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An unexpected property of fractional difference operators: Finite and eventual monotonicity

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We consider the relationship between the sign of the fractional difference $(\Delta^{\alpha}u)(n)$ and the positivity or monotonicity of u. Our focus is on the case in which the fractional difference can be negative, and we show that surprisingly, $(\Delta^{\alpha}u)(n) > -C$, where $C > 0$ is a constant, can still imply that u is increasing or is positive. We also consider the setting of sequential difference operators such as $\Delta^{\beta} \circ \Delta^{\alpha}$ for suitable choices of the parameters α and β. As is demonstrated by explicit examples, our results substantially improve some recent results in the literature and, moreover, shed light on some previous observations that heretofore were only able to be investigated by numerical simulations. We also provide applications of our results to an analysis of fractional-order initial value problems.

KEYWORDS
convolution, discrete fractional calculus, monotonicity, negative lower bound, positivity

MSC CLASSIFICATION
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1 | INTRODUCTION

For given functions $a, b \in L^1((0,\infty))$, define the finite convolution of a and b at $t \geq 0$, denoted $(a * b)(t)$, by

$$(a * b)(t) := \int_0^t a(t-s)b(s)\,ds.$$

The finite convolution operator plays an important role in both pure and applied mathematics. A principal application is to the theory of fractional calculus. For example, if, for $\alpha > 0$, we define the function $a : (0,\infty) \to \mathbb{R}$ by

$$a(t) := \frac{1}{\Gamma(\alpha)} t^{\alpha-1},$$

then $(a * b)(t)$ is the α-th order Riemann-Liouville fractional integral at $t > 0$—see [1–3] for some recent advances in the theory of fractional calculus together with the classic reference by Podlubny [4].

A discrete analogue of the finite convolution operator can be defined as follows. Let $a \in \mathbb{R}$ be fixed. For a sequences u, $v \in s(\mathbb{N}_a;\mathbb{R})$, where henceforth $s(\mathbb{N}_a;\mathbb{R})$ denotes the collection of all sequences from $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$ into $\mathbb{R}$, define the finite convolution of u and v by

$$(u * v)(n) := \sum_{j=0}^{n} u(n-j)v(j), n \in \mathbb{N}_0. \tag{1.1}$$

As with the continuous version, definition (1.1) creates a *nonlocal* operator in the sense that the map $n \mapsto (u * v)(n)$ depends not only on the values of u and v at n but on all their values on the set $\{0, 1, \dots, n\}$.

A natural question is as follows: given the nonlocal nature of the convolution operator $(u, v) \mapsto u * v$, what sorts of qualitative information can be gleaned about u and v from knowledge of the convolution? More precisely, if, say, we know something about $(u * v)(n)$ together with some information about one of the sequences in the convolution, can we then deduce information about the other sequence in the convolution—and, if so, what information? This question is related to questions posed in the 1960s by Ehrenpreis [5]—namely, given a and v, what u satisfy $a * u = v$? In this paper, we consider this general question, but in the case where $*$ is a discrete convolution as in (1.1) above and, moreover, where we consider a convolutional inequality

$$a * u \geq b,$$

for a and b given sequences selected in a special way to be detailed momentarily. We are interested in deducing qualitative information about the factors in the convolution (e.g., positivity and monotonicity).

With this general context in mind, let us now explain more precisely what types of discrete convolutions we will be analyzing in this paper. In order to properly fix the broader context, let us recall that the first-order forward (or delta) difference of u, denoted Δu, is the operator $\Delta : s(\mathbb{N}_a; \mathbb{R}) \to s(\mathbb{N}_a; \mathbb{R})$ defined pointwise by

$$(\Delta u)(n) := u(n+1) - u(n).$$

One obvious property of the forward difference operator is that it is strongly connected to the monotonicity of the function u—that is, if $(\Delta u)(n) \geq 0$ for each $n \in \mathbb{N}_a$, then evidently u must be a monotone increasing sequence on $\mathbb{N}_a$. A direct consequence of this relationship is that if $(\Delta u)(n) < 0$, then u cannot be monotone increasing at n. Thus, there is a clear relationship between the sign of $(\Delta u)(n)$ and the monotone behavior of u, summarized as follows.

$$\begin{aligned} (\Delta u)(n) > 0 &\iff u \text{ is increasing at } n \\ (\Delta u)(n) < 0 &\iff u \text{ is decreasing at } n. \end{aligned} \tag{1.2}$$

Fundamentally, that this relationship holds is a consequence of the *local* nature of the operator Δ.

A natural question, then, is what happens if in (1.2) we replace the local operator $(\Delta u)(n)$ by something involving the nonlocal convolution operator $(u * v)(n)$ for some specific choice of v. If (see Section 2 for more details) we set

$$k^{\alpha}(j) := \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)}, j \in \mathbb{N}_0,$$

where $\alpha \in \mathbb{R}$ is such that $m - 1 < \alpha < m$ for a positive integer m, then

$$(\Delta^{\alpha} u)(n) := \left(\Delta^m \circ \left(k^{m-\alpha} * u\right)\right)(n) = \Delta^m \left[\sum_{j=0}^{n} \frac{\Gamma(\alpha - m + n - j)}{\Gamma(\alpha - m)\Gamma(n - j + 1)} u(j)\right]$$

is known as the α-th order fractional difference of u. This fractional difference is, therefore, a nonlocal operator expressible as a convolution-type operator. Indeed, we notice at once that the operator Δ^{α} is nonlocal since it involves a linear combination of the values $\{u(k)\}_{k=0}^{n+m}$. While there are many different "fractional difference operators," each of them is nonlocal, the nonlocality being the essential property of "fractional" operators—compare [6]. Part of the practical importance of these nonlocal operators is that they are able to retain information about the "memory" of the dynamics where the operator acts. For additional fundamental theoretical results regarding the discrete fractional difference and sum operators, one can consult papers by Abadias, Lizama, Miana, and Velasco [7], Atici and Eloe [8, 9], Holm [10], Lizama [11], Lizama and Murillo-Arcila [12], or Lizama, Murillo-Arcila, and Peris [13].

So, keeping in mind that, when $\alpha = 1$ (i.e., the non-convolutional and, thus, local setting), we obtain a very nice geometrical relationship summarized by (1.2), there is a growing literature trying to capture optimal conditions on the initial data of u that are able to determine the geometrical character of u (e.g., positivity, monotonicity, and convexity, among others) under the hypothesis $(\Delta^{\alpha} u)(n) \geq 0$, which as $(\Delta^{\alpha} u)(n) = (\Delta^m \circ (k^{m-\alpha} * u))(n)$, is, in fact, a convolutional inequality. This is a relevant problem because it constitutes an important ingredient in determining the same qualitative property for solutions of nonlinear problems in the form:

$$(\Delta^{\alpha}u)(n) = f(n, u(n)), \ n \in \mathbb{N},$$

under the natural hypothesis $f(n,x) \geq 0$. Obviously, the answer to this problem covers many nonlinear systems. However, it is much less obvious what happens in cases where f is not necessarily positive. For example, for the nonlinear equation

$$(\Delta^{\alpha}u)(n) = \sin(u(n)) \ \ n \in \mathbb{N},$$

we wonder if the non-local character of the fractional-order discrete operator Δ^{α} could capture the positivity or monotonicity of u in any extent.

As described momentarily, our main goal in this paper is to try to understand at a deep level the conditions under which a bound of the form

$$(\Delta^{\alpha}u)(n) > -M,$$

for some $M > 0$, implies that u is either positive or monotone increasing. In other words, at a conceptual level, we are looking to discover theorems that have the following general form:

$$\begin{cases} (\Delta^{\alpha}u)(n) \geq -M \\ \text{Some Auxiliary Conditions} \end{cases} \implies u \text{ is positive and/or monotone,}$$

where $M > 0$. Evidently, such highly anomalous behavior requires an explanation in light of what happens when α is an integer (i.e., the classical setting). Indeed, such a result can only be achieved due to the memory property of the nonlocal operator Δ^{α}. Our results, consequently, further develop the program outlined in Ehrenpreis' work mentioned earlier.

In order to understand the broader importance of this program, let us sketch the relevant literature. The first investigation of the relationship between the monotonicity of u at n and the associated sign of $(\Delta^{\alpha}u)(n)$ was conducted by Dahal and Goodrich [14], which was then subsequently followed by an analogous study of convexity by Goodrich [15]. Numerous related results and improvements then quickly followed, and these include contributions by Abdeljawad and Abdalla [16], Abdeljawad and Baleanu [17, 18], Atici and Uyanik [19], Bravo, Lizama, and Rueda [20], Du, Jia, Erbe, and Peterson [21], Goodrich and Jonnalagadda [22], Jia, Erbe, and Peterson [23–26], Liu, Du, Anderson, and Jia [27], Mohammed, Abdeljawad, and Hamasalh [28, 29], Suwan, Abdeljawad, and Jarad [30], Suwan, Owies, and Abdeljawad [31, 32], and Suwan, Owies, Abussa, and Abdeljawad [33]. We note that each of these papers investigated *non-sequential fractional operators*, by which we mean a single fractional difference. By contrast, a different direction that has been pursued is to consider *sequential fractional operators* (see [34]), by which we mean a composition of fractional differences such as $\Delta^{\beta} \circ \Delta^{\alpha}$ for suitable choices of the parameters α and β. Some representative papers in this direction have been written by Dahal and Goodrich [35], Goodrich [36], and Goodrich and Muellner [37]. For a very comprehensive analysis of some of the available results (in both the sequential and non-sequential settings) together with their application, we mention the recent papers by Goodrich and Lizama [38, 39]. Moreover, for some corresponding analysis in the continuous setting, very scarce though such results are, one may consult either the paper by Diethelm [40] or part of the paper by Goodrich and Lizama [39].

A very important comment regarding the preceding collection of papers is that in each case, the authors assume a *nonnegative* lower bound on the fractional difference. That is to say, in each case, the result (whether positivity, monotonicity, or convexity) has the following general form:

$$\begin{cases} (\Delta^{\alpha}u)(n) \geq M \\ \text{Some Auxiliary Conditions} \end{cases} \implies u \text{ is positive and/or monotone and/or convex,}$$

where $M \geq 0$. Thus, the condition $(\Delta^{\alpha}u)(n) \geq M$ is a nonnegative lower bound on the fractional difference. Of course, this is a preeminently reasonable condition. Indeed, in the non-fractional case and as discussed above, if one wants u to be increasing, then the condition $(\Delta u)(n) \geq 0$ is imposed—not the condition $(\Delta u)(n) \leq 0$, which would imply that u is *non*-increasing.

However, in a series of very recent papers first initiated by Goodrich, Lyons, and Velcsov [41] and then expanded upon by Dahal and Goodrich [42, 43] and Goodrich, Lyons, Scapellato, and Velcsov [44], the possibility of either the positivity, monotonicity, or convexity of u being implied even when a suitable fractional-order difference of u is ***negative*** has been explored—that is, the so-called "negative lower bound" case. As was mentioned at the beginning of this section, this sort of behavior absolutely cannot occur in the integer-order setting. But the fractional setting turns out to be a completely different situation.

Our goal in this paper, then, is to make a comprehensive and complete analysis of when $(\Delta^\alpha u)(n) > -M$ can possibly imply either the positivity or monotonicity of u. While the aforementioned papers [41–44] have made some important strides toward achieving this understanding, the results of these papers leave some important gaps. For example, no analysis of the sharpness of the results was considered in those papers; here, we fill this crucial gap. Moreover, much of the analysis in those papers was supplemented by numerical simulations, for which there was not much intuition. That is to say, certain relationships were suggested by means of numerical methods, but these relationships were unable to be stated precisely or proved in any meaningful way; here, again, we fill these gaps by describing in a very precise way why the particular numerical curiosities observed in those papers exist. Consequently, this paper will provide for a comprehensive answer of some of the open questions raised in [41–44].

