have:
$$\int_0^t T(s)\, y\Delta s = T(t)x - x.$$
- Let $x \in X$. Then $x \in D(A)$, if and only if

$$y = \begin{cases} \dfrac{T(\sigma(0))x - x}{\mu(0)} & \text{if 0 is right-scattered,} \\ \lim\limits_{t\to 0} \dfrac{T(t)x - x}{t} & \text{if 0 is right-dense,} \end{cases} \tag{1.2}$$

  exists. In this case, $Ax = y$.

Also, we exhibit some applications of our results to the study of the well–posedness of the abstract Cauchy problem given below

$$\begin{cases} u^\Delta(t) = Au(t) & \text{for } t \in \mathbb{T}_0^+, \\ u(0) = x. \end{cases} \tag{1.3}$$

More precisely, we show that the following properties hold.

- For all $x \in X$, the function $u_x(t) = T(t)x$, $t \in \mathbb{T}_0^+$, is a mild solution of the homogeneous Cauchy problem (1.3).
- $T(\,\cdot\,)x$ is a classical solution of the homogeneous Cauchy problem (1.3) if and only if $x \in D(A)$.

We will prove the remarkable fact that the behavior of the graininess function μ at zero determines the boundedness of the infinitesimal generator, namely, if $\mu(0) > 0$ and the homogeneous Cauchy problem (1.3) has a solution, then A is a bounded linear map (Theorem 4.12).

In addition, we will show that a direct definition by means of the functional equation (1.1) is not possible for a wide class of graininess functions, more precisely, if $\mu(0) = 0$ and $T(\,\cdot\,)$ satisfies (1.1), then $\mathbb{T}_0^+ = [0, +\infty)$ (see Theorem 4.16).

Section 5 is devoted to study the nonhomogeneous abstract Cauchy problem

$$\begin{cases} u^\Delta(t) = Au(t) + f(t) & \text{for } t \in \mathbb{T}_0^+, \\ u(0) = x. \end{cases} \tag{1.4}$$

In the sixth section, we will study abstract Cauchy problems in Frechét spaces. Finally, in the last section, we will present some examples to illustrate our definition of C_0-semigroup depending on the chosen time scale. A part of them will be showned only for some specific case.

2. Preliminaries

In this section, we present some basic concepts and properties concerning time scales which will be essential to prove our main results. The reader can find in references [6], [7] all the notation and concepts that follow.

Let $\mathbb{T}$ be a time scale, that is, $\mathbb{T}$ is a closed and nonempty subset of $\mathbb{R}$. For every $t \in \mathbb{T}$, we define the *forward* and *backward jump operators* $\sigma, \rho\colon \mathbb{T} \to \mathbb{T}$, respectively, as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T},\ s < t\}.$$

In this definition, we consider $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say that t is *right-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*. Analogously, if $\rho(t) < t$, then t is called *left-scattered*, whereas if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is *left-dense*.

We also define the *graininess* and the *backward graininess function* $\mu, \nu\colon \mathbb{T} \to [0, +\infty)$, respectively, by:

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = t - \rho(t).$$

Definition 2.1. Let X be a Banach space. A function $f\colon \mathbb{T} \to X$ is called *rd-continuous* if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$. We denote the class of all rd-continuous functions $f\colon \mathbb{T} \to X$ by $\mathcal{C}_{\mathrm{rd}} = \mathcal{C}_{\mathrm{rd}}(\mathbb{T}) = \mathcal{C}_{\mathrm{rd}}(\mathbb{T}, X)$. If the function $f\colon \mathbb{T} \to X$ is continuous at each right-dense point and each left-dense point, then the function f is said to be *continuous* on $\mathbb{T}$.

Remark 2.2. Notice that if $\mathbb{T} = \mathbb{R}$, then the definition of rd-continuous functions coincides with the definition of continuous functions, since all the points are right-dense in $\mathbb{R}$.

Given a pair of numbers a, $b \in \mathbb{T}$, the symbol $[a, b]_{\mathbb{T}}$ will be used to denote a closed interval in $\mathbb{T}$, that is,$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T};\ a \le t \le b\}$. On the other hand, $[a, b]$ is the usual closed interval on the real line, that is, $[a, b] = \{t \in \mathbb{R};\ a \le t \le b\}$. To define delta-derivatives, we need the set $\mathbb{T}^\kappa$ which is defined as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.3. For $y\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the *delta-derivative* (*or Hilger-derivative*) of y to be the number (if it exists) $y^\Delta(t)$ with the following property: given $\varepsilon > 0$, there exists a neighbourhood U of t such that

$$\big|y(\sigma(t)) - y(s) - y^\Delta(t)[\sigma(t) - s]\big| \le \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

Definition 2.4. A *partition* of $[a, b]_{\mathbb{T}}$ is a finite sequence of points

$$\{t_0, t_1, \ldots, t_m\} \subset [a, b]_{\mathbb{T}}, \quad a = t_0 < t_1 < \ldots < t_m = b.$$

Given such a partition, we put $\Delta t_i = t_i - t_{i-1}$. A tagged partition $\mathcal{P}$ consists of a partition and a sequence of tags $\{\xi_1, \ldots, \xi_m\}$ such that $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$, $\xi_i \in \mathbb{T}$, for all $i \in \{1, \ldots, m\}$. The set of all tagged partitions of $[a,b]_{\mathbb{T}}$ will be denoted by the symbol $D(a,b)$.

If $\delta > 0$, then $D_\delta(a,b)$ denotes the set of all tagged partitions of $[a,b]_{\mathbb{T}}$ such that for every $i \in \{1, \ldots, m\}$, either $\Delta t_i \leq \delta$, or $\Delta t_i > \delta$ and $\sigma(t_{i-1}) = t_i$. Note that, in the last case, the only way to choose a tag in $[t_{i-1}, t_i)_{\mathbb{T}}$ is to take $\xi_i = t_{i-1}$.

In the sequel, we present the definition of the Riemann Δ-integral. The reader may consult [6], [7] for details.

Definition 2.5. We say that $f\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is *Riemann Δ-integrable* on $[a,b]_{\mathbb{T}}$, if there exists a vector $I \in \mathbb{R}^n$ with the following property: for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sum_i f(\xi_i)(t_i - t_{i-1}) - I \right| < \varepsilon,$$

for all $P \in D_\delta(a,b)$ independently of $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}}$ for $1 \leq i \leq n$. It is clear that such a number I is unique. The number I is called the Riemann Δ-integral of f from a to b . As usual, we denote $I = \int_a^b f(t)\Delta t$.

Next, we present an important property of the Riemann Δ-integral. A proof of this result can be found in [6, Theorem 1.75].

Theorem 2.6. *If $f \in C_{\mathrm{rd}}$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t) f(t).$$

In what follows, we present a concept of regressive functions.

Definition 2.7. We say that a function $p\colon \mathbb{T} \to \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0, \quad \text{for all } t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Suppose that $p, q \in \mathcal{R}$, then we define $p \oplus q$ and $\ominus p$ by

$$\begin{aligned} (p \oplus q)(t) &:= p(t) + q(t) + \mu(t)p(t)q(t), && \text{for all } t \in \mathbb{T}^\kappa, \\ (\ominus p)(t) &:= \frac{-p(t)}{1 + \mu(t)p(t)}, && \text{for all } t \in \mathbb{T}^\kappa. \end{aligned}$$

It is known that $(\mathcal{R}, \oplus)$ is an Abelian group See, for instance, [6].

In what follows, we present the *Hilger complex plane*. For more details, the reader may want to consult [6].

DEFINITION 2.8. For $h > 0$, we define the *Hilger complex numbers*, the *Hilger real axis*, the *Hilger alternating axis* and the *Hilger imaginary circle* respectively by:

$$\begin{aligned}\mathbb{C}_h &:= \left\{ z \in \mathbb{C} : \ z \neq -\frac{1}{h} \right\},\\ \mathbb{R}_h &:= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h} \right\},\\ \mathbb{A}_h &:= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h} \right\},\\ \mathbb{I}_h &:= \left\{ z \in \mathbb{C}_h : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\}.\end{aligned}$$

For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}$, $\mathbb{R}_0 := \mathbb{R}$, $\mathbb{I}_0 = i\mathbb{R}$, and $\mathbb{A}_0 := \emptyset$.

DEFINITION 2.9. Let $h > 0$ and $z \in \mathbb{C}_h$. We define the *Hilger real part of* z by

$$\mathrm{Re}_h(z) := \frac{|zh + 1| - 1}{h}$$

and the *Hilger imaginary part of* z by

$$\mathrm{Im}_h(z) := \frac{\arg(zh + 1)}{h},$$

where $\arg(z)$ denotes the principal argument of z (i.e. $-\pi < \arg(z) \leq \pi$).

We point out that $\mathrm{Re}_h(z)$ and $\mathrm{Im}_h(z)$ satisfy

$$-\frac{1}{h} < \mathrm{Re}_h(z) < \infty \quad \text{and} \quad -\frac{\pi}{h} < \mathrm{Im}_h(z) \leq \frac{\pi}{h},$$

respectively. See, for instance, [6].

DEFINITION 2.10. Let $-\pi/h < \omega \leq \pi/h$. We define the *Hilger purely imaginary number* $i\omega$ by

$$\overset{\circ}{\imath}\,\omega = \frac{e^{i\omega h} - 1}{h}.$$

For $z \in \mathbb{C}_h$, we have that $\overset{\circ}{\imath}\,\mathrm{Im}_h(z) \in \mathbb{I}_h$. It is not difficult to prove that

$$\lim_{h \to 0} \left[\mathrm{Re}_h z + \overset{\circ}{\imath}\,\mathrm{Im}_h z \right] = \mathrm{Re}(z) + i\mathrm{Im}(z).$$

See, for instance, [6].

REMARK 2.11. In the sequel, we present the following notation that we will use in the rest of the paper:

$$\mathrm{Re}_{\mu(t)}(z) = \mathrm{Re}_{\mu}(z)(t)$$

and clearly, we have $\mathrm{Re}_0(z) = \mathrm{Re}(z)$ in the usual sense.

THEOREM 2.12 ([6, Theorem 2.7]). *If we define the "circle plus" addition $\oplus$ on $\mathbb{C}_h$ by*

$$z \oplus \omega := z + \omega + z\omega h,$$

then $(\mathbb{C}_h, \oplus)$ is an Abelian group.

Theorem 2.13 ([6, Theorem 2.10]). *For $z \in \mathbb{C}_h$ we have*

$$z = \mathrm{Re}_h z \oplus \mathring{\imath}\, \mathrm{Im}_h z.$$

Now, we are able to define the *generalized exponential function* $e_p(t,s)$.

