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A TRANSFERENCE PRINCIPLE FOR NONLOCAL OPERATORS USING A CONVOLUTIONAL APPROACH: FRACTIONAL MONOTONICITY AND CONVEXITY

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ABSTRACT

We utilize a new definition for the fractional delta operator and prove that it is equivalent by translation to the more commonly used operator. By means of the convolution operation we demonstrate that this new operator is strongly connected to the positivity, monotonicity, and convexity of the functions on which it operates. We also analyze the case of compositions of discrete fractional operators. Finally, since the operator we study here is translationally related to the more commonly used discrete fractional operators, we are able to establish many new results for all types of discrete fractional differences, and we explicitly demonstrate that our results improve all known existing results in the literature.

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1. Introduction

In recent years the notion of a "discrete fractional calculus" has been developed as a counterpart to the classical integer-order calculus. Using the notation

$$\mathbb{N}_a := \{a, a+1, a+2, \dots\},$$

for some $a \in \mathbb{R}$, recall that given a function $f : \mathbb{N}_a \to \mathbb{R}$ the first-order forward (or delta) difference of f at $t \in \mathbb{N}_a$, denoted $(\Delta f)(t)$, is defined by

$$(\Delta f)(t) := f(t+1) - f(t).$$

Then one may define iteratively the higher order differences Δ^n, for $n \in \mathbb{N}$, by writing

$$(\Delta^n f)(t) := (\Delta \circ \Delta^{n-1} f)(t).$$

By contrast, a fractional forward difference of order $\alpha > 0$ may be defined by (see Gray and Zhang [35] or Atici and Eloe [11, 12, 13])

$$(\Delta_a^\nu f)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s), \quad t \in \mathbb{N}_{a+N-\nu}, \tag{1.1}$$

where $N \in \mathbb{N}_1$ is the unique integer satisfying $N-1 < \nu < N$, and the map $t \mapsto t^{\underline{\nu}}$ is defined by

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}.$$

We observe that the operator $f \mapsto \Delta_a^\nu f$ defined by (1.1) is nonlocal since the quantity $\Delta_a^\nu f(t)$ involves a linear combination of the collection $\{f(s)\}_{s=a}^{t+\nu}$. Due to the nonlocal nature of the fractional difference operator, analyzing its qualitative properties remains a challenging avenue of study. For example, the relationship between the sign of $(\Delta_a^\nu f)(t)$ and the monotonicity or convexity of f is far more complicated than when ν is an integer.

Recently, Lizama [44] proposed an alternative definition to (1.1) by setting

$$(\Delta^\alpha f)(n) := \Delta^N \left[\sum_{j=0}^{n} k^{N-\alpha}(n-j) f(j) \right], \tag{1.2}$$

where $N-1<\alpha<N$, $N\in\mathbb{N}$, and the kernel $n\mapsto k^\alpha(n)$ is defined by

$$k^\alpha(n):=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}.$$

It turns out that (1.2) behaves extremely nicely with respect to convolution since, as explained in Section 2, (1.2) can be recast as

$$\Delta^N(k^\alpha * f)(n),$$

where $*$ is the discrete convolution operator. The definition (1.2) has been used in some recent papers that opened new lines of research: maximal regularity characterizations for abstract fractional difference equations in Lebesgue spaces of sequences [43, 45, 46, 42, 47, 41]; existence and uniqueness of qualitative properties of solutions for abstract fractional difference equations [2, 8]; existence and uniqueness of solutions for nonlinear fractional difference equations in the abstract setting [48, 36]; and generalized Césaro sums and their interplay with algebra homomorphisms and other subjects of interest in harmonic and functional analysis [4, 1, 3]. However, it has not been used to study either the monotonicity or convexity of the function f.

While definitions (1.1) and (1.2) appear to be somewhat dissimilar, in Section 4 we prove that definitions (1.1) and (1.2) are strongly related by a sort-of translation property, which is captured by the fact that the diagram below is commutative.

$$\begin{array}{ccc} s(\mathbb{N}_0;\mathbb{R}) & \xrightarrow{\Delta^\alpha} & s(\mathbb{N}_0;\mathbb{R}) \\ {\scriptstyle\tau_a}\uparrow & & \uparrow{\scriptstyle\tau_{a+1-\alpha}} \\ s(\mathbb{N}_a;\mathbb{R}) & \xrightarrow[\Delta_a^\alpha]{} & s(\mathbb{N}_{a+1-\alpha};\mathbb{R}) \end{array}$$

Since the diagram commutes this allows us to deduce results about the operator Δ^α and then readily deduce the corresponding property for the operator Δ_a^α. We call this property "transference" since by the commutative diagram above we are able literally to transfer properties about Δ^α to those about Δ_a^α. In this sense, theorems that we are able to deduce regarding the new operator Δ^α may be readily translated into theorems regarding the old operator Δ_a^α.

This notion of transference is of deep importance, as we demonstrate in this article, since due to the fact that Lizama's definition (1.2) for the fractional difference is easily realizable as a convolution of k^α with f, we can use the simple operational properties of $*$ in order to produce elegant and straightforward calculations involving Δ^α. Succinctly, our typical plan of attack, which we are able to repeat again and again in a variety of contexts (see Sections 5, 6, and 7), is as follows.

(1) We consider a question regarding the entire class of discrete fractional operators—for example, whether there is a connection between the sign of the operator when $1 < \alpha < 2$ and the monotonicity of the function f on which it acts.
(2) We use the properties of convolution together with definition (1.2) in order to answer our question in a computationally clean and elegant manner—something that would be quite difficult if we dealt with definition (1.1) directly.
(3) Finally, by the property of transference and the commutative diagram above we translate the result in the language of the operator Δ^α to the language of the operator Δ_a^α, thereby attaining a result that may have been very difficult to deduce directly from (1.1).

The power of transference is, therefore, that by using the simple properties of convolution, we can obtain a result about Δ^α, a result that would be difficult to deduce by using the original definition (1.1) directly, and then by transference obtain the corresponding result for the operator Δ_a^α. In this sense, the inherent complexity of (1.1) is avoided.

As already intimated, one of our main goals is to understand more clearly the connections between the sign of $(\Delta^\alpha f)(t)$ and either the positivity, monotonicity, or convexity of f. We are especially interested in the relationship when we consider sequential fractional differences—that is, compositions of fractional differences in a particular order such as $\Delta^\beta \circ \Delta^\alpha$ with the orders α and β in a particular range; these were first considered by Goodrich [25] in the context of a boundary value problem. As we will explain momentarily, discrete sequential fractional differences display particularly complex qualitative behavior. By great contrast, in the integer-order setting the order of composition is irrelevant since the delta operators are identical and of integer order—and, especially, they commute in the sense that

$$\Delta^m \circ \Delta^n \equiv \Delta^n \circ \Delta^m$$

for all m, $n \in \mathbb{N}$. Moreover, there is a straightforward connection between the sign of $(\Delta f)(t)$ and of $(\Delta^2 f)(t)$ and, respectively, the monotonicity and convexity of the function f—see, in the case of convexity, the article by Atici and Yaldız [15] for more details.

When considering a fractional-order forward difference, however, the simplicity of the integer-order setting vanishes. More specifically, in recent papers by Atici and Uyanik [14], Baoguo et al. [16, 17], Dahal and Goodrich [19, 20, 21], Goodrich [26, 29, 30, 31, 32], and Jia et al. [38, 39] various connections between the sign of an appropriate discrete fractional operator, the corresponding monotonicity and convexity of f was considered—see also the survey article by Erbe et al. [22]. In particular, it has been shown that the condition $\Delta_a^\nu f(t) \geq 0$, in case $\nu \in (1,2)$, is not sufficient to guarantee the monotonicity of f on its domain; this is in great contrast to the integer-order case. Likewise the condition $\Delta_a^\nu f(t) \geq 0$, in case $\nu \in (2,3)$, is not sufficient to guarantee the convexity of f on its domain, and this also contrasts greatly with the integer-order setting.

In addition, the qualitative properties of discrete fractional operators can be studied for their mathematical interest, and, indeed, the case of sequential operators is particularly interesting and has only been recently considered. Not only is this construction of potential interest since in, say, the sequential operator $\Delta^\nu \circ \Delta^\mu$, the number μ does not necessarily equal the number ν (again, unlike in the integer-order case), it is also of interest due to the following result of Holm [37], which shows us that discrete fractional operators are, in general, noncommutative.

Theorem 1.1: *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $\nu, \mu > 0$, with $N-1 < \nu \leq N$ and $M-1 < \mu \leq M$, where M, $N \in \mathbb{N}_1$. Then for $t \in \mathbb{N}_{a+M-\mu+N-\nu}$ it holds that*

$$\begin{aligned}(1.3)\qquad &\Delta_{a+M-\mu}^\nu \Delta_a^\mu f(t)\\ &=\Delta_a^{\nu+\mu} f(t) - \sum_{j=0}^{M-1} h_{-\nu-M+j}(t-M+\mu, a)\Delta_a^{j-M+\mu} f(a+M-\mu),\end{aligned}$$

where $N-1<\nu<N$. If $\nu = N$, then (1.3) simplifies to

$$\Delta_{a+M-\mu}^\nu \Delta_a^\mu f(t) = \Delta_a^{\nu+\mu} f(t), \quad t \in \mathbb{N}_{a+M-\mu}.$$

In particular, this renders reduction of the order of fractional difference equations impossible.

An interesting and surprising aspect of the sequential case is that in the results obtained so far the type of result obtained seems to depend on the specific choice of μ and ν. To see what is meant by this consider the following result, which can be found in [32, Theorem 1.1, Corollary 1.2].

THEOREM 1.2: *Consider the following conditions:*

(1) $f(a) \geq 0$,
(2) $\Delta f(a) \geq 0$,
(3) $\Delta^{\nu}_{1+a-\mu}\Delta^{\mu}_{a} f(t) \geq 0$,

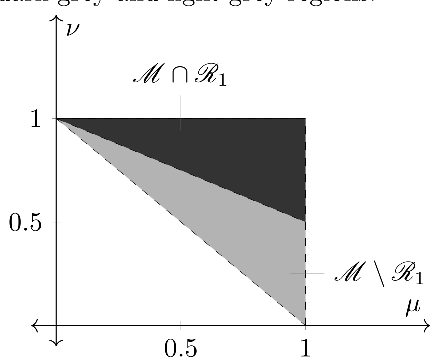
for each $t \in \mathbb{N}_{2+a-\mu-\nu} := \{2+a-\mu-\nu, 3+a-\mu-\nu, \ldots\}$.

In addition, define the sets $\mathscr{M}$, $\mathscr{R}_1$, $\mathscr{R}_2 \subseteq [0,1] \times [0,1]$ *as follows.*

$$\mathscr{M} := \{(\mu,\nu) \in [0,1] \times [0,1] : 1 < \mu + \nu < 2\},$$
$$\mathscr{R}_1 := \left\{(\mu,\nu) \in [0,1] \times [0,1] : \nu \geq -\frac{1}{2}(\mu - 2)\right\},$$
$$\mathscr{R}_2 := \left\{(\mu,\nu) \in [0,1] \times [0,1] : \nu < -\frac{1}{2}(\mu - 2) - \frac{1}{2}\mu^2\right\}.$$

(1) *Suppose that* $f : \mathbb{N}_a \to \mathbb{R}$ *and that each of conditions (1)–(3) above holds. If* $(\mu,\nu) \in \mathscr{M} \cap \mathscr{R}_1$, *then* f *is monotone increasing on* $\mathbb{N}_a$*—i.e.,* $\Delta f(t) \geq 0$ *for each* $t \in \mathbb{N}_a$.
(2) *Assume that conditions (1)–(3) above hold. Then for each pair* $(\mu,\nu) \in \mathscr{M} \cap \mathscr{R}_2$ *there exists a function* $f : \mathbb{N}_a \to \mathbb{R}$ *such that* f *is not monotone increasing on* $\mathbb{N}_a$*—i.e., there exists a point* $t_0 \in \mathbb{N}_{a+1}$ *such that* $\Delta f(t_0) < 0$.