We would also like to emphasize that our results are very general, being as they apply to a variety of discrete fractional operators (e.g., both the delta and nabla operators are captured by our results). One way in which we are able to accomplish this is by using the notion of "transference" that we introduced in [38, Theorem 4.1]. This is the idea, roughly speaking, that one can transfer a result from the fractional difference Δ^α to a result for another type of fractional difference. Here (see Section 3), we prove a version of this idea that relates results for the fractional difference Δ^α to the fractional nabla difference ∇_a^α of Riemann-Liouville type. The principle is succinctly stated by noting that the following diagram commutes.

$$\begin{array}{ccc} s(\mathbb{N}_0;\mathbb{R}) & \overset{\Delta^\alpha}{\to} & s(\mathbb{N}_0;\mathbb{R}) \\ \uparrow \tau_a & & \uparrow \tau_{a+m} \\ s(\mathbb{N}_a;\mathbb{R}) & \overset{\nabla_a^\alpha}{\to} & s(\mathbb{N}_{a+m};\mathbb{R}) \end{array}$$

Note that in the commutative diagram, τ_a is the translation operator—see Section 2 for additional details.

Finally, we would like to point out that the aforementioned results have some interesting implications when considered in the context of applications. For example, the discrete fractional calculus has been used to model tumor growth [45–48], cryptographic techniques [49], and image restoration [50]. Connections have also been demonstrated to entropy calculations [51, 52], inequality theory [53], numerical analysis [39, 54], partial differential equations [55], and the theory of special functions [56]. As is well known to any calculus student, that the derivative (or difference) detects whether a function is increasing or decreasing is essential in a variety of applications. Since this clean relationship is lost in the fractional setting, this has important implications for the use of fractional calculus in applications, and so, a thorough understanding of these connections seems to be essential. Moreover, it also has substantial implications for detecting extrema with fractional differences. While extrema can be easily detected with the first-order difference, it would seem that developing an analogous theory with respect to the fractional difference is likely a far more difficult endeavor. In fact, we are aware of only paper in this direction (see [57]), and this is likely telling. Thus, we hope that the present results will continue to develop the qualitative theory of nonlocal discrete operators and that this will be useful in their application.

2 | PRELIMINARIES

In this section, we recall some preliminary results regarding both difference operators and the discrete fractional calculus. We refer the reader to the recent textbook by Goodrich and Peterson [58] for a comprehensive account of the discrete fractional calculus and numerous additional results well beyond what is presented here; we will also recall some relevant results from [38]. Recall throughout that the finite convolution of $u, v \in s(\mathbb{N}_0;\mathbb{R})$, the pointwise value of which is denoted $(u * v)(n)$, is defined by

$$(u * v)(n) := \sum_{j=0}^{n} u(n-j)v(j), n \in \mathbb{N}_0.$$

We first recall the function that forms the kernel of the fractional difference and sum.

Definition 2.1. For any $\alpha \in \mathbb{C}$, define the function $k^\alpha : \mathbb{N}_0 \to \mathbb{R}$ by

$$k^\alpha(0) := 1 \text{ and } k^\alpha(n) := \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}, n \in \mathbb{N}_1.$$

Remark 2.2. Note that in case $\alpha \in \mathbb{C}\backslash\{\dots,-2,-1,0\}$, the function k^α in Definition 2.1 reduces to

$$k^\alpha(j) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)}, j \in \mathbb{N}_0.$$

Next, we recall the versions of the fractional sum and difference that we use in this paper.

Definition 2.3. For any $\alpha > 0$ and $u \in s(\mathbb{N}_0;\mathbb{R})$, the α-th order fractional sum is the operator $\Delta^{-\alpha} : s(\mathbb{N}_0;\mathbb{R}) \to s(\mathbb{N}_0;\mathbb{R})$ defined by

$$(\Delta^{-\alpha}u)(n) := (k^\alpha * u)(n).$$

And for any $\alpha > 0$ and $u \in s(\mathbb{N}_0;\mathbb{R})$, the α-th order fractional difference is the operator $\Delta^\alpha : s(\mathbb{N}_0;\mathbb{R}) \to (\mathbb{N}_0;\mathbb{R})$ defined by

$$(\Delta^\alpha u)(n) := \left(\Delta^m \circ \Delta^{\alpha-m}u\right)(n),$$

where $m \in \mathbb{N}$ satisfies $m-1 < \alpha < m$.

The following translation operator will be helpful in much of what follows.

Definition 2.4. For any given $a \in \mathbb{R}$, the translation operator $\tau_a : s(\mathbb{N}_a;\mathbb{R}) \to (\mathbb{N}_0;\mathbb{R})$ is defined by

$$(\tau_a u)(n) := u(a+n).$$

Remark 2.5. In many studies on discrete fractional operators (see, for example, Atici and Eloe [9]), the operator $\Delta_a^\alpha : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_{a+1-\nu};\mathbb{R})$, which is defined by

$$(\Delta_a^\nu u)(t) := \frac{\Gamma(1-\nu)}{\Gamma(-\nu)} \sum_{s=0}^{n+1} \frac{1}{n-s+1} k^{1-\nu}(n-s)u(a+s), t \in \mathbb{N}_{a+1-\nu},$$

is used. However, we note that by [38, §4], it is known that the operator Δ^α and the Δ_a^α are related by the simple formula

$$\tau_{a+1-\alpha} \circ \Delta_a^\alpha u \equiv \Delta^\alpha \circ \tau_a u.$$

This is the notion of "transference" whereby theorems proved about the operator Δ^α can be transferred back to the (somewhat more complicated) operator Δ_a^α. In fact, we will make frequent use of this idea in the subsequent sections of this paper. (See Goodrich and Lizama [38] for additional details on this notion of "transference.")

We next recall some facts about the corresponding nabla (or backwards) difference operator—note that some additional fundamental results on the nabla fractional calculus can be found in [8, 59, 60]. Recall that given $f \in s(\mathbb{N}_a;\mathbb{R})$, where $a \in \mathbb{R}$, then for $\mu > 0$, we define the nabla sum operator $\nabla_a^{-\mu} : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_a;\mathbb{R})$ by

$$\left(\nabla_a^{-\mu} f\right)(t) = \frac{1}{\Gamma(\mu)} \sum_{s=a}^{t} (t-s+1)^{\overline{\mu-1}} f(s),\ t \in \mathbb{N}_a,$$

with $t^{\overline{\mu}} := \frac{\Gamma(t+\mu)}{\Gamma(t)}$. Given $m-1 < \mu \le m$, the nabla difference operator $\nabla_a^\mu : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_{a+m};\mathbb{R})$ is defined as

$$\left(\nabla_a^\mu f\right)(t) := \nabla_a^m \circ \nabla_a^{-(m-\nu)} f(t),\ t \in \mathbb{N}_{a+m},$$

where $\nabla_a : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_{a+1};\mathbb{R})$ is defined as $(\nabla_a f)(t) = f(t) - f(t-1)$, $t \in \mathbb{N}_{a+1}$, and inductively $\nabla_a^m = \nabla_a \circ \nabla_a^{m-1}$, $m \in \mathbb{N}_2$. Note that $\nabla_a^m : s(\mathbb{N}_a;\mathbb{R}) \to s(\mathbb{N}_{a+m};\mathbb{R})$, and, hence, the definition of the composition of operators $\nabla_a^\mu := \nabla_a^m \circ \nabla_a^{-(m-\nu)}$ make sense.

Next, we recall some facts about the kernel k^α defined earlier in this section. The following results may be found, respectively, in [38, Proposition 2.9], [38, Lemma 3.2], and [38, Proposition 3.1]. These lemmata will be especially useful in Sections 4 and 5.

Lemma 2.6. *The following properties hold:*

(i) *For any* $a \in s(\mathbb{N}_0;\mathbb{R})$ *we have* $\Delta(k * a)(n) = a(n+1)$.
(ii) *For any* $0 < \alpha < 2$ *and* $b \in s(\mathbb{N}_0;\mathbb{R})$, *we have* $(\Delta \circ \Delta^\alpha b)(n) = (\Delta^{\alpha+1} b)(n)$.
(iii) *For any* $0 < \alpha < 1$ *and* $b \in s(\mathbb{N}_0;\mathbb{R})$, *we have*

$$(\Delta^\alpha \circ \Delta b)(n) = (\Delta \circ \Delta^\alpha b)(n) - \Delta k^{1-\alpha}(n+1) b(0).$$

(iv) *For any* $a, b \in s(\mathbb{N}_0;\mathbb{R})$, *we have* $\Delta(a * b)(n) = (\Delta a * b)(n) + b(n+1)a(0)$ *and*

$$(\Delta a * b)(n) = (a * \Delta b)(n) + a(n+1)b(0) - a(0)b(n+1).$$

Lemma 2.7. *For any* $\alpha > 0$, *the following identity hold:*

$$\left(\Delta^k k^\alpha\right)(n) = \left(\prod_{j=1}^{k} \frac{\alpha - j}{n + j}\right) k^\alpha(n), \qquad k \in \mathbb{N}.$$

Lemma 2.8. *The following properties hold:*

(i) *For* $\alpha > 0$, $k^\alpha(n+1) = \frac{\alpha+n}{n+1} k^\alpha(n)$, $n \in \mathbb{N}_0$.
(ii) *For* $\alpha > 0$, $k^\alpha(n) > 0$, $n \in \mathbb{N}_0$.
(iii) *For* $0 < \alpha < 1$, $k^\alpha(n)$ *is decreasing and* $k^\alpha(n) \to 0$ *as* $n \to \infty$.
(iv) *For* $\alpha > 1$, $k^\alpha(n)$ *is increasing.*
(v) *The following generation formula holds:*

$$\sum_{j=0}^{\infty} k^\alpha(j) z^j = \frac{1}{(1-z)^\alpha} \text{ for all } z \in \mathbb{C},\ |z| < 1.$$

(vi) *For* $\alpha, \beta \in \mathbb{C}$, *we have the semigroup property:*

$$k^{\alpha+\beta}(n) = \sum_{j=0}^{n} k^\alpha(n-j) k^\beta(j) =: (k^\alpha * k^\beta)(n),\ n \in \mathbb{N}_0.$$

Finally, the following lemma recalls a property of finite convolutions. It will prove useful in Section 5—specifically, in the proof of Theorem 5.11. This result also can be found in [38, Lemma 2.3].

Lemma 2.9. *Let* $f, g \in s(\mathbb{N}_0;\mathbb{R})$ *be sequences. Then for each* $p = 1, 2, \ldots$, *we have*

$$(f * \tau_p g)(n) = \tau_p(f * g)(n) - \sum_{j=0}^{p-1} \tau_p f(n-j) g(j).$$

3 | A TRANSFERENCE PRINCIPLE FOR THE NABLA OPERATOR

In this brief section, we present a transference principle for the nabla operator. The main result of this section, Theorem 3.1, can be summarized in the following commutative diagram.

$$\begin{array}{ccc} s(\mathbb{N}_0;\mathbb{R}) & \overset{\Delta^\alpha}{\to} & s(\mathbb{N}_0;\mathbb{R}) \\ \uparrow \tau_a & & \uparrow \tau_{a+m} \\ s(\mathbb{N}_a;\mathbb{R}) & \overset{\nabla_a^\alpha}{\to} & s(\mathbb{N}_{a+m};\mathbb{R}) \end{array}$$

In other words, by using the shift operator, τ_a, we are able to deduce results for the fractional operator Δ^α and then *transfer* such results to the nabla Riemann-Liouville fractional operator ∇_a^α by means of Theorem 3.1. This allows us to work mainly within the convolution setting described in Section 2, and this has significant analytical advantages as will be seen in the following sections. The result of Theorem 3.1 may be compared with that of [38, Theorem 4.1].

