ALMOST AUTOMORPHIC SOLUTIONS OF NONAUTONOMOUS DIFFERENCE EQUATIONS

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Abstract. In the present paper, we study the non-autonomous difference equations given by $u(k+1) = A(k)u(k) + f(k)$ and $u(k+1) = A(k)u(k) + g(k, u(k))$ for $k \in \mathbb{Z}$, where $A(k)$ is a given non-singular $n \times n$ matrix with elements $a_{ij}(k)$, $1 \leq i, j \leq n$, $f : \mathbb{Z} \to E^n$ is a given $n \times 1$ vector function, $g : \mathbb{Z} \times E^n \to E^n$ and $u(k)$ is an unknown $n \times 1$ vector with components $u_i(k)$, $1 \leq i \leq n$. We obtain the existence of a discrete almost automorphic solution for both the equations, assuming that $A(k)$ and $f(k)$ are discrete almost automorphic functions and the associated homogeneous system admits an exponential dichotomy. Also, assuming the function g satisfies a global Lipschitz type condition, we prove the existence and uniqueness of an almost automorphic solution of the nonlinear difference equation.

1. Introduction

In the present work, our aim is to investigate the existence of almost automorphic solutions of non-autonomous linear difference equations. Also, we investigate conditions to ensure the existence and uniqueness of almost automorphic solutions of non-autonomous semilinear difference equations.

Our interest to study difference equations comes from the fact that these equations are a powerful tool for applications and to describe interesting models. The investigations concerning the qualitative properties of their solutions have been increased a lot, specially related to their periodicity. For instance, the study of periodic solutions for these equations were developed by several authors (see [1], [2], [3], [11], [14], [18], [25], [28], for instance). Also, several papers have been treated about almost periodic solutions of these equations (see [3], [19], [21], [26], [27], [28], for instance).

On the other hand, the literature concerning discrete almost automorphic functions is still really scarce. Recently, the concept of discrete almost automorphic functions was formally introduced and developed by Araya et al [4], after previous work of Minh, Naito and N'Guérékata [24]. In their paper, they investigate the existence of almost automorphic solutions of autonomous nonlinear difference equations.

Later, in 2009, Caraballo and Cheban (see [9] and [10]) proved a result concerning existence of almost automorphic solutions of non-autonomous linear difference equations. Their result says that if $(A, f) \in C(\mathbb{Z}, \mathcal{B}(X)) \times C(\mathbb{Z}, X)$ is Poisson stable and almost automorphic (where

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X is a Banach space) and if the equation

$$u(k+1) = A(k)u(k) + f(k) \tag{1.1}$$

admits a relatively compact solution $\psi(k, u_0, (A, f))$ and all the relatively compact solutions of

$$u(k+1) = A(k)u(k) \tag{1.2}$$

tend to zero as the time k tends to ∞, then they prove that the equation (1.1) has an almost automorphic solution. See Corollary 4.14 from [9]. However, although Caraballo-Cheban's treatment is very general and well-established, their conditions seem not to be simple to check in practice, e.g., in finite dimensional cases.

In the present work, we take a new approach assuming different conditions to ensure the existence of an almost automorphic solution of (1.1). More precisely, we will assume dichotomy of the homogeneous linear system (1.2). We remark that exponential dichotomies in the discrete case have been studied extensively by Agarwal [1] and Elaydi [14](see also the references therein). In this way, our conditions are simpler than the ones presented in [9] and thus, are easier to check.

Further, we prove the existence and uniqueness of an almost automorphic solution of the non-autonomous semilinear difference equation given by

$$u(k+1) = A(k)u(k) + f(k, u(k)), \quad k \in \mathbb{Z}, \tag{1.3}$$

where $A(k)$ is a given non-singular $n \times n$ matrix with elements $a_{ij}(k)$, $1 \leq i, j \leq n$, $f : \mathbb{Z} \times E^n \to E^n$ is a given $n \times 1$ vector function and $u(k)$ is an unknown $n \times 1$ vector with components $u_i(k)$, $1 \leq i \leq m$. We remark that this problem appear not to be studied in the existing literature, except very recently by Diagana [13].

Our paper is organized as follows. The second section is devoted to present the basic concepts and results concerning discrete almost automorphic functions and difference equations. In the third section, we prove a result concerning existence of an almost automorphic solution of the non-autonomous linear difference equation and present an example to illustrate it. Finally, the last section is devoted to prove the existence and uniqueness of almost automorphic solution of the non-autonomous semilinear difference equation (1.3). More precisely, if $A(k)$ is discrete almost automorphic and a non-singular matrix and

$$\sup_{k \in \mathbb{Z}} \|A^{-1}(k)\| < \infty.$$

Also, if the equation (1.2) admits an exponential dichotomy on $\mathbb{Z}$ with positive constants η, ν, β, α and the function $f : \mathbb{Z} \times E^n \to E^n$ is discrete almost automorphic in k for each u in E^n, and Lipschitz with constant $L > 0$ satisfying the following condition:

$$0 < L < \frac{(1 - e^{-\alpha})(e^{\beta} - 1)}{\eta(e^{\beta} - 1) + \nu(1 - e^{-\alpha})}.$$

Then, the semilinear system (1.3) has a unique almost automorphic solution. Also, we present some examples to illustrate it as well.

2. Preliminaries

In this section, we present some basic concepts and results which will be essential to prove our main results.

We start by recalling the definition of a discrete almost automorphic function. See [4].

Definition 2.1. Let X be a (real or complex) Banach space. A function $u : \mathbb{Z} \to X$ is said to be *discrete almost automorphic* if for every integer sequence (k'_n), there exists a subsequence (k_n) such that

$$\lim_{n\to\infty} u(k+k_n) =: \bar{u}(k) \tag{2.1}$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{u}(k-k_n) = u(k) \tag{2.2}$$

for each $k \in \mathbb{Z}$.