Theorem 1.2 demonstrates that with regard to condition (1)–(3), these conditions are sufficient to imply the monotonicity of f only if the parameters (μ,ν) live in a specific region of the parameter space $[0,1] \times [0,1]$. More specifically, when $(\mu,\nu) \in \mathscr{M} \cap \mathscr{R}_1$ conditions (1)–(3) are sufficient to deduce the monotonicity of f, whereas when $(\mu,\nu) \in \mathscr{M} \setminus \mathscr{R}_1$ conditions (1)–(3) are insufficient to deduce the monotone behavior of f. This is represented in the drawing below by, respectively, the dark grey and light grey regions.



Thus, with respect to the definition of the discrete fractional difference as given by (1.1) we see that the discrete sequential difference exhibits a complexity that appears to be absent in the non-sequential case. For additional results, similar to Theorem 1.2, in the sequential setting, we direct the reader to the papers [21, 29, 30, 31, 32].

In this paper we want to give a qualitative and quantitative jump in this investigation into the connection between the sign of an appropriate fractional difference operator and its subsequent relationship to monotonicity and convexity. Indeed, in contrast to the previously mentioned references, in this work we take a different approach, which yields new results and insights. In fact, there are a couple of novelties to the work we pursue in this paper.

First of all, and as already mentioned, here we consider a newer definition (1.2) for the fractional difference, which differs from the more extensively used definition in (1.1). As we have already explained at the beginning of this section, this newer definition involves an appropriate discrete convolution, and it has some appealing properties, which we exploit in this paper. We are not aware of any papers that treat monotonicity and convexity results (either sequential or non-sequential) for this new definition. Consequently, in light of the burgeoning literature for definition (1.1), it seems interesting to see what changes, if any, are encountered when trying to transfer results, such as Theorem 1.1, to this new setting. Due to the fact that we do not directly use definition (1.1) here, the proofs of our monotonicity and convexity results are quite different than those given in other papers. Hence, there is, we believe, interest in not only the results we produce but also the proof methodology.

Second, regarding the specific results we provide here we note that they are, in many cases, cleaner. For example, as will be seen in Sections 5–7, our results apply on the full range of the parameter space—this is in contrast to the situation described earlier in this section regarding the existing results for definition (1.1). This reason for this improvement is a consequence of our novel proof methodology yielding slightly different hypotheses than those that have been previously used. So, the method of proof introduced in this paper yields better insight into the range of results one can recover and, more precisely, the relationship between sequential discrete fractional operators and monotonicity and convexity.

Finally, we would like to mention also that one important task that arises in the study of fractional difference equations of the form

$$\Delta_a^\nu u(n) = f(n+a+N-\nu, u(n+a+N-\nu)), \quad n \in \mathbb{N}_0, \quad N-1 < \nu < N,$$

is to understand how properties on the forcing term f and initial conditions of the equation affect the qualitative properties of the solution u, whenever it exists. Indeed, there are many papers in the literature that analyze precisely this type of problem in the context of both initial and boundary value problems. So, a potential application of the results we present here is to the theory of fractional difference equations as well as any discrete fractional models that involve difference equations.

We conclude by mentioning that, in addition to the references already mentioned, there exists now a wide and increasing literature in the discrete fractional calculus. We do not attempt to give an exhaustive accounting here, but wish to mention that there are many other active lines of research presently being pursued. For example, much work has been completed in the area of boundary value problems—see, for example, papers by Ferreira [24], Goodrich [27], Lv, Gong and Chen [49], Sitthiwirattham, Tariboon and Ntouyas [51] and Sitthiwirattham [50]. In addition, many papers have provided an analysis of various abstract properties of discrete fractional operators—see, for example, the papers by Abdeljawad [5, 6], Anastassiou [9], Atici and Acar [10], Bastos, Mozyrska and Torres [18], Ferreira [23], Lizama [43], Lizama and Murillo-Arcila [45], and Xu and Zhang [52]. All in all, there no exist many interesting research directions as concerns the discrete fractional calculus, and we hope that this work will continue to further this research, especially as concerns the relatively new definition of the fractional difference that we employ here.

2. Preliminaries

In this section we collect some preliminary results that will be used throughout the paper. For additional background on the discrete fractional calculus and related topics we direct the interested reader to the recent monograph by Goodrich and Peterson [33].

For any $\alpha \in \mathbb{C}$ we define

$$k^\alpha(n) := \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}, \quad n \in \mathbb{N}_1$$

and

$$k^\alpha(0) := 1.$$

In case $\alpha = 1$ we denote $k^1(n) \equiv k(n)$. For $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, we have

$$k^\alpha(j) = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)\Gamma(j+1)} = \binom{j+\alpha-1}{\alpha-1} = (-1)^n \binom{-\alpha}{j}, \quad j \in \mathbb{N}_0. \tag{2.1}$$

Remark 2.1: We note that the kernels $k^\alpha(j)$ have appeared in the literature in connection with summability of Fourier series [53, Chapter 3] among others. It plays the same role that the kernel

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad \beta > 0$$

plays in the continuous case. This connection is reinforced by the fact that the following identity holds

$$k^\alpha(n) = \int_0^\infty p_n(t) g_\alpha(t) dt,$$

where

$$p_n(t) := e^{-t}\frac{t^n}{n!}$$

is the Poisson distribution; see [44, Example 3.3]. The integral on the right hand side is named the Poisson transformation of g_α and was defined in [44]. Some important properties of the kernels $k^\alpha(j)$ are described in the next section.

Let $a \in \mathbb{R}$ be given. In what follows we denote by $s(\mathbb{N}_a; \mathbb{R})$ the vectorial space that consists of all sequences $f : \mathbb{N}_a \to \mathbb{R}$. The definition of the α-th fractional sum on the set $\mathbb{N}_0$ is given by:

Definition 2.2: For each $\alpha > 0$ and $f \in s(\mathbb{N}_0; \mathbb{R})$, we define the fractional sum of order α as follows:

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^\alpha(n-j) f(j), \quad n \in \mathbb{N}_0.$$

For example, in case $\alpha = 1$ we have $k^1(j) \equiv 1$ for all $j \in \mathbb{N}_0$ and, hence, $\Delta^{-1} f(n) = \sum_{j=0}^n f(j)$, $n \in \mathbb{N}_0$. More generally, for $\alpha = m \in \mathbb{N}$ we have

$$\Delta^{-m} f(n) = \sum_{j=0}^{n} \frac{(n-j+m-1)!}{(m-1)!(n-j)!} f(j),$$

for all $n \in \mathbb{N}_0$.

Definition 2.2 corresponds to the definition of the nabla fractional sum operator, $\nabla_a^{-\alpha}$, due to Atici and Eloe, in a very special case (consider [12, Definition (2)] or [11, Definition (2.1)] with $a = 0$). In such a case, the definition of the operator

$$\nabla_0^{-\alpha} \equiv \Delta^{-\alpha}$$

admits a convolutional form that turns out to be more flexible and appropriate for mathematical analysis. This special case of the nabla operator was highlighted by Lizama in [44] and used, for instance, by Abadías et al. in [4] where the algebraic structure of Cesáro sums and their relationship with the theory of fractional sums and fractional semigroups of operators are investigated. In such paper it is shown how the connection provides new insight into the properties of Cesáro sums and their interplay with algebra homomorphisms. For further information on this line of research see the work of Abadías [1], Abadias and Miana [3] and Lizama [43, 48, 45, 47]. Historically, we note that by editorial delay in the publication, the paper [44] appeared after the above mentioned references.

Given $a \in \mathbb{R}$, we define the translation (by $a \in \mathbb{R}$) operator

$$\tau_a : s(\mathbb{N}_a; \mathbb{R}) \to s(\mathbb{N}_0; \mathbb{R})$$

by

$$\tau_a f(n) := f(a+n), \quad n \in \mathbb{N}_0. \tag{2.2}$$

Note that

$$\tau_a^{-1} = \tau_{-a} \quad \text{and} \quad \tau_{a+b} = \tau_a \circ \tau_b = \tau_b \circ \tau_a.$$

We also recall that the finite convolution $*$ of two sequences $f, g \in s(\mathbb{N}_0; \mathbb{R})$ is defined by

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j)g(j), \quad n \in \mathbb{N}_0.$$

Therefore, by definition, we have

$$\Delta^{-\alpha} f(n) = (k^\alpha * f)(n),\ n \in \mathbb{N}_0,$$

where the convolution operator $*$ enjoys algebraic properties like commutativity and associativity, which will be very useful to simplify and better understand the proof of some results. In this context, a very useful result is the following.

Lemma 2.3: *Let $f, g \in s(\mathbb{N}_0;\mathbb{R})$ be sequences. Then for each $p = 1, 2, \ldots$ we have*

$$(f * \tau_p g)(n) = \tau_p(f * g)(n) - \sum_{j=0}^{p-1} \tau_p f(n-j)g(j).$$

In particular, for $p = 1$ we have

$$(f * \tau_1 g)(n) = (f * g)(n+1) - f(n+1)g(0),$$

and

$$(f * \tau_2 g)(n) = (f * g)(n+2) - f(n+2)g(0) - f(n+1)g(1)$$

in case $p = 2$ and

$$(f * \tau_3 g)(n) = (f * g)(n+3) - f(n+3)g(0) - f(n+2)g(1) - f(n+1)g(2)$$

in case $p = 3$.

An immediate consequence of the definition of the operator of a fractional sum, is the well known law of exponents and the power rule. We note that compared with [11, Theorem 2.2 and Lemma 2.3] and [12, Theorem 2.1] (case $a = 0$) our one-line proofs are simpler.

Corollary 2.4: (a) *For α, $\beta > 0$ and $f \in s(\mathbb{N}_0, \mathbb{R})$ we have*

$$\Delta^{-\alpha}(\Delta^{-\beta} f)(n) = \Delta^{-(\alpha+\beta)} f(n) = \Delta^{-\beta}(\Delta^{-\alpha} f)(n), \quad \forall n \in \mathbb{N}_0.$$

(b) *For α, $\beta > 0$, we have*

$$\Delta^{-\alpha} k^{\beta}(n) = k^{\alpha+\beta}(n), \quad \forall n \in \mathbb{N}_0.$$

Proof. (a) For all $f \in s(\mathbb{N}_0; X)$, we have

$$\Delta^{-\alpha}(\Delta^{-\beta} f) = \Delta(k^{\beta} * f) = k^{\alpha} * (k^{\beta} * f) = (k^{\alpha} * k^{\beta}) * f = k^{\alpha+\beta} * f = \Delta^{-(\alpha+\beta)} f.$$

Interchanging the role of α and β, we obtain the claim.

(b) For $\alpha, \beta > 0$, we get

$$\Delta^{-\alpha} k^{\beta} = k^{\alpha} * k^{\beta} = k^{\alpha+\beta},$$

obtaining the desired result. ∎

The following definitions have been used in several recent papers. They were introduced by Lizama in the article [44] to study the abstract fractional Cauchy problem on the time scale $\mathbb{N}_0$ and used, for the first time, in the work [43] for the analysis of linear abstract difference equations in the form

$$\Delta^\alpha u(n) = Tu(n) + f(n), \quad n \in \mathbb{N}_0,$$

where T is a bounded linear operator defined on a Banach space X. In [43] the maximal regularity property on Lebesgue sequence spaces was studied by Lizama. By using Blunck's operator-valued Fourier multiplier theorem, the author completely characterized the existence and uniqueness of solutions in Lebesgue sequence spaces with fractional order. Further research in the same vein are the papers of Lizama and Murillo [45, 47] and Leal et al. [41]. See also the monograph of Agarwal, Cuevas and Lizama [7] for more information on the method used in the above-cited references, as well as applications in the setting of abstract difference equations. After these works, in [48], Lizama and Velasco studied existence and uniqueness of solutions for the abstract problem

$$\Delta^\alpha u(n) = Tu(n) + f(n, u(n)), \quad n \in \mathbb{N}_0.$$

Additional research on this nonlinear problem is found in the paper by He et al. [36].