Definition 2.14. For $p \in \mathcal{R}$, we define the *generalized exponential function* by

$$e_p(t,s) = \exp\bigg(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\bigg) \quad \text{for } s,t \in \mathbb{T}.$$

Here the cylinder transformation $\xi_h \colon \mathbb{C}_h \to \mathbb{Z}_h$ is given by:

$$\xi_h(z) = \frac{1}{h}\log(1+zh),$$

where log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Theorem 2.15 ([6, Theorem 2.36]). *If $p, q \in \mathcal{R}$, then*

(a) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(b) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(c) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(d) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(e) $e_p(t,s)e_p(s,r) = e_p(s,r)$;

(f) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(g) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

(h) $\bigg(\dfrac{1}{e_p(\,\cdot\,,s)}\bigg)^{\Delta} = -\dfrac{p(t)}{e_p^{\sigma}(\,\cdot\,,s)}$.

Following the calculations from [6, p. 64] and using the properties of the generalized exponential function, we have the following result.

Lemma 2.16. *If $p \in \mathcal{R}$, then*

$$pe_{\ominus p}(\sigma(t),c) = -e_{\ominus p}^{\Delta}(t,c), \quad \textit{for every } t \in \mathbb{T}_0^+.$$

Proof. Notice that

$$\begin{aligned} pe_{\ominus p}(\sigma(t),c) &= p(t)[1+\mu(t)\ominus p(t)]e_{\ominus p}(t,c) = p(t)\bigg[1 - \frac{\mu(t)p(t)}{1+\mu(t)p(t)}\bigg]e_{\ominus p}(t,c) \\ &= p(t)\frac{1}{1+\mu(t)p(t)}e_{\ominus p}(t,c) = -(\ominus p)(t)e_{\ominus p}(t,c) = -e_{\ominus p}^{\Delta}(t,c), \end{aligned}$$

proving the result. □

REMARK 2.17. Assuming that $\mathbb{T}$ is a time scale with bounded graininess function and suppose that $\lambda \in \mathbb{H}$, where $\mathbb{H}$ denotes the *Hilger circle* given by

$$\mathbb{H} = \mathbb{H}_t = \left\{ z \in \mathbb{C} : 0 < \left| z + \frac{1}{\mu(t)} \right| < \frac{1}{\mu(t)} \right\},$$

when t is a right-scattered point of $\mathbb{T}$, we obtain $\lim_{t\to\infty} e_\lambda(t, 0) = 0$. See [24].

We use the following notation for the rest of the paper

$$[e_p(c,t)]^\sigma = e_p^\sigma(c,t) = e_p(c,\sigma(t)) \quad \text{and} \quad [e_p(t,c)]^\sigma = e_p^\sigma(t,c) = e_p(\sigma(t),c).$$

The next result can be found in [20, Lemma 5.1].

LEMMA 2.18. *Let $\alpha > 0$. Then, for any fixed $s \in \mathbb{T}$,*

$$e_{\ominus\alpha}(t,s) \to 0 \quad \text{as } t \to +\infty.$$

The next theorem will be essential to prove our main results. For a proof, we refer to [6, Theorem 2.39].

THEOREM 2.19. *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c,t)]^\Delta = -p[e_p(c,t)]^\sigma$$

and

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

Next, we present a very important property of generalized exponential function, which also will be fundamental to prove our main results. For a proof of this result, see [6, Theorem 2.44].

THEOREM 2.20. *Assume $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. If $1 + \mu p > 0$ on $\mathbb{T}^\kappa$, then*

$$e_p(t,t_0) > 0 \quad \text{for all } t \in \mathbb{T}.$$

As an immediate consequence of this result, and using the properties of the generalized exponential function, we obtain the following result.

COROLLARY 2.21. *Assume $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. If $1 + \mu p > 0$ on $\mathbb{T}^\kappa$, then*

$$e_p^\sigma(t,t_0) > 0 \quad \text{and} \quad e_{\ominus p}^\sigma(t,t_0) > 0 \quad \text{for all } t \in \mathbb{T}.$$

3. Laplace transform on time scales

In this section, we are concerned with the notion of Laplace transform on time scales and we establish some important properties which will be essential to our purposes. Let us denote by $\mathbb{T}_0$ a time scale satisfying

$$0 \in \mathbb{T}_0 \quad \text{and} \quad \sup \mathbb{T}_0 = +\infty.$$

Also, let us denote by $\mathbb{T}_0^+ = \mathbb{T}_0 \cap [0, +\infty)$. Note that, if we assume that $\lambda \in \mathcal{R}$ is constant, then $\ominus\lambda \in \mathcal{R}$ and $e_{\ominus\lambda}(t, 0)$ is well-defined on $\mathbb{T}_0$ (see [6]).

We begin by recalling the definition of the Laplace transform on time scales.

Definition 3.1 ([6]). Assume that $x\colon \mathbb{T}_0 \to \mathbb{R}$ is a regulated function. Then the *Laplace transform* of x is defined by

$$\widehat{x}(\lambda) = \mathcal{L}\{x\}(\lambda) := \int_0^\infty x(t)e^\sigma_{\ominus\lambda}(t,0)\Delta t, \tag{3.1}$$

for $\lambda \in \mathcal{D}\{x\}$, where $\mathcal{D}\{x\}$ consists of all complex numbers $\lambda \in \mathcal{R}$ for which the improper integral exists.

In what follows, we present some properties of the Laplace transform.

Theorem 3.2 (Linearity, [6]). *Assume f and g are regulated functions on $\mathbb{T}_0$ and α and β are constants. Then, for $\lambda \in \mathcal{D}\{x\} \cap \mathcal{D}\{y\}$,*

$$\mathcal{L}\{\alpha x + \beta y\}(\lambda) = \alpha\mathcal{L}\{x\}(\lambda) + \beta\mathcal{L}\{y\}(\lambda).$$

Theorem 3.3 ([6, Lemma 3.85]). *If $\lambda \in \mathbb{C}$ is regressive, then*

$$e^\sigma_{\ominus\lambda}(t,0) = \frac{e_{\ominus\lambda}(t,0)}{1+\mu(t)\lambda} = -\frac{(\ominus\lambda)(t)}{\lambda}e_{\ominus\lambda}(t,0).$$

The following concept is essential in the study of Laplace transform.

Definition 3.4 ([10]). The function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be of *exponential type* II if there exist constants $M, c > 0$ such that $|f(t)| \le Me_c(t,0)$.

The next result can be found in [10, Theorem 1.1].

Theorem 3.5 (Domain of the transform). *Assume that $f\colon \mathbb{T}_0 \to \mathbb{R}$ is a regulated function of exponential type* II *with exponential constant $c > 0$. Then the integral $\int_0^\infty e^\sigma_{\ominus\lambda'}(t,0)f(t)\Delta t$ converges absolutely for*

$$\lambda' \in D = \{\lambda \in \mathbb{C} : \mathrm{Re}_\mu(\lambda)(t) > c, \textit{ for all } t \in \mathbb{T}_0\}.$$

Remark 3.6. The previous concepts and results can be extended in a standard way to functions $f\colon \mathbb{T}_0^+ \to X$, where $(X, \|\cdot\|)$ is a Banach space endowed with the strong topology. In particular, we denote by $\mathcal{C}_{\mathrm{rd}}(\mathbb{T}_0^+, X)$ the set of rd-continuous functions from $\mathbb{T}_0^+$ to X.

In what follows, X, Y denote Banach spaces, and $\mathcal{L}(X,Y)$ denotes the space consisting of all bounded linear operators from X into Y endowed with the norm of operators. Moreover, X^* denotes the dual space of X.

Let us remember an important property of linear operators. See [23].

Theorem 3.7. *If X, Y are Banach spaces and $T\colon X \to Y$ is a linear map, then the following statements are equivalent*:

(a) *T is continuous.*
(b) *T is continuous at 0.*
(c) *T is bounded.*

In the sequel, we present the following characterization of a closed operator. See [23].

THEOREM 3.8. *An operator* $A\colon D(A) \subset X \to X$ *is a closed linear operator if and only if, for each sequence* $x_n \to x_0$ *as* $n \to \infty$ *with* $Ax_n \to y$ *as* $n \to \infty$*, we have* $x \in D(A)$ *and* $Ax = y$*.*

Proceeding as in the continuous case, we can establish the next result.

THEOREM 3.9. *Let* $A\colon D(A) \subseteq X \to Y$ *be a closed linear operator and* $f\colon [a,b]_{\mathbb{T}} \to X$ *be an rd-continuous function. Suppose that* $f(t) \in D(A)$ *for all* $t \in [a,b]_{\mathbb{T}}$ *and* $Af\colon [a,b]_{\mathbb{T}} \to Y$ *is an rd-continuous function. Then* $\int_a^b f(s)\Delta s \in D(A)$ *and*

$$\int_a^b Af(t)\Delta t = A\int_a^b f(t)\Delta t.$$

As a consequence of the Theorem 3.9, we have the following result.

COROLLARY 3.10. *Let* $f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}_0^+, X)$ *and* $T \in \mathcal{L}(X,Y)$*. Then* $T \circ f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}_0^+, Y)$*. Furthermore, let* $\lambda \in \mathbb{C} \setminus \{0\} \cap \mathcal{R}$ *be such that if* $\widehat{f}(\lambda)$ *exists, then* $\widehat{T \circ f}(\lambda)$ *exists and* $\widehat{T \circ f}(\lambda) = T(\widehat{f}(\lambda))$*.*

Next, we present a result which is an immediate consequence of Theorem 3.9.

COROLLARY 3.11. *Let* $f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}_0^+, X)$ *and* A *be a closed linear operator in* X*. Suppose that* $f(t) \in D(A)$ *for all* $t \in \mathbb{T}$ *and* $A \circ f \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}_0^+, X)$*. Let* $\lambda \in \mathbb{C} \setminus \{0\} \cap \mathcal{R}$ *be such that both* $\widehat{f}(\lambda)$ *and* $\widehat{A \circ f}(\lambda)$ *exist, then* $\widehat{f}(\lambda) \in D(A)$ *and* $\widehat{A \circ f}(\lambda) = A(\widehat{f}(\lambda))$*.*

PROOF. Using Theorems 3.7 and 3.9, we get

$$\int_0^\tau e^\sigma_{\ominus\lambda}(t,0)(A \circ f)(t)\Delta t = A\biggl(\int_0^\tau e^\sigma_{\ominus\lambda}(t,0)f(t)\Delta t\biggr).$$

Since A is a closed operator, the second assertion follows by taking the limit above as $\tau \to \infty$. $\square$

The proof of the next result is inspired by the proof of [6, Theorem 3.87].