Theorem 3.1. *For each $a \in \mathbb{R}$ and $0 < \alpha < 1$, we have*

$$(\tau_{a+1} \circ \nabla_a^\alpha f)(n) = (\Delta^\alpha \circ \tau_a f)(n), \text{ where } n \in \mathbb{N}_0 \text{ and } f \in s(\mathbb{N}_a; \mathbb{R}).$$

Proof. For each $t := a + n \in \mathbb{N}_a$, $n \in \mathbb{N}_0$, we have by definition

$$\begin{aligned}(\nabla_a^{-\alpha} f)(a+n) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{s=a+n} (a+n-s+1)^{\overline{\alpha-1}} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{n} (n-p+1)^{\overline{\alpha-1}} f(a+p) \\ &= \sum_{p=0}^{n} \frac{\Gamma(n-p+\alpha)}{\Gamma(\alpha)\Gamma(n-p+1)} f(a+p) = \sum_{p=0}^{n} k^\alpha(n-p)\tau_a f(p).\end{aligned}$$

Therefore, we obtain the identity

$$(\tau_a \circ \nabla_a^{-\alpha} f)(n) = (k^\alpha * \tau_a f)(n), \ n \in \mathbb{N}_0.$$

Then, for each $n \in \mathbb{N}_0$, we have

$$\begin{aligned}(\tau_{a+1} \circ \nabla_a^\alpha f)(n) &= (\nabla_a^\alpha f)(n+a+1) \\ &= (\nabla_a \circ \nabla_a^{-(1-\alpha)} f)(n+a+1) = \left(\nabla_a^{-(1-\alpha)} f\right)(n+a+1) - \left(\nabla_a^{-(1-\alpha)} f\right)(n+a) \\ &= \left(\tau_a \circ \nabla_a^{-(1-\alpha)} f\right)(n+1) - \left(\tau_a \circ \nabla_a^{-(1-\alpha)} f\right)(n) \\ &= \left(k^{-(1-\alpha)} * \tau_a f\right)(n+1) - \left(k^{-(1-\alpha)} * \tau_a f\right)(n) \\ &= \left(\Delta^{-(1-\alpha)} \tau_a f\right)(n+1) - \left(\Delta^{-(1-\alpha)} \tau_a f\right)(n) = \left(\Delta \circ \Delta^{-(1-\alpha)} \tau_a f\right)(n) \\ &= (\Delta^\alpha \circ \tau_a f)(n),\end{aligned}$$

proving the theorem. □

Remark 3.2. In the general case, that is, $m - 1 < \alpha < m$, $m \in \mathbb{N}$, we have:

$$(\tau_{a+m} \circ \nabla_a^\alpha f)(n) = (\Delta^\alpha \circ \tau_a f)(n), \ n \in \mathbb{N}_0, \ f \in s(\mathbb{N}_a; \mathbb{R}).$$

4 | POSITIVITY

Let $0 < \alpha < 1$. In [38, Theorem 5.4], it was proved that if $(\Delta^\alpha u)(n) \geq 0$ and $u(0) \geq 0$, then $u(n) \geq 0$ for all $n \in \mathbb{N}$ (and, in fact, that u is α-monotone increasing, too). The first property follows easily from the identity [38, p. 554, line 17]:

$$u(n+1) = (k^\alpha * \Delta^\alpha u)(n) + k^\alpha(n+1)u(0). \tag{4.1}$$

However, the converse is not true without additional conditions such as, for instance, that u is increasing [61, Remark 4.5] (which does not imply that u is α-monotone). Therefore, a complete characterization is missing and remains an open problem.

Suppose now that $(\Delta^\alpha u)(n) \leq 0$ with $u(0) \geq 0$. The identity (4.1) proves that $u(n+1) \leq k^\alpha(n+1)u(0)$, i.e., $k^\alpha(n+1)u(0)$ is a upper bound. But this does not exclude the possibility that $u(n) \geq 0$ as the following example shows. Indeed, Example 4.1 shows that $u(n) > 0$ can hold even if $(\Delta^\alpha u)(n) < 0$, which is quite surprising.

Example 4.1. Let $u(n) := k^{\beta}(n)$, $n \in \mathbb{N}_0$ where $0 < \beta < \alpha < 1$. Then

(i) $(\Delta^{\alpha} u)(n) \le 0$;
(ii) $u(0) > 0$; and
(iii) $u(n) \ge 0$ for all $n \in \mathbb{N}$.

Indeed, note that $n \to k^{1+\beta-\alpha}(n)$ is decreasing because $0 < 1+\beta-\alpha < 1$. Then we have

$$(\Delta^{\alpha} k^{\beta})(n) = (k^{1-\alpha} * k^{\beta})(n+1) - (k^{1-\alpha} * k^{\beta})(n) = \left(\Delta k^{1+\beta-\alpha}\right)(n) \le 0.$$

Moreover, it is clear that $u(0) = k^{\beta}(0) = 1$ and $u(n) = k^{\beta}(n) \ge 0$ for all $n \in \mathbb{N}$. Thus, we conclude that u is always nonnegative in spite of its α-th order fractional difference being always nonpositive.

This example shows an unexpected property of fractional difference operators. Therefore, we wish to characterize more precisely the conditions under which u can be positive in spite of $(\Delta^{\alpha} u)(n) < 0$. The following theorem gives us the answer.

Theorem 4.2. *Let $0 < \alpha < 1$. Suppose that there exists $\varepsilon \ge 0$ such that $(\Delta^{\alpha} u)(n) \ge -\varepsilon u(0)$ for all $n \in \mathbb{N}$, and $u(0) > 0$. Then $u(j) \ge 0$ for all $j = 1, 2, \dots, n_0$ if $0 \le \varepsilon \le \frac{\alpha}{n_0+1}$.*

Proof. Note that $k^1(n) \equiv 1$. From the inequality of the hypothesis, and after convolution with the positive sequence $n \mapsto k^{\alpha}(n)$, we obtain the inequality

$$(k^{\alpha} * \Delta^{\alpha} u)(n) \ge -\varepsilon (k^{\alpha} * k^1)(n) u(0) = -\varepsilon k^{1+\alpha}(n) u(0), \ \forall n \in \mathbb{N}_0.$$

Then the identity (4.1) shows

$$u(n+1) \ge k^{\alpha}(n+1) u(0) - \varepsilon k^{1+\alpha}(n) u(0), \ \forall n \in \mathbb{N}_0.$$

Since $u(0) > 0$, in order to have positivity it is enough to impose the condition $k^{\alpha}(n+1) \ge \varepsilon k^{1+\alpha}(n)$ in the above inequality. We note that the identity $x\Gamma(x) = \Gamma(x+1)$ implies

$$\frac{k^{\alpha}(n+1)}{k^{1+\alpha}(n)} = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha)(n+1)} = \frac{\alpha}{n+1}, \ n \in \mathbb{N}_0.$$

So, we obtain:

$$k^{\alpha}(n+1) - \varepsilon k^{1+\alpha}(n) \ge 0 \ \text{ if and only if } \ 0 \le \varepsilon \le \frac{\alpha}{n+1}.$$

Finally, because $\frac{1}{n+1}$ is decreasing, we obtain the desired conclusion. □

Note in Theorem 4.2 that the condition $0 \le \varepsilon \le \frac{\alpha}{n+1}$ implies that $u(j) \ge 0$ for all $j \in \mathbb{N}$ only if $\varepsilon = 0$, which recovers [38, Theorem 5.4]. Using the transference principle [38, Theorem 4.1], we find the following result.

Corollary 4.3. *Let $0 < \mu < 1$, $a \in \mathbb{R}$, $\varepsilon \ge 0$ and $f \in s(\mathbb{N}_a; \mathbb{R})$. Suppose that*

(i) $(\Delta_a^{\mu} f)(t) \ge -\varepsilon f(a)$ for all $t \in \mathbb{N}_{a+1-\mu}$;
(ii) $f(a) \ge 0$; and
(iii) there exists $n_0 \in \mathbb{N}$ such that $0 \le \varepsilon \le \frac{\alpha}{n_0+1}$.

Then $f(a+j) \ge 0$ for all $j = 1, 2, \dots, n_0$.

Proof. Define $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = f(a+n) = (\tau_a f)(n)$. Then (i) and the transference principle imply that

$$(\Delta^{\mu} u)(n) = (\tau_{a+1-\mu} \circ \Delta_a^{\mu} \circ \tau_{-a} u)(n) = (\Delta_a^{\mu} f)(n+a-1-\mu) = \left(\Delta_a^{\mu} f\right)(t) \ge -\varepsilon f(a) = -\varepsilon u(0),$$

for all $t := n+a-1-\mu \in \mathbb{N}_{a+1-\mu}$. Since also $u(0) = f(a) \ge 0$, and in view of (iii), Theorem 4.2 implies $u(j) = f(a+j) \ge 0$ for all $j = 1, 2, \dots, n_0$. □

We can easily deduce the corresponding result for the nabla operator using the transference principle Theorem 3.1.

Corollary 4.4. *Let* $0 < \mu < 1$, $a \in \mathbb{R}$, *and* $f \in s(\mathbb{N}_a;\mathbb{R})$. *Suppose that*

(i) there exists $\varepsilon \geq 0$ *such that* $(\nabla_a^{\mu} f)(t) \geq -\varepsilon f(a)$ *for all* $t \in \mathbb{N}_{a+1}$;
(ii) $f(a) \geq 0$; *and*
(iii) there exists $n_0 \in \mathbb{N}$ *such that* $0 \leq \varepsilon \leq \frac{\alpha}{n_0+1}$.

Then $f(a+j) \geq 0$ *for all* $j = 1, 2, \ldots, n_0$.

Proof. Define $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = f(a+n) = (\tau_a f)(n)$. Then (i) and the transference principle for the nabla operator (Theorem 3.1) imply that

$$(\Delta^{\mu} u)(n) = \left(\tau_{a+1} \circ \nabla_a^{\mu} \circ \tau_{-a} u\right)(n) = \left(\nabla_a^{\mu} f\right)(n+a+1) = \left(\nabla_a^{\mu} f\right)(t) \geq -\varepsilon f(a) = -\varepsilon u(0),$$

for all $t := n+a+1 \in \mathbb{N}_{a+1}$. Since also $u(0) = f(a) \geq 0$, and in view of (iii), Theorem 4.2 implies $u(j) = f(a+j) \geq 0$ for all $j = 1, 2, \ldots, n_0$. □

Remark 4.5. Regarding Corollary 4.4, it is interesting to note that the result is no better in the nabla case than in the delta case—compare Corollary 4.3. This is perhaps mildly surprising because it has been shown that in the case of a zero lower bound, the nabla case behaves rather better than the delta case—see [24].

In case of sequential operators, we have from [38, Theorem 5.8]: If $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \geq \frac{\beta}{2}(1-\beta)u(0)$ and $u(0) \geq 0$, then u is positive. We upgrade this result in the following theorem.

Theorem 4.6. *Let* $0 < \alpha < 1$, $0 < \beta < 1$ *where* $0 < \alpha + \beta < 1$. *Suppose that*

(1) there exists $\varepsilon \geq 0$ *such that* $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \geq -\varepsilon u(0)$, $\forall n \in \mathbb{N}_0$;
(2) $u(0) \geq 0$; *and*
(3) there exists $n_0 \in \mathbb{N}_0$ *such that* $0 \leq \varepsilon \leq \frac{\alpha}{n_0+1} \frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)} \frac{\Gamma(1+\alpha+n_0)}{\Gamma(1+\alpha+\beta+n_0)}$.