Definition 2.5: The fractional difference operator $\Delta^\alpha : s(\mathbb{N}_0;\mathbb{R}) \to s(\mathbb{N}_0;\mathbb{R})$ of order $\alpha > 0$ (in the sense of Riemann–Liouville) is defined by

$$\Delta^\alpha f(n) := \Delta^m \circ \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0, \tag{2.3}$$

where $m - 1 < \alpha < m$, $m := \lceil \alpha \rceil$, the least integer that is greater than or equal to α.

As has been pointed out by Lizama in the reference [44], the operator Δ^α corresponds to a special case of the nabla fractional difference operator of order $\alpha > 0$. We refer to the papers by Atici and Eloe [11] and [12] for more information on the nabla operator. We note that the symbol Δ^α instead of ∇_0^α has been used in several earlier papers, e.g., [4, 8, 11, 41, 42, 44, 43, 48, 45].

Remark 2.6: We observe that the above definition is given in [44] in the context of the vector-valued space of sequences $s(\mathbb{N}_0, X)$ where X is a Banach space. Since in this paper we will study positivity, monotonicity and convexity, we consider $X = \mathbb{R}$. However, all of the results in this paper remain valid if we replace X by any suitable ordered Banach space.

Definition 2.7: We define the fractional difference (in the sense of Caputo) of order $\alpha > 0$ by

$$_C\Delta^\alpha f(n) := \Delta^{-(m-\alpha)}(\Delta^m f)(n), \quad n \in \mathbb{N}_0,$$

where $m-1 < \alpha < m$, $m := \lceil \alpha \rceil$.

In both definitions, for $\alpha = 0$ we define $\Delta^0 = I$, which is the identity operator on $s(\mathbb{N}_0; \mathbb{R})$.

Remark 2.8: Concerning the case of sequences defined on the set $\mathbb{Z}$, we note that a suitable definition of fractional difference in the Weyl-like sense was introduced by Abadias and Lizama [2], as follows: We consider the weighted Lebesgue space $\ell^1_\rho(\mathbb{Z}, X)$ where $\rho(n) = |n|^{\alpha-1}$, $\alpha > 0$, $n \in \mathbb{Z}$, and for $f \in \ell^1_\rho(\mathbb{Z}, X)$ we define the α-th fractional sum

$$\Delta^{-\alpha} f(n) = \sum_{j=-\infty}^{n} k^\alpha(n-j) f(j), \quad n \in \mathbb{Z}, \quad \alpha > 0,$$

and then the definition of fractional difference of order $\alpha > 0$ follows the same rule as that in (2.3) but for sequences in $\ell^1_\rho(\mathbb{Z}, X)$. With such notion, in the paper [2] the authors achieved the existence and uniqueness of almost automorphic solutions for abstract difference equations in the form

$$\Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z},$$

where A is the generator of a C_0-semigroup. Further research on this line of investigation are the papers by Alvarez and Lizama [8], Lizama and Murillo [46] and Leal et al. [42].

We conclude by listing some properties of the operator of fractional difference. These will be useful to us on several occasions in the sequel.

Proposition 2.9: *The following properties hold:*

(i) *For any* $a \in s(\mathbb{N}_0; \mathbb{R})$ *we have* $\Delta(k * a)(n) = a(n+1)$.
(ii) *For any* $0 < \alpha < 1$ *and* $b \in s(\mathbb{N}_0; \mathbb{R})$ *we have* $(\Delta \circ \Delta^\alpha b)(n) = (\Delta^{\alpha+1} b)(n)$.
(iii) *For any* $1 < \alpha < 2$ *and* $b \in s(\mathbb{N}_0; \mathbb{R})$ *we have* $(\Delta \circ \Delta^\alpha b)(n) = (\Delta^{\alpha+1} b)(n)$.
(iv) *For any* $0 < \alpha < 1$ *and* $b \in s(\mathbb{N}_0; \mathbb{R})$ *we have*

$$\Delta^\alpha \circ \Delta b(n) = \Delta \circ \Delta^\alpha b(n) - \Delta k^{1-\alpha}(n+1) b(0).$$

(v) *For any* $a, b \in s(\mathbb{N}_0; \mathbb{R})$ *we have* $\Delta(a * b)(n) = (\Delta a * b)(n) + b(n+1)a(0)$ *and*

$$(\Delta a * b)(n) = (a * \Delta b)(n) + a(n+1)b(0) - a(0)b(n+1).$$

Proof. (i) is clear from the definition, since $k(n) \equiv k^1(n)$. To prove (ii) we note that by definition

$$\Delta^\alpha b(n) = \Delta(k^{1-\alpha} * b)(n) \quad \text{and} \quad \Delta^{\alpha+1} b(n) = \Delta^2(k^{2-(\alpha+1)} * b)(n) = \Delta^2(k^{1-\alpha} * b)(n).$$

Therefore

$$\Delta \circ \Delta^\alpha b(n) = \Delta \circ \Delta(k^{1-\alpha} * b)(n) = \Delta^2(k^{1-\alpha} * b)(n) = \Delta^{\alpha+1} b(n).$$

To prove (iii) we note that by definition

$$\Delta^\alpha b(n) = \Delta^2(k^{2-\alpha} * b)(n) \quad \text{and} \quad \Delta^{\alpha+1} b(n) = \Delta^3(k^{3-(\alpha+1)} * b)(n) = \Delta^3(k^{2-\alpha} * b)(n).$$

Therefore

$$\Delta \circ \Delta^\alpha b(n) = \Delta \circ \Delta^2(k^{2-\alpha} * b)(n) = \Delta^3(k^{2-\alpha} * b)(n) = \Delta^{\alpha+1} b(n).$$

To show (iv) we first note that by definition $\Delta^\alpha b(n) = \Delta(k^{1-\alpha} * b)(n)$ which imply $\Delta \circ \Delta^\alpha b(n) = (\Delta^2 k^{1-\alpha} * b)(n)$. Now, using definition of Δ^α and Lemma 2.3 we get

$$\begin{aligned}
\Delta^\alpha \circ \Delta b(n) &= \Delta(k^{1-\alpha} * \Delta b)(n) \\
&= \Delta(k^{1-\alpha} * [\tau_1 b - b])(n) \\
&= \Delta(k^{1-\alpha} * \tau_1 b)(n) - \Delta(k^{1-\alpha} * b)(n) \\
&= \Delta(k^{1-\alpha} * b)(n+1) - \Delta k^{1-\alpha}(n+1) b(0) - \Delta(k^{1-\alpha} * b)(n) \\
&= \Delta^2(k^{1-\alpha} * b)(n) - \Delta k^{1-\alpha}(n+1) b(0) \\
&= \Delta \circ \Delta^\alpha b(n) - \Delta k^{1-\alpha}(n+1) b(0),
\end{aligned}$$

which completes the proof. Finally, we check (v). We have by Lemma 2.3 and commutativity

$$\begin{aligned}
(\Delta a * b)(n) &= ([\tau_1 a - a] * b)(n) = (\tau_1 a * b)(n) - (a * b)(n) \\
&= (b * a)(n+1) - b(n+1) a(0) - (a * b)(n) \\
&= \Delta(a * b)(n) - b(n+1) a(0).
\end{aligned}$$

The last line in (iv) is a consequence of the above identity and the commutativity of the convolution. ∎

3. Properties of the kernel k^α

In this section we wish to collect some important properties of the map

$$n \mapsto k^\alpha(n).$$

These will be essential in what follows. To begin we mention that the kernel k^α satisfies each of the following ten properties.

PROPOSITION 3.1: *The following properties hold:*

(i) *For* $\alpha > 0$, $k^\alpha(n+1) = \frac{\alpha+n}{n+1} k^\alpha(n)$, $n \in \mathbb{N}_0$.
(ii) *For* $\alpha > 0$, $k^\alpha(n) > 0$, $n \in \mathbb{N}_0$.
(iii) *For* $0 < \alpha < 1$, $k^\alpha(n)$ *is decreasing and* $k^\alpha(n) \to 0$ *as* $n \to \infty$.
(iv) *For* $\alpha > 1$, $k^\alpha(n)$ *is increasing.*
(v) *The generation formula:*

$$\sum_{j=0}^{\infty} k^\alpha(j) z^j = \frac{1}{(1-z)^\alpha} \quad \text{for all } z \in \mathbb{C}, \quad |z| < 1.$$

(vi) *The identity*

$$k^\alpha(j) = (-1)^j \frac{\Gamma(1-\alpha)}{\Gamma(1+j)\Gamma(1-\alpha-j)}, \quad j \in \mathbb{N}_0.$$

(vii) *For* $\alpha, \beta \in \mathbb{C}$, *we have the semigroup property*

$$k^{\alpha+\beta}(n) = \sum_{j=0}^{n} k^\alpha(n-j) k^\beta(j) =: (k^\alpha * k^\beta)(n), \quad n \in \mathbb{N}_0.$$

(viii) $k^\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1 + O(\frac{1}{n}))$, $n \in \mathbb{N}$.
(ix) $k^\alpha(n) \le k^\beta(n)$ *for* $0 < \alpha \le \beta$, $n \in \mathbb{N}_0$.
(x) $\frac{(n+1)^{\alpha-1}}{\Gamma(\alpha)} < k^\alpha(n) < \frac{n^{\alpha-1}}{\Gamma(\alpha)}$, $n \in \mathbb{N}$, $0 < \alpha < 1$.

Proof. (i) follows from the definition and the recurrence property for the Gamma function. (ii) is a consequence of the definition and the first part of (iii) follows from (i). The second part follows from (viii) which, in turn, is proved in [53, Vol. I, p. 77 (1.18)] and indicated in [4]. In the last reference, also the property (vii) is shown. (iv) and (ix) are straightforward to check. (x) a is consequence of the Gautschi inequality [4]. The generation formula (v) can be proved as follows: Recall the following identity:

$$(3.1)\quad (1+w)^{\beta} = \sum_{j=0}^{\infty} \binom{\beta}{j} w^j = \sum_{j=0}^{\infty} \frac{\Gamma(1+\beta)}{\Gamma(1+j)\Gamma(\beta-j+1)} w^j \quad \text{for all } |w|<1,$$

which is valid for all $\beta \in \mathbb{C} \setminus \mathbb{N}_0$ (see [34, Formula 1.110]). Using (3.1) with $w=-z$ and $\beta=-\alpha$ it follows that for all $\alpha>0$ the coefficients in the power series of $1/(1-z)^{\alpha}$ are

$$(-1)^j \frac{\Gamma(1-\alpha)}{\Gamma(1+j)\Gamma(1-\alpha-j)}, \quad j \in \mathbb{N}_0,$$

whenever $|z|<1$. Now, using the identity $\Gamma(1-z)\Gamma(z)=\pi/\sin(\pi z)$ for $z=\alpha+j$ and for $z=\alpha$, respectively, we get

$$(3.2)\qquad \Gamma(1-\alpha-j)\Gamma(\alpha+j) = \frac{\pi}{\sin\pi(\alpha+j)}$$

and

$$(3.3)\qquad \Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin\pi\alpha}.$$

Then, using (3.2) and (3.3), we obtain

$$\begin{aligned}\frac{(-1)^j\Gamma(1-\alpha)}{\Gamma(1-\alpha-j)\Gamma(j+1)} &= \frac{\sin(\pi\alpha)}{\sin(\pi\alpha)(-1)^j}\frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha-j)\Gamma(j+1)}\\ &= \frac{\sin(\pi\alpha)\Gamma(1-\alpha)}{\pi}\frac{1}{\Gamma(1-\alpha-j)\Gamma(j+1)}\frac{\pi}{\sin\pi(\alpha+j)}\\ &= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha-j)\Gamma(j+1)}\frac{\pi}{\sin\pi(\alpha+j)}\\ &= \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha+j)}{\Gamma(j+1)} = k^{\alpha}(j),\end{aligned}$$

for all $j \in \mathbb{N}_0$, proving (v). In particular, the above identities prove (vi). ∎

Other properties that will be important in the development of this article are described in the following Lemma.