Throughout the paper, we denote by $AA_d(X)$ the set of discrete almost automorphic functions taking values on X. As it was showed in [4], the discrete almost automorphicity is a more general concept than discrete almost periodicity.

The next result brings the properties of discrete almost automorphic functions. It can be found in [4], Theorem 2.4.

Theorem 2.2. *Let u, v be discrete almost automorphic functions, then the following assertions are valid.*

(i) *$u + v$ is discrete almost automorphic;*
(ii) *cu is discrete almost automorphic for every scalar c;*
(iii) *For each fixed $l \in \mathbb{Z}$, the function $u_l : \mathbb{Z} \to X$ defined by $u_l(k) := u(k+l)$ is discrete almost automorphic;*
(iv) *The function $\hat{u} : \mathbb{Z} \to X$ defined by $\hat{u}(k) := u(-k)$ is discrete almost automorphic;*
(v) *$\sup_{k\in\mathbb{Z}} \|u(k)\| < \infty$, that is, u is a bounded function;*
(vi) *$\sup_{k\in\mathbb{Z}} \|\bar{u}(k)\| \le \sup_{k\in\mathbb{Z}} \|u(k)\|$, where*

$$\lim_{n\to\infty} u(k+k_n) = \bar{u}(k) \quad \textit{and} \quad \lim_{n\to\infty} \bar{u}(k-k_n) = u(k).$$

We consider the space of discrete almost automorphic functions provided with the norm

$$\|u\|_d := \sup_{k\in\mathbb{Z}} \|u(k)\|.$$

Remark 2.3. Notice that $AA_d(X)$ with this norm is a Banach space (see [4]).

In what follows, we recall the definition of a discrete almost automorphic function $u : \mathbb{Z} \times X \to X$. It is an important concept for applications to nonlinear difference equations. See [4].

Definition 2.4. A function $u : \mathbb{Z} \times X \to X$ is said to be *discrete almost automorphic* in k for each $x \in X$, if for every sequence of integers numbers (k'_n), there exists a subsequence (k_n) such that

$$\lim_{n\to\infty} u(k+k_n, x) =: \bar{u}(k, x) \tag{2.3}$$

is well defined for each $k \in \mathbb{Z}$, $x \in X$ and

$$\lim_{n\to\infty} \bar{u}(k - k_n, x) = u(k, x) \tag{2.4}$$

for each $k \in \mathbb{Z}$ and $x \in X$.

The next result can be found in [4], Theorem 2.9. It describes the properties of discrete almost automorphic functions $u : \mathbb{Z} \times X \to X$.

Theorem 2.5. *If $u, v : \mathbb{Z} \times X \to X$ are discrete almost automorphic functions in k for each x in X, then the following assertions are true.*

- (i) *$u + v$ is discrete almost automorphic in k for each x in X.*
- (ii) *cu is discrete almost automorphic in k for each x in X, where c is an arbitrary scalar.*
- (iii) *$\sup_{k\in\mathbb{Z}} \|u(k, x)\| = M_x < \infty$, for each x in X.*
- (iv) *$\sup_{k\in\mathbb{Z}} \|\bar{u}(k, x)\| = N_x < \infty$, for each x in X, where $\bar{u}$ is the function in Definition 2.4.*

The next result will be essential to prove our main result. It can be found in [4], Theorem 2.10.

Theorem 2.6. *Let $u : \mathbb{Z} \times X \to X$ be discrete almost automorphic in k for each x in X, satisfying a Lipschitz condition in x uniformly in k, that is,*

$$\|u(k, x) - u(k, y)\| \le L\|x - y\|, \quad \textit{for all} \;\; x, y \in X.$$

Suppose $\varphi : \mathbb{Z} \to X$ is discrete almost automorphic, then the function $U : \mathbb{Z} \to X$ defined by $U(k) = u(k, \varphi(k))$ is discrete almost automorphic.

In the sequel we use the letter E to stand for either $\mathbb{R}$ or $\mathbb{C}$, so E^n then represents either $\mathbb{R}^n$ or $\mathbb{C}^n$, and $E^{n\times n}$ either $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$.

Now, consider the following system of non-autonomous linear difference equations

$$u(k + 1) = A(k)u(k) + f(k), \quad k \in \mathbb{Z} \tag{2.5}$$

where $A(k)$ are given non-singular $n \times n$ matrices with elements $a_{ij}(k)$, $1 \le i, j \le n$, $f : \mathbb{Z} \to E^n$ is a given $n \times 1$ vector function and $u(k)$ is an unknown $n \times 1$ vector with components $u_i(k)$, $1 \le i \le n$.

Its associated homogeneous linear difference equation is given by

$$u(k + 1) = A(k)u(k), \quad k \in \mathbb{Z}. \tag{2.6}$$

Definition 2.7. The matrix $U(k, k_0)$ which satisfies the equation (2.6) and $U(k_0, k_0) = I$ is called *principal fundamental matrix*. We will denote $U(k, k_0)$ by $U(k)U^{-1}(k_0)$.

We recall that any $n \times n$ matrix $V(k)$ whose columns are linearly independent solutions of the system (2.6) is called a *fundamental matrix*. If $V(k)$ is a fundamental matrix of the system (2.6) and C is any nonsingular matrix, then $V(k)C$ is also a fundamental matrix. See [1].

Definition 2.8. A matrix function $A(k)$ is said to be *discrete almost automorphic* if each entry of the matrix is discrete almost automorphic function. In the other words, for every sequence of integers numbers (k'_n), there exists a subsequence (k_n) such that

$$\lim_{n\to\infty} A(k + k_n) =: \bar{A}(k) \tag{2.7}$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{A}(k - k_n) = A(k) \tag{2.8}$$

for each $k \in \mathbb{Z}$.

