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A Characterization of Uniform Well Posedness for Degenerate Second-Order Abstract Differential Equations

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Abstract. We completely characterize the uniform well-posedness of the Dirichlet boundary value problem for degenerate second order abstract differential equations in Hilbert and Banach spaces. Our characterization is given solely in terms of spectral properties of the data and uniform boundedness properties of an appropriate resolvent operator.

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1. Introduction

Let $A : D(A) \to X$ and $M : D(M) \to X$ be closed linear operators defined on a Banach space X satisfying $D(A) \subset D(M)$. In this article we are concerned with the following second-order degenerate equation

$$Mu''(t) + Au(t) = 0,\ t \in [0, \pi], \tag{1}$$

together with Dirichlet boundary conditions

$$Mu(0) = x_0 \text{ and } Mu(\pi) = x_\pi, \tag{2}$$

where x_0 and x_π are elements of X.

In case $M = I$, the identity operator in X, and $A = -\Delta$ the negative Laplacian operator, (1) is the well-known wave equation. If $M \neq I$ the model

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(1) corresponds to a degenerate partial differential equation. Degenerate equations appear in many problems of physics and mechanics. For example, choosing $Mf(x) = xf(x)$ and $Af(x) = f''(x)$ Eq. (1) corresponds to the Tricomi equation [23]

$$xu_{tt}(t, x) + u_{xx}(t, x) = 0.$$

Tricomi's equation plays a relevant role in the study of transonic flow. A generalization that considers $Mf(x) = m(x)f(x)$ where $m(x)$ represents the velocity of the flow, which is positive at subsonic and negative at supersonic speeds, is the so-called Chapligyn equation [17]. We note that the Chapligyn equation is also relevant for the study the deformations of prestressed linear elastic solids. A study of spatial decay estimates for the solutions of the Chapligyn equation has been carried out by Quintanilla [21]. See also the reference [13] for a recent overview. On the other hand, abstract degenerate differential equations of second order in time in the form of (1) have been studied in a systematic way by Favini and Yagi in the monograph [12, Chapter VI].

One of the most intriguing and difficult problems from a mathematical point of view is to find necessary and sufficient conditions for well-posedness in some sense, only in terms of the data and the structure of abstract differential equations. In the case where $M = I$, this question for the *initial value* problem corresponding to (1) (that is, with $u(0)$ and $u'(0)$ specified) was solved in full generality a long time ago, and is well known as the starting point for the theory of strongly continuous cosine families of operators [1, Chapter 3]. The $M \neq I$ case is much more subtle [12] and in many cases depends on the geometrical structure of the Banach space under consideration.

Recently, G. Dore [11] studied the *Dirichlet boundary value* problem (1)–(2) in case $M = I$ and A a closed and densely defined linear operator in a Banach space X. He gave necessary conditions for the uniform well-posedness in terms of the resolvent operator of A. In particular, he proved that if the Dirichlet boundary value problem (1)–(2) is uniformly well-posed then

(i) $\mathbb{N}^2 \subset \rho(A)$, the resolvent set of A,
(ii) $\sup_{k\in\mathbb{N}} \|k(k^2I - A)^{-1}\| < \infty$, and
(iii) $\sum_{k=1}^{\infty}(k^2I - A)^{-1}$ converges in operator norm.

However, he did not give sufficient conditions for uniform well-posedness, except in the situation where the operator A is bounded [11, Theorem 5.2]. Therefore, the following questions arise:

- Can we find sufficient conditions for uniform well-posedness of (1)–(2) in the case $M = I$ and A unbounded?.
- More generally, is it possible to find necessary and sufficient conditions for uniform well-posedness of (1)–(2) in the case $M \neq I$?.

The objective of this article is to give a positive answer to these open questions. As a result, we are able to generalize Dore's results, and gain new insights into the behavior of the model (1) in the cases $M = I$ and $M \neq I$ as well as a new characterization of the uniform well-posedness of the Dirichlet boundary value problem (1)–(2) which is provided here for the first time.

We note that characterizations of well-posedness for abstract equations like (1) have appeared previously in the literature. For example, Arendt and Bu [2, Theorem 6.3] proved that in a Hilbert space H and $M = I$ the following conditions are equivalent

(a) For all $f \in L^p(0,\pi;H)$ there exists a unique $u \in L^p(0,\pi;D(A)) \cap H^{2,p}(0,\pi;H)$ satisfying

$$\begin{cases} u''(t) + Au(t) = f(t),\ t \in [0,\pi], \\ u(0) = 0,\ u(\pi) = 0 \end{cases}$$

(b) $\mathbb{N}^2 \subset \rho(A)$ and $\sup_{k\in\mathbb{N}} \|k^2(k^2 I - A)^{-1}\| < \infty$.

Analogously, they treat the Neumann problem in [2, Theorem 6.4]. Later, Bu [7, Theorems 2.7 and 2.8] and Bu and Cai [8, Theorem 2.3] proved an analogous result for the non homogenous problem (1) but with boundary conditions $u(0) = u(2\pi), (Mu')(0) = (Mu')(2\pi)$ that works under the conditions

(c) $0 \in \rho(A)$,

(d) $\mathbb{N}^2 \subset \rho_M(A)$, the M-resolvent of A, and

$$\sup_{k\in\mathbb{N}} \|k(k^2M - A)^{-1}\| < \infty \text{ and } \sup_{k\in\mathbb{N}} \|k^2M(k^2M - A)^{-1}\| < \infty,$$

see also the references [3–5, 9, 15, 18, 19].