Lemma 3.2: *For any $\alpha>0$, the following identities hold:*

(i) $\Delta k^{\alpha}(n) = (\alpha-1)\frac{k^{\alpha}(n)}{n+1}$.

(ii) $\Delta^2 k^{\alpha}(n) = (\alpha-2)(\alpha-1)\frac{k^{\alpha}(n)}{(n+1)(n+2)}$.

(iii) $\Delta^3 k^{\alpha}(n) = (\alpha-3)(\alpha-2)(\alpha-1)\frac{k^{\alpha}(n)}{(n+1)(n+2)(n+3)}$.

Proof. The identity (i) follows from the equality

$$k^\gamma(n+1) = \frac{\gamma+n}{n+1}k^\gamma(n) \tag{3.4}$$

valid for all $\gamma > 0$. To prove (ii) we use the identity (3.4) to compute

$$\begin{aligned}\Delta\Big[\frac{k^\alpha(n)}{n+1}\Big] &= \frac{k^\alpha(n+1)}{n+2} - \frac{k^\alpha(n)}{n+1} = \frac{\alpha+n}{(n+1)(n+2)}k^\alpha(n) - \frac{k^\alpha(n)}{n+1} \\ &= \frac{k^\alpha(n)}{(n+1)}\Big[\frac{\alpha+n}{n+2} - 1\Big] = (\alpha-2)\frac{k^\alpha(n)}{(n+1)(n+2)}.\end{aligned}$$

Hence, by (i) we obtain

$$\Delta^2 k^\alpha(n) = (\alpha-1)\Delta\Big[\frac{k^\alpha(n)}{n+1}\Big] = (\alpha-1)(\alpha-2)\frac{k^\alpha(n)}{(n+1)(n+2)}.$$

Finally, we compute

$$\begin{aligned}\Delta\Big[\frac{k^\alpha(n)}{(n+1)(n+2)}\Big] &= \frac{k^\alpha(n+1)}{(n+2)(n+3)} - \frac{k^\alpha(n)}{(n+1)(n+2)} \\ &= \frac{\alpha+n}{(n+1)(n+2)(n+3)}k^\alpha(n) - \frac{k^\alpha(n)}{(n+1)(n+2)} \\ &= \frac{k^\alpha(n)}{(n+1)(n+2)}\Big[\frac{\alpha+n}{n+3} - 1\Big] \\ &= (\alpha-3)\frac{k^\alpha(n)}{(n+1)(n+2)(n+3)}.\end{aligned}$$

By the use of (ii) we conclude from the above identity that

$$\begin{aligned}\Delta^3 k^\alpha(n) &= (\alpha-1)(\alpha-2)\Delta\Big[\frac{k^\alpha(n)}{(n+1)(n+2)}\Big] \\ &= (\alpha-1)(\alpha-2)(\alpha-3)\frac{k^\alpha(n)}{(n+1)(n+2)(n+3)}.\end{aligned}$$ ∎

As an immediate consequence, we obtain

Corollary 3.3: *We have:*

(i) $\Delta k^\alpha(n) \le 0$ *if and only if* $0 \le \alpha \le 1$.
(ii) $\Delta^2 k^\alpha(n) \le 0$ *if and only if* $1 \le \alpha \le 2$.
(iii) $\Delta^3 k^\alpha(n) \le 0$ *if and only if* $\alpha \in [0,1] \cup [2,3]$.

Remark 3.4: By simple induction, we observe that the above Lemma and its Corollary can be generalized to

$$\Delta^N k^\alpha(n) = \prod_{j=1}^{N}\Big(\frac{\alpha-j}{n+j}\Big)k^\alpha(n), \quad \alpha>0,\ n\in\mathbb{N}_0,\ N\in\mathbb{N}_1,$$

and

$$\Delta^N k^\alpha(n)\le 0 \text{ for all } n\in\mathbb{N}_0 \text{ if and only if } \prod_{j=1}^{N}(\alpha-j)\le 0, \quad N\in\mathbb{N}_1.$$

4. Transference

In this section we relate definition (1.1) with definition (2.3) by means of the operator of translation (2.2). The next result has important consequences, since it highlights the fact that it is possible to transfer properties from one definition to another with a clear advantage of the definition (2.3) because it allows the use of simpler, cleaner and transparent algebraic manipulations that only involves the convolution of given sequences with the distinguished sequence kernel k^α. Our main result in this section is the following.

Theorem 4.1: *Let $0<\alpha<1$ and $a\in\mathbb{R}$. For each sequence $f\in s(\mathbb{N}_a;\mathbb{R})$ we have*

$$\tau_{a+1-\alpha}\circ\Delta_a^\alpha f = \Delta^\alpha\circ\tau_a f.$$

In other words, the following diagram is commutative:

$$\begin{array}{ccc} s(\mathbb{N}_0;\mathbb{R}) & \xrightarrow{\ \Delta^\alpha\ } & s(\mathbb{N}_0;\mathbb{R}) \\ \tau_a\Big\uparrow & & \Big\uparrow\tau_{a+1-\alpha} \\ s(\mathbb{N}_a;\mathbb{R}) & \xrightarrow[\ \Delta_a^\alpha\ ]{} & s(\mathbb{N}_{a+1-\alpha};\mathbb{R}). \end{array}$$

Proof. Given $n\in\mathbb{N}_0$ we set $t=a+1-\alpha+n\in\mathbb{N}_{a+1-\alpha}$. Then a simple manipulation of the definition shows that

$$\Delta_a^\alpha f(t) = \frac{\Gamma(1-\alpha)}{\Gamma(-\alpha)}\sum_{p=0}^{n+1}\frac{1}{n-p+1}k^{1-\alpha}(n-p)f(a+p).$$

On the other hand, we note that

$$\begin{aligned} &\Delta_a^{-(1-\alpha)} f(t+1) - \Delta_a^{-(1-\alpha)} f(t) \\ &\quad = \sum_{p=0}^{n+1} k^{1-\alpha}(n+1-p) f(a+p) - \sum_{p=0}^{n} k^{1-\alpha}(n-p) f(a+p). \end{aligned} \tag{4.1}$$

Using the fact that $k^{1-\alpha}(-1) = 0$, where we use the convention that $1/\Gamma(0) := 0$ in (2.1), we obtain

$$\begin{aligned} &\Delta_a^{-(1-\alpha)} f(t+1) - \Delta_a^{-(1-\alpha)} f(t) \\ &\quad = \sum_{p=0}^{n+1} [k^{1-\alpha}(n+1-p) - k^{1-\alpha}(n-p)] f(a+p). \end{aligned}$$

A computation using the definition of k^β shows that

$$k^{1-\alpha}(n+1-p) - k^{1-\alpha}(n-p) = \frac{-\alpha}{(n+1-p)} k^{1-\alpha}(n-p).$$

Consequently, using the identity $z\Gamma(z) = \Gamma(z+1)$ we obtain that

$$\Delta_a^\alpha f(t) = \Delta_a^{-(1-\alpha)} f(t+1) - \Delta_a^{-(1-\alpha)} f(t).$$

Finally, we have by definition (2.3) and the identity (4.1)

$$\begin{aligned} (\Delta^\alpha \circ \tau_a f)(n) =& \Delta^\alpha(\tau_a f)(n) \\ =& (k^{1-\alpha} * \tau_a f)(n+1) - (k^{1-\alpha} * \tau_a f)(n) \\ =& \Delta_a^\alpha f(t) = (\tau_{a+1-\alpha} \circ \Delta_a^\alpha f)(n), \end{aligned}$$

for each $n \in \mathbb{N}_0$, proving the theorem. ∎

Corollary 4.2: *Let $0 < \alpha < 1$ and $\beta, a \in \mathbb{R}$. We have*

$$\tau_{1-\beta} \circ \Delta_{a+1-\beta}^\alpha = \Delta_a^\alpha \circ \tau_{1-\beta}.$$

Proof. By Theorem 4.1 we have $\Delta_b^\gamma = \tau_{-b-1+\gamma} \circ \Delta^\gamma \circ \tau_b$ for any $0 < \gamma < 1$ and $b \in \mathbb{R}$. Therefore,

$$\begin{aligned} \tau_{1-\beta} \circ \Delta_{a+1-\beta}^\alpha =& \tau_{1-\beta} \circ \tau_{-a-1+\beta-1+\alpha} \circ \Delta^\alpha \circ \tau_{a+1-\beta} \\ =& (\tau_{-a-1+\alpha} \circ \Delta^\alpha \circ \tau_a) \circ \tau_{1-\beta} = \Delta_a^\alpha \circ \tau_{1-\beta}. \end{aligned}$$ ∎

More generally, we can prove the following result. We omit the proof.

Theorem 4.3: *Let $N-1<\alpha<N$, $N\in\mathbb{N}_1$ and $a,\beta\in\mathbb{R}$. For each sequence $f\in s(\mathbb{N}_a;\mathbb{R})$ we have*

$$\tau_{a+N-\alpha}\circ\Delta_a^\alpha f=\Delta^\alpha\circ\tau_a f,$$

and

$$\tau_{N-\beta}\circ\Delta_{a+N-\beta}^\alpha f=\Delta_a^\alpha\circ\tau_{N-\beta}f.$$

Remark 4.4: Let $\alpha>0$ and $a\in\mathbb{R}$ be given. By definition, between the operators $\nabla_a^\alpha : s(\mathbb{N}_a;\mathbb{R})\to s(\mathbb{N}_a;\mathbb{R})$ and $\Delta^\alpha : s(\mathbb{N}_0;\mathbb{R})\to s(\mathbb{N}_0;\mathbb{R})$ we have the following relation:

$$\tau_a\circ\nabla_a^\alpha=\Delta^\alpha\circ\tau_a.$$

From this identity and Theorem 4.3 we retrieve the following identity stated in [12, Lemma 2.1 (i)]:

$$\tau_{a+N-\alpha}\circ\Delta_a^\alpha=\tau_a\circ\nabla_a^\alpha,$$

where $N-1<\alpha<N$.

5. Positivity and α-monotonicity

In this section we prove several interrelated results. Because of the transference property identified in Section 4, our strategy is to prove results for the operator Δ^α, which allows us to take advantage of its good operational properties with respect to convolution, and then to transfer these results to the operator Δ_a^α. Our main goal is to show that under certain reasonable conditions on either $(\Delta^\alpha u)(t)$ or $(\Delta^\beta\circ\Delta^\alpha u)(t)$ one may deduce the positivity of u as well as, respectively, the α- or $(\alpha+\beta)$-monotonicity—see Definition 5.1 for the definition of α-monotonicity, which is a weaker form of monotonicity suitable for the fractional setting (see also Atici and Uyanik [14]).

More specifically, our first result, which is Theorem 5.4, is both a positivity and an α-monotonicity result in the case of a non-sequential fractional difference Δ^α. Then by transference we recover a corresponding result, namely Corollary 5.6, for the operator Δ_a^α. We note that Theorem 5.4 is an analogue of [14, Theorem 3.5]. In this latter result, Atici and Uyanik proved that if, for a function $y:\mathbb{N}_0\to\mathbb{R}$, it holds that $y(0)\geq 0$ and $\Delta_0^\nu y(t)\geq 0$ for each $t\in\mathbb{N}_{1-\nu}$, then y is ν-increasing on $\mathbb{N}_0$, by which it is meant that $y(t+1)\geq\nu y(t)$, for each $t\in\mathbb{N}_0$.