In what follows, we prove a result which describes a property of almost automorphic matrix functions.

Lemma 2.9. *Suppose $A(k)$ is discrete almost automorphic and a non-singular matrix on $\mathbb{Z}$. Also, suppose that the set $\{A^{-1}(k)\}_{k\in\mathbb{Z}}$ is bounded. Then $A^{-1}(k)$ is almost automorphic on $\mathbb{Z}$, that is, for every sequence of integers numbers (k'_n), there exists a subsequence (k_n) such that*

$$\lim_{n\to\infty} A^{-1}(k + k_n) =: \bar{A}^{-1}(k) \tag{2.9}$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{A}^{-1}(k - k_n) = A^{-1}(k) \tag{2.10}$$

for each $k \in \mathbb{Z}$

Proof. Let (k'_n) be a sequence of integers numbers. Since $A(k)$ is discrete almost automorphic, there exists a subsequence (k_n) such that

$$\lim_{n\to\infty} A(k + k_n) =: \bar{A}(k)$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{A}(k - k_n) = A(k)$$

for each $k \in \mathbb{Z}$.

Fix $k \in \mathbb{Z}$ and define $A_n := A(k+k_n)$, $n \in \mathbb{N}$. By hypothesis, the set $\{A_n^{-1}\}_{n\in\mathbb{N}}$ is bounded. Using the identity

$$A_n^{-1} - A_m^{-1} = A_n^{-1}(A_m - A_n)A_m^{-1}$$

and the fact that $\{A_n\}$ is a Cauchy sequence, it follows that $\{A_n^{-1}\}$ is a Cauchy sequence. Hence, there exists a matrix T (for each $k \in \mathbb{Z}$ fixed) such that

$$A_n^{-1} \to T(k).$$

Taking the limit of $A_n A_n^{-1} = A_n^{-1} A_n = I$, where I denotes the identity matrix, we obtain that $\bar{A}(k)$ is invertible and $\bar{A}^{-1}(k) = T(k)$ for each $k \in \mathbb{Z}$. Since the map $A \to A^{-1}$ is continuous on the set of non-singular matrices, it follows that

$$\lim_{n\to\infty} A^{-1}(k + k_n) = \bar{A}^{-1}(k)$$

for each $k \in \mathbb{Z}$. Analogously, one can prove that

$$\lim_{n\to\infty} \bar{A}^{-1}(k - k_n) = A^{-1}(k),$$

for each $k \in \mathbb{Z}$. □

Now, we present a result which describes the solution of (2.6). This result can be found in [1], Theorem 2.4.1.

Theorem 2.10. *The following holds*

$$U(k)U^{-1}(k_0) = \begin{cases} \displaystyle\prod_{l=k_0}^{k-1} A(k_0 + k - 1 - l), & \text{for all } k_0 \le k \in \mathbb{Z}, \\ \displaystyle\prod_{l=k}^{k_0-1} A^{-1}(l), & \text{for all } k_0 \ge k \in \mathbb{Z}, \end{cases} \tag{2.11}$$

In what follows, we recall the definition of exponential dichotomy and ordinary dichotomy for the system (2.6) (see [1, Chapter 5, section 5.8]).

Definition 2.11. Let $U(k)$ be the principal fundamental matrix of the difference system (2.6). The system (2.6) is said to possess an *exponential dichotomy* if there exists a projection P, which commutes with $U(k)$, and positive constants η, ν, α, β such that for all $k, l \in \mathbb{Z}$, we have

$$\left\|U(k)PU^{-1}(l)\right\| \le \eta e^{-\alpha(k-l)}, \quad k \ge l$$
$$\left\|U(k)(I-P)U^{-1}(l)\right\| \le \nu e^{-\beta(l-k)}, \quad l \ge k.$$

On the other hand, it is said to possess an *ordinary dichotomy* if the inequalities above hold for $\alpha = \beta = 0$. In the other words, if we have the following inequalities

$$\left\|U(k)PU^{-1}(l)\right\| \le \eta, \quad k \ge l$$
$$\left\|U(k)(I-P)U^{-1}(l)\right\| \le \nu, \quad l \ge k.$$

Now, we present the following result which will be essential to prove our main result. It can be found in [23, Lemma 2.13] for a general time scale $\mathbb{T}$. We repeat the proof here for the particular case $\mathbb{T} = \mathbb{Z}$ for reader's convenience.

Theorem 2.12. *If the system* (2.6) *possesses an exponential dichotomy and the function* $f : \mathbb{Z} \to E^n$ *is bounded, then the system* (2.5) *has a bounded solution which is given by*

$$u(k) = \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)f(j) - \sum_{j=k}^{\infty} U(k)(I-P)U^{-1}(j+1)f(j), \tag{2.12}$$

where $U(k)$ *is a fundamental matrix of* (2.6).

Proof. Let $U(k)$ be a fundamental matrix of (2.6), then

$$u(k+1) - A(k)u(k) =$$

$$= \sum_{j=-\infty}^{k} U(k+1)PU^{-1}(j+1)f(j) - \sum_{j=k+1}^{\infty} U(k+1)(I-P)U^{-1}(j+1)f(j)$$
$$-A(k) \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)f(j) + A(k) \sum_{j=k}^{\infty} U(k)(I-P)U^{-1}(j+1)f(j)$$
$$= \sum_{j=-\infty}^{k} U(k+1)PU^{-1}(j+1)f(j) - \sum_{j=k+1}^{\infty} U(k+1)(I-P)U^{-1}(j+1)f(j)$$

$$
\begin{aligned}
&-\sum_{j=-\infty}^{k-1} U(k+1)PU^{-1}(j+1)f(j)+\sum_{j=k}^{\infty} U(k+1)(I-P)U^{-1}(j+1)f(j) \\
&\quad = U(k+1)PU^{-1}(k+1)f(k)+U(k+1)(I-P)U^{-1}(k+1)f(k) \\
&\quad = U(k+1)(P+I-P)U^{-1}(k+1)f(k)=f(k)
\end{aligned}
$$

and we obtain the desired result.