In this paper, for $M = I$ case, we show that if X is a Hilbert space, conditions (i), (ii) are also sufficient for uniform well-posedness, provided the operator $-A$ is the generator of a strongly continuous cosine operator family and $0 \in \rho(A)$. We note that strongly continuous cosine families play a significant role in the study of abstract Cauchy problems of second order [1].

For general M, we find in the context of a general Banach space a necessary and sufficient condition for uniform well-posedness of (1)–(2) that requires the above mentioned condition (d), plus a compatibility condition between the operators A and M, but that in contrast to the case $M = I$, does not require that A be a generator nor $0 \in \rho(A)$. It is notable that our characterization is not trivial only for $M \neq I$. Otherwise, the operator A must be bounded, which resembles the above mentioned result of Dore [11].

2. Preliminaries

Let A and M be closed linear operators with domains $D(A)$ and $D(M)$ defined in a Banach space X such that $D(A) \subseteq D(M)$ and define the set $M(D(A)) := \{x \in X : My = x, \text{ for some } y \in D(A)\}$. We introduce the following definition of solution.

Definition 2.1. We call solution of Eq. (1) a function $v : [0,\pi] \to X$ such that

(1) $v \in C^2([0,\pi], D(M)) \cap C([0,\pi], D(A))$;

(2) for all $t \in [0,\pi]$, $Mv''(t) + Av(t) = 0$.

We call solution of problem (1)–(2) a solution v of Eq. (1) such that $Mv(0) = x_0$ and $Mv(\pi) = x_\pi$.

Obviously a solution of the problem with the condition (2) can exist only if x_0, $x_\pi \in M(D(A))$.

We introduce the notion of well-posedness as follows.

Definition 2.2. We say that problem (1)–(2) is uniformly well-posed if

(1) for all x_0, $x_\pi \in M(D(A))$, problem (1)–(2) has solution;

(2) there exists $C > 0$ such that for any solution v of Eq. (1) we have

$$\sup_{t\in[0,\pi]} \|v(t)\| \le C(\|Mv(0)\| + \|Mv(\pi)\|).$$

Remark 2.3. From (2) in Definition 2.2, it follows that if a solution exists, it is unique.

For later use we recall from [11, Lemma 3.2] the following Lemma.

Lemma 2.4. *The series of functions* $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kt)$ *converges pointwise for* $t \in [0,\pi]$, *the sequence of partial sums is uniformly bounded and*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kt) = \begin{cases} \frac{1}{2}t, & \text{if } 0 \le t < \pi, \\ 0, & \text{if } t = \pi. \end{cases}$$

Analogously we have the following result [11, Lemma 5.1].

Lemma 2.5. *The series of functions* $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \sin(kt)$ *converges uniformly for* $t \in [0,\pi]$ *and*

$$\frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \sin(kt) = \frac{\pi}{6}t - \frac{1}{6\pi}t^3.$$

3. Main Results

The following result characterizes the uniform well-posedness of Problem (1)–(2) in terms of the existence of a one-parameter family of strongly continuous operators that satisfy certain properties. In case $M = I$ and if we further assume that $-A$ generates a cosine family, the family used here is formally related to the associated sine family (see [1, Section 3.15]).

Theorem 3.1. *Let A and M be closed linear operators in a Banach space X with $D(A) \subseteq D(M)$ and $M(D(A))$ dense in X. Problem (1)–(2) is uniformly well-posed if and only if there exists $S : [0,\pi] \to \mathcal{L}(X)$ such that*

(i) *for all $x \in M(D(A))$, the function $S(\cdot)x$ is solution of Eq. (1);*

(ii) *for all $x \in X$ the function $S(\cdot)x$ is continuous;*

(iii) *for all $x \in X$, $S(0)x$, $S(\pi)x \in D(M)$ and $MS(0)x = 0$, $MS(\pi)x = x$;*

(iv) *if* $v : [0, \pi] \to X$ *is a solution of Eq.* (1)*, then*

$$v(t) = S(t)Mv(\pi) + S(\pi - t)Mv(0).$$

Proof. Suppose that problem (1)–(2) is uniformly well-posed. Let $x \in M(D(A))$ and consider the problem

$$\begin{cases} Mu''(t) + Au(t) = 0,\ t \in [0, \pi], \\ Mu(0) = 0 \text{ and } Mu(\pi) = x. \end{cases} \tag{3}$$

Then there exists a unique solution v_x of Eq. (3) such that

$$v_x \in C^2([0, \pi], D(M)) \cap C([0, \pi], D(A)).$$

For $t \in [0, \pi]$ put $\tilde{S}(t)x = v_x(t)$. Then $\tilde{S}(t)$ is a linear operator in X and, because of the uniform well-posedness, there exists $C > 0$ such that $\|\tilde{S}(t)x\| \leq C\|x\|$, for every $x \in M(D(A))$. Since $M(D(A))$ is dense in X, $\tilde{S}(t)$ can be extended to a bounded linear operator $S(t)$ in X and $\|S(t)\| \leq C$.

If $x \in M(D(A))$, then $S(\cdot)x = \tilde{S}(\cdot)x = v_x$ where v_x is the unique solution of the problem (3). So, $S(\cdot)x$ is a solution of Eq. (1). Therefore (i) is satisfied.