Then $u(j) \geq 0$ *for all* $j = 1, 2, \ldots, n_0$.

Proof. We have the identity [38, Theorem 5.7]

$$\left(\Delta^{\alpha+\beta} u\right)(n+1) = (\Delta^{\beta} \circ \Delta^{\alpha} u)(n) + \left(\Delta k^{1-\beta}\right)(n+1)u(0), \ \forall n \in \mathbb{N}_0.$$

Hence,

$$\left(\Delta^{\alpha+\beta} u\right)(n+1) \geq -\varepsilon u(0) + \left(\Delta k^{1-\beta}\right)(n+1)u(0), \ \forall n \in \mathbb{N}_0$$

or, equivalently,

$$\left(\Delta^{\alpha+\beta} u\right)(n) \geq -\varepsilon u(0) + \left(\Delta k^{1-\beta}\right)(n)u(0), \ \forall n \in \mathbb{N}.$$

Convolving with $k^{\alpha+\beta}$, we obtain

$$\left(k^{\alpha+\beta} * (\Delta^{\alpha+\beta} u)\right)(n) \geq -\varepsilon k^{1+\alpha+\beta}(n)u(0) + \left(k^{\alpha+\beta} * \Delta k^{1-\beta}\right)(n)u(0),$$

for each $n \in \mathbb{N}$. Since $0 < \alpha + \beta < 1$, from (4.1), we obtain

$$u(n+1) = \left(k^{\alpha+\beta} * \Delta^{\alpha+\beta} u\right)(n) + k^{\alpha+\beta}(n+1)u(0), \ \forall n \in \mathbb{N}_0.$$

Therefore,

$$u(n+1) \geq -\varepsilon k^{1+\alpha+\beta}(n)u(0) + k^{\alpha+\beta}(n+1)u(0) + \left(k^{\alpha+\beta} * \Delta k^{1-\beta}\right)(n)u(0), \ \forall n \in \mathbb{N}_0,$$

where an application of Lemma 2.6 shows that

$$\left(k^{\alpha+\beta} * \Delta k^{1-\beta}\right)(n) = \Delta\left(k^{1-\beta} * k^{\alpha+\beta}\right)(n) - k^{\alpha+\beta}(n+1)k^{1-\beta}(0) = \left(\Delta k^{1+\alpha}\right)(n) - k^{\alpha+\beta}(n+1),$$

for each $n \in \mathbb{N}_0$, which implies that

$$u(n+1) \geq -\varepsilon k^{1+\alpha+\beta}(n)u(0) + \left(\Delta k^{1+\alpha}\right)(n)u(0), \ \forall n \in \mathbb{N}_0.$$

Using Lemma 2.7, we obtain, equivalently,

$$u(n+1) \geq \left[-\varepsilon k^{1+\alpha+\beta}(n) + \frac{\alpha}{n+1}k^{1+\alpha}(n)\right]u(0), \ \forall n \in \mathbb{N}_0.$$

Therefore, $u(n_0+1) \geq 0$ provided that

$$\varepsilon < \frac{\alpha}{n_0+1}\frac{k^{1+\alpha}(n_0)}{k^{1+\alpha+\beta}(n_0)} = \frac{\alpha}{n_0+1}\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)}\frac{\Gamma(1+\alpha+n_0)}{\Gamma(1+\alpha+\beta+n_0)}.$$

We finally observe that the sequence $\left\{\frac{k^{1+\alpha}(n)}{k^{1+\alpha+\beta}(n)}\right\}_{n=0}^{\infty}$ is decreasing, which allows us to complete the proof. □

We remark that in case $\beta = 0$, the above theorem reduces to Theorem 4.2. We now observe that the particular case $\varepsilon = 0$ produces the following result, which improves [38, Theorem 5.8].

Theorem 4.7. *Let $0 < \alpha < 1$, $0 < \beta < 1$ where $0 < \alpha+\beta < 1$. Suppose that $(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) \geq 0$, $\forall n \in \mathbb{N}_0$, and $u(0) \geq 0$. Then u is positive on $\mathbb{N}_0$.*

We also obtain the following corollary.

Corollary 4.8. *Let $0 < \nu < 1$, $0 < \mu < 1$, $a \in \mathbb{R}$ and $f \in s(\mathbb{N}_a;\mathbb{R})$. Suppose that*

(i) there exists $\varepsilon \geq 0$ such that $(\Delta^{\nu}_{1+a-\mu} \circ \Delta^{\mu}_{a}f)(t) \geq -\varepsilon f(a)$ for all $t \in \mathbb{N}_{2+a-\mu-\nu}$;
(ii) $f(a) \geq 0$; and
(iii) there exists $n_0 \in \mathbb{N}_0$ such that $0 \leq \varepsilon < \frac{\mu}{n_0+1}\frac{\Gamma(1+\mu+\nu)}{\Gamma(1+\mu)}\frac{\Gamma(1+\mu+n_0)}{\Gamma(1+\mu+\nu+n_0)}$.

Then $f(a+j) \geq 0$ for all $j = 1, 2, \ldots, n_0$.

Proof. We can follow the proof of [38, Corollary 5.10], and so, we omit the proof. □

The result for the nabla operator reads as follows.

Corollary 4.9. *Let $0 < \nu < 1$, $0 < \mu < 1$, $a \in \mathbb{R}$ and $f \in s(\mathbb{N}_a;\mathbb{R})$. Suppose that*

(i) there exists $\varepsilon \geq 0$ such that $\left(\nabla^{\nu}_{a+1} \circ \nabla^{\mu}_{a}f\right)(t) \geq -\varepsilon f(a)$ for all $t \in \mathbb{N}_{a+2}$;
(ii) $f(a) \geq 0$; and
(iii) there exists $n_0 \in \mathbb{N}_0$ such that $0 \leq \varepsilon < \frac{\mu}{n_0+1}\frac{\Gamma(1+\mu+\nu)}{\Gamma(1+\mu)}\frac{\Gamma(1+\mu+n_0)}{\Gamma(1+\mu+\nu+n_0)}$.

Then $f(a+j) \geq 0$ for all $j = 1, 2, \ldots, n_0$.

Proof. Define $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = f(a+n) = (\tau_a f)(n)$ and observe that

$$\nabla^{\nu}_{a} \circ \tau_1 = \tau_1 \circ \nabla^{\nu}_{a+1}. \tag{4.2}$$

Indeed, by the transference principle we have $\nabla^{\nu}_{a} = \tau_{-(a+1)} \circ \Delta^{\nu} \circ \tau_a$. Then $\nabla^{\nu}_{a} \circ \tau_1 = \tau_{-(a+1)} \circ \Delta^{\nu} \circ \tau_{a+1}$ and also $\tau_1 \circ \nabla^{\nu}_{a+1} = \tau_1 \circ \tau_{-(a+2)} \circ \Delta^{\nu} \circ \tau_{a+1} = \tau_{-(a+1)} \circ \Delta^{\nu} \circ \tau_{a+1}$, proving the claim.

Using (i), Theorem 3.1, and identity (4.2), we obtain

$$\begin{aligned}(\Delta^{\nu} \circ \Delta^{\mu}u)(n) &= \left(\tau_{a+1} \circ \nabla^{\nu}_{a} \circ \tau_{-a} \circ \tau_{a+1} \circ \nabla^{\mu}_{a} \circ \tau_{-a}u\right)(n) = \left(\tau_{a+1} \circ \nabla^{\nu}_{a} \circ \tau_1 \circ \nabla^{\mu}_{a}f\right)(n)\\ &= \left(\nabla^{\nu}_{a} \circ \tau_1 \circ \nabla^{\mu}_{a}f\right)(n+a+1) = \left(\tau_1 \circ \nabla^{\nu}_{a+1} \circ \nabla^{\mu}_{a}f\right)(n+a+1)\\ &= \left(\nabla^{\nu}_{a+1} \circ \nabla^{\mu}_{a}f\right)(n+a+2) = \left(\nabla^{\nu}_{a+1} \circ \nabla^{\mu}_{a}f\right)(t)\\ &\geq -\varepsilon f(a) = -\varepsilon u(0),\end{aligned}$$

for all $t := n + a + 2 \in \mathbb{N}_{a+2}$. Since also $u(0) = f(a) \geq 0$, and in view of (iii), Theorem 4.6 implies $f(a + j) = u(j) \geq 0$ for all $j = 1, 2, \ldots, n_0$. □

5 | MONOTONICITY

In this section, we consider the case in which either a non-sequential or a sequential fractional difference can be nonpositive and yet a monotonicity-type result can still be recovered. We consider in the first instance a result for non-sequential fractional differences. For convenience in what follows, we will use the notation

$$\mathbb{N}_a^b := \{a, a + 1, \ldots, b\},$$

for any $a, b \in \mathbb{R}$ satisfying $b - a \in \mathbb{N}_0$.

Theorem 5.1. *Let $1 < \alpha < 2$. Suppose that each of the following conditions is satisfied.*

1. *There exists $\varepsilon \geq 0$ such that $(\Delta^\alpha u)(n) \geq -\varepsilon u(0)$ for each $n \in \mathbb{N}_0$;*
2. *$(\Delta u)(0) \geq 0$; and*
3. *$u(0) \geq 0$.*

If for some $n_0 \in \mathbb{N}_0$ it holds that

$$0 \leq \varepsilon \leq \frac{(\alpha - 1)(2 - \alpha)}{n_0 + 2},$$

then $(\Delta u)(n) \geq 0$ for each $n \in \mathbb{N}_0^{n_0+1}$—that is, u is finitely monotone.

Proof. As in the proof of Theorem 4.2 we recall that $k^1(n) \equiv 1$. As such, condition (1) can be rewritten as

$$(\Delta^\alpha u)(n) \geq (-\varepsilon u(0))\, k^1(n),$$

for each $n \in \mathbb{N}_0$. Now convolve both sides of inequality (5) with $k^{\alpha-1}$ to obtain

$$\left(k^{\alpha-1} * \Delta^\alpha u\right)(n) \geq (-\varepsilon u(0)) \left(k^{\alpha-1} * k^1\right)(n) = (-\varepsilon u(0))\, k^\alpha(n), \tag{5.1}$$

for each $n \in \mathbb{N}_0$. So, recalling from [38, (6.2)] that

$$(\Delta u)(n + 1) = \left(k^{\alpha-1} * \Delta^\alpha u\right)(n) + k^{\alpha-1}(n + 2)u(0) + k^{\alpha-1}(n + 1)\left[u(1) - \alpha u(0)\right], \tag{5.2}$$

for each $n \in \mathbb{N}_0$, it follows upon combining inequalities (5.1) and (5.2) that

$$\begin{aligned}(\Delta u)(n + 1) &= \left(k^{\alpha-1} * \Delta^\alpha u\right)(n) + k^{\alpha-1}(n + 2)u(0) + k^{\alpha-1}(n + 1)\left[u(1) - \alpha u(0)\right] \\ &\geq (-\,\varepsilon u(0))\, k^\alpha(n) + k^{\alpha-1}(n + 2)u(0) + k^{\alpha-1}(n + 1)\left[u(1) - \alpha u(0)\right] \\ &= (-\,\varepsilon u(0))\, k^\alpha(n) + k^{\alpha-1}(n + 2)u(0) + (1 - \alpha)k^{\alpha-1}(n + 1)u(0) + \underbrace{k^{\alpha-1}(n + 1)\left[u(1) - u(0)\right]}_{\geq 0} \\ &= \left(-\,\varepsilon k^\alpha(n) + k^{\alpha-1}(n + 2) + (1 - \alpha)k^{\alpha-1}(n + 1)\right) u(0),\end{aligned} \tag{5.3}$$

where we have used assumption (2) to deduce that

$$k^{\alpha-1}(n + 1)\left[u(1) - u(0)\right] = ((\Delta u)(0))\, k^{\alpha-1}(n + 1) \geq 0.$$

Consequently, from inequality (5.3), we see that it is sufficient that

$$-\,\varepsilon k^\alpha(n) + k^{\alpha-1}(n + 2) + (1 - \alpha)k^{\alpha-1}(n + 1) \geq 0, n \in \mathbb{N}_0^{n_0}, \tag{5.4}$$

in order for u to be monotone increasing on $\mathbb{N}_0^{n_0}$ for the given $n_0 \in \mathbb{N}_0$ in the statement of the theorem.