THEOREM 3.12. *Let* $f\colon \mathbb{T}_0^+ \to X$ *be an rd-continuous and* Δ*-differentiable function such that* f^Δ *is regulated. Let* $\lambda \in \mathbb{C} \setminus \{0\} \cap \mathcal{R}$ *be such that*

$$\lim_{t\to\infty} f(t)e_{\ominus\lambda}(t,0) = 0$$

and $\widehat{f^\Delta}(\lambda)$ *exists, then* $\widehat{f}(\lambda)$ *exists and* $\widehat{f^\Delta}(\lambda) = \lambda\widehat{f}(\lambda) - f(0)$*.*

Proof. Applying integration by parts and Theorem 3.3, we obtain

$$\begin{aligned}\widehat{f^{\Delta}}(\lambda) &= \int_0^\infty f^{\Delta}(t)e^{\sigma}_{\ominus\lambda}(t,0)\Delta t\\ &= \big[f(t)e_{\ominus\lambda}(t,0)\big]_{t=0}^{t\to\infty} - \int_0^\infty f(t)(\ominus\lambda)e_{\ominus\lambda}(t,0)\Delta t\\ &= -f(0) + \lambda\int_0^\infty f(t)e^{\sigma}_{\ominus\lambda}(t,0)\Delta t = -f(0)+\lambda\widehat{f}(\lambda),\end{aligned}$$

and the desired result follows as well. □

As an immediate consequence of this theorem, we have the following result.

Corollary 3.13. *Let $f \in \mathcal{C}_{\rm rd}(\mathbb{T}_0^+, X)$ be Δ-differentiable such that f^{Δ} is regulated and $F(t) = \int_0^t f(s)\Delta s$. Let $\lambda \in \mathbb{C}\setminus\{0\}\cap\mathcal{R}$ be such that $\widehat{f}^{\Delta}(\lambda)$ exists and $\lim_{t\to\infty} f(t)e_{\ominus\lambda}(t,0) = 0$, then $\widehat{F}(\lambda) = \widehat{f}(\lambda)/\lambda$.*

Proof. By definition, we have $F^{\Delta}(t) = f(t)$. Let $\lambda \in \mathbb{C}\setminus\{0\}\cap\mathcal{R}$ be such that $\widehat{f}^{\Delta}(\lambda)$ exists and $\lim_{t\to\infty} f(t)e_{\ominus\lambda}(t,0) = 0$, then by using Theorem 3.12, we get

$$\widehat{F^{\Delta}}(\lambda) = \lambda\widehat{F}(\lambda) - F(0) = \lambda\widehat{F}(\lambda).$$

Hence, we obtain

$$\widehat{F}(\lambda) = \frac{\widehat{F^{\Delta}}(\lambda)}{\lambda} = \frac{\widehat{f}(\lambda)}{\lambda},$$

which completes the proof. □

The next result ensures the uniqueness of Laplace transform on time scales.

Theorem 3.14 (Uniqueness). *Let $f, g \in \mathcal{C}_{\rm rd}(\mathbb{T}_0^+, X)$ and suppose that $\widehat{f}(\lambda) = \widehat{g}(\lambda)$ for $\lambda \in D\{f\}\cap D\{g\}$, where $D\{f\}\cap D\{g\}$ is the set of all regressive and complex λ such that the improper integral exists. Then $f(t) = g(t)$ for all $t \in \mathbb{T}_0^+$.*

Proof. Let $h \in C_{\rm rd}(\mathbb{T}_0^+, X)$ such that $\widehat{h}(\lambda) = 0$ for all regressive $\lambda \in D\{h\}$. Let $x^* \in X^*$, then $\mathcal{L}(\langle x^*, h\rangle)(\lambda) = 0$ for $\lambda \in D\{h\}$. Applying the results in [17], we can prove that $\langle x^*, h(t)\rangle = 0$ for all $t \in \mathbb{T}_0^+$. Finally, using the Hahn–Banach theorem, we conclude that $h(t) = 0$ for $t \in \mathbb{T}_0^+$. Applying this property to $h = f - g$, we obtain that $f(t) = g(t)$ for all $t \in \mathbb{T}_0^+$. □

Theorem 3.15. *Let A be a closed linear operator in X, let $f, g \in L^1_{\rm loc}(\mathbb{T}_0^+, X)$ such that $\omega \in D\{f\}\cap D\{g\}$. Then, the following assertions are equivalent:*

(a) *$f(t) \in D(A)$ and $Af(t) = g(t)$ a.e. on $\mathbb{T}_0^+$.*
(b) *$\widehat{f}(\lambda) \in D(A)$ and $A\widehat{f}(\lambda) = \widehat{g}(\lambda)$ whenever $Re_\mu(\lambda)(t) > Re_\mu(\omega)(t)$ for all $t \in \mathbb{T}_0^+$ and $\lambda \in D\{f\}\cap D\{g\}$.*

Proof. (a) ⇒ (b). It follows immediately from Theorem 3.11.

(b) ⇒ (a). Let $G(A)$ be the graph of A, which is a closed subspace of $X \times X$, and let $q\colon X\times X \to (X\times X)/G(A)$ be the quotient map. Define

$h\colon \mathbb{T}_0^+ \to (X \times X)/G(A)$ by $h(t) = q(f(t), g(t))$. Since q is a bounded linear map, we have $\widehat{h}(\lambda) = q(\widehat{f}(\lambda), \widehat{g}(\lambda)) = 0$ whenever $\mathrm{Re}_\mu(\lambda)(t) > \mathrm{Re}_\mu(\omega)(t)$ for all $t \in \mathbb{T}_0^+$ by (b). By the uniqueness theorem, $h(t) = 0$. This proves (a). $\square$

The proof of our next result follows the same lines as the proof for bounded operators. For this reason, we omit its proof.

THEOREM 3.16. *Let A be a linear operator in X with nonempty resolvent set, and let $T \in \mathcal{L}(X)$. The following assertions are equivalent*:

(a) $(\lambda - A)^{-1}T = T(\lambda - A)^{-1}$ *for all* $\lambda \in \rho(A)$.
(b) $(\lambda - A)^{-1}T = T(\lambda - A)^{-1}$ *for some* $\lambda \in \rho(A)$.
(c) *For all* $x \in D(A)$, $Tx \in D(A)$ *and* $ATx = TAx$.

4. C_0-semigroups and the abstract Cauchy problem

In this section, our aim is to introduce the concept of C_0-semigroup on time scales, and to present some results concerning this theory and its application to the abstract Cauchy problem on time scales. For the classical theory of C_0-semigroups, we refer the reader to the excellent books [3], [12] and references therein.

Let $\mathbb{T}_0$ be a time scale such that $0 \in \mathbb{T}$ and $\sup \mathbb{T}_0 = +\infty$ and let A be a closed linear operator in a Banach space X. Our goal is to study the existence of solutions of the abstract Cauchy problem

$$\begin{cases} u^\Delta(t) = Au(t) & \text{for } t \in \mathbb{T}_0^+, \\ u(0) = x, \end{cases} \tag{4.1}$$

where $x \in X$. By a *classical solution* of (4.1), we understand a function $u \in C^1_{\mathrm{rd}}(\mathbb{T}_0^+, X)$ such that $u(t) \in D(A)$ for all $t \in \mathbb{T}_0^+$ and (4.1) holds.

We point out that if a classical solution of (4.1) exists, then it follows that $x = u(0) \in D(A)$. It will be important to find a weaker notion of solution where $u(0) = x$ may be arbitrary. This can be done by integrating the equation. Assume that u is a classical solution of (4.1). Since A is closed, it follows from Theorem 3.9 that

$$\int_0^t u(s)\Delta s \in D(A) \quad \text{and} \quad A\int_0^t u(s)\Delta s = u(t) - x, \quad t \in \mathbb{T}_0^+. \tag{4.2}$$

DEFINITION 4.1. A function $u \in C_{\mathrm{rd}}(\mathbb{T}_+, X)$ is called a *mild solution* of (4.1) if (4.2) holds.

The following assertion shows that mild and classical solutions of (4.1) differ merely by regularity.

THEOREM 4.2. *A mild solution u of (4.1) is a classical solution if and only if $u \in C^1_{\mathrm{rd}}(\mathbb{T}_0^+, X)$.*

Proof. If u is a classical solution of (4.1), then it follows by the definition that $u \in \mathcal{C}^1_{\mathrm{rd}}(\mathbb{T}_0^+, X)$.

Conversely, assume that $u \in C^1_{\mathrm{rd}}(\mathbb{T}_0^+, X)$ is a mild solution of (4.1). Let $t \in \mathbb{T}_0^+$ and consider two cases, that is, if t is right-dense or if t is right-scattered. Suppose initially that t is right-dense, we have that there exists a sequence $t_h > t$, $t_h \in \mathbb{T}_0^+$, for each $h = 1/n$, $n \in \mathbb{N}$, such that $t_h \to t$ when $h \to 0^+$. Then using (4.2)

$$\frac{1}{t_h - t}(u(t_h) - u(t)) = \frac{1}{t_h - t} A \int_t^{t_h} u(s)\Delta s. \tag{4.3}$$

On the other hand, we have

$$u(t) = \lim_{h \to 0^+} \frac{1}{t_h - t} \int_t^{t_h} u(s)\Delta s,$$

which implies that $u \in D(A)$, since A is closed. Also, from (4.3)

$$u^{\Delta}(t) = \lim_{h \to 0^+} \frac{1}{t_h - t}(u(t_h) - u(t)) = \left(\lim_{h \to 0^+} \frac{1}{t_h - t} A \int_t^{t_h} u(s)\Delta s \right)$$

we infer that

$$u^{\Delta}(t) = Au(t),$$

since A is closed. Therefore, $u(\,\cdot\,)$ is a classical solution.

Suppose now that t is right-scattered, then by Theorem 2.6, we have

$$\frac{1}{\mu(t)} \int_t^{\sigma(t)} u(s)\Delta s = \frac{1}{\mu(t)} u(t)\mu(t) = u(t) \tag{4.4}$$

and, by relation (4.2), we get

$$\frac{1}{\mu(t)} A \int_t^{\sigma(t)} u(s)\Delta s = \frac{1}{\mu(t)}(u(\sigma(t)) - u(t)).$$

Since A is closed, it follows that

$$u(t) = \frac{1}{\mu(t)} \int_t^{\sigma(t)} u(s)\Delta s \in D(A),$$

and

$$u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)} = \frac{1}{\mu(t)} A \int_t^{\sigma(t)} u(s)\Delta s = A\left(\frac{1}{\mu(t)} \int_t^{\sigma(t)} u(s)\Delta s \right).$$

Thus, by (4.4), we have $u^{\Delta}(t) = Au(t)$, which implies that u is a classical solution of (4.1) and the desired result follows. $\square$

Let $u \in C_{\mathrm{rd}}(\mathbb{T}_0^+, X)$ be a function which satisfies the following inequality

$$\left\| \int_0^t u(s)\Delta s \right\| \le M e_c(t, 0) \tag{4.5}$$

for some constants $M, c > 0$, where $c \in \mathcal{R}$. Using Theorem 3.5, we know that the Laplace transform

$$\widehat{u}(\lambda) := \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)u(t)\Delta t \tag{4.6}$$

exists whenever $\mathrm{Re}_\mu(\lambda)(t) > c$ for all $t \in \mathbb{T}_0$ and $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$. In the sequel, we prove an important equivalence.