Since for all $x \in M(D(A))$ the function $S(\cdot)x$ is a solution of Eq. (1), it follows that $S(\cdot)x$ is continuous for all $x \in M(D(A))$. Now, for $x \in X$, there exist a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M(D(A))$ such that $x_n \to x$. The functions $S(\cdot)x_n$ are continuous. Since

$$\begin{aligned} \|S(t)x - S(t_0)x\| &= \|S(t)x - S(t)x_n + S(t)x_n - S(t_0)x_n + S(t_0)x_n - S(t_0)x\| \\ &\leq \|S(t)x - S(t)x_n\| + \|S(t)x_n - S(t_0)x_n\| + \|S(t_0)x_n - S(t_0)x\| \\ &= \|S(t)(x - x_n)\| + \|S(t)x_n - S(t_0)x_n\| + \|S(t_0)(x_n - x)\| \\ &\leq C\|x - x_n\| + \|S(t)x_n - S(t_0)x_n\| + C\|x_n - x\|, \end{aligned}$$

we see that $S(\cdot)x$ is continuous. Therefore (ii) is satisfied.

Given that for all $x \in M(D(A))$ the function $S(\cdot)x$ is solution of the Eq. (3), then $MS(0)x = 0$ and $MS(\pi)x = x$ for all $x \in M(D(A))$. Now, for $x \in X$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M(D(A))$ such that $x_n \to x$. Since $S(0)$ and $S(\pi)$ are bounded operators, then $S(0)x_n \to S(0)x$ and $S(\pi)x_n \to S(\pi)x$. Since $MS(0)x_n \to 0$ and $MS(\pi)x_n \to x$ and M is closed, then $S(0)x$, $S(\pi)x \in D(M)$ and $MS(0)x = 0$, $MS(\pi)x = x$. Therefore (iii) is satisfied.

Let v be a solution of Eq. (1). Then it is a solution of the Dirichlet problem

$$\begin{cases} Mu''(t) + Au(t) = 0,\ t \in [0, \pi], \\ Mu(0) = Mv(0) \text{ and } Mu(\pi) = Mv(\pi). \end{cases}$$

Since $Mv(\pi)$, $Mv(0) \in M(D(A))$, by (i), the function $t \mapsto S(t)Mv(\pi) + S(\pi - t)Mv(0)$ is solution of the same problem, hence, by the uniqueness of the solution, we have $v(t) = S(t)Mv(\pi) + S(\pi - t)Mv(0)$. Therefore (iv) is satisfied.

Conversely, suppose that there exists S satisfying conditions (i) to (iv). Since, for all $x \in X$, the function $S(\cdot)x$ is continuous, it is bounded, hence by the Uniform Boundedness Principle, the set $\{S(t) : t \in [0, \pi]\}$ is bounded. Therefore, there exists $C > 0$ such that $\|S(t)\| \leq C$, for all $t \in [0, \pi]$.

For x_0, $x_\pi \in M(D(A))$ define $v : [0, \pi] \to X$ by $v(t) = S(t)x_\pi + S(\pi - t)x_0$. From (i) we have that $v \in C^2([0, \pi], D(M)) \cap C([0, \pi], D(A))$ and for all $t \in [0, \pi]$, $Mu''(t) + Au(t) = 0$. By (iii) $Mv(0) = x_0$ and $Mv(\pi) = x_\pi$. Hence problem (1)–(2) has a solution.

Let v be a solution of Eq. (1). Then by (iv) $v(t) = S(t)Mv(\pi) + S(\pi - t)Mv(0)$. Since $\|S(t)\| \leq C$, for all $t \in [0, \pi]$, we have

$$\begin{aligned} \|v(t)\| &= \|S(t)Mv(\pi) + S(\pi - t)Mv(0)\| \\ &\leq \|S(t)\|\|Mv(\pi)| + \|S(\pi - t)\|\|Mv(0)\| \\ &\leq C(\|Mv(\pi)\| + \|Mv(0)\|). \end{aligned}$$

Hence problem (1)–(2) is uniformly well-posed. □

Remark 3.2. Note that if problem (1)–(2) is uniformly well-posed then, by Theorem 3.1-(iii), M must have a bounded right inverse.

We define the M-resolvent set of A, $\rho_M(A)$, by

$$\rho_M(A) = \{\lambda \in \mathbb{C} : \lambda M - A : D(A) \to X \text{ is bijective and } (\lambda M - A)^{-1} \in \mathcal{L}(X)\}$$

Thus, $\lambda \in \rho_M(A)$ if and only if the linear operator $(\lambda M - A)^{-1}$ is a continuous isomorphism from X onto $D(A)$. Here, we consider $D(A)$ and $D(M)$ as normed spaces equipped with their respective graph norms. These are Banach spaces since the linear operators A and M are closed. Since $D(A) \subset D(M)$, we have $M(\lambda M - A)^{-1}, A(\lambda M - A)^{-1} \in \mathcal{L}(X)$ by the Closed Graph Theorem and the closedness of M and A. Moreover, $\rho_M(A)$ is an open subset of $\mathbb{C}$, the function $R_M(\lambda) := (\lambda M - A)^{-1}$ is analytic and $\frac{d}{d\lambda}R_M(\lambda) = -R_M(\lambda)MR_M(\lambda)$. For a proof, see [6, Lemma 3]. Note that $\rho_I(A) = \rho(A)$, the resolvent set of A. Here, I denotes the identity operator in X.

The following result establishes necessary conditions for well-posedness.