Our next collection of results concerns fractional sequential operators of the form either $\Delta^\beta \circ \Delta^\alpha$ or $\Delta^\beta_{1-a-\alpha} \circ \Delta^\alpha_a$, the latter being obtained by transference of the results regarding the former operator. The first result in this collection is Theorem 5.8. As far as we know, a result of the type given by Theorem 5.8, that is, a positivity-type result in the sequential setting, has never been formally stated. Moreover, the $(\alpha+\beta)$-th monotonicity likewise has never been considered, so far as we are aware. We emphasize that once we have the result for $\Delta^\beta \circ \Delta^\alpha$, which allows us to utilize the convolution operator, then it is easy (see Corollary 5.10) to transfer the result to the case of the sequential operator $\Delta^\beta_{1-a-\alpha} \circ \Delta^\alpha_a$.

We begin by introducing the following definition:

Definition 5.1: Let $0 \le \alpha \le 1$. We say that a sequence $u \in s(\mathbb{N}_0, \mathbb{R})$ is α-monotone increasing (decreasing) if

$$u(n+1) \ge \alpha u(n)$$

(respectively, $u(n+1) \le \alpha u(n)$) for all $\mathbb{N}_0$.

When $\alpha = 1$ we recover the classical notion of monotonicity. Otherwise, the notion of α-monotonicity is more general. Note that 0-monotone increasing means that a sequence is positive.

Remark 5.2: We have the following properties that can be directly checked from the definition.

(i) Subordination: If $0 \le \alpha \le \beta \le 1$, then β-monotone increasing implies α-monotone increasing; in particular, each monotone increasing sequence is α-monotone.
(ii) If u is α-monotone increasing and $u(0) \ge 0$, then u is positive.

Corresponding properties for α-monotone decreasing sequences hold.

Example 5.3: By (i) of Proposition 3.1 we have that the sequence $k^\beta(n)$ is β-monotone increasing for all $0 \le \beta \le 1$. However, for α fixed in the same range $0 < \alpha < 1$, $k^\alpha(n)$ is decreasing (see (iii) of Proposition 3.1). Observe that $k^0(n) \equiv 0$, using that $1/\Gamma(0) = 0$, and $k^1(n) \equiv 1$. This example shows that the notion of α-monotonicity reflects, in some sense, a measure of the curvature to the distribution of the points in a given sequence, in terms of α (compare with (ix) of Proposition 3.1).

Theorem 5.4: *Let $0 \leq \alpha \leq 1$ and $u \in s(\mathbb{N}_0;\mathbb{R})$ be a given sequence. Suppose that*

(i) $(\Delta^\alpha u)(n) \geq 0$ *for all* $n \in \mathbb{N}_0$,
(ii) $u(0) \geq 0$.

Then u is positive and α-monotone increasing on $\mathbb{N}_0$.

Proof. The cases $\alpha = 0$ and $\alpha = 1$ are trivial because $\Delta^0 \equiv I$ and $\Delta^1 \equiv \Delta$, respectively. Suppose $0 < \alpha < 1$. We have

$$\begin{aligned}\Delta^\alpha u(n) =&\Delta(k^{1-\alpha} * u)(n) = (k^{1-\alpha} * u)(n+1) - (k^{1-\alpha} * u)(n)\\ =&(k^{1-\alpha} * \tau_1 u)(n) + k^{1-\alpha}(n+1)u(0) - (k^{1-\alpha} * u)(n)\\ =&(k^{1-\alpha} * [\tau_1 u - u])(n) + k^{1-\alpha}(n+1)u(0)\\ =&(k^{1-\alpha} * \Delta u)(n) + \tau_1 k^{1-\alpha}(n)u(0), \quad n \in \mathbb{N}_0.\end{aligned} \tag{5.1}$$

Convolving with k^α and using the semigroup property and Lemma 2.3 we obtain the identity

$$\begin{aligned}u(n+1) - u(0) =&(k^\alpha * \Delta^\alpha u)(n) - (k^\alpha * \tau_1 k^{1-\alpha})(n)u(0)\\ =&(k^\alpha * \Delta^\alpha u)(n) - u(0) + k^\alpha(n+1)u(0).\end{aligned}$$

Therefore

$$u(n+1) = (k^\alpha * \Delta^\alpha u)(n) + k^\alpha(n+1)u(0),$$

for all $n \in \mathbb{N}_0$. From (i) and (ii) we conclude that u is positive on $\mathbb{N}_0$.

On the other hand, using (i) of Proposition 2.9 and (5.1) we have

$$\begin{aligned}0 \leq \Delta^\alpha u(n) &= \sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)u(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)u(j)\\ &= \sum_{j=0}^{n}[k^{1-\alpha}(n+1-j) - k^{1-\alpha}(n-j)]u(j) + u(n+1)\\ &= \sum_{j=0}^{n} \Delta k^{1-\alpha}(n-j)u(j) + u(n+1)\\ &= \sum_{j=0}^{n} \frac{-\alpha}{n-j+1} k^{1-\alpha}(n-j)u(j) + u(n+1).\end{aligned}$$

Therefore, using the fact that u is positive on $\mathbb{N}_0$, we obtain

$$\begin{aligned} u(n+1) \geq& \alpha \sum_{j=0}^{n} \frac{1}{n-j+1} k^{1-\alpha}(n-j)u(j) \\ =& \alpha u(n) + \alpha \sum_{j=0}^{n-1} \frac{1}{n-j+1} k^{1-\alpha}(n-j)u(j) \geq \alpha u(n), \end{aligned}$$

for all $n \in \mathbb{N}_0$. This proves the theorem. ∎

For a given $c \in \mathbb{R}$, we denote $\mathbf{c} : \mathbb{N}_0 \to X$ by $\mathbf{c}(n) = c$ for all $n \in \mathbb{N}_0$.

Example 5.5: Let us consider the equation

$$\Delta^{\alpha} u(n) = \mathbf{c}, \quad 0 < \alpha \leq 1,$$

with initial condition $u(0)$. If we assume $c > 0$ and $u(0) \geq 0$, then we deduce from Theorem 5.4 that the unique solution must be positive and α-monotone increasing on $\mathbb{N}_0$. This can be also directly checked from the following explicit representation of the solution:

$$u(n) = k^{\alpha}(n)\Big[u(0) + \frac{cn}{\alpha}\Big]; \quad n \in \mathbb{N}_0.$$

From the transference Theorem 4.1 we obtain the following corollary.

Corollary 5.6: *Let $0 < \alpha < 1$, $a \in \mathbb{R}$ and $v \in s(\mathbb{N}_a; \mathbb{R})$ be a given sequence. Suppose that*

(i) $(\Delta_a^{\alpha} v)(t) \geq 0$ *for all* $t \in \mathbb{N}_{a+1-\alpha}$,
(ii) $v(a) \geq 0$.

Then v is positive and $v(t+1) \geq \alpha v(t)$ for all $t \in \mathbb{N}_a$.

Proof. Define $u : \mathbb{N}_0 \to X$ by $u(n) := (\tau_a v)(n)$. Then (i) implies

$$\begin{aligned} \Delta^{\alpha} u(n) = (\tau_{a+1-\alpha} \circ \Delta_a^{\alpha} \circ \tau_{-a} u)(n) = (\Delta_a^{\alpha} v)(n + a - 1 - \alpha) = (\Delta_a^{\alpha} v)(t) \geq 0, \\ t := n + a - 1 - \alpha \in \mathbb{N}_{a-1-\alpha}. \end{aligned}$$

Also $u(0) = v(a) \geq 0$. Therefore, Theorem 5.4 gives $v(t+1) \geq \alpha v(t)$ for all $t \in \mathbb{N}_a$. ∎

Concerning the composition of fractional difference operators, we prove the following important connection between the operators $\Delta^{\beta} \circ \Delta^{\alpha}$ and $\Delta^{\alpha+\beta}$:

THEOREM 5.7: *Assume* $0 < \alpha < 1$ *and* $0 < \beta < 1$. *If* $0 < \alpha + \beta < 1$, *then*

$$(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) = (\Delta^{\alpha+\beta} u)(n+1) - (\Delta k^{1-\beta})(n+1)u(0).$$

Proof. Since $0 < \beta < 1$, we have

$$\begin{aligned}(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) =& \Delta^{\beta}(\Delta^{\alpha} u)(n) \\ =& (k^{1-\beta} * \Delta^{\alpha} u)(n+1) - (k^{1-\beta} * \Delta^{\alpha} u)(n),\end{aligned} \tag{5.2}$$

where, using Lemma 2.3,

$$\begin{aligned}(\Delta^{\alpha} u)(j) &= (k^{1-\alpha} * u)(j+1) - (k^{1-\alpha} * u)(j) \\ &= (k^{1-\alpha} * \tau_1 u)(j) + k^{1-\alpha}(j+1)u(0) - (k^{1-\alpha} * u)(j) \\ &= (k^{1-\alpha} * \tau_1 u)(j) + \tau_1 k^{1-\alpha}(j)u(0) - (k^{1-\alpha} * u)(j),\end{aligned}$$

because $0 < \alpha < 1$. Convolving with $k^{1-\beta}$ we obtain

$$\begin{aligned}&(k^{1-\beta} * \Delta^{\alpha} u)(n) \\ &\quad =(k^{1-\beta} * k^{1-\alpha} * \tau_1 u)(n) + (k^{1-\beta} * \tau_1 k^{1-\alpha})(n)u(0) - (k^{1-\beta} * k^{1-\alpha} * u)(n).\end{aligned}$$

Applying again Lemma 2.3 and the semigroup property:

$$\begin{aligned}&(k^{1-\beta} * \Delta^{\alpha} u)(n) \\ &\quad =(k^{2-(\alpha+\beta)} * \tau_1 u)(n) + (k^{1-\beta} * \tau_1 k^{1-\alpha})(n)u(0) - (k^{2-(\alpha+\beta)} * u)(n) \\ &\quad =[(k^{2-(\alpha+\beta)} * u)(n+1) - k^{2-(\alpha+\beta)}(n+1)u(0)] \\ &\qquad + [(k^{1-\beta} * k^{1-\alpha})(n+1) - k^{1-\beta}(n+1)k^{1-\alpha}(0)]u(0) - (k^{2-(\alpha+\beta)} * u)(n) \\ &\quad =(k^{2-(\alpha+\beta)} * u)(n+1) - k^{1-\beta}(n+1)u(0) - (k^{2-(\alpha+\beta)} * u)(n) \\ &\quad =(k * k^{1-(\alpha+\beta)} * u)(n+1) - k^{1-\beta}(n+1)u(0) - (k * k^{1-(\alpha+\beta)} * u)(n) \\ &\quad =[k * \tau_1(k^{(1-(\alpha+\beta)} * u)](n) + k(n+1)(k^{1-(\alpha+\beta)} * u)(0) \\ &\qquad - k^{1-\beta}(n+1)u(0) - (k * k^{1-(\alpha+\beta)} * u)(n) \\ &\quad =k * [\tau_1 k^{1-(\alpha+\beta)} * u - k^{1-(\alpha+\beta)} * u](n) + (1 - k^{1-\beta}(n+1))u(0),\end{aligned}$$

because $k(n+1) \equiv 1$ and $(k^{1-(\alpha+\beta)} * u)(0) = u(0)$ (since $(a * b)(0) = a(0)b(0)$).