It remains to prove that $u(k)$ is a bounded function, for every $k \in \mathbb{Z}$. Then, by the exponential dichotomy and boundedness of f, we have

$$
\begin{aligned}
\|u(k)\| &= \left\| \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)f(j)-\sum_{j=k}^{\infty} U(k)(I-P)U^{-1}(j+1)f(j)\right\| \\
&\le \sum_{j=-\infty}^{k-1} \eta e^{-\alpha(k-j-1)}\|f(j)\|+\sum_{j=k}^{\infty} \nu e^{-\beta(j+1-k)}\|f(j)\| \\
&= \sum_{j=0}^{\infty} \eta e^{-\alpha j}\|f(j)\|+\sum_{j=1}^{\infty} \nu e^{-\beta j}\|f(j)\| \\
&\le \sum_{j=0}^{\infty} \eta e^{-\alpha j}C+\sum_{j=1}^{\infty} \nu e^{-\beta j}C=\eta C\frac{1}{1-e^{-\alpha}}+\nu C\frac{e^{-\beta}}{1-e^{-\beta}} =: M,
\end{aligned}
$$

where $C=\sup_{j\in\mathbb{Z}}\|f(j)\|$. Thus, for every $k \in \mathbb{Z}$, we obtain

$$\|u(k)\| \le M,$$

where $M=\eta C\frac{1}{1-e^{-\alpha}}+\nu C\frac{e^{-\beta}}{1-e^{-\beta}}$ and the desired result follows as well. □

3. Almost automorphic solutions of non-autonomous difference equations

In this section, consider the following non-autonomous linear difference equation

$$u(k+1)=A(k)u(k)+f(k), \quad k \in \mathbb{Z} \tag{3.1}$$

where $A(k)$ are given non-singular $n\times n$ matrices with elements $a_{ij}(k)$, $1 \le i,j \le n$, $f : \mathbb{Z}\times E^n \to E^n$ is a given $n\times 1$ vector function and $u(k)$ is an unknown $n\times 1$ vector with components $u_i(k)$, $1 \le i \le n$. Its associated homogeneous equation is given by

$$u(k+1)=A(k)u(k), \quad k \in \mathbb{Z}. \tag{3.2}$$

We start by proving a result concerning existence of an almost automorphic solution of the equation (3.1).

Theorem 3.1. *Suppose $A(k)$ is discrete almost automorphic and a non-singular matrix and the set $\{A^{-1}(k)\}_{k\in\mathbb{Z}}$ is bounded. Also, suppose the function $f:\mathbb{Z}\to E^n$ is a discrete almost automorphic function and the equation* (3.2) *admits an exponential dichotomy with positive constants ν,η,β and α. Then, the system* (3.1) *has an almost automorphic solution on $\mathbb{Z}$.*

Proof. Since $A(k)$ is discrete almost automorphic, then for every sequence $(k'_n) \in \mathbb{Z}$, there exists a subsequence $(k''_n) \subset (k'_n)$ such that

$$\lim_{n\to\infty} A(k+k''_n) =: \bar{A}(k) \tag{3.3}$$

exists and is well-defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{A}(k-k''_n) = A(k) \tag{3.4}$$

for each $k \in \mathbb{Z}$. Also, since $f(k)$ is a discrete almost automorphic function, there exists a subsequence $(k_n) \subset (k''_n)$ such that

$$\lim_{n\to\infty} f(k+k_n) =: \bar{f}(k) \tag{3.5}$$

exists and is well-defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{f}(k+k_n) = f(k) \tag{3.6}$$

for each $k \in \mathbb{Z}$.

Consider the following system

$$v(k+1) = \bar{A}(k)v(k), \tag{3.7}$$

where $\bar{A}(k)$ is defined by (3.3). Remember that by hypothesis, and Lemma 2.9, the matrix $\bar{A}(k)$ is non-singular. Consider also that its non-homogenous system is given by

$$v(k+1) = \bar{A}(k)v(k) + \bar{f}(k), \tag{3.8}$$

where $\bar{f} : \mathbb{Z} \to E^n$ is given by (3.5).

Let $V(k)$ be a fundamental matrix of (3.7). Then, by Theorem 2.10, we get

$$V(k)V^{-1}(m) = \prod_{l=m}^{k-1} \bar{A}(m+k-1-l)$$

for $k \geq m$, $k, m \in \mathbb{Z}$ and

$$\begin{aligned} U(k+k_n)U^{-1}(m+k_n) &= \prod_{l=m+k_n}^{k+k_n-1} A(m+k_n+k+k_n-1-l) \\ &= \prod_{l=m}^{k-1} A(m+k_n+k+k_n-1-(l+k_n)) \\ &= \prod_{l=m}^{k-1} A(m+k_n+k-1-l). \end{aligned}$$

Thus, for every $k \geq m$ and $k, m \in \mathbb{Z}$, we have

$$\begin{aligned} &\left\| U(k+k_n)U^{-1}(m+k_n) - V(k)V^{-1}(m) \right\| \\ &\leq \left\| \prod_{l=m}^{k-1} A(m+k+k_n-1-l) - \prod_{l=m}^{k-1} \bar{A}(m+k-1-l) \right\| \overset{n\to\infty}{\to} 0, \end{aligned}$$

by automorphicity of A, using the fact that $k+m-1+l \in \mathbb{Z}$ and $(v)-(vi)$ of Theorem 2.2. Thus,

$$\lim_{n\to\infty} \left\|U(k+k_n)U^{-1}(m+k_n) - V(k)V^{-1}(m)\right\| = 0. \tag{3.9}$$

It is clear from equation (3.9) that

$$\lim_{n\to\infty} \left\|U(k+k_n)PU^{-1}(m+k_n) - V(k)PV^{-1}(m)\right\| = 0, \tag{3.10}$$

for every $k, m \in \mathbb{Z}$ and $k \geq m$, using the fact that P commutes with $U(k)$. By the same reason, we have

$$\lim_{n\to\infty} \left\|U(k+k_n)(I-P)U^{-1}(m+k_n) - V(k)(I-P)V^{-1}(m)\right\| = 0, \tag{3.11}$$

for each $k, m \in \mathbb{Z}$ and $k \leq m$.