Theorem 3.3. *Let A and M be closed linear operators defined in a Banach space X. Assume that any of the following conditions holds:*

(i) *$D(A) \subseteq D(M)$ and $M(D(A)) = X$,*
(ii) *M is bounded and $M(D(A))$ is dense in X.*

Then if problem (1)–(2) is uniformly well-posed, we have $k^2 \in \rho_M(A)$ for all $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}} \|k(k^2M - A)^{-1}\| < \infty$.

Proof. Let S be the operator valued function whose existence is guaranteed by Theorem 3.1 and $C = \sup_{t\in[0,\pi]} \|S(t)\|$. For $k \in \mathbb{N}$, the operator $k^2M - A$ is injective. Indeed if $x \in D(A)$ is such that $k^2Mx = Ax$, it is easy to check that the function $t \mapsto \sin(kt)x$ is solution of the problem

$$\begin{cases} Mu''(t) + Au(t) = 0,\ t \in [0, \pi], \\ Mu(0) = 0 \text{ and } Mu(\pi) = 0. \end{cases}$$

Since zero is also a solution of this problem, by uniqueness we conclude that $x = 0$. For $k \in \mathbb{N}$ and $x \in X$ let

$$N(k)x := \frac{(-1)^{k+1}}{k} \int_0^\pi \sin(ks)S(s)xds.$$

Then $N(k)$ defines a bounded linear operator on X for every $k \in \mathbb{N}$. Moreover,

$$\|kN(k)x\| \le \int_0^\pi |\sin(ks)|\|S(s)x\|ds \le C\|x\| \int_0^\pi |\sin(ks)|ds = 2C\|x\|.$$

So, we obtain $\sup_{k\in\mathbb{N}} \|kN(k)\| < \infty$.

Let $x \in M(D(A))$. Since $S(\cdot)x \in D(A)$ and A is closed, then $N(k)x \in D(A)$ and

$$\begin{aligned} AN(k)x &= \frac{(-1)^{k+1}}{k} \int_0^\pi \sin(ks)AS(s)xds \\ &= \frac{(-1)^{k}}{k} \int_0^\pi \sin(ks)MS''(s)xds \\ &= -x + k^2 MN(k)x. \end{aligned}$$

In case (i) $AN(k)x = -x + k^2 MN(k)x$ for all $x \in X = M(D(A))$. In case (ii), given $x \in X$, there exists $\{x_n\}_{n\in\mathbb{N}} \subseteq M(D(A))$ such that $x_n \to x$. Since M and $N(k)$ are bounded, then

$$N(k)x_n \to N(k)x \text{ and } AN(k)x_n = -x_n + k^2 MN(k)x_n \to -x + k^2 MN(k)x.$$

By the closedness of A, we have $N(k)x \in D(A)$ and $AN(k)x = -x + k^2 MN(k)x$. Hence, for all $x \in X$, $(k^2M - A)N(k)x = x$.

In both cases $AN(k)x = -x + k^2 MN(k)x$ for all $x \in X$. Therefore, $N(k)$ is the right inverse of $(k^2M - A)$. For $x \in D(A)$, we have

$$(k^2M - A)N(k)(k^2M - A)x = (k^2M - A)x.$$

Since $(k^2M - A)$ in injective, then $N(k)(k^2M - A)x = x$. Hence, $k^2M - A$ is invertible and $N(k) = (k^2M - A)^{-1}$. Therefore, $k^2 \in \rho_M(A)$ for all $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}} \|k(k^2M - A)^{-1}\| < \infty$. □

We remark that the necessary condition found in the above theorem, was also considered in connection with second order operators of elliptic type by Gorbachuk and Knyazyuk [14]. If $M = I$, then the operator $-A$ and is named π-positive [14] and implies that the sequence $R_N = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}(k^2 - A)^{-1}$ is bounded in $\mathcal{L}(X)$, see [10, Proposition 2.3]. This observation, motivates the following result that extends [11, Theorem 3.3] to $M \ne I$.

Theorem 3.4. *Let A and M be closed linear operators in a Banach space X. Assume that (i) or (ii) of Theorem 3.3 holds. If problem (1)–(2) is uniformly well-posed, then the series $\sum_{k=1}^{\infty}(k^2M-A)^{-1}$ converges in operator norm. Moreover, if $S(\cdot)$ is the operator-valued function introduced in Theorem 3.3, then*

$$\sum_{k=1}^{\infty}(k^2M-A)^{-1}x=\frac{1}{2}\int_0^{\pi} sS(s)xds, \text{ for all } x\in X.$$

Proof. From the proof of Theorem 3.3 we know that for all $k\in\mathbb{N}$, $x\in X$.

$$(k^2M-A)^{-1}x=\frac{(-1)^{k+1}}{k}\int_0^{\pi}\sin(ks)S(s)xds$$

Then

$$\begin{aligned}&\left\|\sum_{k=1}^{n}(k^2M-A)^{-1}x-\frac{1}{2}\int_0^{\pi} tS(s)x\right\|\\ &\quad\le\int_0^{\pi}\left\|\left(\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sin(ks)-\frac{1}{2}s\right)S(s)x\right\|ds\\ &\quad\le C\|x\|\int_0^{\pi}\left\|\left(\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sin(ks)-\frac{1}{2}s\right)\right\|ds.\end{aligned}$$

By Lemma 2.4 the series of functions $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(kt)$ converges pointwise to $\frac{1}{2}t$ and the partial sums are uniformly bounded. Therefore, by the dominated convergence theorem, the right hand side in the above inequality converges to 0 as n tends to ∞. □

We arrive at the main result of this article that completely characterize the well-posedness of Dirichlet boundary value problem (1)–(2) under an additional condition involving A and M. It generalizes [11, Theorem 5.2] to the situation that A is unbounded and $M\neq I$. In what follows we recall that an operator $B\in\mathcal{L}(X)$ is a bounded right inverse of M if $MBx=x$ for all $x\in X$. A standard example is $Mf(t):=f'(t)$ and $Bg(t):=\int_0^t g(s)ds$ in $X=C([0,1])$.