To obtain the conclusion of Theorem 5.1 we rewrite the left-hand side of inequality (5.4). To this end note that

$$k^{\alpha-1}(n+2)=\frac{\alpha+n}{n+2}k^{\alpha-1}(n+1)=\frac{(\alpha+n)(\alpha-1+n)}{(n+2)(n+1)}k^{\alpha-1}(n), \tag{5.5}$$

where we repeatedly use Lemma 2.8. Similarly,

$$k^{\alpha-1}(n+1)=\frac{\alpha-1+n}{n+1}k^{\alpha-1}(n). \tag{5.6}$$

At the same time we note that

$$k^{\alpha}(n)=\frac{\alpha+n-1}{\alpha-1}k^{\alpha-1}(n). \tag{5.7}$$

Therefore, putting (5.5)–(5.7) into the left-hand side of (5.4), we obtain that

$$-\varepsilon k^{\alpha}(n)+k^{\alpha-1}(n+2)+(1+\alpha)k^{\alpha-1}(n+1)=\left[-\varepsilon\frac{\alpha+n-1}{\alpha-1}+\frac{(\alpha+n)(\alpha-1+n)}{(n+2)(n+1)}+(1-\alpha)\frac{\alpha-1+n}{n+1}\right]k^{\alpha-1}(n)\geq 0$$

if and only if

$$\begin{aligned}\varepsilon&\leq\frac{(\alpha+n)(\alpha-1+n)+(1-\alpha)(\alpha-1+n)(n+2)}{(n+1)(n+2)}\cdot\frac{\alpha-1}{\alpha+n-1}\\&=\frac{(\alpha+n)(\alpha-1)-(\alpha-1)^2(n+2)}{(n+1)(n+2)}=\frac{(\alpha-1)(2-\alpha)}{n+2}.\end{aligned} \tag{5.8}$$

Furthermore, we notice that

$$n\mapsto\frac{(\alpha-1)(2-\alpha)}{n+2}$$

defines a decreasing sequence in $n\in\mathbb{N}_0$. Therefore, we conclude that if inequality (5.8) holds for $n=n_0$, then inequality (5.8) holds for each $n\in\mathbb{N}_0^{n_0}$, and so, it follows that $(\Delta u)(n)\geq 0$ for each $n\in\mathbb{N}_0^{n_0}$, as claimed. □

Remark 5.2. Notice that

$$\lim_{n\to\infty}\frac{(\alpha-1)(2-\alpha)}{n+2}=0,$$

which unsurprisingly suggests that for very large n_0 we need ε to be very small in order for the theorem to be applicable. In other words, if we want u to increase for many time steps in spite of a negative fractional difference, then Theorem 5.1 suggests that the fractional difference cannot be too negative. Furthermore, notice that

$$\sup_{\alpha\in(1,2)}\frac{(\alpha-1)(2-\alpha)}{n+2}=\left.\frac{(\alpha-1)(2-\alpha)}{n+2}\right|_{\alpha=\frac{3}{2}}=\frac{1}{4(n+2)}.$$

In other words, for any given n_0, the maximal upper bound for ε occurs when $\alpha=\frac{3}{2}$.

Remark 5.3. Define the function $g:(1,2)\times\mathbb{N}_0\to[0,+\infty)$ by

$$g(\alpha,n):=\frac{(\alpha-1)(2-\alpha)}{n+2}.$$

An interesting observation is that

$$\lim_{\alpha\to 1^+}g(\alpha,n)=0\ \text{ and }\ \lim_{\alpha\to 2^-}g(\alpha,n)=0,$$

in each case for each fixed $n\in\mathbb{N}_0$. This implies that when α is close to but larger than one and when α is close to but smaller than two, the bound on ε is very close to zero (no matter than choice of n). Thus, we would expect Theorem 5.1 not to apply for very many time steps when either $\alpha\approx 1$ or $\alpha\approx 2$. Incidentally, this is consistent with the observation noticed in an earlier paper by Dahal and Goodrich [42].

Remark 5.4. It is worth comparing the result of Theorem 5.1 with the recent result [42, Corollary 3.4], which is a monotonicity result phrased in terms of a nabla difference in case of a negative fractional difference (again of nabla type). In fact, the result of Theorem 5.1 is significantly better. In [42, Corollary 3.4] the authors used an upper bound for ε of the form

$$\varepsilon \leq -\frac{\Gamma(-\alpha+n_0+3)}{\Gamma(-\alpha+1)\Gamma(n_0+3)} = -\frac{1}{(n_0+2)!}\prod_{j=1}^{n_0+2}(j-\alpha). \tag{5.9}$$

For example, say $n_0 = 5$—that is, we wish to apply Theorem 5.1 in the case of (possibly) six time steps of monotonicity, which is to say $(\Delta u)(k) \geq 0$ for $k \in \mathbb{N}_0^6$. (Note that in the case of [42, Theorem 3.4] this, in fact, would correspond only to *three* time steps of monotonicity.) Then the bound from [42]—that is, from (5.9)—is

$$\varepsilon \leq -\frac{1}{5040}(7-\alpha)(6-\alpha)(5-\alpha)(4-\alpha)(3-\alpha)(2-\alpha)(1-\alpha),$$

which is a worse bound than the bound

$$\varepsilon \leq \frac{1}{7}(\alpha-1)(2-\alpha),$$

which is the bound generated by Theorem 5.1. Indeed, if we pick $\alpha = \frac{5}{4}$, then the former bound is

$$\varepsilon \leq \frac{4807}{262144},$$

whereas the latter bound is

$$\varepsilon \leq \frac{3}{112},$$

and

$$\frac{4807}{262144} - \frac{3}{112} = -\frac{15503}{1835008} \approx -0.00845.$$

So, in this particular case, we obtain an improvement on the upper bound of ε just shy of 0.01. (Although this might seem to be a trivial improvement, because ε, in general, must be quite small, this is actually an appreciable improvement in the context of this problem.)

In fact, we can prove that the bound in Theorem 5.1 is *always* better than the bound in [42]. Indeed, this is true if and only if

$$-\frac{1}{(n_0+2)!}\prod_{j=1}^{n_0+2}(j-\alpha) < \frac{(\alpha-1)(2-\alpha)}{n_0+2}$$

if and only if

$$\frac{1}{(n_0+1)!}\prod_{j=3}^{n_0+2}(j-\alpha) < 1$$

if and only if

$$(n_0+2-\alpha)(n_0+1-\alpha)(n_0-\alpha)\cdots(3-\alpha) < (n_0+1)(n_0)(n_0-1)\cdots(2),$$

which is true for any $n_0 \in \mathbb{N}_0$ and any $\alpha \in (1, 2)$. Therefore, the desired claim follows at once.

Remark 5.5. From (5.4), the condition

$$\varepsilon < \frac{k^{\alpha-1}(n+2) + (1-\alpha)k^{\alpha-1}(n+1)}{k^{\alpha}(n)}$$

is sufficient for finite monotonicity. It should be compared with the condition

$$\varepsilon < \frac{k^{\alpha}(n+1)}{k^{\alpha+1}(n)}$$

needed for finite positivity. It can be shown that the right-hand side of these inequalities are

$$\frac{(\alpha-1)(2-\alpha)}{n+2}, \qquad 1<\alpha<2, \tag{5.10}$$

and

$$\frac{\alpha}{n+1}, \qquad 0<\alpha<1, \tag{5.11}$$

respectively, as we essentially already know from our previous calculations.

The identities in (5.10)–(5.11) are consequence of the following Lemma 5.6. The easy proof is left as an exercise. The conclusion of Lemma 5.6 will be useful in what follows.

Lemma 5.6. *For any $\gamma > 0$ we have*

(i) $k^{\gamma}(n+1) = \frac{\gamma}{n+1}k^{\gamma+1}(n),\ n \in \mathbb{N}_0$;
(ii) $k^{\gamma}(n+2) = \frac{(\gamma+n+1)\gamma}{(n+1)(n+2)}k^{\gamma+1}(n),\ n \in \mathbb{N}_0$.

In case $\varepsilon = 0$ we recover [38, Theorem 6.3] with a less restrictive initial condition, namely, instead of $u(1) \geq \alpha u(0)$ we merely assume $u(1) \geq u(0)$. Thus, we obtain the following corollary to Theorem 5.1.

Corollary 5.7. *Let $1<\alpha<2$. Suppose that each of the following conditions is satisfied.*

1. $(\Delta^{\alpha}u)(n) \geq 0$ *for each* $n \in \mathbb{N}_0$;
2. $(\Delta u)(0) \geq 0$; *and*
3. $u(0) \geq 0$.

Then $(\Delta u)(n) \geq 0$ *for each* $n \in \mathbb{N}_0$.

Assuming the strongest condition, namely $u(1) \geq \alpha u(0)$, we obtain the following result, which complements Theorem 5.1.

Theorem 5.8. *Let $1<\alpha<2$. Suppose that each of the following conditions is satisfied.*

1. *There exists* $\varepsilon \geq 0$ *such that* $(\Delta^{\alpha}u)(n) \geq -\varepsilon u(0)$ *for each* $n \in \mathbb{N}_0$;
2. $u(1) \geq \alpha u(0)$; *and*
3. $u(0) \geq 0$.

If for some $n_0 \in \mathbb{N}_0$ *it holds that*

$$0 \leq \varepsilon \leq \frac{(\alpha+n_0)(\alpha-1)}{(n_0+2)(n_0+1)},$$

then u is finitely monotone—that is, $(\Delta u)(n) \geq 0$ *for each* $n \in \mathbb{N}_0^{n_0+1}$.

Proof. Since we can follow the same steps as in the proof of Theorem 5.1, we omit the proof of the theorem. □

Remark 5.9. Taking $\varepsilon = 0$ in Theorem 5.8, we recover [38, Theorem 3.6]. Also note that

$$\frac{(\alpha-1)(2-\alpha)}{n_0+2} \leq \frac{(\alpha+n_0)(\alpha-1)}{(n_0+2)(n_0+1)},$$

which suggests that Theorem 5.8 imposes a weaker restriction on the value of ε than does Theorem 5.1. That Theorem 5.8 is "better" in this sense than Theorem 5.1 is quite sensible considering that the condition $u(1) \geq \alpha u(0)$, with $1<\alpha<2$, is a stronger condition than the corresponding condition $(\Delta u)(0) \geq 0$ appearing in the statement of Theorem 5.1.

By the transference principle, we obtain the following corollary.

Corollary 5.10. *Let* $1 < \alpha < 2$, $a \in \mathbb{R}$, *and* $u \in s(\mathbb{N}_a; \mathbb{R})$. *Suppose that each of the following conditions is satisfied.*

1. *There exists* $\varepsilon \geq 0$ *such that* $\left(\Delta_a^\alpha u\right)(n) \geq -\varepsilon u(a)$ *for each* $n \in \mathbb{N}_{a+2-\alpha}$;
2. $(\Delta u)(a) \geq 0$; *and*
3. $u(a) \geq 0$.