Now, define the following functions:

$$M(k) = \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)f(j)$$

and

$$\bar{M}(k) = \sum_{j=-\infty}^{k-1} V(k)PV^{-1}(j+1)\bar{f}(j).$$

Then, we have

$$\begin{aligned}
\|M(k+k_n) - \bar{M}(k)\| &= \left\| \sum_{j=-\infty}^{k+k_n-1} U(k+k_n)PU^{-1}(j+1)f(j) - \sum_{j=-\infty}^{k-1} V(k)PV^{-1}(j+1)\bar{f}(j) \right\| \\
&= \left\| \sum_{j=-\infty}^{k-1} U(k+k_n)PU^{-1}(j+k_n+1)f(j+k_n) - \sum_{j=-\infty}^{k-1} V(k)PV^{-1}(j+1)\bar{f}(j) \right\| \\
&\leq \sum_{j=-\infty}^{k-1} \left\|[U(k+k_n)PU^{-1}(j+k_n+1) - V(k)PV^{-1}(j+1)]\bar{f}(j)\right\| \\
&\quad + \sum_{j=-\infty}^{k-1} \left\|[U(k+k_n)PU^{-1}(j+k_n+1)[f(j+k_n) - \bar{f}(j)]\right\| \\
&\leq \sum_{j=-\infty}^{k-1} C \left\|U(k+k_n)PU^{-1}(j+k_n+1) - V(k)PV^{-1}(j+1)\right\| \\
&\quad + \sum_{j=-\infty}^{k-1} \eta e^{-\alpha(k-j-1)} \left\|f(j+k_n) - \bar{f}(j)\right\|,
\end{aligned}$$

where $C = \sup_{z\in\mathbb{Z}} \|\bar{f}(z)\|$. Thus, it is clear by (3.10) and by automorphicity of f, that

$$\|M(k+k_n) - \bar{M}(k)\| \overset{n\to\infty}{\to} 0. \tag{3.12}$$

Therefore, $\lim_{n\to\infty} M(k+k_n) = \bar{M}(k)$, for each $k \in \mathbb{Z}$. Similarly, we can prove that

$$\lim_{n\to\infty} \bar{M}(k-k_n) = M(k), \tag{3.13}$$

for each $k \in \mathbb{Z}$.

Now, define the following functions:

$$N(k) = \sum_{j=k}^{\infty} U(k)(I-P)U^{-1}(j+1)f(j)$$

and

$$\bar{N}(k) = \sum_{j=k}^{\infty} V(k)(I-P)V^{-1}(j+1)\bar{f}(j).$$

The same way as before and using (3.11), one can prove that for every $k \in \mathbb{Z}$, we get

$$\lim_{n\to\infty} N(k+k_n) = \bar{N}(k) \tag{3.14}$$

and

$$\lim_{n\to\infty} \bar{N}(k-k_n) = N(k). \tag{3.15}$$

By Theorem 2.12, the system (3.1) has a bounded solution which is given by

$$y(k) = \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)f(j) - \sum_{j=k}^{\infty} U(k)(I-P)U^{-1}(j+1)f(j).$$

Then, using the equations (3.12), (3.13), (3.14) and (3.15), we obtain

$$\lim_{n\to\infty} y(k+k_n) = \lim_{n\to\infty} M(k+k_n) + N(k+k_n) = \bar{M}(k) + \bar{N}(k) = (\bar{M}+\bar{N})(k) =: \bar{y}(k)$$

is well defined for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{y}(k-k_n) = \lim_{n\to\infty} (\bar{M}+\bar{N})(k-k_n) = \lim_{n\to\infty} \bar{M}(k-k_n) + \bar{N}(k-k_n) = y(k)$$

for each $k \in \mathbb{Z}$. Thus, it follows that $y : \mathbb{Z} \to E^n$ is an almost automorphic function and we have the desired result. □

Example 3.2. Let $A(k) = A$ be a non-singular $n \times n$-matrix and suppose that the intersection $\sigma_p(A) \cap S^1$ is empty, where $\sigma_p(A)$ denotes the set of eigenvalues of A. Let $f(k)$ be an almost automorphic function. Then, the system

$$u(k+1) = Au(k) + f(k) \tag{3.16}$$

has an almost automorphic solution.

In fact, if the intersection $\sigma_p(A) \cap S^1$ is empty, then it follows that the equation

$$u(k+1) = Au(k)$$

admits an exponential dichotomy (see [1, Section 5.8]).

Thus, all the hypothesis of Theorem 3.1 are satisfied, then the system (3.16) has an almost automorphic solution.

4. Almost automorphic solutions for semilinear nonautonomous difference equations

In this section, we consider the following system

$$u(k+1) = A(k)u(k) + f(k, u(k)), \quad k \in \mathbb{Z}, \tag{4.1}$$

where $A(k)$ is a given non-singular $n \times n$ matrix with elements $a_{ij}(k)$, $1 \leq i, j \leq n$, $f : \mathbb{Z} \times E^n \to E^n$ is a given $n \times 1$ vector function and $u(k)$ is an unknown $n \times 1$ vector with components $u_i(k)$, $1 \leq i \leq n$.