Theorem 3.5. *Let A and M be closed linear operators in a Banach space X such that $D(A)\subseteq D(M)$ and M has a bounded right inverse B with $B(X):=\{y\in X: Bx=y,\ x\in X\}\subseteq D(A)$. The following assertions are equivalent.*

- (a) *Problem (1)–(2) is uniformly well-posed;*
- (b) $\mathbb{N}^2\subseteq\rho_M(A)$, $\sup_{k\in\mathbb{N}}\|k(k^2M-A)^{-1}\|<\infty$ *and* $\sup_{k\in\mathbb{N}}\|k^2M(k^2M-A)^{-1}\|<\infty$.

Proof. First note that since B is a bounded right inverse of M and $B(X)\subseteq D(A)\subseteq D(M)$, then $M(D(A))=X$.

Assume that problem (1)–(2) is uniformly well-posed, then by Theorem 3.3-(i) we have that $\mathbb{N}^2 \subseteq \rho_M(A)$ and $\sup_{k\in\mathbb{N}} \|k(k^2M-A)^{-1}\| < \infty$. From the proof of Theorem 3.3, we have that for $k \in \mathbb{N}$ and $x \in X$,

$$(k^2M - A)^{-1}x = \frac{(-1)^{k+1}}{k}\int_0^{\pi} \sin(ks)S(s)x ds.$$

Therefore, for $x \in M(D(A)) = X$, we have that $S(\cdot)x \in C^2([0,\pi], D(M)) \cap C([0,\pi], D(A))$ and by integration by parts, we obtain

$$\begin{aligned}\|k^2M(k^2M-A)^{-1}x\| &= \left\|(-1)^{k+1}k\int_0^{\pi} \sin(ks)MS(s)x ds\right\| \\ &= \left\|x - \frac{(-1)^{k+1}}{k}\int_0^{\pi} \sin(ks)MS''(s)x ds\right\| \\ &\le \|x\| + C_x.\end{aligned}$$

So, $\sup_{k\in\mathbb{N}} \|k^2M(k^2M-A)^{-1}x\| < \infty$ for every $x \in X$. Then by the Uniform Boundedness Principle $\sup_{k\in\mathbb{N}} \|k^2M(k^2M-A)^{-1}\| < \infty$.

Now we prove the converse. We will use Theorem 3.1. Since A is closed and B is bounded, then AB is closed linear operator. Since $B(X) \subseteq D(A)$, then $D(AB) = X$. Now by the Closed Graph Theorem AB is bounded. Let $t \in [0,\pi]$ and $x \in X$. Define

$$\begin{aligned}S(t)x = \ & \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}\sin(kt)(k^2M-A)^{-1}(AB)^2x \\ &+ \left(\frac{\pi}{6}t - \frac{1}{6\pi}t^3\right)B(AB)x + \frac{t}{\pi}Bx.\end{aligned} \tag{4}$$

By hypothesis (b), there exists a constant $C > 0$ such that

$$\|k(k^2M-A)^{-1}\| \le C, \quad k \in \mathbb{N}. \tag{5}$$

Then there exists a constant $C' > 0$ such that

$$\|S(t)x\| \le \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^4}C\|AB\|^2\|x\| + \frac{\pi^2}{9\sqrt{3}}\|B\|\|AB\|\|x\| + \frac{t}{\pi}\|B\|\|x\| \le C'\|x\|.$$

This estimate shows that the series (4) is uniformly convergent. Hence $S(t)x$ is well defined, $S(t) \in \mathcal{L}(X)$ and the function $S(\cdot)x$ is continuous. Hence Theorem 3.1-(ii) is satisfied.

Now, observe that $S(0)x \in D(M)$ and $MS(0)x = 0$ because $S(0)x = 0$. Since M is closed and for any $x \in X$ we have $(k^2\,M - A)^{-1}(AB)^2x \in D(A) \subseteq D(M)$ and $B(AB)x, Bx \in B(X) \subseteq D(A) \subseteq D(M)$, we obtain that $S(t)x \in$

$D(A) \subseteq D(M)$ for any $t \geq 0$ and

$$MS(t)x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\sin(kt)M(k^2M-A)^{-1}(AB)^2x + \left(\frac{\pi}{6}t - \frac{1}{6\pi}t^3\right)(AB)x + \frac{t}{\pi}x.$$

where we have used the fact that $MB = I$. In particular, for each $x \in X$ we have $S(\pi)x \in D(M)$ and $MS(\pi)x = x$. Therefore, Theorem 3.1-(iii) is satisfied.

From the estimate (5) we also obtain that the series

$$S'(t)x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\cos(kt)(k^2M-A)^{-1}(AB)^2x + \left(\frac{\pi}{6} - \frac{1}{2\pi}t^2\right)B(AB)x + \frac{1}{\pi}Bx,$$

$$S''(t)x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\sin(kt)(k^2M-A)^{-1}(AB)^2x - \frac{1}{\pi}tB(AB)x,$$

are uniformly convergent. A similar argument as for $S(t)$ shows that $S'(t)x$, $S''(t)x \in D(M)$ for any $t \geq 0$ and $x \in X$.