If for some $n_0 \in \mathbb{N}_0$ *it holds that*

$$0 \leq \varepsilon \leq \frac{(\alpha-1)(2-\alpha)}{n_0+2},$$

then u is finitely monotone—that is, $(\Delta u)(n) \geq 0$ *for each* $n \in \mathbb{N}_a^{a+n_0+1}$.

Proof. By appealing to [38, Corollary 6.9], one can give an argument similar to the proof of Corollary 4.3. So, we omit the details here. □

Our second collection of results concern monotonicity results for *sequential* fractional differences satisfying a negative lower bound. Each of the following results restricts the numbers α and β to the region in which $0 < \alpha < 1$ and $0 < \beta < 1$. Our first such result, Theorem 5.11, further restricts the admissible parameter space to those numbers $\alpha, \beta \in (0,1)$ satisfying $1 < \alpha + \beta < 2$, $2\beta + \alpha > 2$ and $\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} \leq \frac{3(\alpha+\beta)^\beta}{2\alpha+3\beta-1}$. This region is depicted in Figure 1. We note that for $\alpha = 1$, the value of β is approximately $0.636658693908\ldots$.

Before presenting Theorem 5.11, we recall from [38, Theorem 6.12] that if $\left(\Delta^\beta \circ \Delta^\alpha u\right)(n) \geq \frac{\beta}{2}(1-\beta)u(0)$, $u(1) \geq (\alpha+\beta)u(0)$, and $u(0) \geq 0$, then $(\Delta u)(n) \geq 0$. The following theorem answers what happens when the quantity $\left(\Delta^\beta \circ \Delta^\alpha u\right)(n)$ decreases below the bound $\frac{\beta}{2}(1-\beta)u(0)$.

Theorem 5.11. *Let* $0 < \alpha < 1$ *and* $0 < \beta < 1$ *be such that* $1 < \alpha + \beta < 2$. *Suppose that each of the following conditions is satisfied.*

(1) *There exists* $\varepsilon \geq 0$ *such that* $\left(\Delta^\beta \circ \Delta^\alpha u\right)(n) \geq -\varepsilon u(0)$ *for each* $n \in \mathbb{N}_0$;
(2) $(\Delta u)(0) \geq 0$;
(3) $u(0) \geq 0$;
(4) $2\beta + \alpha > 2$; *and*
(5) $\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} \leq \frac{3(\alpha+\beta)^\beta}{2\alpha+3\beta-1}$.

If for some $n_0 \in \mathbb{N}_0$ *it holds that*

$$\varepsilon \leq \frac{1-\alpha}{n_0+1}\left[(\alpha+\beta-1) - \frac{1}{n_0+2}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n_0+1)}{\Gamma(\alpha+\beta+n_0)}\right], \tag{5.12}$$

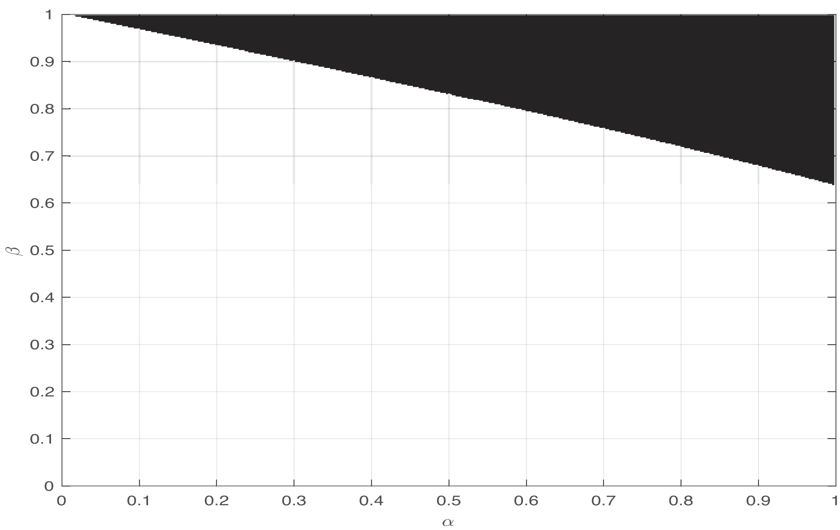


FIGURE 1 Admissible parameter space for Theorem 5.11.

then u is finitely monotone—that is, $(\Delta u)(n) \geq 0$ *for all* $n = 1, 2, \ldots, n_0 + 1$.

Proof. We first observe that due to [38, Theorem 6.11], we know that

$$\left(\Delta^{\alpha+\beta} u\right)(n) = \left(\Delta^{\beta} \circ \Delta^{\alpha} u\right)(n) + \left(\Delta k^{1-\beta}\right)(n+1)u(0), \tag{5.13}$$

for each $n \in \mathbb{N}_0$. Therefore, from assumption (1), it follows from equality (5.13) that

$$\left(\Delta^{\alpha+\beta} u\right)(n) \geq -\varepsilon u(0) + \left(\Delta k^{1-\beta}\right)(n+1)u(0) = -\varepsilon u(0) + \left(\tau_1 \circ \Delta k^{1-\beta}\right)(n)u(0), \tag{5.14}$$

for each $n \in \mathbb{N}_0$. In addition, from [38, (6.2), p. 561], we have that

$$\begin{aligned}(\Delta u)(n+1) &= \left(k^{\alpha+\beta-1} * \Delta^{\alpha+\beta} u\right)(n) + k^{\alpha+\beta-1}(n+2)u(0) + k^{\alpha+\beta-1}(n+1)\left[u(1) - (\alpha+\beta)u(0)\right] \\ &= \left(k^{\alpha+\beta-1} * \Delta^{\alpha+\beta} u\right)(n) + k^{\alpha+\beta-1}(n+2)u(0) + (1-(\alpha+\beta))\, k^{\alpha+\beta-1}(n+1)u(0) + k^{\alpha+\beta-1}(n+1)\left[u(1) - u(0)\right] \\ &\geq \left(k^{\alpha+\beta-1} * \Delta^{\alpha+\beta} u\right)(n) + k^{\alpha+\beta-1}(n+2)u(0) + (1-(\alpha+\beta))\, k^{\alpha+\beta-1}(n+1)u(0)\end{aligned} \tag{5.15}$$

for each $n \in \mathbb{N}_0$, keeping in mind that $1 < \alpha + \beta < 2$ and $u(1) \geq u(0)$.

Consequently, combining identity (5.15) with inequality (5.14), we arrive at the inequality

$$(\Delta u)(n+1) \geq -\varepsilon k^{\alpha+\beta}(n)u(0) + \left(k^{\alpha+\beta-1} * \left(\tau_1 \circ \Delta k^{1-\beta}\right)\right)(n)u(0) + k^{\alpha+\beta-1}(n+2)u(0) + (1-(\alpha+\beta))\, k^{\alpha+\beta-1}(n+1)u(0), \tag{5.16}$$

for each $n \in \mathbb{N}_0$. Note that by Lemma 2.9 together with part (iv) of Lemma 2.6, we have

$$\begin{aligned}\left(k^{\alpha+\beta-1} * \left(\tau_1 \circ \Delta k^{1-\beta}\right)\right)(n) &= \left(k^{\alpha+\beta-1} * \Delta k^{1-\beta}\right)(n+1) - k^{\alpha+\beta-1}(n+1)\left(\Delta k^{1-\beta}\right)(0) \\ &= \left(\Delta\left(k^{\alpha+\beta-1} * k^{1-\beta}\right)\right)(n+1) - k^{\alpha+\beta-1}(n+2)k^{1-\beta}(0) + \beta k^{\alpha+\beta-1}(n+1) \\ &= k^{\alpha}(n+2) - k^{\alpha}(n+1) - k^{\alpha+\beta-1}(n+2) + \beta k^{\alpha+\beta-1}(n+1).\end{aligned} \tag{5.17}$$

Inserting (5.17) in (5.16), we obtain

$$\begin{aligned}(\Delta u)(n+1) &\geq \big[-\varepsilon k^{\alpha+\beta}(n) + k^{\alpha}(n+2) - k^{\alpha}(n+1) - k^{\alpha+\beta-1}(n+2) \\ &\quad + \beta k^{\alpha+\beta-1}(n+1) + k^{\alpha+\beta-1}(n+2) + (1-(\alpha+\beta))\, k^{\alpha+\beta-1}(n+1)\big]\, u(0) \\ &= \left[-\varepsilon k^{\alpha+\beta}(n) + k^{\alpha}(n+2) - k^{\alpha}(n+1) + (1-\alpha)k^{\alpha+\beta-1}(n+1)\right] u(0),\end{aligned} \tag{5.18}$$

for each $n \in \mathbb{N}_0$. Therefore, $(\Delta u)(n_0+1) \geq 0$ if the condition

$$\varepsilon \leq \frac{\left(\Delta k^{\alpha}\right)(n_0+1) + (1-\alpha)k^{\alpha+\beta-1}(n_0+1)}{k^{\alpha+\beta}(n_0)}$$

holds. Note that by Lemma 2.7, the quantity $\left(\Delta k^{\alpha}\right)(n+1) = \frac{\alpha-1}{n+2}k^{\alpha}(n+1)$ is negative. Using (i) of Lemma 5.6 with $\gamma := \alpha + \beta - 1$, the right side of (5) equals to

$$\begin{aligned}\frac{\left(\Delta k^{\alpha}\right)(n+1) + (1-\alpha)k^{\alpha+\beta-1}(n+1)}{k^{\alpha+\beta}(n)} &= (1-\alpha)\frac{k^{\alpha+\beta-1}(n+1)}{k^{\alpha+\beta}(n)} - \frac{(1-\alpha)}{n+2}\frac{k^{\alpha}(n+1)}{k^{\alpha+\beta}(n)} \\ &= \frac{(1-\alpha)(\alpha+\beta-1)}{n+1} - \frac{(1-\alpha)}{(n+2)(n+1)}\frac{\alpha k^{\alpha+1}(n)}{k^{\alpha+\beta}(n)} \\ &= \frac{(1-\alpha)(\alpha+\beta-1)}{n+1} - \frac{(1-\alpha)}{(n+1)(n+2)}\frac{\alpha\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\beta+n)} \\ &= \frac{1-\alpha}{n+1}\left[(\alpha+\beta-1) - \frac{1}{n+2}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\beta+n)}\right],\end{aligned} \tag{5.19}$$

for each $n \in \mathbb{N}_0$, which in case $n = n_0$ is precisely the right-hand side of inequality (5.12) in the statement of the theorem.

Observe that in (5.12) depending on the values of α and β, the quantity at the right-hand side could be either positive or negative. Our goal is to analyze this quantity. So, put $\gamma := \alpha + \beta - 1$. Then the right-hand side of inequality (5.19) takes the form

$$\frac{1-\alpha}{n}\left[\gamma - \frac{1}{(n+1)}\frac{\Gamma(\gamma+1)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)}\right] = (1-\alpha)\gamma\left[\frac{1}{n} - \frac{1}{n(n+1)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)}\right],\ n \in \mathbb{N}.$$

Note, moreover, that by hypothesis, we have $0 < \alpha/2 < \gamma \le \alpha < 1$. We define the sequence $\{h(n)\}_{n=1}^{\infty}$ by

$$h(n) := \frac{1}{n} - \frac{1}{n(n+1)}\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)},\ n \in \mathbb{N}.$$

Observe that to complete the proof is enough to prove that

(A) $nh(n) \ge 0$ for all $n \in \mathbb{N}$; and
(B) $h(n) \ge h(n+1)$ for all $n \in \mathbb{N}$.

This is because we want $n \mapsto h(n)$ to be nonnegative and decreasing.