Its associated homogeneous linear difference equation is given by

$$u(k+1) = A(k)u(k), \quad k \in \mathbb{Z}. \tag{4.2}$$

Here, we will restrict ourselves for the following concept of solution for the equation (4.2).

Definition 4.1. We say that $u : \mathbb{Z} \to \mathbb{E}^n$ is a *solution* of (4.1), if it satisfies

$$u(k) = \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)f(j, u(j)) - \sum_{j=k}^{\infty} U(k)(I-P)U^{-1}(j+1)f(j, u(j)), \tag{4.3}$$

for every $k \in \mathbb{Z}$.

Remark 4.2. The previous definition makes sense, since $u(k)$ given by (4.3) satisfies the equation (4.1). Indeed, notice that

$$u(k+1) - A(k)u(k) =$$

$$\begin{aligned}
&= \sum_{j=-\infty}^{k} U(k+1)PU^{-1}(j+1)f(j, u(j)) - \sum_{j=k+1}^{\infty} U(k+1)(I-P)U^{-1}(j+1)f(j, u(j)) \\
&-A(k) \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)f(j, u(j)) + A(k)\sum_{j=k}^{\infty} U(k)(I-P)U^{-1}(j+1)f(j, u(j)) \\
&= \sum_{j=-\infty}^{k} U(k+1)PU^{-1}(j+1)f(j, u(j)) - \sum_{j=k+1}^{\infty} U(k+1)(I-P)U^{-1}(j+1)f(j, u(j)) \\
&- \sum_{j=-\infty}^{k-1} U(k+1)PU^{-1}(j+1)f(j, u(j)) + \sum_{j=k}^{\infty} U(k+1)(I-P)U^{-1}(j+1)f(j, u(j)) \\
&= U(k+1)PU^{-1}(k+1)f(k, u(k)) + U(k+1)(I-P)U^{-1}(k+1)f(k, u(k)) \\
&= U(k+1)(P+I-P)U^{-1}(k+1)f(k, u(k)) = f(k, u(k)),
\end{aligned}$$

which implies

$$u(k+1) = A(k)u(k) + f(k, u(k)).$$

Now, let us prove the following result which ensures the existence and uniqueness of an almost automorphic solution of (4.1).

Theorem 4.3. *Suppose $A(k)$ is discrete almost automorphic and a non-singular matrix and the set $\{A^{-1}(k)\}_{k\in\mathbb{Z}}$ is bounded. Also, assume the equation (4.2) admits an exponential dichotomy on $\mathbb{Z}$ with positive constants η, ν, β, α and the function $f : \mathbb{Z}\times E^n \to E^n$ is discrete almost automorphic in k for each u in E^n, satisfying the following condition:*

(i) *There exists a constant $0 < L < \dfrac{(1-e^{-\alpha})(e^{\beta}-1)}{\eta(e^{\beta}-1)+\nu(1-e^{-\alpha})}$ such that*

$$\|f(k,u)-f(k,v)\| \leq L\|u-v\|, \tag{4.4}$$

for every $u, v \in E^n$ and $k \in \mathbb{Z}$.

Then, the system (4.1) has a unique almost automorphic solution.

Proof. Define an operator $T : AA_d(E^n) \to AA_d(E^n)$ as follows:

$$(Tu)(k) = \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)f(j,u(j)) - \sum_{j=k}^{\infty} U(k)(I-P)U^{-1}(j+1)f(j,u(j)),$$

for all $u \in AA_d(E^n)$ and U is the fundamental matrix of (4.1).

Let $u(k) \in AA_d(E^n)$, then since the function f satisfies the global Lipschitz-type condition, we obtain by Theorem 2.6 that $f(\cdot, u(\cdot))$ is in $AA_d(E^n)$.

Thus, by the automorphicity of u, it follows that for every sequence of integers numbers (k_n''), there exists a subsequence (k_n') such that

$$\lim_{n\to\infty} u(k+k_n') = \bar{u}(k)$$

for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{u}(k-k_n') = u(k),$$

for every $k \in \mathbb{Z}$. And by the automorphicity of f, there exists a subsequence $(k_n) \subset (k_n')$ such that

$$\lim_{n\to\infty} f(k+k_n, u(k+k_n)) = \bar{f}(k, \bar{u}(k))$$

for each $k \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \bar{f}(k-k_n, \bar{u}(k-k_n) = f(k, u(k)),$$

for every $k \in \mathbb{Z}$.

Also, using the same arguments as Theorem 3.1, one can prove that

$$\lim_{n\to\infty} \left\|U(k+k_n)PU^{-1}(m+k_n) - V(k)PV^{-1}(m)\right\| = 0, \tag{4.5}$$

for every $k, m \in \mathbb{Z}$ such that $k \geq m$. Also,

$$\lim_{n\to\infty} \left\|U(k+k_n)(I-P)U^{-1}(m+k_n) - V(k)(I-P)V^{-1}(m)\right\| = 0, \tag{4.6}$$

for each $k, m \in \mathbb{Z}$ such that $k \leq m$, where V is the fundamental matrix of the equation

$$v(k+1) = \bar{A}(k)v(k), \quad k \in \mathbb{Z}.$$

Now, we define the following function $h : \mathbb{Z} \to E^n$:

$$h(k) := \sum_{j=-\infty}^{k-1} V(k)PV^{-1}(j+1)\bar{f}(j,\bar{u}(j)) - \sum_{j=k}^{\infty} V(k)(I-P)V^{-1}(j+1)\bar{f}(j,\bar{u}(j)),$$

where V, $\bar{f}$ and $\bar{u}$ are defined as above.