Since by hypothesis $MB = I$ and the set $\{k^2 M(k^2 M - A)^{-1}\}_{k\in\mathbb{N}}$ is uniformly bounded, we obtain that the series

$$MS'(t)x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\cos(kt)M(k^2M-A)^{-1}(AB)^2x + \left(\frac{\pi}{6} - \frac{1}{2\pi}t^2\right)(AB)x + \frac{1}{\pi}x,$$

$$MS''(t)x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k}\sin(kt)M(k^2M-A)^{-1}(AB)^2x - \frac{1}{\pi}t(AB)x,$$

are uniformly convergent. Finally, the identity $A(k^2M - A)^{-1} = -I + k^2M(k^2M - A)^{-1}$ shows that the set $\{A(k^2M - A)^{-1}\}_{k\in\mathbb{N}}$ is uniformly bounded and, since $S(t)x \in D(A)$, we conclude that the series

$$AS(t)x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\sin(kt)A(k^2M-A)^{-1}(AB)^2x + \left(\frac{\pi}{6}t - \frac{1}{6\pi}t^3\right)(AB)^2x + \frac{t}{\pi}ABx,$$

is uniformly convergent too. Hence $S(\cdot)x \in C^2([0,\pi], D(M)) \cap C([0,\pi], D(A))$ and, for all $t \in [0,\pi]$,

$$
\begin{aligned}
&MS''(t)x + AS(t)x \\
&\quad = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\sin(kt)k^2M(k^2M-A)^{-1}(AB)^2x - \frac{t}{\pi}ABx \\
&\qquad + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}\sin(kt)A(k^2M-A)^{-1}(AB)^2x \\
&\qquad + \left(\frac{\pi}{6}t - \frac{1}{6\pi}t^3\right)(AB)^2x + \frac{t}{\pi}ABx \\
&\quad = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\sin(kt)(AB)^2x + \left(\frac{\pi}{6}t - \frac{1}{6\pi}t^3\right)(AB)^2x = 0,
\end{aligned}
$$

where the last equality follows from Lemma 2.5. Therefore, Theorem 3.1-(i) is satisfied.

To prove Theorem 3.1-(iv) we first show the uniqueness of the solution of problem (1)–(2). Let $v \in C^2([0,\pi], D(M)) \cap C([0,\pi], D(A))$ be a solution of the problem

$$
\begin{cases} Mu''(t) + Au(t) = 0,\ t \in [0,\pi], \\ Mu(0) = 0 \text{ and } Mu(\pi) = 0. \end{cases}
$$

For $k \in \mathbb{N}$, we define $v(k) := \int_0^\pi \sin(ks)v(s)ds$. Then

$$
Av(k) = \int_0^\pi \sin(ks)Av(s)ds = -\int_0^\pi \sin(ks)Mv''(s)ds = k^2Mv(k).
$$

Hence $(k^2M - A)^{-1}v(k) = 0$. Since $(k^2M - A)^{-1}$ is injective, this proves that $v(k) = 0$. Therefore all the Fourier coefficients of the 2π-periodic extension of the odd extension of v are null, hence $v(t) = 0$ for a.e. t. Since v is continuous, $v(t) = 0$ for all t.

The function $v(t) = S(t)Mv(\pi) + S(\pi - t)Mv(0)$ is solution of problem (1), but the solution is unique, hence every solution of problem (1) coincides with that function. Therefore Theorem 3.1-(iv) is satisfied. □

Remark 3.6. When $M = I$ the identity $k(k^2 - A)^{-1} = \frac{1}{k}k^2(k^2 - A)^{-1}$, $k \in \mathbb{N}$ shows that we need only the assumption $\sup_{k\in\mathbb{N}} \|k^2(k^2 - A)^{-1}\| < \infty$. Moreover, in such case we have $B(X) = X$ which implies that $D(A) = X$ and hence A must be bounded.

4. Case $M = I$: A Spectral Characterization

Before we give our main result of this section, we recall the following concepts about cosine operator functions: if $-A$ generates a strongly continuous cosine operator function $\mathrm{Cos}(t)$, then by definition, $\mathrm{Sin(t)} := \int_0^t \mathrm{Cos}(s)ds$ is the associated sine family. In this case there exists a unique Banach space V such that $D(A) \hookrightarrow V \hookrightarrow X$ and the operator matrix $\begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$ with domain $D(A)\times V$ generates a C_0-semigroup in $V \times X$. The space $V \times X$ is called the phase space associated with $-A$, see [1, Section 3.14].

Our main result of this section provides a converse to [11, Theorem 3.1] on Hilbert spaces when $-A$ is the generator of a cosine function.

Theorem 4.1. *Let H be a Hilbert space and suppose that $-A$ generates a cosine function $\{Cos(t)\}_{t\in\mathbb{R}}$ on H and $0 \in \rho(A)$. Then the following assertions are equivalent:*

(a) *The problem*

$$\begin{cases} u''(t) + Au(t) = 0,\ t \in (0, \pi), \\ u(0) = x_0,\ u(\pi) = x_\pi. \end{cases} \tag{6}$$

is uniformly well-posed.