THEOREM 4.3. *Let $u \in \mathcal{C}_{\mathrm{rd}}(\mathbb{T}_0^+, X)$ be such that (4.5) is satisfied. Then, the following assertions are equivalent*:

(a) *u is a mild solution of (4.1).*
(b) *$\widehat{u}(\lambda) \in D(A)$ and $\lambda\widehat{u}(\lambda) - A\widehat{u}(\lambda) = x$ whenever $\mathrm{Re}_\mu(\lambda)(t) > c$ for all $t \in \mathbb{T}_0^+$ and $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$.*

PROOF. (a) $\Rightarrow$ (b). Let u be a mild solution of (4.1) and $\mathrm{Re}_\mu(\lambda)(t) > c$ for all $t \in \mathbb{T}_0^+$ and $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$. Using integration by parts in (4.6), we obtain

$$\widehat{u}(\lambda) = -\ominus\lambda \int_0^\infty e_{\ominus\lambda}(t,0)\int_0^t u(s)\Delta s\Delta t = \lambda\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t u(s)\Delta s\Delta t.$$

Since A is closed, it follows from Corollary 3.11 that $\widehat{u}(\lambda) \in D(A)$ and

$$A\widehat{u}(\lambda) = A\biggl(\lambda\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t u(s)\Delta s\Delta t\biggr).$$

Applying now that u is a mild solution, we have

$$A\biggl(\lambda\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t u(s)\Delta s\Delta t\biggr) = \lambda\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)[u(t)-x]\Delta t = \lambda\widehat{u}(\lambda) - x$$

and (b) follows. Conversely, suppose that (b) is satisfied. Let $v(t) := \int_0^t u(s)\Delta s$. Then, using again integration by parts, we obtain

$$\begin{aligned}\widehat{u}(\lambda) &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)u(t)\Delta t\\ &= \lambda\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\int_0^t u(s)\Delta s\Delta t\lambda\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)v(t)\Delta t = \lambda\widehat{v}(\lambda).\end{aligned}$$

Thus, $\widehat{v}(\lambda) = \widehat{u}(\lambda)/\lambda \in D(A)$ and by hypothesis, we have

$$A\widehat{v}(\lambda) = A\,\frac{\widehat{u}(\lambda)}{\lambda} = \widehat{u}(\lambda) - \frac{x}{\lambda},$$

for $\mathrm{Re}_\mu(\lambda)(t) > c$ for all $t \in \mathbb{T}_0^+$ and $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$. Define $f(t) := u(t) - x$ for $t \in \mathbb{T}_0^+$. We claim that $\widehat{f}(\lambda) = \widehat{u}(\lambda) - x/\lambda$, whenever $\mathrm{Re}_\mu(\lambda)(t) > c$ for all

$t \in \mathbb{T}_0^+$ and $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$. Indeed,

$$\begin{aligned}\widehat{f}(\lambda) &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0) f(t)\Delta t = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)[u(t)-x]\Delta t \\ &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)u(t)\Delta t - \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)x\Delta t \\ &= \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)u(t)\Delta t - \frac{1}{\lambda}\int_0^\infty \lambda e^\sigma_{\ominus\lambda}(t,0)x\Delta t\end{aligned}$$

by Lemma 2.16

$$= \widehat{u}(\lambda) + \lim_{s\to\infty} \frac{1}{\lambda}\big[e_{\ominus\lambda(t,0)}x\big]_{t=0}^{t=s}$$

by Lemma 2.18

$$= \widehat{u}(\lambda) - \frac{x}{\lambda}.$$

Since $A\widehat{v}(\lambda) = \widehat{f}(\lambda)$ whenever $\mathrm{Re}_\mu(\lambda)(t) > c$ for all $t \in \mathbb{T}_0^+$, it follows from Theorem 3.15 that $u(t) \in D(A)$ and $Av(t) = f(t) = u(t) - x$ for all $t \in \mathbb{T}_0^+$, which means that u is a mild solution of (4.1). $\square$

Let u be a mild solution of (4.1) satisfying (4.5). Assume that $c < \mathrm{Re}_\mu(\lambda)(t)$ for every $t \in \mathbb{T}_0^+$ and $\lambda \in \rho(A) \cap \mathcal{R} \cap \mathbb{C} \setminus \{0\}$. Then, it follows from Theorem 4.3 that $(\lambda - A)\widehat{u}(\lambda) = x$ and thus, $\widehat{u}(\lambda) = (\lambda - A)^{-1}x$.

Definition 4.4. We say that $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$ is *strongly continuous* if T satisfies $\|T(t)x - x\|_X \to 0$ as $t \to 0^+$ for each $x \in X$.

We use this property to establish our concept of C_0-semigroup on time scales.

Definition 4.5. We say that $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$ is a *C_0-semigroup with infinitesimal generator A* if the following conditions are satisfied:

(a) $T(0) = I$ and for every $x \in X$, the function $t \mapsto T(t)x$ is strongly continuous.

(b) There exists a λ_0 such that $(\lambda_0, \infty) \subset \rho(A)$, $\lambda \in D\{T\} \cap \mathcal{R}$ and

$$\widehat{T}(\lambda)x = (\lambda - A)^{-1}x, \quad \text{for } x \in X,$$

for all $Re_\mu(\lambda)(t) > \lambda_0$ for $t \in \mathbb{T}_0^+$.

As a consequence of Theorem 3.14, we can establish the following immediate result.

Corollary 4.6. *Let $A\colon D(A) \subset X \to X$ be a closed linear operator. Then there exists at most a unique C_0-semigroup with infinitesimal generator A.*

Lemma 4.7. *If $f\colon \mathbb{T}_0^+ \to X$ be a continuous function at 0 and $|f(t) - f(0)| \le M$, then*

$$\lim_{\lambda\to\infty} \lambda\widehat{f}(\lambda) = f(0).$$

PROOF. Notice that

$$\left|\lambda \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)f(t)\Delta t - f(0)\right| = \left|\lambda \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)[f(t)-f(0)]\Delta t\right|,$$

where this last equality follows, since

$$-\lambda \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)f(0)\Delta t = \lim_{s\to+\infty}\left[e_{\ominus\lambda}(t,0)f(0)\right]_{t=0}^{t=s} = -f(0),$$

using Lemmas 2.16 and 2.18. On the other hand, by the continuity of f, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t)-f(0)| < \varepsilon \quad \text{for } t \in (0,\delta)_{\mathbb{T}_0^+}.$$

Then, we have again by Lemmas 2.16 and 2.18

$$\begin{aligned}
(4.7)\quad & \left|\lambda \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)[f(t)-f(0)]\Delta t\right| \\
&\le \lambda\left[\int_0^\delta e^\sigma_{\ominus\lambda}(t,0)|f(t)-f(0)|\Delta t + \int_\delta^\infty e^\sigma_{\ominus\lambda}(t,0)|f(t)-f(0)|\Delta t\right] \\
&< \lambda \int_0^\delta e^\sigma_{\ominus\lambda}(t,0)\varepsilon\Delta t + \lambda \int_\delta^\infty e^\sigma_{\ominus\lambda}(t,0)M\Delta t \\
&= \left[-e_{\ominus\lambda}(t,0)\varepsilon\right]_0^\delta + \lim_{s\to+\infty}\left[-e_{\ominus\lambda}(t,0)M\right]_\delta^s \\
&= \left[-e_{\ominus\lambda}(\delta,0)\varepsilon\right] + \varepsilon + \left[e_{\ominus\lambda}(\delta,0)M\right] = (M-\varepsilon)e_{\ominus\lambda}(\delta,0) + \varepsilon.
\end{aligned}$$

We claim that for each fixed $t > 0$, $\lim_{\lambda\to\infty} e_{\ominus\lambda}(t,0) = 0$. Indeed,

$$\begin{aligned}
e_{\ominus\lambda}(t,0) &= \exp\left(\int_0^t \frac{1}{\mu(\tau)}\log(1+\mu(\tau)\ominus\lambda)\Delta\tau\right) \\
&= \exp\left(\int_0^t \frac{1}{\mu(\tau)}\log\left(\frac{1}{1+\mu(\tau)\lambda}\right)\Delta\tau\right) \\
&= \exp\left(\int_0^t \frac{1}{\mu(\tau)}(-\log(1+\mu(\tau)\lambda))\Delta\tau\right) \\
&= \exp\left(\int_0^t \frac{1}{\mu(\tau)}(-\log(1+\mu(\tau)\lambda))\Delta\tau\right) \\
&\le \exp\left(-\log(1+\overline{\mu}\lambda)\int_0^t \frac{1}{\overline{\mu}}\Delta\tau\right) = \exp\left(-\log(1+\overline{\mu}\lambda)\frac{t}{\overline{\mu}}\right)
\end{aligned}$$

where $\overline{\mu} = \min_{\tau\in[0,t]_{\mathbb{T}}} \mu(\tau)$ if $\overline{\mu} \neq 0$. Notice that, if $\lambda \to +\infty$, then $-\log(1+\overline{\mu}\lambda) \to -\infty$, since $\overline{\mu} > 0$. It implies that

$$\lim_{\lambda\to\infty} \exp\left(-\log(1+\overline{\mu}\lambda)\frac{t}{\overline{\mu}}\right) = 0,$$

which yields

$$\lim_{\lambda\to\infty} e_{\ominus\lambda}(t,0) = 0. \tag{4.8}$$

On the other hand, for the case $\mu(t) \equiv 0$, the equality (4.8) follows immediately. Also, the case where $\overline{\mu} = 0$, but $\mu(t)$ is not identically zero, follows by combining the previous cases.

Using inequality (4.7), we obtain

$$\left|\lambda \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)[f(t) - f(0)]\Delta t\right| \to 0,$$

where $\lambda \to +\infty$, obtaining the desired result. □

Corollary 4.8. *Let $A\colon D(A) \subset X \to X$ be a closed linear operator that generates a C_0-semigroup $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$. Then $D(A)$ is dense in X.*

In the sequel, we present a result which describes an important property of C_0-semigroup on time scales.