Let us prove that $Tu \in AA_d(E^n)$. In fact,

$$\|Tu(k+k_n) - h(k)\| \leq$$

$$\leq \left\| \sum_{j=-\infty}^{k+k_n-1} U(k+k_n)PU^{-1}(j+1)f(j,u(j)) - \sum_{j=-\infty}^{k-1} V(k)PV^{-1}(j+1)\bar{f}(j,\bar{u}(j)) \right\| +$$

$$+ \left\| \sum_{j=k+k_n}^{\infty} U(k+k_n)(I-P)U^{-1}(j+1)f(j,u(j)) - \sum_{j=k}^{\infty} V(k)(I-P)V^{-1}(j+1)\bar{f}(j,\bar{u}(j)) \right\|$$

$$= \left\| \sum_{j=-\infty}^{k-1} [U(k+k_n)PU^{-1}(j+k_n+1)f(j+k_n,u(j+k_n)) - V(k)PV^{-1}(j+1)\bar{f}(j,\bar{u}(j))] \right\| +$$

$$+ \left\| \sum_{j=k}^{\infty} [U(k+k_n)(I-P)U^{-1}(j+k_n+1)f(j+k_n,u(j+k_n)) - V(k)(I-P)V^{-1}(j+1)\bar{f}(j,\bar{u}(j))] \right\|$$

$$\leq \sum_{j=-\infty}^{k-1} \left\| U(k+k_n)PU^{-1}(j+k_n+1) - V(k)PV^{-1}(j+1) \right\| \left\| \bar{f}(j,\bar{u}(j)) \right\| +$$

$$+ \sum_{j=-\infty}^{k-1} \left\| U(k+k_n)PU^{-1}(j+k_n+1) \right\| \left\| f(j+k_n,u(j+k_n)) - \bar{f}(j,\bar{u}(j)) \right\| +$$

$$+ \sum_{j=-\infty}^{k-1} \left\| U(k+k_n)(I-P)U^{-1}(j+k_n+1) - V(k)(I-P)V^{-1}(j+1) \right\| \left\| \bar{f}(j,\bar{u}(j)) \right\| +$$

$$+ \sum_{j=-\infty}^{k-1} \left\| U(k+k_n)(I-P)U^{-1}(j+k_n+1) \right\| \left\| f(j+k_n,u(j+k_n)) - \bar{f}(j,\bar{u}(j)) \right\| +$$

Then, using the exponential dichotomy and the fact that $f(\cdot,u(\cdot)) \in AA_d(E^n)$, we obtain

$$\lim_{n\to\infty} \|Tu(k+k_n) - h(k)\| = 0,$$

by the equations (4.5) and (4.6).

Therefore, we obtain

$$\lim_{n\to\infty} Tu(k+k_n) = h(k),$$

for every $k \in \mathbb{Z}$. Analogously, one can prove that

$$\lim_{n\to\infty} h(k-k_n) = Tu(k),$$

for every $k \in \mathbb{Z}$.

It follows that $Tu(k) \in AA_d(E^n)$ and thus, T is well-defined. Now, let us prove that $T : AA_d(E^n) \to AA_d(E^n)$ is a contraction.

$$\begin{aligned}\|Tu-Tv\|_\infty &= \| \sum_{j=-\infty}^{k-1} U(k)PU^{-1}(j+1)[f(j,u(j))-f(j,v(j))] \\ &- \sum_{j=k}^{+\infty} U(k)(I-P)U^{-1}(j+1)[f(j,u(j))-f(j,v(j))]\|_\infty.\end{aligned}$$

Then, by the exponential dichotomy and Lipschitz condition, we have

$$\begin{aligned}\|Tu-Tv\|_\infty &\le \sum_{j=-\infty}^{k-1} \eta e^{-\alpha(k-j-1)} L\|u-v\|_\infty + \sum_{j=k}^{+\infty} \nu e^{-\beta(j+1-k)} L\|u-v\|_\infty \\ &\le \eta \sum_{j=0}^{+\infty} e^{-\alpha j} L\|u-v\|_\infty + \sum_{j=1}^{+\infty} \nu e^{-\beta j} L\|u-v\|_\infty \\ &\le \eta \frac{1}{1-e^{-\alpha}} L\|u-v\|_\infty + \nu \frac{e^{-\beta}}{1-e^{-\beta}} L\|u-v\|_\infty \\ &= \left[\eta \frac{1}{1-e^{-\alpha}} + \nu \frac{e^{-\beta}}{1-e^{-\beta}}\right] L\|u-v\|_\infty.\end{aligned}$$

Thus, using the fact that $L < \dfrac{(1-e^{-\alpha})(e^{\beta}-1)}{\eta(e^{\beta}-1)+\nu(1-e^{-\alpha})}$, we obtain that T is a contraction. Then by Banach Fixed-Point Theorem, T has a unique fixed point. By the definition of T, Definition 4.1 and Theorem 2.12, it follows that the system (4.1) has a unique solution which is almost automorphic. □

We finish this section presenting an example which illustrates the Theorem 4.3.

Example 4.4. Let $A(k) = A$ be a non-singular matrix such that $\sigma_p(A) \cap S^1 = \emptyset$. Also, let $f(k,u) = h(k)g(u)$ be a function such that $h \in AA_d(E)$ and g is a Lipschitz function, that is, for every $u, v \in E^n$, there exists a constant $L > 0$ such that

$$\|g(u)-g(v)\| \le L\|u-v\|.$$

Since $\sigma_p(A) \cap S^1 = \emptyset$, the system

$$u(k+1) = Au(k)$$

admits a exponential dichotomy with positive constants α, β, ν and η (see [1, Section 5.8], for instance). If we suppose that $L < \dfrac{(1-e^{-\alpha})(e^{\beta}-1)}{\eta(e^{\beta}-1)+\nu(1-e^{-\alpha})}$, then the system

$$u(k+1) = Au(k) + f(k,u)$$

has a unique almost automorphic solution by Theorem 4.3.