(b) $\{k^2 : k \in \mathbb{N}\} \cup \{0\} \subseteq \rho(A)$ *and* $\sup_{k\in\mathbb{N}} \|kR(k^2; A)\| < \infty$*.*

Proof. (a) implies (b) follows from [11, Theorem 3.1] and the hypothesis $0 \in \rho(A)$. Suppose (b) holds. Since H is a Hilbert space, by [10, Theorem 3.2] we conclude that $1 \in \rho(\mathrm{Cos}(2\pi))$.

Now, by [10, Theorem 2.1, (c)], (see also [20]), we have $\{k^2 : k \in \mathbb{N}\} \cup \{0\} \subseteq \rho(A)$ and

$$P := \lim_{N\to\infty} R_N, \qquad Q := \lim_{N\to\infty} S_N$$

define bounded operators in X, where

$$R_N := \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} R(-k^2; -A) \tag{7}$$

and

$$S_N := -\frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} AR(-k^2; -A).$$

Moreover, the proof of [10, Theorem 2.1], formulas (10) and (11) shows that

$$R_N(1 - \mathrm{Cos}(2\pi)) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} \int_0^{2\pi} e^{-iks}\mathrm{Sin}(s)ds$$

and

$$S_N(1 - \mathrm{Cos}(2\pi)) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} \int_0^{2\pi} e^{-iks}\frac{(s - 2\pi)\mathrm{Cos}(s) - \mathrm{Sin}(s)}{2}ds,$$

respectively. By Fejér's theorem (see e.g. [10, Theorem 1.1]), we obtain

$$P(I - \mathrm{Cos}(2\pi)) = \pi \mathrm{Sin}(2\pi),$$

and

$$Q(I - \mathrm{Cos}(2\pi)) = -\frac{2\pi^2 I + \pi \mathrm{Sin}(2\pi)}{2}.$$

It follows that $I = \frac{1}{2\pi^2}(2Q - P)(I - \mathrm{Cos}(2\pi))$. From the cosine identity $\mathrm{Cos}(t+s) - \mathrm{Cos}(t-s) = -2A\mathrm{Sin}(t)\mathrm{Sin}(s)$ taking $t = s = \pi$ we obtain that $(I - \mathrm{Cos}(2\pi)) = 2A\mathrm{Sin}(\pi)\mathrm{Sin}(\pi)$. Thus,

$$I = \frac{1}{\pi^2}(2Q - P)A\mathrm{Sin}(\pi)\mathrm{Sin}(\pi).$$

From this, it follows that $\mathrm{Sin}(\pi)$ is injective.

Since $\mathrm{Sin}(\pi)$ is a bounded operator which commutes with R_N and S_N, we conclude that it also commutes with P and Q. Let $x \in V$. Then by [22, Proposition 2.2, formula (2.18)] or [1, Theorem 3.4.11], we have that $\mathrm{Sin}(\pi)x \in D(A)$ and

$$x = \frac{1}{\pi^2}(2Q - P)\mathrm{Sin}(\pi)A\mathrm{Sin}(\pi)x = \mathrm{Sin}(\pi)\frac{1}{\pi^2}(2Q - P)A\mathrm{Sin}(\pi)x.$$

This proves that $\mathrm{Sin}(\pi) : X \to V$ is onto. Then, as a consequence of the open mapping theorem, we conclude that the operator $\mathrm{Sin}^{-1}(\pi) : V \to X$ is bounded. Now, recalling from [22, Proposition 2.2, formula (2.17)] or [1, Theorem 3.14.11] that $\mathrm{Sin}(s)x \in V$ for all $x \in X$ and all $s \geq 0$, we conclude that $\mathrm{Sin}(\pi)^{-1}\mathrm{Sin}(s)$ are well defined and bounded operators in X for all $s \in [0, \pi]$. Then the function u defined by

$$u(t) = \mathrm{Sin}(\pi)^{-1}\mathrm{Sin}(t)x_\pi + \mathrm{Sin}(\pi)^{-1}\mathrm{Sin}(\pi - t)x_0. \tag{8}$$

is the unique solution of (6) and the result follows. □

We end this article with the following examples showing how our abstract results apply.

Example 1. Consider the problem

$$\begin{cases} \dfrac{\partial^3 u}{\partial x \partial t^2}(t,x) = \dfrac{\partial u}{\partial x}(t,x), \text{ for all } (t,x) \in [0,\pi] \times [0,1], \\ \\ \dfrac{\partial u}{\partial x}(0,x) = \varphi_0(x) \text{ and } \dfrac{\partial}{\partial x}u(\pi,x) = \varphi_\pi(x) \text{ for all } x \in [0,1], \\ \\ u(t,0) = 0 \text{ for all } t \in [0,\pi] \end{cases} \tag{9}$$

where $\varphi_0,\ \varphi_\pi \in X := C([0,1])$.

Let $M := -\dfrac{\partial}{\partial x}$ with $D(M) = C^1([0,1])$ and $A := \dfrac{\partial}{\partial x}$ with $D(A) = \{u \in C^1([0,1]) : u(0) = 0\}$. Note that A and M are closed linear operators in X

such that $D(A) \subseteq D(M)$. Define $Bf(x) = -\int_0^x f(s)ds$ then $B \in \mathcal{L}(X)$ is the right inverse of M and clearly $B(X) = D(A)$.