In fact, to prove (A), we note that by hypothesis (4) in the statement of the theorem we have $(\gamma + (1-\alpha))\,n + (2\gamma - \alpha) \ge 0$ for all $n \in \mathbb{N}$. Hence, a calculation shows that

$$\frac{\Gamma(\gamma+n)}{\Gamma(\gamma+n+1)}\frac{1}{n+2} \le \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+1)}\frac{1}{n+1},\ n \in \mathbb{N}.$$

This proves that $(n+1)h(n+1) \ge nh(n)$, $n \in \mathbb{N}$. Since $h(1) = 1 - \frac{\alpha}{2\gamma} \ge 0$, we have (A).

To prove (B), note that a simple calculation shows that it is equivalent to proving the following inequality:

$$\frac{1}{n}\frac{\Gamma(\gamma+n)}{\Gamma(\alpha+n)} \ge \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\left[\frac{1}{n} - \frac{\alpha+n}{(\gamma+n)(n+2)}\right] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\left[\frac{(1+\beta)n+2\gamma}{n(n+2)(\gamma+n)}\right],\ n \in \mathbb{N}.$$

or, equivalently,

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)} \le \frac{n+2}{2\gamma+(1+\beta)n}\frac{\Gamma(\gamma+n+1)}{\Gamma(\alpha+n)} = \frac{n+2}{2\gamma+(1+\beta)n}\frac{\Gamma(\gamma+n+1)}{\Gamma(\gamma+n+(1-\beta))},\ n \in \mathbb{N}. \tag{5.20}$$

Since $0 < \beta < 1$, Gautschi's inequality shows that

$$\frac{\Gamma(\gamma+n+1)}{\Gamma(\gamma+n+(1-\beta))} \ge (\gamma+n)^{1-(1-\beta)} = (\gamma+n)^{\beta},\ n \in \mathbb{N}.$$

Therefore, to prove (5.20), it is enough to show that

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)} \le \frac{(n+2)(\gamma+n)^{\beta}}{2\gamma+(1+\beta)n} =: \psi(n),\ n \in \mathbb{N}. \tag{5.21}$$

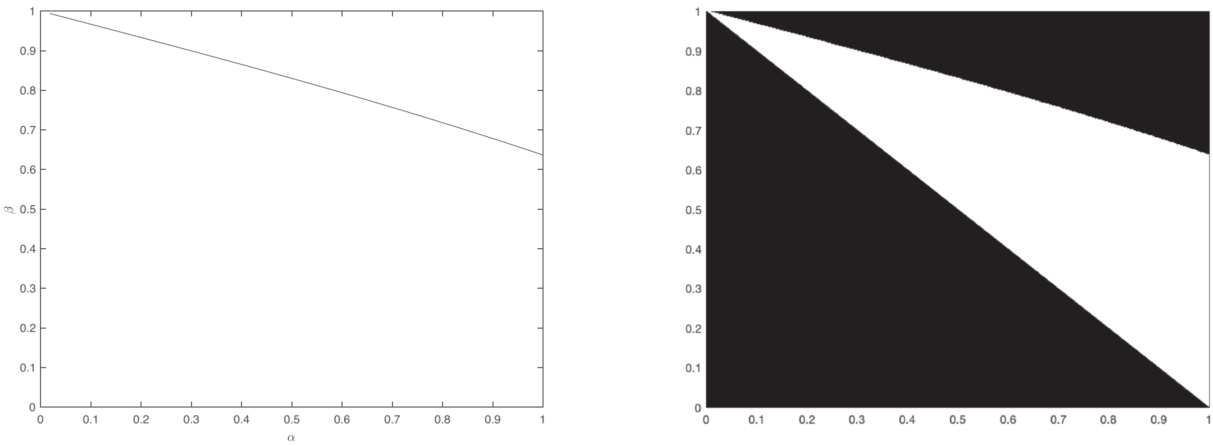
Note that $\psi(1) = \frac{3(\alpha+\beta)^{\beta}}{2\alpha+3\beta-1}$, which occurs in the hypothesis (5) in the statement of the theorem—that is, $\frac{\Gamma(\gamma)}{\Gamma(\alpha)} \le \psi(1)$. Therefore, in order to show (5.21), we have to prove that $\{\psi(n)\}_{n=1}^{\infty}$ is an increasing sequence on $\mathbb{N}$. Indeed, defining its continuous counterpart

$$\Psi(x) := \frac{(x+2)(\gamma+x)^{\beta}}{2\gamma+(1+\beta)x},\ x \ge 1,$$

we obtain after a computation,

$$\frac{d}{dx}\Psi(x) = \frac{(\gamma+x)^{\beta-1}\left[\beta(1+\beta)x^2 + 2(1+\beta)(\gamma+\beta-1)x + 2\gamma(\gamma+\beta-1)\right]}{(2\gamma+(1+\beta)x)^2} \ge 0,$$

where in the last inequality we use that $\gamma + \beta - 1 = \alpha + 2\beta - 2 \ge 0$ which follows from hypothesis (4). This proves statement (B) and finishes the proof. □



(A) bordeline for condition (5) (B) Set Ω

FIGURE 2 Conditions on the pair (α, β).

A picture of the admissible values of α, β in condition (5) is illustrated in Figure 2. Figure 2A corresponds to $\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} = \frac{3(\alpha+\beta)^\beta}{2\alpha+3\beta-1}$, and the dashed region in Figure 2B is the set $\Omega = \left\{ (\alpha, \beta) \in [0,1] \times [0,1] \, : \, \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} \leq \frac{3(\alpha+\beta)^\beta}{2\alpha+3\beta-1} \right\}$.

We can take $\varepsilon = 0$ in Theorem 5.11, and we note from the proof of the theorem that to have positivity of the right hand side in (5.12) it is enough to have hypothesis (4)—that is, hypothesis (5) in the statement of Theorem 5.11 is unnecessary in case $\varepsilon = 0$. Hence, we obtain the following result that complements and improves [38, Theorem 6.12], and it also "rediscovers" [36, Theorem 2.5].

Corollary 5.12. *Let* $0 < \alpha < 1$ *and* $0 < \beta < 1$ *be such that* $1 < \alpha + \beta < 2$. *Suppose that each of the following conditions is satisfied.*

(1) $\left(\Delta^\beta \circ \Delta^\alpha u\right)(n) \geq 0$ *for each* $n \in \mathbb{N}_0$;
(2) $(\Delta u)(0) \geq 0$;
(3) $u(0) \geq 0$;
(4) $2\beta + \alpha > 2$; *and*

Then $(\Delta u)(n) \geq 0$ *for each* $n \in \mathbb{N}_0$.

We have the following interesting counterpart, which is complementary to Theorem 5.11. As illustrated by Figure 3, note that Theorem 5.13 is valid on a larger subset of the (α, β)-parameter space.

Thus, Theorem 5.13 is applicable on the maximal (α, β)-parameter space—that is, $0 < \alpha < 1$, $0 < \beta < 1$, and $1 < \alpha + \beta < 2$. As sort-of compensation for this increased parameter space applicability, we note that Theorem 5.13 yields *eventual* monotonicity—see Remark 5.14. Interestingly, we are not aware of any results in the existing literature on "eventual monotonicity" in the context of a fractional difference with negative lower bound, and so, in this regard, Theorem 5.13 demonstrates a new property of discrete fractional difference operators. It may also be compared to [22, Theorem 3.9], which is an eventual monotonicity result but in the context of a *nonnegative* lower bound for the fractional difference.

Theorem 5.13. *Let* $0 < \alpha < 1$ *and* $0 < \beta < 1$ *be such that* $1 < \alpha + \beta < 2$. *Suppose that each of the following conditions is satisfied.*

(1) There exists $\varepsilon \geq 0$ *such that* $\left(\Delta^\beta \circ \Delta^\alpha u\right)(n) \geq -\varepsilon u(0)$ *for each* $n \in \mathbb{N}_0$;
(2) $(\Delta u)(0) \geq 0$; *and*
(3) $u(0) \geq 0$;
(4) $\varepsilon < \alpha + \beta - 1$.

Then u is eventually monotone—that is, there exists $n_0 \in \mathbb{N}_0$ *such that* $(\Delta u)(n) \geq 0$ *for each* $n \geq n_0$.

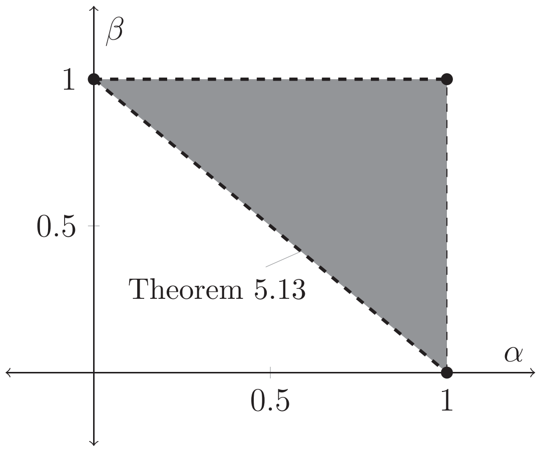


FIGURE 3 The parameter space region considered in Theorem 5.13.

Proof. Since $0 < \beta < 1$, Gautschi's inequality implies that

$$\frac{1}{n+2}(\alpha+n)^{1-\beta} < \frac{1}{n+2}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n+\beta)} < \frac{1}{n+2}(\alpha+n+1)^{1-\beta},\ n \in \mathbb{N}.$$

Hence,

$$g_\varepsilon(n) := -\varepsilon + (\alpha+\beta-1) - \frac{1}{n+2}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\beta+n)} \to \underbrace{-\varepsilon+\alpha+\beta-1}_{>0} \text{ as } n \to +\infty,$$

where by hypothesis $\varepsilon < \alpha+\beta-1 < 1$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $g_\varepsilon(n) \geq 0$ for all $n \geq n_0$. From (5.18) and using (i) of Lemma 5.6, we obtain

$$\begin{aligned}
(\Delta u)(n+1) &\geq \left[-\varepsilon k^{\alpha+\beta}(n) + (1-\alpha)k^{\alpha+\beta-1}(n+1) + \Delta k^{\alpha}(n+1)\right]u(0)\\
&= k^{\alpha+\beta}(n)\left[-\varepsilon + (1-\alpha)\frac{k^{\alpha+\beta-1}(n+1)}{k^{\alpha+\beta}(n)} + \frac{\Delta k^{\alpha}(n+1)}{k^{\alpha+\beta}(n)}\right]u(0)\\
&= k^{\alpha+\beta}(n)\left[-\varepsilon + (1-\alpha)\frac{k^{\alpha+\beta-1}(n+1)}{k^{\alpha+\beta}(n)} - \frac{(1-\alpha)}{n+2}\frac{k^{\alpha}(n+1)}{k^{\alpha+\beta}(n)}\right]u(0)\\
&= k^{\alpha+\beta}(n)\left[-\varepsilon + \frac{(1-\alpha)(\alpha+\beta-1)}{n+1} - \frac{(1-\alpha)}{(n+2)(n+1)}\frac{\alpha k^{\alpha+1}(n)}{k^{\alpha+\beta}(n)}\right]u(0)\\
&= k^{\alpha+\beta}(n)\left[-\varepsilon + \frac{(1-\alpha)(\alpha+\beta-1)}{n+1} - \frac{(1-\alpha)}{(n+1)(n+2)}\frac{\alpha\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\beta+n)}\right]u(0)\\
&= \frac{1-\alpha}{n+1}k^{\alpha+\beta}(n)\left[-\varepsilon + (\alpha+\beta-1) - \frac{1}{n+2}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\beta+n)}\right]u(0)\\
&= \frac{1-\alpha}{n+1}k^{\alpha+\beta}(n)g_\varepsilon(n)u(0).
\end{aligned}$$

The conclusion follows. □

Remark 5.14. An interesting observation regarding Theorem 5.13 is that it extends in a nontrivial way the conclusions of [35, Theorem 2.5], [35, Corollary 2.10], and [36, Theorem 2.5]. In those results, Theorem 5.13 was proved under the assumption that $\varepsilon = 0$. Moreover, those results also showed that if $\varepsilon = 0$, then in the shaded region depicted in Figure 4, monotonicity could not be deduced on the basis of conditions (1)–(3) in the statement of Theorem 5.13.