Remark 4.5. After this work was submitted for publication, we learn that a paper by T. Diagana [13] has recently been published on the same topic of the present article, but in which the singular case is studied and then applied to study the existence of globally attracting almost automorphic solutions to some higher-order difference equations in a Banach space.

In particular, Theorem 3.3 from [13] extends our Theorem 3.1 to the singular case. Also, the results in [13] are in the context of general Banach spaces.

We remark that the results in the present work can be stated in the more general context of Banach spaces, and hence consider bounded linear operators $A(k)$ rather than matrices, taking into consideration the definitions given in the reference [13].

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References

[1] R. P. Agarwal, *Difference equations and inequalities: Theory, Methods and Applications*, 2nd edition, Marcel Dekker, New York, 1992.

[2] R. P. Agarwal and J. Popenda, *On periodic solutions of first order linear difference equations*, Math. Comput. Modelling, 22 (1) (1995), 11-19.

[3] R. P. Agarwal, D. O' Regan and P. J. Y. Wong, *Constant-sign periodic and almost periodic solutions of a system of difference equations*, Computers and Mathematics with Applications, vol. 50, no 1012 (2005), 1725-1754.

[4] D. Araya, R. Castro, C. Lizama, *Almost automorphic solutions of difference equations*, Advances in Difference Equations, Vol 2009, Article ID 591380, 15 pages, doi:10.1155/2009/591380.

[5] S. Bochner, *Continuous mappings of almost automorphic and almost periodic functions*, Proc. Nat. Acad. Sci. USA, 52 (1964), 907-910.

[6] S. Bochner, *Uniform convergence of monotone sequences of functions*, Proc. Nat. Acad. Sci. USA, 47 (1961), 582-585.

[7] S. Bochner, *A new approach in almost-periodicity*, Proc. Nat. Acad. Sci. USA, 48 (1962), 2039-2043.

[8] S. Bochner and J. von Neumann, *On compact solutions of operational-differential equations*, I, Ann. Math., 36 (1935), 255-290.

[9] T. Caraballo and D. Cheban, *Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. I*, J. Diff. Equations, 246(1) (2009), 108-128.

[10] T. Caraballo and D. Cheban, *Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. II*, J. Diff. Equations, 246(3) (2009), 1164-1186.

[11] C. Corduneanu, *Almost periodic discrete processes*, Libertas Math. 2 (1982), 159-169.

[12] T. Diagana, *Existence of pseudo-automorphic mild solutions to some nonautonomous partial evolution equations*, Advances in Difference Equations, 2011, Article ID 895079, 23 pages, doi: 10.1155/2011/895079.

[13] T. Diagana, *Existence of globally attracting almost automorphic solutions to some nonautonomous higher-order difference equations,* Appl. Math. Comput. 219 (12) (2013), 6510–6519.

[14] S. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, 3rd ed. Springer Verlag, 2005.

[15] J. A. Goldstein and G. M. N. Guérékata, *Almost automorphic solutions of semilinear evolution equations*, Proc. Amer. Math. Soc., 133(8) (2005), 2401-2408.

[16] G. M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/Plenium Publishers, New York, 2001.

[17] G. M. N'Guérékata, *Topics in Almost Automorphy*, Springer-Verlag, New York-Boston-London-Moscow, 2005.

[18] A. Halanay, *Solution periodiques et presque periodiques des systems d'equationcs aux difference finies*, Arch. Rat. Mech. 12 (1963), 134-149.

[19] Y. Hamaya, *Existence of an almost periodic solution in a difference equation with infinite delay*, Journal of Difference Equations and Applications, 9(2), (2003), 227-237.

[20] M. Hu and L. Wang, *Unique Existence Theorem of Solution of Almost Periodic Differential Equations on Time Scales*, Discrete Dynamics in Nature and Society, Volume 2012 (2012), Article ID 240735, 8 pages, doi:10.1155/2012/240735.

[21] A. O. Ignatyev and O. A. Ignatyev, *On the stability in periodic and almost periodic difference systems*, J. Math. Anal. Appl., 313(2) (2006), 678-688.

[22] H. Jialin and Y. Rong, *The existence of almost periodic solutions for a class of differential equations with piecewise constant argument*, Nonlinear Analysis, 28(8) (1997), 1439-1450.

[23] Yongkun Li and Chao Wang, *Almost Periodic Functions on Time Scales and Applications*, Discrete Dynamics in Nature and Society, Vol. 2011, 20 pages.

[24] N. V. Minh, T. Naito, and G. N'Guérékata, *A spectral countability condition for almost automorphy of solutions of differential equations*, Proc. Amer. Math. Soc., 134(11) (2006), 3257-3266.

[25] P. Y. H. Pang and R. P. Agarwal, *On periodicity of difference equations of a general type*, J. Difference Equations and Applications, 2 (1996), 271-286.

[26] Y. Song, *Almost periodic solutions of discrete Volterra equations*, J. Math. Anal. Appl., 314(1) (2006), 174-194.

[27] Y. Song, *Periodic and almost periodic solutions of functional difference equations with infinite delay*, Advances in Difference Equations, Article ID 68023 (2007).

[28] S. Sugiyama, *On periodic solutions of difference equations*, Bull. Sci. Eng. Research Lab. Waseda Univ. 52 (1971), 89-94.

[29] S. Zaidman, *Almost automorphic solutions of some abstract evolution equations*, Instituto Lombardo di Sci. e Lett., 110 (1976), 578-588.

[30] S. Zaidman, *Existence of asymptotically almost periodic and of almost automorphic solutions for some classes of abstract differential equations*, Ann. Sc. Math. Québec, 13 (1989), 79-88.

[31] S. Zaidman, *Almost automorphic solutions of certain abstract differential equations*, Annali di Mat. Pura ed Appl., series 4, 101 (1974), 91-114.

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