A computation shows that $(\lambda M - A)^{-1} f = \frac{1}{\lambda+1} Bf$ for all $\lambda \in \mathbb{C}\backslash\{-1\}$ and $f \in C([0,1])$. Therefore, we have that $\mathbb{N}^2 \subseteq \rho_M(A)$ and

$$\begin{aligned}\sup_{k\in\mathbb{N}} \|k(k^2M - A)^{-1}\| &\leq \sup_{k\in\mathbb{N}} \sup_{\|f\|\leq 1} \sup_{x\in[0,1]} \left|\frac{k}{k^2+1} Bf(x)\right| \\ &\leq \sup_{\|f\|\leq 1} \sup_{x\in[0,1]} |Bf(x)| \\ &\leq \sup_{\|f\|\leq 1} \|f\| \leq 1,\end{aligned}$$

as well as $\sup_{k\in\mathbb{N}} \|k^2 M(k^2 M - A)^{-1}\| \leq 1$. Therefore, by Theorem 3.5, the problem (9) is uniformly well-posed. Furthermore, using formula (4) we obtain that

$$S(t)f(x) = \left[\frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3(k^2+1)} \sin(kt) + \left(\frac{\pi}{6}t - \frac{1}{6\pi}t^3\right) - \frac{t}{\pi}\right] \int_0^x f(s)ds,$$

and that the function

$$u(t,x) = S(t)\varphi_\pi(x) + S(\pi - t)\varphi_0(x), \quad (t,x) \in [0,\pi] \times [0,1],$$

is the unique solution of problem (9). In particular, observe that $u(0,x) = S(\pi)\varphi_0(x) = -\int_0^x \varphi_0(s)ds$ and $u(\pi,x) = S(\pi)\varphi_\pi(x) = -\int_0^x \varphi_\pi(s)ds$ imply $Mu(0,x) = -\frac{\partial}{\partial x}(u(0,x)) = \varphi_0(x)$ and $Mu(\pi,x) = -\frac{\partial}{\partial x}(u(\pi,x)) = \varphi_\pi(x)$, respectively.

Example 2. Let $X = l^2(\mathbb{N},\mathbb{C})$. Choose α such that $0 < \alpha < 1$ and let A be the operator in X defined by

$$D(A) = \{x = (x_n)_{n\in\mathbb{N}} : (n^2 x_n)_{n\in\mathbb{N}} \in X\},\ (Ax)_n = (n-\alpha)^2 x_n.$$

Define B on X by

$$D(B) = \{x = (x_n)_{n\in\mathbb{N}} : (nx_n)_{n\in\mathbb{N}} \in X\},\ (Bx)_n = i(n-\alpha)x_n.$$

Then B generates the C_0-group U given by $U(t)x = (e^{i(n-\alpha)t}x_n)_{n\in\mathbb{N}}$. Therefore, by [1, Corollary 3.16.8.] and [1, Example 3.14.15.], we have that $B^2 = -A$ generates the cosine function $\mathrm{Cos}(t)x = (\cos((n-\alpha)t)x_n)_{n\in\mathbb{N}}$. The associated sine operator function is given by: $\mathrm{Sin}(t)x = \left(\frac{\sin((n-\alpha)t)}{n-\alpha}x_n\right)_{n\in\mathbb{N}}$. Note that $\mathrm{Sin}(\pi)x = \left(\frac{\sin((n-\alpha)\pi)}{n-\alpha}x_n\right)_{n\in\mathbb{N}}$. Therefore $\mathrm{Sin}^{-1}(\pi)x = \left(\frac{n-\alpha}{\sin((n-\alpha)\pi)}x_n\right)_{n\in\mathbb{N}}$. If $k \in \mathbb{N}_0$ we can check that $k^2 - A$ is invertible and, for all $x \in X$, $n \in \mathbb{N}$, we have $((k^2I - A)^{-1}x)_n = \frac{x_n}{k^2 - (n-\alpha)^2}$.

Recall that for a multiplication operator T on $l^2(\mathbb{N})$ given by $Te_n = a_n e_n$, $n \in \mathbb{N}$ (where $(a_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N})$), the norm of T is given by $\|T\| = \|(a_n)\|_{l^\infty(\mathbb{N})}$. Therefore, for each $k \in \mathbb{N}$, we obtain after a computation

$$\|k(k^2 - A)^{-1}\| = \sup_{n\in\mathbb{N}} \frac{k}{|k^2 - (n-\alpha)^2|} \leq \max\left\{\frac{1}{\alpha(2-\alpha)}, \frac{1}{2(1-\alpha)}\right\}.$$

Thus, we have show that $\{k^2 : k \in \mathbb{N}\} \cup \{0\} \subseteq \rho(A)$ and $\sup_{k\in\mathbb{N}} \|k(k^2 - A)^{-1}\| < \infty$. Therefore, by Theorem 4.1, we have that the semidiscrete problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,n) + (n-\alpha)^2 u(t,n) = 0, \quad t \in (0,\pi), \quad n \in \mathbb{N}, \\ \\ u(0,n) = x_0(n), \quad u(\pi,n) = x_\pi(n), \end{cases}$$

is uniformly well-posed. Moreover, the unique solution is given by

$$\begin{aligned} u(t,n) &= \mathrm{Sin}^{-1}(\pi)\mathrm{Sin}(t)x_\pi(n) + \mathrm{Sin}^{-1}(\pi)\mathrm{Sin}(\pi - t)x_0(n) \\ &= \frac{\sin((n-\alpha)t)}{\sin((n-\alpha)\pi)} x_\pi(n) + \frac{\sin((n-\alpha)(\pi-t))}{\sin((n-\alpha)\pi)} x_0(n), \quad t \in (0,\pi), \quad n \in \mathbb{N}. \end{aligned}$$

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