Theorem 4.9. *Let $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$ be a C_0-semigroup with infinitesimal generator A. Let (4.5) be satisfied and $c \in D\{T\}$ for every $t \in \mathbb{T}_0^+$. Then, the following conditions hold:*

(a) *Let $B \in \mathcal{L}(X)$ such that $B\widehat{T}(\lambda) = \widehat{T}(\lambda)B$ whenever $Re_\mu(\lambda)(t) > c$ for every $t \in \mathbb{T}_0^+$ and $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$. Then $BT(t) = T(t)B$ for all $t \in \mathbb{T}_0^+$.*
(b) *In particular, $T(t)T(s) = T(s)T(t)$ for all $t, s \in \mathbb{T}_0^+$.*

Proof. We start by proving item (a). For $x \in X$ and $Re_\mu(\lambda)(t) > c$ for $t \in \mathbb{T}_0^+$, we have

$$\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)T(t)Bx\Delta t = \widehat{T}(\lambda)Bx = B\widehat{T}(\lambda)x = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)BT(t)x\Delta t,$$

which implies by the uniqueness theorem that $T(t)Bx = BT(t)x$ for all $t \in \mathbb{T}_0^+$.

Now, let us prove item (b). We fix $\omega \in \mathbb{C} \setminus \{0\} \cap \mathcal{R}$ such that $\mathrm{Re}_\mu(\omega)(t) > c$ for every $t \in \mathbb{T}_0^+$. Since $(\omega - A)^{-1}(\lambda - A)^{-1} = (\lambda - A)^{-1}(\omega - A)^{-1}$, whenever $\mathrm{Re}_\mu(\lambda)(t) > c$, for every $t \in \mathbb{T}_0^+$ and $\lambda \in \mathcal{R}$, we obtain by (a) that

$$(\omega - A)^{-1}T(t) = T(t)(\omega - A)^{-1} \quad \text{for all } t \in \mathbb{T}_0^+.$$

We can repeat this argument with $B = T(t)$ and using again (a), we have

$$T(s)T(t) = T(t)T(s), \quad \text{for all } t, s \in \mathbb{T}_0^+.$$ □

The next result describes a property of the delta integral of a C_0-semigroup T.

Theorem 4.10. *Let $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$ be a C_0-semigroup, then the following equalities hold:*

(a) *If t is right-scattered, then*

$$\frac{1}{\mu(t)} \int_t^{\sigma(t)} T(\tau)x\Delta\tau = T(t)x, \quad \textit{for } x \in X.$$

(b) *If t is right-dense, then*

$$\lim_{s\to t^+} \frac{1}{s-t}\int_t^s T(\tau)x\Delta\tau = T(t)x, \quad \textit{for } x \in X.$$

PROOF. First, let us prove assertion (a). This property is an immediate consequence of Theorem 2.6, because

$$\frac{1}{\mu(t)}\int_t^{\sigma(t)} T(\tau)x\Delta\tau = \frac{1}{\mu(t)}\,\mu(t)T(t)x = T(t)x.$$

(b) Define $f(t) := \int_0^t T(\tau)x\Delta\tau$. Then, we have

$$\lim_{s\to t^+} \frac{1}{s-t}\int_t^s T(\tau)x\Delta\tau = \lim_{s\to t^+} \frac{f(s)-f(t)}{s-t} = f^{\Delta}(t) = T(t)x,$$

by the Fundamental Theorem of Calculus for Δ-integrals (see [7]). □

Our next objective is to characterize the infinitesimal generator of the semigroup $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$. To this object, we begin by introducing the operator B as follows:

(a) If 0 is right-scattered, then

$$Bx = \left.\frac{T(\sigma(t))x - x}{\mu(t)}\right|_{t=0} = \frac{T(\sigma(0))x - x}{\mu(0)}.$$

(b) If 0 is right-dense, then

$$Bx = \lim_{t\to 0} \frac{T(t)x - x}{t},$$

on the domain $D(B)$ consisting of all $x \in X$ for which the limit exists.

In our next result, we establish that our definition of C_0-semigroup is equivalent to the existence of mild solutions of an abstract Cauchy problem. In what follows, let us consider that $A\colon D(A) \subseteq X \to X$ is a closed linear operator and λ_0 is the constant involved in Definition 4.5.

THEOREM 4.11. *Let $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$ be a C_0-semigroup on X and let A be its generator. Then the following properties hold*:

(a) *For all $x \in X$, the function $u_x(t) = T(t)x$, $t \in \mathbb{T}_0^+$, is a mild solution of (4.1), where u_x satisfies (4.5).*
(b) *$(\lambda - A)^{-1}T(t) = T(t)(\lambda - A)^{-1}$ for all $\lambda \in (\lambda_0, \infty) \subset \rho(A)$, where $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$, for all $t \in \mathbb{T}_0^+$.*
(c) *If $x \in D(A)$, then $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$, for all $t \in \mathbb{T}_0^+$.*
(d) *For all $x \in X$ and $t \in \mathbb{T}_0^+$*

$$\int_0^t T(s)x\Delta s \in D(A) \quad \textit{and} \quad A\int_0^t T(s)x\Delta s = T(t)x - x.$$

(e) *Let $x, y \in X$. Then $x \in D(A)$ and $Ax = y$, if and only if,*

$$\int_0^t T(s)y\Delta s = T(t)x - x \quad \textit{for all } t \in \mathbb{T}_0^+.$$

(f) $A = B$.

(g) *$T(\cdot)x$ is a classical solution of (4.1), if and only if, $x \in D(A)$.*

Proof. (a) It follows from Definition 4.5 that $\widehat{u}_x(\lambda) = (\lambda I - A)^{-1}x$ exists whenever $\mathrm{Re}_\mu(\lambda)(t) > \lambda_0$ for all $t \in \mathbb{T}_0^+$ and $\lambda \in \mathcal{R} \cap \mathbb{C} \setminus \{0\}$. Therefore, $\widehat{u}_x(\lambda) \in D(A)$ and $\lambda\widehat{u}_x(\lambda) - A\widehat{u}_x(\lambda) = x$. Using Theorem 4.3, we obtain the assertion.

(b) This assertion is an immediate consequence of Theorem 4.9(i).

(c) This property is a direct consequence of (b) and Theorem 3.16.

(d) This assertion follows immediately from item (a) and by our definition of mild solution.

(e) Let $x \in D(A)$ and $Ax = y$. Then, by (c), we have that $AT(t)x = T(t)Ax$. Also, by (d), we obtain:

$$\int_0^t T(s)Ax\Delta s = A\int_0^t T(s)x\Delta s = T(t)x - x, \quad \text{for } t \in \mathbb{T}_0^+.$$

Reciprocally, let $x, y \in X$ be such that

$$\int_0^t T(s)y\Delta s = T(t)x - x \quad \text{for all } t \in \mathbb{T}_0^+.$$

Since

$$(\lambda - A)^{-1}y = \int_0^\infty e^\sigma_{\ominus\lambda}(t,0)\,T(t)y\Delta t,$$

using integration by parts, we obtain

$$\begin{aligned}
(\lambda - A)^{-1}y &= -\int_0^\infty \ominus\lambda e_{\ominus\lambda}(t,0)\int_0^t T(s)y\Delta s \\
&= \int_0^\infty \lambda e^\sigma_{\ominus\lambda}(t,0)(T(t)x - x)\Delta t \\
&= \lambda\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)T(t)x\Delta t - \lambda\int_0^\infty e^\sigma_{\ominus\lambda}(t,0)x\Delta t \\
&= \lambda(\lambda - A)^{-1}x - e_{\ominus\lambda}(0,0)x = \lambda(\lambda - A)^{-1}x - x,
\end{aligned}$$

where we have used Lemmas 2.18 and 2.16. Consequently, $x \in D(A)$ and

$$y = \lambda x - (\lambda - A)x = Ax.$$

(f) Initially, we consider that $t = 0$ is right-dense. For $x \in D(A)$, applying (e), we obtain

$$\frac{1}{t}\,(T(t)x - x) = \frac{1}{t}\int_0^t T(s)Ax\Delta s \to Ax$$

as $t \to 0$. This implies that $x \in D(B)$ and $Bx = Ax$. Conversely, let $x \in D(B)$, then

$$Bx = \lim_{t\to 0} \frac{1}{t}\,(T(t)x - x) = \lim_{t\to 0} \frac{1}{t}\, A \int_0^t T(s)x\Delta s.$$

Since A is a closed operator, it follows that $x \in D(A)$ and $Ax = Bx$.

On the other hand, if $t = 0$ is right-scattered, then by Theorem 2.6 and (e), we have

$$Bx = \frac{1}{\mu(0)}\,(T(\sigma(0))x - x) = \frac{1}{\mu(0)} \int_0^{\sigma(0)} T(s)Ax\Delta s = \frac{1}{\mu(0)}\, T(0)\mu(0)Ax = Ax,$$

for all $x \in D(A) \cap D(B)$.

(g) Let $x \in D(A)$, then by assertion (e), we obtain

$$T(t)x = x + \int_0^t T(s)x\Delta s, \quad \text{for } t \in \mathbb{T}_0^+.$$

Thus, clearly $T(\,\cdot\,) \in C^1_{\mathrm{rd}}(\mathbb{T}_0^+, X)$ and the result follows from Theorem 4.2. Conversely, if $T(\,\cdot\,)x$ is a classical solution, then $x = T(0)x \in D(A)$ by the definition. □

Consequently, our definition of C_0-semigroup is directly related with the existence of mild solutions to the abstract Cauchy problem (4.1). For this reason, we now discuss the existence of mild solutions to problem (4.1). Initially, we characterize the existence of solutions when $\mu(0) > 0$.

THEOREM 4.12. *Assume that $\mu(0) > 0$. Suppose further that there exists a subspace $D \subseteq X$ dense in X having the following property: for each $x \in D$, there exists a unique classical solution $u(\,\cdot\,)$ of (4.1). Then A is a bounded linear operator on X.*

PROOF. Since

$$u^{\Delta}(0) = \frac{u(\sigma(0)) - x}{\mu(0)} = Ax,$$

it follows that A is continuous on D. Since D is dense in X, A has a unique extension to X as a bounded linear operator. □

THEOREM 4.13. *Let $A \in \mathcal{L}(X)$, then (4.1) has a unique classical solution.*

PROOF. We fix $a \in \mathbb{T}_0^+$ and define the operator

$$\Gamma\colon C_{\mathrm{rd}}([0,a]_{\mathbb{T}_0^+}, X) \to C_{\mathrm{rd}}([0,a]_{\mathbb{T}_0^+}, X)$$

by

$$(\Gamma u)(t) = x + A\int_0^t u(s)\Delta s, \quad \text{for } t \in \mathbb{T}_0^+.$$

It is not difficult to prove that Γ has a unique fixed point $u(\,\cdot\,)$ which is the solution of (4.1). Moreover, since

$$u(t) = x + A\int_0^t u(s)\Delta s = x + \int_0^t Au(s)\Delta s, \quad \text{for } t \in \mathbb{T}_0^+$$

we have that $u^\Delta(t) = Au(t)$ and $u(0) = x$, which implies that $u(\,\cdot\,)$ is a classical solution of (4.1). □

We next introduce a class of problems defined by unbounded linear operators for which the associated abstract Cauchy problem has mild solutions, but no classical solutions.