Observe that $0 < \alpha + \beta < 1$ implies by definition

$$\begin{aligned}(\Delta^{\alpha+\beta} u)(n) &= (k^{1-(\alpha+\beta)} * u)(n+1) - (k^{1-(\alpha+\beta)} * u)(n) \\ &= \tau_1(k^{1-(\alpha+\beta)} * u)(n) - (k^{1-(\alpha+\beta)} * u)(n).\end{aligned}$$

That is,

$$\Delta^{\alpha+\beta}u = \tau_1 k^{1-(\alpha+\beta)} * u - k^{1-(\alpha+\beta)} * u.$$

Replacing in the above identity we obtain

$$(k^{1-\beta} * \Delta^{\alpha}u)(n) = k * \Delta^{\alpha+\beta}u(n) + (1 - k^{1-\beta}(n+1))u(0).$$

Hence, replacing the preceding identity in (5.2) we obtain

$$\begin{aligned}(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) =& k * \Delta^{\alpha+\beta}u(n+1) + (1 - k^{1-\beta}(n+2))u(0)\\ &- k * \Delta^{\alpha+\beta}u(n) - (1 - k^{1-\beta}(n+1))u(0)\\ =&\Delta(k * \Delta^{\alpha+\beta}u)(n) - \Delta k^{1-\beta}(n+1)u(0).\end{aligned}$$

Since $k(n) \equiv 1$, a simple calculation shows that

$$\Delta(k * \Delta^{\alpha+\beta}u)(n) = \Delta^{\alpha+\beta}u(n+1),$$

thus proving the theorem. ∎

Our main result concerning positivity is the following theorem. As mentioned at the beginning of this section, this sort of positivity result in the sequential setting has not, to the best of our knowledge, been reported for any particular type of fractional difference.

THEOREM 5.8: *Let $0 < \alpha < 1$ and $0 \le \beta < 1$ be given and assume that*

(i) $0 < \alpha + \beta < 1$;
(ii) $(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) \ge \frac{\beta}{2}(1-\beta)u(0)$, *for all $n \in \mathbb{N}_0$;*
(iii) $u(0) \ge 0$;
(iv) $u(1) \ge (\alpha+\beta)u(0)$.

Then u is positive and $(\alpha+\beta)$-monotone increasing on $\mathbb{N}_0$.

Proof. The case $\beta = 0$ is Theorem 5.4 because $\Delta^{\alpha}u(0) \ge 0$ is the same as (iv). Since $k^{\gamma}(n) \ge 0$ and decreasing for $0 < \gamma < 1$ we have

$$\Delta k^{1-\beta}(n+1) = -\beta\frac{k^{1-\beta}(n+1)}{n+2} \ge -\frac{\beta}{2}k^{1-\beta}(1) = -\frac{\beta}{2}(1-\beta).$$

Hence, by Theorem 5.7, and hypotheses (ii) and (iii) we obtain

$$\begin{aligned}(\Delta^{\alpha+\beta}u)(n+1) =&(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) + \Delta k^{1-\beta}(n+1)u(0)\\ \ge&(\Delta^{\beta} \circ \Delta^{\alpha}u)(n) - \frac{\beta}{2}(1-\beta)u(0) \ge 0\end{aligned} \tag{5.3}$$

for all $n \in \mathbb{N}_0$. Observe that for any $0 < \gamma < 1$ we have

$$\Delta^\gamma u(0) = \Delta(k^{1-\gamma} * u)(0) = u(1) - \gamma u(0).$$

Hence (i) and (iv) together with (5.3) imply

$$(\Delta^{\alpha+\beta} u)(n) \geq 0$$

for all $n \in \mathbb{N}_0$. Using (i) and Theorem 5.4 the conclusion follows. ∎

We can simplify hypothesis (ii) assuming stronger conditions on the initial data, and obtain a new result in the whole sector $[0,1] \times [0,1]$, such as the following result shows.

Theorem 5.9: *Let $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$ be given, and assume that*

(i) $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
(ii) $u(1) \geq \alpha u(0)$;
(iii) $u(0) \geq 0$.

Then u is positive and α-monotone increasing on $\mathbb{N}_0$.

Proof. Let $v = \Delta^\alpha u$. By hypotheses (i) and (ii) we have $\Delta^\beta v(n) \geq 0$ for all $n \in \mathbb{N}_0$ and $v(0) \geq 0$, respectively. From Theorem 5.4 we obtain that v is positive and β-monotone increasing on $\mathbb{N}_0$. Since $u(0) \geq 0$, a new application of Theorem 5.4 to u shows that u is positive and α-monotone increasing on $\mathbb{N}_0$. ∎

By using the transference Theorem 4.1 and its corollary, we can prove the following result.

Corollary 5.10: *Let $0 < \alpha < 1$ and $0 \leq \beta < 1$ be given and assume that*

(i) $0 < \alpha + \beta < 1$;
(ii) $(\Delta^\beta_{1-a-\alpha} \circ \Delta^\alpha_a v)(t) \geq \frac{\beta}{2}(1-\beta)v(a)$, *for all* $t \in \mathbb{N}_{a+2-\alpha-\beta}$;
(iii) $v(a) \geq 0$;
(iv) $v(a+1) \geq (\alpha+\beta)v(a)$.

Then v is positive and $v(t+1) \geq (\alpha+\beta)v(t)$ for all $t \in \mathbb{N}_{a+1}$.

Proof. Define $u : \mathbb{N}_0 \to X$ by

$$u(n) := (\tau_a v)(n).$$

By Theorem 4.1 we have $\Delta^\gamma = \tau_{b+1-\gamma} \circ \Delta_b^\gamma \circ \tau_{-b}$ for all $0 < \gamma < 1$ and $b \in \mathbb{R}$. Then (ii) and Corollary 4.2 implies that

$$\begin{aligned}
\Delta^\beta \circ \Delta^\alpha u(n) &= (\tau_{a+1-\beta} \circ \Delta_a^\beta \circ \tau_{-a} \circ \tau_{a+1-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u)(n) \\
&= (\tau_{a+1-\beta} \circ (\Delta_a^\beta \circ \tau_{1-\alpha}) \circ \Delta_a^\alpha \circ \tau_{-a} u)(n) \\
&= (\tau_{a+1-\beta} \circ (\tau_{1-\alpha} \circ \Delta_{1+a-\alpha}^\beta) \circ \Delta_a^\alpha \circ \tau_{-a} u)(n) \\
&= (\tau_{a+2-\alpha-\beta} \circ \Delta_{1+a-\alpha}^\beta \circ \Delta_a^\alpha v)(n) \\
&= (\Delta_{1+a-\alpha}^\beta \circ \Delta_a^\alpha v)(a+2-\alpha-\beta+n) \\
&= (\Delta_{1+a-\alpha}^\beta \circ \Delta_a^\alpha v)(t) \\
&\geq \frac{\beta}{2}(1-\beta)v(a) \\
&= \frac{\beta}{2}(1-\beta)u(0),
\end{aligned}$$

for all $t = a+2-\alpha-\beta+n \in \mathbb{N}_{a+2-\alpha-\beta}$. Also

$$u(0) = v(a) \geq 0 \quad \text{and} \quad u(1) = v(a+1) \geq (\alpha+\beta)v(a) = (\alpha+\beta)u(0).$$

Therefore, Theorem 5.8 gives $u(n+1) \geq (\alpha+\beta)u(n)$ for all $n \in \mathbb{N}_0$ and hence $v(n+1+a) \geq v(n+a)$ for all $n \in \mathbb{N}_0$ which means $v(t+1) \geq (\alpha+\beta)v(t)$ for all $t \in \mathbb{N}_{a+1}$. ∎

6. Monotonicity and α-convexity

In this section we assume $1 < \alpha + \beta < 2$, and we will consider the following cases:

(a) $0 < \alpha < 1$ and $0 < \beta < 1$.
(b) $1 \leq \alpha < 2$ and $0 \leq \beta < 1$.
(c) $0 < \alpha < 1$ and $1 < \beta < 2$.

We first introduce the following concept.

Definition 6.1: Let $1 \leq \alpha \leq 2$. We say that a sequence $u \in s(\mathbb{N}_0, \mathbb{R})$ is α-convex (resp. α-concave) if

$$u(n+2) - \alpha u(n+1) + (\alpha-1)u(n) \geq 0, \quad n \in \mathbb{N}_0$$

(with obvious modification in the case of α-concave). When $\alpha = 2$ we recover the notion of convexity and when $\alpha = 1$ the concept of monotonicity (increasing) on the set $\mathbb{N}_1$.

Remark 6.2: The function u is α-convex if and only if Δu is $(\alpha-1)$-monotone increasing.

We start with the following result concerning the case $\beta = 0$ in (b)—that is to say, we consider first the (trivial) sequential operator

$$\Delta^0 \circ \Delta^\alpha \equiv \Delta^\alpha.$$

Regarding the existing results in the literature, we mention that Theorem 6.3 recovers in our setting a more general form of a result of Dahal and Goodrich [19]—see also Dahal and Goodrich [20] and Jia et al. [38]. The result of Dahal and Goodrich states that if $\Delta_0^\nu y(t) \geq 0$, for each $t \in \mathbb{N}_{2-\nu}$, and if $\Delta y(0) \geq 0$, then y is monotone increasing on $\mathbb{N}_0$. Since, in the statement of Theorem 6.3, we require that

$$u(1) \geq \alpha u(0),$$

it is seen that this is a weaker condition than the theorems of Dahal and Goodrich in [19] or Jia et al. [38].

Theorem 6.3: *Let $1 \leq \alpha < 2$ be given and assume that*

- (i) $(\Delta^\alpha u)(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
- (ii) $u(1) \geq \alpha u(0)$;
- (iii) $u(0) \geq 0$.

Then u is monotone increasing on $\mathbb{N}_1$ and positive on $\mathbb{N}_0$. Moreover, u is α-convex, inclusive in case $\alpha = 2$.

Proof. The cases $\alpha = 1$ and $\alpha = 2$ are trivial by (i). Assume $1 < \alpha < 2$. By definition and Lemma 2.3 we have

$$\begin{aligned}\Delta^\alpha u(n) =&(k^{2-\alpha} * u)(n+2) - 2(k^{2-\alpha} * u)(n+1) + (k^{2-\alpha} * u)(n)\\ =&(k^{2-\alpha} * \tau_2 u)(n) + k^{2-\alpha}(n+2)u(0) + k^{2-\alpha}(n+1)u(1)\\ &- [(k^{2-\alpha} * \tau_1 u)(n) + k^{2-\alpha}(n+1)u(0)] + (k^{2-\alpha} * u)(n)\\ =&(k^{2-\alpha} * \Delta^2 u)(n) + k^{2-\alpha}(n+1)[u(1) - 2u(0)]\\ &+ k^{2-\alpha}(n+2)u(0).\end{aligned} \tag{6.1}$$

Then,

$$(k^{2-\alpha} * \Delta^2 u)(n) = \Delta^\alpha u(n) - \tau_1 k^{2-\alpha}(n)[\Delta u(0) - u(0)] - \tau_2 k^{2-\alpha}(n)u(0).$$

Convolving with $k^{\alpha-1}$ we obtain

$$\begin{aligned}(k^{\alpha-1}*k^{2-\alpha} * \Delta^2)u(n)\\ =&(k^{\alpha-1} * \Delta^\alpha u)(n) - (k^{\alpha-1} * \tau_1 k^{2-\alpha})(n)[\Delta u(0) - u(0)]\\ &- (k^{\alpha-1} * \tau_2 k^{2-\alpha})(n)u(0)\\ =&(k^{\alpha-1} * \Delta^\alpha u)(n)\\ &- [k^{\alpha-1}*k^{2-\alpha}(n+1) - k^{\alpha-1}(n+1)k^{2-\alpha}(0)][\Delta u(0) - u(0)]\\ &- [k^{\alpha-1}*k^{2-\alpha}(n+2) - k^{\alpha-1}(n+2)k^{2-\alpha}(0) - k^{\alpha-1}(n+1)k^{2-\alpha}(1)]u(0).\end{aligned}$$