So, Theorem 5.13 shows us that whereas [35, Theorem 2.5], [35, Corollary 2.10], and [36, Theorem 2.5] collectively demonstrate that *everywhere* monotonicity cannot be guaranteed on the shaded region in Figure 4, *eventual* monotonicity can be guaranteed on this region.

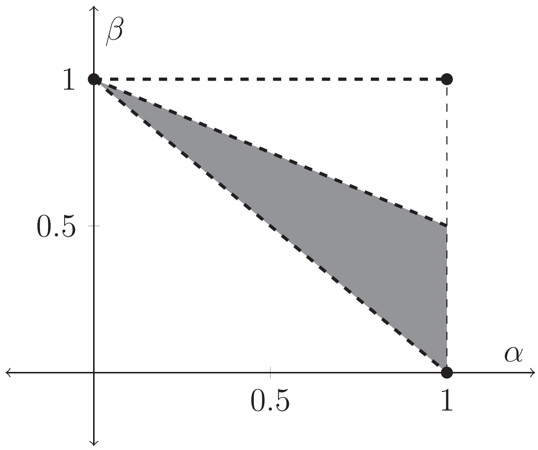


FIGURE 4 The parameter space region on which everywhere monotonicity cannot be guaranteed but on which eventual monotonicity can be guaranteed.

Remark 5.15. We observe that for positivity, we need the condition

$$\varepsilon < \frac{\alpha}{n+1}\frac{k^{1+\alpha}(n)}{k^{1+\alpha+\beta}(n)},$$

whereas for monotonicity, we require

$$\varepsilon \leq \frac{\Delta k^{\alpha}(n+1)+(1-\alpha)k^{\alpha+\beta-1}(n+1)}{k^{\alpha+\beta}(n)}.$$

Assuming conditions (4) and (5) in the statement of Theorem 5.11, we deduce from (5.19) the following interesting inequality

$$k^{\alpha+\beta-1}(n) \geq \frac{1}{n+1}k^{\alpha}(n), \;\; n \in \mathbb{N}_0.$$

Here, we also assume $0 < \alpha < 1$, $0 < \beta < 1$ and $1 < \alpha+\beta < 2$.

Our next remark details some of the ways in which Theorem 5.11 can be used to explain in a mathematically precise way several of the observations and predictions in [41], which in that paper were only observed via numerical simulations.

Remark 5.16. Define the function $g : (0,1)\times(0,1)\times\mathbb{N}_0 \to [0,+\infty)$ by

$$g(\alpha,\beta,n) := \frac{1-\alpha}{n+1}\left[(\alpha+\beta-1) - \frac{1}{n+2}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\beta+n)}\right].$$

Notice both that

$$\lim_{\alpha\to 1^-} g(\alpha,\beta,n) = 0 \tag{5.22}$$

and that

$$\lim_{\alpha+\beta\to 2^-} g(\alpha,\beta,n) = 0, \tag{5.23}$$

where in (5.23), we use the fact that if $\alpha+\beta \to 2^-$, then $\alpha \to 1^-$. From these observations, we can give clear explanations as to some of the peculiarities in the heat maps presented in [41, Figures 6 and 7].

For example, the authors there noted that when $\alpha+\beta \approx 2$, their monotonicity result [41, Corollary 3.4] seemed to be applicable only for $\varepsilon \approx 0$. The calculation in (5.23) shows why this phenomenon is observed. And this provides a concise analytical explanation for the numerical phenomenon observed in [41, Figures 6 and 7]. Similarly, in [41, Figures 6 and 7], the authors noticed that when $\alpha \approx 1$ (even when $\alpha+\beta$ was not especially close to 2), their monotonicity result could only be expected to hold for ε extremely small. The calculation in (5.22) provides an analytical explanation for this phenomenon, which in [41] was only observed via numerical simulation.

To conclude this section, we present an application of the transference principle to Theorem 5.11, and we thus obtain the following corollary.

Corollary 5.17. *Let* $0 < \alpha < 1$ *and* $0 < \beta < 1$ *be such that* $1 < \alpha + \beta < 2$ *and let* $u \in s(\mathbb{N}_a; \mathbb{R})$ *be a given sequence. Suppose that each of the following conditions is satisfied.*

(1) There exists $\varepsilon \geq 0$ *such that* $\left(\Delta^{\beta}_{1+a-\alpha} \circ \Delta^{\alpha}_{a} u\right)(n) \geq -\varepsilon u(0)$ *for each* $n \in \mathbb{N}_{2+a-\alpha-\beta}$*;*
(2) $(\Delta u)(a) \geq 0$*;*
(3) $u(a) \geq 0$*;*
(4) $2\beta + \alpha > 2$*; and*
(5) $\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} \leq \frac{3(\alpha+\beta)^{\beta}}{2\alpha+3\beta-1}$.

If for some $n_0 \in \mathbb{N}_0$ *it holds that*

$$\varepsilon \leq \frac{1-\alpha}{n_0+1}\left[(\alpha+\beta-1) - \frac{1}{n_0+2}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n_0+1)}{\Gamma(\alpha+\beta+n_0)}\right],$$

then $(\Delta u)(n) \geq 0$ *for all* $n = 1, 2, \ldots, n_0 + 1$.

Proof. Since the proof of this corollary is similar to earlier proofs, we omit it. □

6 | APPLICATIONS

In this final section, we provide an application of our results to an analysis of scalar and vectorial initial value problems in discrete fractional calculus. These examples also both complement and extend [35, Corollary 2.5] and [35, Corollary 2.6].

Example 6.1. The nonlinear equations $y'(t) = \sin(y(t))$ and $y''(t) = \sin(y(t))$, $t > 0$, arise in physics, engineering and other fields. They can be used to model oscillatory phenomena, such as the motion of a pendulum or the behavior of certain electrical circuits (in the first order case) or the motion of a mass-spring system or the behavior of certain mechanical systems (in the second order case). They also arise in the study of nonlinear dynamics and chaos theory, where these equations exhibit complex and interesting behavior, including limit cycles and bifurcations [62]. A fractional generalization of the previous models is the equation

$$D^{\alpha} y(t) = \sin(y(t)) \tag{6.1}$$

where D^{α} denotes the Caputo fractional derivative of order $\alpha > 0$.

The convolution quadrature of Lubich presents one versatile framework for developing time-stepping schemes for the above models. Considering convoluted quadrature generated by backward Euler with a constant time step size 1 for Equation (6.1) we obtain the following fractional nonlinear difference equation [54, Section 3]:

$$\begin{cases} (\Delta^{\alpha} u)(n) = \sin u(n), & n \in \mathbb{N}, \\ u(0) = u_0 > 0. \end{cases}$$

Suppose $0 < \alpha < 1$. Then for $\varepsilon := \frac{1}{u_0}$, we have

$$(\Delta^{\alpha} u)(n) \geq -1 = -\varepsilon u(0), \quad n \in \mathbb{N}.$$

Therefore, Theorem 4.2 implies that $u(j) \geq 0$ for all $j = 1, 2, \ldots, n$ where $n+1 \leq \alpha u_0$. This shows that even when the nonlinear term is oscillating, we can preserve the positivity of the solution in some extent, which is highly dependent on the size of u_0.

For the same problem, in case $1 < \alpha < 2$, we can apply Theorem 5.1. Then under the additional hypothesis:

$$u(1) \geq u(0),$$

we can conclude that $u(j+1) \geq u(j)$ for all $j = 1, 2, \ldots, n+1$ whenever $n+2 \leq (\alpha-1)(2-\alpha)u_0$. Again, the amount of monotonicity is dependent on the size of u_0.

A little different condition is given if one assume the strongest condition

$$u(1) \geq \alpha u(0).$$

By Theorem 5.8, we obtain that $u(j+1) \geq u(j)$ for all $j = 1, 2, 3, \ldots, n+1$ whenever

$$\frac{(n+2)(n+1)}{\alpha+n} \leq (\alpha-1)u_0.$$

Example 6.2. As a second example, we consider the following fractional system:

$$\begin{cases} (\Delta^{\alpha} v)(n) = \frac{e^{-n}}{1+u(n)^2}, \; n \in \mathbb{N}, \; 0 < \alpha < 1; \\ \left(\Delta^{\beta} u\right)(n) = v(n), \; n \in \mathbb{N}, \; 0 < \beta < 1; \\ u(1) \geq u(0) \geq 0. \end{cases}$$

Assuming $1 < \alpha + \beta < 2$, we obtain by Theorem 5.13 with $\varepsilon = 0$ that there exists $n_0 \in \mathbb{N}_0$ such that

$$u(n+1) \geq u(n) \text{ for each } n = n_0, n_0+1, \cdots. \tag{6.2}$$

On the other hand, if we assume $u(1) \geq \beta u(0)$, then

$$v(0) = \left(\Delta^{\beta} u\right)(0) = u(1) - \beta u(0) \geq 0,$$

and hence, an application of Theorem 4.2 with $\varepsilon = 0$ shows that $v(n) \geq 0$ for all $n \in \mathbb{N}$. Applying now again Theorem 4.2 with $\varepsilon = 0$ to the equation $\left(\Delta^{\beta} u\right)(n) = v(n)$, we deduce that $u(n) \geq 0$ for all $n \in \mathbb{N}_0$.

The last conclusion can also be reached under the additional hypothesis

$$2\alpha + \beta > 2,$$

because in this case, we obtain by the Corollary 5.12 that $n_0 = 0$ in the equation (6.2). We deduce that $u(n) \geq 0$ for all $n \in \mathbb{N}_0$.

Example 6.3. As our last example, we consider the system

$$\begin{cases} \left(\Delta^{\beta} v\right)(n) = a \cos u(n), \; n \in \mathbb{N}, \; a > 0, \; 0 < \beta < 1; \\ (\Delta^{\alpha} u)(n) = v(n), \; n \in \mathbb{N}, \; 0 < \alpha < 1; \\ u(1) \geq u(0) > 0. \end{cases}$$

Assuming again $1 < \alpha + \beta < 2$, and adding the hypothesis $2\beta + \alpha > 2$ and

$$\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)} \leq \frac{3(\alpha+\beta)^{\beta}}{2\alpha+3\beta-1}, \tag{6.3}$$

we obtain from Theorem 5.11 with $\varepsilon = \frac{a}{u(0)}$ that there exists $n_0 \in \mathbb{N}_0$ such that $(\Delta u)(n) \geq 0$ for all $n = 1, 2, \ldots, n_0+1$, whenever

$$a \leq \frac{1-\alpha}{n_0+1}\left[(\alpha+\beta-1) - \frac{1}{n_0+2}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n_0+1)}{\Gamma(\alpha+\beta+n_0)}\right]u_0.$$

If instead of $2\beta + \alpha > 2$ and (6.3), we add the simpler hypothesis $a < (\alpha+\beta-1)u(0)$, then we conclude from Theorem 5.13 that u is eventually monotone. Thus, monotonicity is preserved to some extent even when the nonlinear term changes sign.

AUTHOR CONTRIBUTIONS

Christopher Goodrich: Investigation; conceptualization; validation; software; formal analysis; writing—review and editing; writing—original draft. **Carlos Lizama**: Conceptualization; methodology; formal analysis; validation; investigation; funding acquisition; writing—original draft; writing—review and editing.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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