Example 4.14. We know that the scalar equation

$$y^\Delta(t) = -ay(t), \qquad y(0) = y_0,$$

where $t \in \mathbb{T}_0^+$ and a is a fixed positive constant and $-a$ is regressive, has the solution

$$y(t) = e_{-a}^\sigma(t,0)y_0.$$

In what follows, we assume that the exponential function satisfies the condition

$$\left|e_{-a}^\sigma(t,0)\right| \le \frac{1}{1+at},$$

on the time scale $\mathbb{T}_0^+$. We consider the space $X = L^2([0,\pi])$, and let $A\colon D(A) \subset X \to X$ be the operator given by

$$Ax(\xi) = \frac{d^2x(\xi)}{d\xi^2}, \quad \xi \in [0,\pi]$$

with domain $D(A) = \{x(\,\cdot\,) \in X : x''(\,\cdot\,) \in X,\ x(0) = x(\pi) = 0\}$.

It is well known that A is the diffusion operator, which appears, for instance, in the heat equation for a unidimensional metal bar located on $[0,\pi]$. From another point of view, it is well know that A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by

$$z_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2} \sin n\xi, \quad \xi \in [0,\pi].$$

In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X and, for $w \in D(A)$,

$$Aw = \sum_{n=1}^{\infty} -n^2\langle w, z_n\rangle z_n.$$

We will show that the equation (4.1) has a mild solution for all $x \in X$. Because, in general $x \notin D(A)$, such mild solution is not a classical solution.

Let $x_n = \langle x, z_n\rangle$, $n \in \mathbb{N}$, and $-n^2 \in \mathcal{R}$. Using the previous comments, we can prove that the solution of the scalar equation

$${u_n}^\Delta(t) = -n^2u_n(t), \quad u_n(0) = x_n, \quad t \in \mathbb{T}_0^+$$

has solution $u_n(t) = e^{\sigma}_{-n^2}(t,0)x_n$, which satisfies

$$|u_n(t)| \leq \frac{1}{1+n^2t}\,|x_n|. \tag{4.9}$$

Moreover, it follows from the Fundamental Theorem of Calculus ([7, Theorem 5.34]) that

$$-\,n^2\int_0^t u_n(\tau)\Delta\tau = u_n(t) - x_n, \quad t \in \mathbb{T}_0^+, \tag{4.10}$$

which implies that

$$\left|\int_0^t u_n(\tau)\Delta\tau\right| \leq \frac{1}{n^2}\left(1+\frac{1}{1+n^2t}\right)|x_n|, \quad \text{for } t \in \mathbb{T}_0^+. \tag{4.11}$$

We define

$$u(t) = \sum_{n=1}^{\infty} u_n(t)z_n, \quad \text{for } t \in \mathbb{T}_0^+. \tag{4.12}$$

It follows from (4.9) that the series (4.12) converges uniformly, which implies that

$$\int_0^t u(\tau)\Delta\tau = \sum_{n=1}^{\infty}\int_0^t u(\tau)\Delta\tau z_n, \quad \text{for } t \in \mathbb{T}_0^+.$$

In addition, using (4.11), we obtain

$$\sum_{n=1}^{\infty} n^4\left|\int_0^t u_n(\tau)\Delta\tau\right|^2 \leq \sum_{n=1}^{\infty}\left(1+\frac{1}{1+n^2t}\right)^2|x_n|^2 \leq 4\sum_{n=1}^{\infty}|x_n|^2 < \infty,$$

which yields that

$$\int_0^t u(\tau)\Delta\tau \in D(A).$$

Furthermore,

$$A\int_0^t u(\tau)\Delta\tau = \sum_{n=1}^{\infty} -n^2\int_0^t u(\tau)\Delta\tau z_n = \sum_{n=1}^{\infty}(u_n(t)-x_n)z_n = u(t) - x,$$

for $t \in \mathbb{T}_0^+$, so that $u(\,\cdot\,)$ is a mild solution of (4.1).

As a matter of fact, the previous construction is valid for any self-adjoint with compact resolvent operator A with spectrum included in $\mathbb{R}^-$.

In the sequel, we discuss some important properties of semigroups on time scales.

REMARK 4.15. A natural question which appears is if the usual property of semigroup

$$T(t+s) = T(t)T(s), \quad \text{for } s,t \in \mathbb{T}_0^+, \tag{4.13}$$

is fulfilled. In order to guarantee that condition (4.13) holds, we need to ensure that the time scale $\mathbb{T}$ has the additive property

(C_2) If $a, b \in \mathbb{T}$, then $a + b \in \mathbb{T}$.

However, since we are developing a semigroup theory for a general time scale $\mathbb{T}$, this property has no sense because, in general, $s + t \notin \mathbb{T}$ for $s, t \in \mathbb{T}$. In other words, most time scales do not satisfy the additive property (C_2). For instance, consider the quantum scale $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$. Then, clearly $q^m, q^n \in \mathbb{T}$, but $q^m + q^n \notin \mathbb{T}$. To avoid this difficult, some authors have considered the condition (C_1) mentioned in the introduction. However, it is clear that (C_1) is a group type condition that implies condition (C_2). Nevertheless, we will show that even with this "weaker" condition, the abstract Cauchy problem on time scales does not encompass interesting cases of time scales.

First, we present a result which describes how strong the condition (C_2) is. For instance, if $\mathbb{T}_0^+$ satisfies the property (C_2) and 0 is right-dense, then $\mathbb{T}_0^+ = [0, +\infty)$.

Theorem 4.16. *Assume that* $\mu(0) = 0$ *and* (C_2) *is fulfilled. Then* $\mathbb{T}_0^+ = [0, +\infty)$.

Proof. Since $\mu(0) = 0$, 0 is right-dense. Thus, there exists a sequence $\{t_n\}$ in $\mathbb{T}_0^+$ such that $t_n \to 0$ as $n \to \infty$. Moreover, considering that the condition (C_2) is fulfilled, it is not difficult to see that $\{kt_n : k, n \in \mathbb{N}\} \subset \mathbb{T}_0^+$. From these facts, we obtain

$$[0, \infty) \subset \overline{\{kt_n : k, n \in \mathbb{N}\}} \subset \overline{\mathbb{T}_0^+} = \mathbb{T}_0^+,$$

and we get the desired result. □

Remark 4.17. By Theorem 4.16, it is clear that the quantum time scale $q^{\mathbb{Z}} \cup \{0\}$ does not satisfy the property (C_2), since $\mu(0) = 0$ in this case. Also, all the hybrid continuous-discrete time scales such that $0 \in \mathbb{T}$ and 0 is right-dense do not satisfy the property (C_2).

Consequently, as an immediate consequence of Theorem 4.16, if $\mu(0) = 0$ and $T(\cdot)$ satisfies the classical property of C_0-semigroup in $\mathbb{T}$, then $T(\cdot)$ is a C_0-semigroup in the classical sense.

We can abridge the earlier discussion as follows.

(i) If $\mu(0) = 0$ and condition (C_2) holds, then (4.1) is the classical abstract Cauchy problem and the solution is given by a C_0-semigroup defined on $[0, \infty)$.

(ii) If $\mu(0) > 0$ and (4.1) has a solution $u(t, x)$ for all $x \in X$, then A is a bounded linear map. We define $e_A(t, 0)x = u(t, x)$. It is clear that $e_A(t, 0)\colon X \to X$ is a bounded linear map, strongly continuous and uniformly bounded in bounded intervals.

We finish this section with an application to the behaviour of semigroups on time scales. Assume that A generates the semigroup $T\colon \mathbb{T}_0^+ \to \mathcal{L}(X)$ in the Banach space X, and let $\lambda \in \mathbb{R}$ be an eigenvalue of A with corresponding eigenvector $x \in X$.

THEOREM 4.18. *Under the above conditions, if $\lambda \in \mathbb{R} \cap \mathcal{R}$, then*

$$T(t)x = e_\lambda(t,0)x, \quad \textit{for } t \in \mathbb{T}_0^+.$$

PROOF. Let $u(t) = e_\lambda(t,0)x$. It follows that

$$u^\Delta(t) = e_\lambda(t,0)^\Delta x = \lambda e_\lambda(t,0)x = Au(t), \quad \text{for } t \in \mathbb{T}_0^+,$$

and $u(0) = x$, where $A := \lambda$. From the uniqueness of solutions of the problem (4.1) for the case $A := \lambda$ (see [6]), we conclude that $T(t)x = u(t)$, for $t \in \mathbb{T}_0^+$. □

Assume now that $\{\lambda_n : n \in \mathbb{N}\}$ is a sequence of eigenvalues of A, $\lambda_n \in \mathbb{R} \cap \mathcal{R}$, with corresponding eigenvectors x_n for all $n \in \mathbb{N}$. Assume further that $\{x_n : n \in \mathbb{N}\}$ is a unconditional (Schauder) basis of X.

COROLLARY 4.19. *Under the above conditions, there exists a constant $K > 0$ such that*

$$\|T(t)x\| \leq K \sup_n |e_{\lambda_n}(t,0)| \|x\|, \quad \textit{for } t \in \mathbb{T}_0^+,\ x \in X,$$

where $\lambda_n \in \mathcal{R}$.

PROOF. Assume that $x = \sum_{n=1}^{\infty} a_n x_n$. Since $T(t) \in \mathcal{L}(X)$, for all $t \in \mathbb{T}_0^+$ we have

$$T(t)x = \sum_{n=1}^{\infty} a_n T(t) x_n = \sum_{n=1}^{\infty} a_n e_{\lambda_n}(t,0) x_n, \quad vt \in \mathbb{T}_0^+,$$

by Theorem 4.18 and it follows from [21, Proposition 1.c.7] that

$$\|T(t)x\| \leq K \sup_n |e_{\lambda_n}(t,0)| \left\| \sum_{n=1}^{\infty} a_n x_n \right\| = K \sup_n |e_{\lambda_n}(t,0)| \|x\|,$$

for all $t \in \mathbb{T}_0^+$, obtaining the desired result. □

We point out that this last result generalizes the properties of $e_A(t,\cdot)$ when A is an $n \times n$ matrix (see [6], [7]).