Hence,

$$\begin{aligned}\Delta u(n+1) - \Delta u(0) =&(k^{\alpha-1} * \Delta^\alpha u)(n) - [\Delta u(0) - u(0)]\\ &+ k^{\alpha-1}(n+1)[\Delta u(0) - u(0)]\\ &- u(0) + k^{\alpha-1}(n+2)u(0) + (2-\alpha)k^{\alpha-1}(n+1)u(0).\end{aligned}$$

Therefore, for all $n \in \mathbb{N}_0$ we have

$$\begin{aligned}\Delta u(n+1) =&(k^{\alpha-1} * \Delta^\alpha u)(n) + k^{\alpha-1}(n+2)u(0)\\ &+ k^{\alpha-1}(n+1)[\Delta u(0) - u(0) + (2-\alpha)u(0)]\\ =&(k^{\alpha-1} * \Delta^\alpha u)(n) + k^{\alpha-1}(n+2)u(0)\\ &+ k^{\alpha-1}(n+1)\underbrace{[u(1) + \alpha u(0)]}_{\geq 0},\end{aligned} \tag{6.2}$$

concluding by hypothesis that $\Delta u(m) \geq 0$ for all $m \in \mathbb{N}_1$. Hence, (6.2) proves that u is monotone increasing on $\mathbb{N}_1$ and, by (ii)–(iii), positive on $\mathbb{N}_0$. Next, from (6.1) and (v) of Proposition 2.9 we obtain

$$\begin{aligned}0 \leq \Delta^\alpha u(n) =&(\Delta u * \Delta k^{2-\alpha})(n) + \Delta u(n+1)k^{2-\alpha}(0) - k^{2-\alpha}(n+1)\Delta u(0)\\ &+ k^{2-\alpha}(n+1)[\Delta u(0) - u(0)] + k^{2-\alpha}(n+2)u(0)\\ =&\sum_{j=0}^{n} \Delta u(n-j)\Delta k^{2-\alpha}(j) + \Delta u(n+1) + \Delta k^{2-\alpha}(n+1)u(0)\\ =&\sum_{j=1}^{n} \Delta u(n-j)\Delta k^{2-\alpha}(j) + (1-\alpha)\Delta u(n)\\ &+ \Delta u(n+1) + \Delta k^{2-\alpha}(n+1)u(0),\end{aligned}$$

where we have used that $k^{2-\alpha}(0) = 1$ and $k^{2-\alpha}(1) = 2-\alpha$. Finally, observe that $0 < 2-\alpha < 1$ implies $\Delta k^{2-\alpha}(m) \leq 0$ by Corollary 3.3, and also $\Delta u(m) \geq 0$

on $\mathbb{N}_1$ by what was previously proved. Therefore, the above inequality implies

$$(1-\alpha)\Delta u(n)+\Delta u(n+1) \geq -\sum_{j=1}^{n} \Delta u(n-j)\Delta k^{2-\alpha}(j) - \Delta k^{2-\alpha}(n+1)u(0) \geq 0,$$

for all $n \in \mathbb{N}_0$, which is precisely α-convexity. ∎

Remark 6.4: Note that condition (ii) does not actually require that u satisfy $\Delta u(0) \geq 0$. If it does, then one can strengthen the conclusion to u increasing on all of $\mathbb{N}_0$ rather than $\mathbb{N}_1$.

Remark 6.5: If we replace hypothesis (ii) in Theorem 6.3 by the stronger one

(ii)′ $u(1) \geq u(0)$,

then we can conclude that u is monotone increasing on $\mathbb{N}_0$, positive and α-convex by the following simple argument: We first observe that an application of (ii) and (iv) of Proposition 2.9 shows that the following identity holds:

$$\Delta^{\alpha} u(n) = (\Delta^{(\alpha-1)+1} u)(n) = (\Delta^{\alpha-1} \circ \Delta u)(n) + \Delta k^{2-\alpha}(n+1)u(0).$$

Let $v = \Delta u$. From the above identity and hypothesis (i) we have

$$(\Delta^{\alpha-1} v)(n) = \Delta^{\alpha} u(n) - \Delta k^{2-\alpha}(n+1)u(0) \geq -\Delta k^{2-\alpha}(n+1)u(0) \geq 0,$$

where in the last inequality we have used hypothesis (iii), and item (i) of Corollary 3.3. Since by (ii)′ we also have $v(0) = \Delta u(0) \geq 0$, then, by Theorem 5.4, we conclude that $v = \Delta u$ is positive and $(\alpha-1)$-monotone increasing on $\mathbb{N}_0$. Therefore, u is monotone increasing on $\mathbb{N}_0$, positive, and Remark 6.2 shows that u is α-convex.

Remark 6.6: Condition (ii)′ is precisely what is required in Dahal and Goodrich [19] and Jia et al. [38].

Remark 6.7: We observe that convexity coincides with the notion of α-convexity (which is defined for $1 \leq \alpha < 2$) in the upper limit case $\alpha = 2$. We also note that in the lower limit case $\alpha = 1$ we obtain that u is monotone. This way, the notion of α-convexity interpolates between the concept of convexity and monotonicity for $1 \leq \alpha \leq 2$. Graphically, we can think of a discrete set of points drawing a line with positive slope that can be "continuously deformed," as α goes from 1 to 2, into a convex set of points. In the same way, the notion of α-increasing sequence (defined for $0 \leq \alpha < 1$) interpolates between the notion of positivity (when $\alpha = 0$) and monotonicity (when $\alpha = 1$). Since we are dealing with the concept of fractional differences, it seems to be "natural," in some sense, that an appropriate geometrical interpretation of the fractional order should vary with α. Our findings show that this notable and important property occurs.

Another way to think of this property is that α is a measure of the discrete curvature of the set of points $u(n)$. It is interesting to make a graph of the example below, setting, for instance, $u(0) = u(1) = 1$ and $c = 1$. Then, one can look at the cases $\alpha = 1, \alpha = 1.5$ and $\alpha = 2$, for instance.

For a given $c \in \mathbb{R}$, recall that we denote $\mathbf{c} : \mathbb{N}_0 \to \mathbb{R}$ by $\mathbf{c}(n) = c$ for all $n \in \mathbb{N}_0$.

Example 6.8: We consider the problem

$$\Delta^\alpha u(n) = \mathbf{c}, \quad 1 < \alpha < 2.$$

If we assume $c > 0$ it follows, using Theorem 6.3, that the (unique) solution is monotone increasing and positive on $\mathbb{N}_1$ whenever $u(1) \geq \alpha u(0)$ and $u(0) \geq 0$. This can be directly verified by means of the following explicit representation of the solution:

$$u(n) = k^\alpha(n)\Big[u(0) - (\alpha u(0) - u(1))\frac{n}{\alpha + n - 1} + c\frac{n(n-1)}{(\alpha+n-2)(\alpha+n-1)}\Big], \quad n \in \mathbb{N}_1.$$

Corollary 6.9: *Let $1 \leq \alpha < 2$ and $a \in \mathbb{R}$ be given and assume that*

(i) $(\Delta_a^\alpha v)(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+2-\alpha}$;
(ii) $v(a+1) \geq \alpha v(a)$;
(iii) $v(a) \geq 0$.

Then v is monotone increasing and positive on $\mathbb{N}_{a+1}$. Moreover, v is α-convex— i.e.,

$$v(t+2) - \alpha v(t+1) + (\alpha - 1)v(t) \geq 0, \quad t \in \mathbb{N}_a.$$

Proof. Define $u(n) := \tau_a v(n)$ and note that for each $n \in \mathbb{N}_0$ we have, by (i) and Theorem 4.3 with $N = 2$:

$$\Delta^\alpha u(n) = (\tau_{a+2-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u)(n) = (\tau_{a+2-\alpha} \circ \Delta_a^\alpha v)(n) = \Delta_a^\alpha v(t) \geq 0,$$

for $t := n + a + 2 - \alpha \in \mathbb{N}_{a+2-\alpha}$. Moreover,

$$u(1) = v(a+1) \geq \alpha v(a) = \alpha u(0) \quad \text{and} \quad u(0) = v(a) \geq 0.$$

The conclusion follows from Theorem 6.3. ∎

Remark 6.10: We note that Corollary 6.9 may also be compared directly to [17, Corollary 2.1.1]. In that result and with $\nu \in (1,2)$, Erbe et al. proved that if $\Delta_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+2-\nu}$, $f(a) \geq 0$, and $f(a+1) \geq \frac{\nu}{2} f(a)$, then f was monotone increasing on $\mathbb{N}_{a+1}$. Note, for example, that if $\nu = \frac{3}{2}$, then their result yields the condition

$$f(a+1) \geq \frac{3}{4} f(a).$$

However, in our case above, if we take $\alpha = \frac{3}{2}$, then condition (ii) becomes

$$u(a+1) \geq \frac{1}{2} u(a),$$

which is actually less restrictive. In fact, since

$$\frac{\alpha}{2} \geq (\alpha - 1),$$

whenever $\alpha \leq 2$, it follows that the condition we obtain in our Corollary 6.9 is better than the condition given in Erbe et al. [17, Corollary 2.11].

6.1. Case (a): $0 < \alpha < 1$ and $0 < \beta < 1$. The key result in this case is the following theorem.

Theorem 6.11: *Assume* $0 < \alpha < 1$ *and* $0 < \beta < 1$. *If* $1 < \alpha + \beta < 2$, *then*

$$(\Delta^\beta \circ \Delta^\alpha u)(n) = (\Delta^{\alpha+\beta} u)(n) - (\Delta k^{1-\beta})(n+1) u(0).$$

Proof. Since $0 < \beta < 1$, we have

$$\begin{aligned} (\Delta^\beta \circ \Delta^\alpha u)(n) &= \Delta^\beta (\Delta^\alpha u)(n) \\ &= (k^{1-\beta} * \Delta^\alpha u)(n+1) - (k^{1-\beta} * \Delta^\alpha u)(n), \end{aligned} \tag{6.3}$$

where, using Lemma 2.3,

$$\begin{aligned} (\Delta^\alpha u)(j) &= (k^{1-\alpha} * u)(j+1) - (k^{1-\alpha} * u)(j) \\ &= (k^{1-\alpha} * \tau_1 u)(j) + k^{1-\alpha}(j+1) u(0) - (k^{1-\alpha} * u)(j) \\ &= (k^{1-\alpha} * \tau_1 u)(j) + \tau_1 k^{1-\alpha}(j) u(0) - (k^{1-\alpha} * u)(j), \end{aligned}$$

because $0 < \alpha < 1$. Convolving with $k^{1-\beta}$ we obtain

$$\begin{aligned} &(k^{1-\beta} * \Delta^\alpha u)(n) \\ &\quad = (k^{1-\beta} * k^{1-\alpha} * \tau_1 u)(n) + (k^{1-\beta} * \tau_1 k^{1-\alpha})(n) u(0) - (k^{1-\beta} * k^{1-\alpha} * u)(n). \end{aligned}$$

Applying again Lemma 2.3 and the semigroup property

$$\begin{aligned}
&(k^{1-\beta} * \Delta^\alpha u)(n)\\
&=(k^{2-(\alpha+\beta)} * \tau_1 u)(n) + (k^{1-\beta} * \tau_1 k^{1-\alpha})(n)u(0) - (k^{2-(\alpha+\beta)} * u)(n)\\
&=[(k^{2-(\alpha+\beta)} * u)(n+1) - k^{2-(\alpha+\beta)}(n+1)u(0)]\\
&\quad + [(k^{1-\beta} * k^{1-\alpha})(n+1) - k^{1-\beta}(n+1)k^{1-\alpha}(0)]u(0)\\
&\quad - (k^{2-(\alpha+\beta)} * u)(n)\\
&=(k^{2-(\alpha+\beta)} * u)(n+1) - k^{1-\beta}(n+1)u(0) - (k^{2-(\alpha+\beta)} * u)(n)\\
&=\Delta(k^{2-(\alpha+\beta)} * u)(n) - k^{1-\beta}(n+1)u(0).
\end{aligned}$$