5. Nonhomogeneous abstract Cauchy problem on time scales

Our purpose in this section is to investigate the existence of solutions of the following nonhomogeneous abstract Cauchy problem on time scales

$$\begin{cases} u^\Delta(t) = Au(t) + f(t) & \text{for } t \in \mathbb{T}_0^+, \\ u(0) = x \in X. \end{cases} \tag{5.1}$$

Here, we assume that the values $u(t) \in X$ and $f\colon \mathbb{T}_0^+ \to X$ is an rd-continuous function.

Assume initially that A is the infinitesimal generator of a C_0-semigroup $T\colon \mathbb{T}_0^+ \to X$, and that there exists an rd-continuous function g such that

$$(\lambda I - A)\widehat{g}(\lambda) = \widehat{f}(\lambda), \quad \mathrm{Re}_\mu(\lambda)(t) > \mathrm{Re}_\mu(w)(t) \quad \text{for all } t \in \mathbb{T}_0^+.$$

We start by presenting the definition of mild solution of problem (5.1).

Definition 5.1. We say that an rd-continuous function $u\colon \mathbb{T}_0^+ \to X$ is a *mild solution* of (5.1) if

$$u(t) = x + A\int_0^t u(s)\Delta s + \int_0^t f(s)\Delta s, \quad t \in \mathbb{T}_0^+.$$

In the sequel, we restrict us to consider the operator $A \in \mathcal{L}(X)$. In this case, for every $s \in \mathbb{T}_0^+$, we consider the abstract Cauchy problem given by

$$\begin{cases} u^\Delta(t) = Au(t) \quad \text{for } t \in \mathbb{T}_0^+,\ t \geq s, u(s) = x \in X. \end{cases} \tag{5.2}$$

Definition 5.2. We say that an rd-continuous function $u\colon [s,\infty)_{\mathbb{T}_0^+} \to X$ is a *mild solution* of problem (5.2) if

$$u(t) = x + A\int_s^t u(\tau)\Delta\tau, \quad \text{for } t \in \mathbb{T}_0^+,\ t \geq s.$$

Proceeding as indicated in Theorem 4.13 , we can show that problem (5.2) has a unique solution $u(t,s)$ for all $x \in X$. We define $T(t,s)x = u(t,s)$. It is not difficult to see that $T\colon \{(t,s) : s,t \in \mathbb{T}_0^+,\ t \geq s\} \to \mathcal{L}(X)$ is a strongly continuous map.

We are now in a position to relate the homogeneous abstract Cauchy problem (5.2) with the nonhomogeneous problem (5.1). The following result generalizes [7, Theorem 5.24].

Theorem 5.3. *Let $A \in \mathcal{L}(X)$ and assume that $f\colon \mathbb{T}_0^+ \to X$ is rd-continuous. Then the mild solution $u(t)$ of the problem* (5.1) *is given by*

$$u(t) = T(t,t_0)x + \int_{t_0}^t T(t,\sigma(\tau))f(\tau)\Delta\tau. \tag{5.3}$$

Remark 5.4. Before to prove this result, we need to clarify formula (5.3) that except for very special situations, the abstract Cauchy problem

$$\begin{cases} u^\Delta(t) = Au(t) & \text{for } t \in \mathbb{T}_0^+,\ t \geq t_0, \\ u(t_0) = x \in X, \end{cases} \tag{5.4}$$

in Banach spaces has no sense for $t \leq t_0$. Thus, we only seek the solution for $t \geq t_0$. Hence, $T(t,s)$ is only defined for $t \geq s$. Consequently, for $\tau < t$, we have that $\sigma(\tau) \leq t$ and $T(t,\sigma(\tau))$ i s well-defined. However, for $\tau = t$ and $\sigma(t) > t$, $T(t,\sigma(\tau))$ has no sense in (5.3). Notwithstanding the foregoing, if we consider the definition of integral in (5.3), we can see that the value $T(t,\sigma(t))$

is never used in (5.3). This allows us to extend formally the definition of $T(t,s)$ by $T(t,s) = I$ for $t \le s$. With this definition, the integral in (5.3) is formally correct, and we can proceed to prove (5.3).

PROOF OF THEOREM 5.3. We define the function

$$u(t) := T(t)x + \int_0^t T(t,\sigma(\tau))f(\tau)\Delta\tau, \quad \text{for } t \in \mathbb{T}_0^+, \tag{5.5}$$

and using the results in [5], we calculate

$$\begin{aligned}
x &+ A\int_0^t u(s)\Delta s + \int_0^t f(s)\Delta s \\
&= x + A\int_0^t \left[T(s)x + \int_0^s T(s,\sigma(\tau))f(\tau)\Delta\tau\right]\Delta s + \int_0^t f(s)\Delta s \\
&= x + A\int_0^t T(s)x\Delta s + A\int_0^t\int_0^s T(s,\sigma(\tau))f(\tau)\Delta\tau\Delta s + \int_0^t f(s)\Delta s \\
&= T(t)x + A\int_0^t\int_{\sigma(\tau)}^t T(s,\sigma(\tau))f(\tau)\Delta s\Delta\tau + \int_0^t f(s)\Delta s \\
&= T(t)x + A\int_0^t\int_{\sigma(\tau)}^t T(s,\sigma(\tau))f(\tau)\Delta s\Delta\tau + \int_0^t f(s)\Delta s \\
&= T(t)x + \int_0^t A\int_{\sigma(\tau)}^t T(s,\sigma(\tau))f(\tau)\Delta s\Delta\tau + \int_0^t f(s)\Delta s \\
&= T(t)x + \int_0^t \big[T(t,\sigma(\tau))f(\tau) - f(\tau)\big]\Delta\tau + \int_0^t f(s)\Delta s \\
&= T(t)x + \int_0^t T(t,\sigma(\tau))f(\tau)\Delta\tau = u(t),
\end{aligned}$$

for $t \in \mathbb{T}_0^+$, which implies that $u(\cdot)$ is the mild solution of problem (5.1). ☐

6. Abstract Cauchy problem on time scales in Fréchet space

In this section, we will show that the abstract Cauchy Problem (4.1) can be solved for all x in a sufficiently large subset of X. Let $A\colon D(A) \subseteq X \to X$ be a closed linear operator and $D := \bigcap_{n=1}^{\infty} D(A^n)$. We define on D the norm

$$\Pi_n(x) := \max_{0\le i\le n} \|A^i x\|, \quad n \in \mathbb{N},$$

then D with the locally convex topology induced by the family $\{\Pi_n : n \in \mathbb{N}\}$ is a Fréchet space. Let $a > 0$ and let $C_{\mathrm{rd}}([0,a]_{\mathbb{T}}, D)$ be the space of rd-continuous functions $x\colon [0,a]_{\mathbb{T}_0^+} \to D$ endowed by the topology generated by the family of norms

$$\widetilde{\Pi}_n(x) = \max_{0\le t\le a} \Pi_n(x(t)).$$

Then $C_{\mathrm{rd}}\big([0,a]_{\mathbb{T}_0^+}, D\big)$ is also a Fréchet space.

Lemma 6.1. *Let (Y,τ) be a Fréchet space, where the topology τ is generated by the family of seminorms $\{p_n : n \in \mathbb{N}\}$. Let $\Gamma\colon Y \to Y$ be a map such that*

$$p_n(\Gamma y_1 - \Gamma y_2) \le \alpha p_{n+1}(y_1 - y_2), \quad \text{for all } y_1, y_2 \in Y \text{ and for all } n \in \mathbb{N},$$

for some $0 < \alpha < 1$. Assume that there exists $y_0 \in Y$ such that $\{p_k(\Gamma y_0 - y_0) : k \in \mathbb{N}\}$ is a bounded set. Then $(\Gamma^k y_0)_k$ converges to a fixed point $\overline{y}$ of Γ. Moreover, $\overline{y}$ is the unique fixed point of Γ in the set

$$\Big\{y \in Y : \sup_{k\in\mathbb{N}} p_k(y) < \infty\Big\}.$$

Proof. We proceed as in the proof of Banach's fixed point theorem. We have that, for every $k \in \mathbb{N}$,

$$\begin{aligned} p_k(\Gamma^2 y_0 - \Gamma y_0) &\le \alpha p_{k+1}(\Gamma y_0 - y_0), \\ p_k(\Gamma^n y_0 - \Gamma^{n-1} y_0) &\le \alpha^{n+1} p_{k+n-1}(\Gamma y_0 - y_0). \end{aligned}$$

Therefore, we get

$$\begin{aligned} p_k(\Gamma^{m+n} y_0 - \Gamma^m y_0) &\le p_k(\Gamma^{n+m} y_0 - \Gamma^{n+m-1} y_0) + \ldots + p_k(\Gamma^{m+1} y_0 - \Gamma^m y_0) \\ &\le \alpha^{n+m-1} p_{k+n+m-1}(\Gamma y_0 - y_0) + \ldots + \alpha^m p_{k+m}(\Gamma y_0 - y_0) \\ &\le \alpha^m[\alpha^{n-1} + \ldots + 1]M \le \frac{\alpha^m}{1-\alpha} M \to 0, \end{aligned}$$

as $m \to \infty$, where $M = \sup_{k\in\mathbb{N}} p_k(\Gamma y_0 - y_0)$. This implies that $(\Gamma^n y_0)_n$ is a Cauchy sequence in Y. As a consequence, there exists $\overline{y} = \lim_{n\to\infty} \Gamma^n y_0$. Since Γ is continuous, we get $\Gamma\overline{y} = \overline{y}$. Moreover, for every $k \in \mathbb{N}$, we have

$$\begin{aligned} p_k(\overline{y}) &\le p_k(\overline{y} - \Gamma^n y_0) + p_k(\Gamma^n y_0 - \Gamma^{n-1} y_0) + \ldots + p_k(F y_0 - y_0) \\ &\le 1 + (\alpha^{n-1} + \ldots + 1)M \le 1 + \frac{M}{1-\alpha}, \end{aligned}$$

where we have selected $n \in \mathbb{N}$ such that $p_k(\overline{y} - \Gamma^n y_0) \le 1$. This implies that $\{p_k(\overline{y}) : k \in \mathbb{N}\}$ is bounded.

Let $\overline{y}_1$ be a fixed point of Γ such that $\{p_k(\overline{y}_1) : k \in \mathbb{N}\}$ is bounded. Then

$$\begin{aligned} p_k(\overline{y} - \overline{y}_1) &= p_k(\Gamma\overline{y} - \Gamma\overline{y}_1) \le p_{k+1}(\overline{y} - \overline{y}_1) \\ &\le \alpha^2 p_{k+2}(\overline{y} - \overline{y}_1) \le \alpha^n p_{k+n}(\overline{y} - \overline{y}_1) \to 0 \end{aligned}$$

as $n \to \infty$. Hence, $p_k(\overline{y} - \overline{y}_1) = 0$ for all $k \in \mathbb{N}$. This implies $\overline{y} = \overline{y}_1$. □

The main result in this section is the following theorem.

Theorem 6.2. *Assume that $\sup_{i\in\mathbb{N}} \|A^i x\| < \infty$. Then the problem (4.1) has a solution $u \in \mathcal{C}_{\rm rd}\big([0,\infty)_{\mathbb{T}_0^+}, D\big)$. Moreover, (4.1) has a unique solution $u(\,\cdot\,) \in \mathcal{C}_{\rm rd}([0,\infty)_{\mathbb{T}_0^+}, D)$ such that $\max_{i\in\mathbb{N}} \max_{t\in[0,b]_{\mathbb{T}_0^+}} \|A^i u(t)\| < \infty$ for all $b \in \mathbb{T}_0^+$.*

Proof. We fix $0<a<1$. We define $\Gamma\colon \mathcal{C}_{\mathrm{rd}}\big([0,a]_{\mathbb{T}_0^+},D\big)\to\mathcal{C}_{\mathrm{rd}}\big([0,a]_{\mathbb{T}_0^+},D\big)$ by

$$\Gamma u(t)=x+\int_0^t Au(s)\Delta s,$$

then

$$\begin{aligned}\widetilde{\Pi}_n(\Gamma u_1-\Gamma u_2)&=\max_{t\in[0,a]_{\mathbb{T}_0^+}}\Pi_n(\Gamma u_1(t)-\Gamma u_2(t))\\&=\max_{t\in[0,a]_{\mathbb{T}_0^+}}\max_{0\le i\le n}\|A^i(\Gamma u_1(t)-\Gamma u_2(t))\|\\&=\max_{t\in[0,a]_{\mathbb{T}_0^+}}\max_{0\le i\le n}\left\|\int_0^t A^{i+1}(u_1(s)-u_2(s))\Delta s\right\|\\&\le a\max_{s\in[0,a]_{\mathbb{T}_0^+}}\Pi_{n+1}(u_1(s)-u_2(s))\le a\widetilde{\Pi}_{n+1}(u_1-u_2).\end{aligned}$$

Since $\Gamma 0=x$, applying Lemma 6.1, we obtain that Γ has a fixed point in $C_{\mathrm{rd}}\big([0,a]_{\mathbb{T}_0^+},D\big)$. It is clear that $u(\cdot)$ is a mild solution of (4.1). Moreover, applying again Lemma 6.1, we have that $\sup\limits_{k\in\mathbb{N}}\widetilde{\Pi}(u(\cdot))<\infty$.

For $a\in\mathbb{T}_0^+$ such that $a>0$ arbitrary, we define a subdivision $0=a_0<a_1<\ldots<a_m=a$ such that $a_i\in\mathbb{T}_0^+$ for all $i=0,\ldots,m$ of the interval $[0,a]_{\mathbb{T}_0^+}$ such that either $|a_i-a_{i-1}|=\delta<1$ for $i=1,\ldots,m$, or $|a_i-a_{i-1}|>\delta$ and $\sigma(a_{i-1})=a_i$. Proceeding as above, there exists a solution $u_1(\cdot)$ defined on the interval $[0,a_1]_{\mathbb{T}_0^+}$. Now, we can proceed inductively to construct the solution $u_i(\cdot)$ on the interval $[a_{i-1},a_i]_{\mathbb{T}_0^+}$ for $i=2,\ldots,m$ of the problem

$$\begin{cases}u^\Delta(t)=Au(t),\\u(a_{i-1})=u_{i-1}(a_{i-1}).\end{cases}$$

A finite induction allows us to define $u_1(\cdot),\ldots,u_m(\cdot)$. Defining $u(t)=u_i(t)$ for $t\in[a_{i-1},a_i]_{\mathbb{T}_0^+}$, we complete the construction of a solution $u(\cdot)$ of problem (4.1). □

7. Examples of C_0-semigroups on time scales

In this section, our aim is to present some examples of C_0-semigroups in different time scales to illustrate our definition of C_0-semigroup. We assume that $A\in\mathcal{L}(X)$ and that

$$\mathbb{R}^-\cap\sigma(A)=\{\alpha\in\mathbb{R}:\alpha\le 0\}\cap\sigma(A)=\emptyset,$$

where $\sigma(A)$ denotes the spectrum of A. As a consequence, for every $h\ge 0$, we have $0\notin\sigma(hI+A)$. Since $\sigma(hI+A)$ is a compact set, there exists a closed rectifiable Jordan curve γ such that $\sigma(hI+A)\subset\mathrm{int}(\gamma)$ and $\mathbb{R}^-\cap\gamma=\emptyset$. Following [25], we define

$$\begin{aligned}\log(hI + A) &= \frac{1}{2\pi i}\int_\gamma \log(z)R(z, hI + A)\,dz\\ &= \frac{1}{2\pi i}\int_\gamma \log(z)(z-h)I - A)^{-1}\,dz.\end{aligned}$$

We start by presenting the continuous case, which is the classical one. Let $A \in \mathcal{L}(X)$ be given.

Example 7.1. Let $\mathbb{T} = \mathbb{R}$. Then A generates a C_0-semigroup given by $T(t, 0) = e^{At}$ and also

$$T(t, \sigma(s)) = T(t, s) = e^{A(t-s)}.$$

In the sequel, we present the C_0-semigroup for the discrete case $\mathbb{T} = \mathbb{Z}$. See also [2] for more information on the subject of discrete semigroups.

Example 7.2. Let $\mathbb{T} = \mathbb{Z}$. Then A generates the following C_0-semigroup

$$T(t, 0) = (I + A)^t.$$

Also

$$T(t, \sigma(s)) = (I + A)^{t-s-1}.$$

Example 7.3. Let $\mathbb{T} = h\mathbb{Z}$. Then A generates the following C_0-semigroup

$$T(t, 0) = (I + hA)^{t/h}.$$

Moreover,

$$T(t, \sigma(s)) = (I + hA)^{(t-s-h)/h}.$$

In what follows, let us consider the particular case $A := \lambda > 0$ for the following examples. Notice that for a general A, the relations below may not be true, that is why we restrict ourselves for this specific case. Below, we present the example of quantum time scale, that is $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$.

Example 7.4. Let $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$. Then A generates the following C_0-semigroup

$$\begin{aligned}T(t, 0) &= \exp\bigg(\int_0^t \frac{1}{(q-1)\tau}\log(I + (q-1)\tau\lambda)\Delta\tau\bigg)\\ &= \exp\sum_{k=-\infty}^{m-1}(q-1)q^k\frac{1}{(q-1)q^k}\log(I + (q-1)q^k\lambda)\\ &= \exp\sum_{k=-\infty}^{m-1}\log\big(I + (q-1)q^k\lambda\big) = \prod_{k=-\infty}^{m-1}\big(I + (q-1)q^k\lambda\big),\end{aligned}$$

where $t = q^m$. Moreover,

$$\begin{aligned} T(t,\sigma(s)) &= \exp\left(\int_{qs}^{t} \frac{1}{(q-1)\tau} \log\big(I + (q-1)\tau\lambda\big)\Delta\tau\right) \\ &= \exp \sum_{k=n+1}^{m-1} \frac{1}{(q-1)q^k} (q-1)q^k \log\big(I + (q-1)q^k\lambda\big) \\ &= \exp \sum_{k=n+1}^{m-1} \log\big(I + (q-1)q^k\lambda\big) = \prod_{k=n+1}^{m-1} \big(I + (q-1)q^k\lambda\big), \end{aligned}$$

where $t = q^m$ and $s = q^n$.

$\mathbb{T}$	t_0	A	μ	$T(t,0)$	$T(t,\sigma(s))$
$\mathbb{R}$	0	A	$\mu(0) = 0$	e^{At}	$e^{A(t-s)}$
$\mathbb{Z}$	0	A	$\mu(0) \neq 0$	$(I + A)^t$	$(I + A)^{t-s}$
$h\mathbb{Z}$	0	A	$\mu(0) \neq 0$	$(I + hA)^{t/h}$	$(I + hA)^{(t-s)/h-1}$
$\mathbb{Z}/n$	0	λ	$\mu(0) \neq 0$	$(I + \lambda/n)^{nt}$	$(I + \lambda/n)^{n(t-s)-1}$
$q^{\mathbb{Z}} \cup \{0\},\ q > 1$	0	λ	$\mu(0) = 0$	$\prod_{k=-\infty}^{m-1} (I + (q-1)q^k\lambda)$	$\prod_{k=n+1}^{m-1} (I + (q-1)q^k\lambda)$
$2^{\mathbb{Z}} \cup \{0\}$	0	λ	$\mu(0) = 0$	$\prod_{k=-\infty}^{m-1} (I + 2^k\lambda)$	$\prod_{k=n+1}^{m-1} (I + 2^k\lambda)$
$\mathbb{N}_0^2$	0	λ	$\mu(0) \neq 0$	$\prod_{k=0}^{\sqrt{t}} (I + (2k+1)\lambda)$	$\prod_{k=\sqrt{s}+1}^{\sqrt{t}} (I + (2k+1)\lambda)$

TABLE 1

Finally, we present our last example for the time scale $\mathbb{T} = \mathbb{N}_0^2$.

EXAMPLE 7.5. Note that in this case $\sigma\big(n^2\big) = (n+1)^2$. If we denote $t = n^2$, then

$$\sigma(t) = \big(\sqrt{t} + 1\big)^2 = t + 2\sqrt{t} + 1.$$

Also, we have

$$\mu(t) = \sigma(t) - t = 2\sqrt{t} + 1.$$

Then, it follows that

$$\begin{aligned} T(t,0) &= \exp\left(\int_{0}^{t} \frac{1}{2\sqrt{\tau}+1} \log\big(I + (2\sqrt{\tau}+1)\lambda\big)\Delta\tau\right) \\ &= \exp \sum_{k=0}^{t} \frac{1}{2\sqrt{k}+1} \big(2\sqrt{k}+1\big) \log\big(I + \big(2\sqrt{k}+1\big)\lambda\big) \\ &= \exp \sum_{k=0}^{t} \log\big(I + \big(2\sqrt{k}+1\big)\lambda\big) \\ &= \prod_{k=0}^{t} \Big(I + (2\sqrt{k}+1)\lambda\Big) = \prod_{k=0}^{\sqrt{t}} \big(I + (2k+1)\lambda\big) \end{aligned}$$

for $t = n^2$. Proceeding as before, it is not difficult to see that

$$T(t,\sigma(s)) = \prod_{k=\sqrt{s}+1}^{\sqrt{t}} (I + (2k+1)\lambda).$$

We summarize all the above examples in the Table 1.

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