Replacing the above identity in (6.3) and using that $1 < \alpha + \beta < 2$ we obtain

$$\begin{aligned}
(\Delta^\beta \circ \Delta^\alpha u)(n) =&[\Delta(k^{2-(\alpha+\beta)} * u)(n+1) - k^{1-\beta}(n+2)u(0)]\\
&- [\Delta(k^{2-(\alpha+\beta)} * u)(n) - k^{1-\beta}(n+1)u(0)]\\
=&\Delta^2(k^{2-(\alpha+\beta)} * u)(n) - \Delta k^{1-\beta}(n+1)u(0)\\
=&(\Delta^{\alpha+\beta} u)(n) - (\Delta k^{1-\beta})(n+1)u(0),
\end{aligned}$$

proving the theorem. ∎

With a similar proof to Theorem 5.8 we obtain the main result of this subsection. We note that this theorem is somewhat different (and, in a certain sense, better) than the corresponding result in Dahal and Goodrich [21] and Goodrich [30]. In particular, and as mentioned in Section 1, in those papers the monotonicity result only applies on a proper subset of the parameter space $[0, 1] \times [0, 1]$. In our Theorem 6.12, by contrast, the result applies on the entire parameter space. In part, this is because condition (i) of Theorem 6.12 is different than the corresponding condition in those papers—namely, the right-hand side is simply 0 in [21, 30]. Moreover, if we recall Theorem 5.9 from Section 5, we note that in Theorem 5.9 with simply a zero lower bound on the quantity $(\Delta^\beta \circ \Delta^\alpha u)(n)$, together with the auxiliary conditions $u(1) \geq \alpha u(0)$ and $u(0) \geq 0$, we obtained the α-monotonicity on the entire parameter space $[0, 1] \times [0, 1]$. Therefore, Theorem 5.9 and Theorem 6.12 together with transference yield the following new insights, heretofore unreported:

(1) if we require a nonzero lower bound on the sequential difference, such as

$$(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \frac{\beta}{2}(1-\beta)u(0),$$

then we can deduce both (classical) monotonicity and $(\alpha+\beta)$-convexity on the entire sparameter space $[0,1]\times[0,1]$ when $1 \leq \alpha+\beta < 2$; whereas

(2) if we require a zero lower bound on the sequential difference, that is

$$(\Delta^\beta \circ \Delta^\alpha u)(n) \geq 0,$$

then we can deduce both positivity and $(\alpha+\beta)$-monotonicity on the entire parameter space $[0,1]\times[0,1]$ when $0 \leq \alpha+\beta < 1$.

Theorem 6.12: *Let $0 < \alpha < 1$ and $0 \leq \beta < 1$ be given, and assume both that $1 \leq \alpha+\beta < 2$ and that*

(i) $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq \frac{\beta}{2}(1-\beta)u(0)$, *for all $n \in \mathbb{N}_0$;*
(ii) $u(1) \geq (\alpha+\beta)u(0)$;
(iii) $u(0) \geq 0$.

Then u is monotone increasing and $(\alpha+\beta)$-convex on $\mathbb{N}_0$.

Proof. Observe that the limit case $\beta = 1-\alpha$ coincides with Theorem 5.8. Since $k^\gamma(n) \geq 0$ and decreasing for $0 < \gamma < 1$ we have

$$\Delta k^{1-\beta}(n+1) = -\beta\frac{k^{1-\beta}(n+1)}{n+2} \geq -\frac{\beta}{2}k^{1-\beta}(1) = -\frac{\beta}{2}(1-\beta).$$

Hence, by Theorem 6.11, and (i) and (iii) we obtain

$$\begin{aligned}(\Delta^{\alpha+\beta}u)(n) =&(\Delta^\beta \circ \Delta^\alpha u)(n) + \Delta k^{1-\beta}(n+1)u(0)\\ \geq&(\Delta^\beta \circ \Delta^\alpha u)(n) - \frac{\beta}{2}(1-\beta)u(0) \geq 0\end{aligned}$$

for all $n \in \mathbb{N}_0$. Using (ii) and Theorem 6.3 the conclusion follows. ∎

Remark 6.13: Similar to Remark 6.6, condition (ii) can be replaced by:
(ii)′ $\Delta u(0) \geq u(0)$.

Analogously to Corollary 5.10 we can prove with the help of Theorem 4.3 (case $N = 1$) the following more general result in the context of Definition (1.1).

Corollary 6.14: *Let $a \in \mathbb{R}$, $0 < \alpha < 1$ and $0 \leq \beta < 1$ be given, and assume both that $1 \leq \alpha + \beta < 2$ and that*

(i) $(\Delta^{\beta}_{a+1-\alpha} \circ \Delta^{\alpha}_{a} v)(t) \geq \frac{\beta}{2}(1-\beta)v(a)$, *for all* $t \in \mathbb{N}_{a+2-\alpha-\beta}$;
(ii) $v(1+a) \geq (\alpha+\beta)v(a)$;
(iii) $v(a) \geq 0$.

Then v is monotone increasing and $(\alpha+\beta)$-convex on $\mathbb{N}_a$.

6.2. Case (b): $1 < \alpha < 2$ and $0 < \beta < 1$. We start with a general result in the parameter space $[1, 2) \times [0, 1]$.

Theorem 6.15: *Suppose that $1 \leq \alpha < 2$, $0 \leq \beta < 1$. In addition, suppose that*

(i) $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \geq 0$, *for each* $n \in \mathbb{N}_0$;
(ii) $\Delta^{\alpha} u(0) \geq 0$;
(iii) $\Delta^{\alpha} u(1) \geq \alpha \Delta^{\alpha} u(0)$; *and*
(iv) $u(0) \geq 0$.

Then u is monotone increasing and positive on $\mathbb{N}_1$. Moreover, u is α-convex on $\mathbb{N}_0$, even in case $\alpha = 2$.

Proof. Let $v = \Delta^{\alpha} u$. By hypotheses (i) and (ii) we obtain that the assumptions of Theorem 5.4 are satisfied and, consequently, we obtain that v is positive and β-monotone increasing on $\mathbb{N}_0$. In particular, the positivity together with hypotheses (iii) and (iv) imply that the conditions of Theorem 6.3 are satisfied. We conclude that u is monotone increasing and positive on $\mathbb{N}_1$. Moreover, u is α-convex on $\mathbb{N}_0$, even in case $\alpha = 2$. ∎

A key result for the forthcoming results is the following.

Theorem 6.16: *Suppose that $1 < \alpha < 2$, $0 < \beta < 1$, and $1 < \alpha + \beta < 2$. Then*

$$\begin{aligned}&\Delta^{\beta} \circ \Delta^{\alpha} u(n)\\ &\quad = \Delta^{\alpha+\beta} u(n+1) - \Delta^2 k^{1-\beta}(n+1) u(0) - \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0).\end{aligned}$$

Proof. Since $1 < \alpha < 2$, $0 < \beta < 1$ and $1 < \alpha + \beta < 2$, we obviously have $0 < \alpha - 1 < 1$, $0 < \beta < 1$ and $0 < (\alpha - 1) + \beta < 1$. Therefore, applying Theorem 5.7 we have the identity

$$(\Delta^{\beta} \circ \Delta^{\alpha-1} u)(n) = (\Delta^{\alpha-1+\beta} u)(n+1) - (\Delta k^{1-\beta})(n+1) u(0).$$

Hence,

$$\Delta \circ (\Delta^{\beta} \circ \Delta^{\alpha-1} u)(n) = \Delta \circ (\Delta^{\alpha+\beta-1} u)(n+1) - \Delta \circ (\Delta k^{1-\beta})(n+1) u(0). \tag{6.4}$$

Using item (ii) of Proposition 2.9 we have

$$\Delta \circ \Delta^{\alpha+\beta-1} = \Delta^{\alpha+\beta}.$$

On the other hand, using item (iv) and again item (ii) of Proposition 2.9 we have

$$\begin{aligned}\Delta \circ \Delta^{\beta} \circ \Delta^{\alpha-1} u(n) &= \Delta^{\beta} \circ \Delta \circ \Delta^{\alpha-1} u(n) + \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0) \\ &= \Delta^{\beta} \circ \Delta^{\alpha} u(n) + \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0).\end{aligned} \tag{6.5}$$

Inserting (6.5) into (6.4) we get

$$\Delta^{\beta} \circ \Delta^{\alpha} u(n) = \Delta^{\alpha+\beta} u(n+1) - \Delta^2 k^{1-\beta}(n+1) u(0) - \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0),$$

which completes the proof of the theorem. ∎

Our next result, Theorem 6.17, is our sequential fractional difference monotonicity result in case $1 \le \alpha < 2$, $0 \le \beta < 1$, and $1 \le \alpha + \beta < 2$.

THEOREM 6.17: *Suppose that* $1 \le \alpha < 2$, $0 \le \beta < 1$, *and* $1 \le \alpha + \beta < 2$. *In addition, suppose that*

(i) $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \ge \frac{\beta}{2}(1-\beta)\Delta^{\alpha-1} u(0)$, *for each* $n \in \mathbb{N}_0$;
(ii) $u(1) \ge (\alpha+\beta) u(0)$; *and*
(iii) $u(0) \ge 0$.

Then u *is monotone increasing and* $(\alpha+\beta)$*-convex on* $\mathbb{N}_0$.

Proof. Since $\Delta^0 \equiv I$, the case $\beta = 0$ is Theorem 6.3 and the case $\alpha = 1$ is Theorem 6.12. For $0 < \beta < 1$ and by Theorem 6.16 we have

$$(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) = (\Delta^{\alpha+\beta} u)(n+1) - \Delta^2 k^{1-\beta}(n+1) u(0) - \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0) \tag{6.6}$$

where

$$\Delta k^{1-\beta}(n+1) = -\beta \frac{k^{1-\beta}(n+1)}{n+2} \ge -\frac{\beta}{2} k^{1-\beta}(1) = -\frac{\beta}{2}(1-\beta), \tag{6.7}$$

because $k^{\gamma}(n)$ is decreasing for $0 < \gamma < 1$. Moreover,

$$\Delta^2 k^{1-\beta}(n+1) = \beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)} \geq 0. \tag{6.8}$$

Inserting (6.7) and (6.8) in (6.6) and taking into account (ii) and (iii) we obtain

$$\begin{aligned}(\Delta^{\alpha+\beta}u)(n+1) \geq &(\Delta^{\beta}\circ\Delta^{\alpha}u)(n) + \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0)\\ \geq &(\Delta^{\beta}\circ\Delta^{\alpha}u)(n) - \frac{\beta}{2}(1-\beta)\Delta^{\alpha-1}u(0),\end{aligned}$$

for all $n \in \mathbb{N}_0$. Moreover, (ii) implies $\Delta^{\alpha+\beta}u(0) \geq 0$. We conclude the result from (i) and Theorem 6.3. ∎

Remark 6.18: According to Remark 6.6, condition (ii) can be replaced by:
(ii)′ $\Delta u(0) \geq u(0)$.

By using the transference property we can recast Theorem 6.17 into the following Corollary 6.19. One may compare Corollary 6.19 to a recent paper of Goodrich [31], in which the author considered monotonicity-type results for sequential discrete fractional differences of the type (1.1). More specifically, there condition (i) in our Theorem 6.17 was replaced by the condition $\Delta^{\nu}_{1-\mu}\Delta^{\mu}_{0}f(t) \geq 0$, for $t \in \mathbb{N}_{3-\mu-\nu}$. Assuming that $\mu + \nu \in (1,2)$, it was shown that when $\mu \in (0,1)$ and $\nu \in (1,2)$ this condition, in addition to $f(0) \geq 0$, $\Delta f(0) \geq 0$, and $\Delta f(1) \geq 0$, were sufficient to deduce the monotonicity of f on $\mathbb{N}_0$. On the other hand, when $\mu \in (1,2)$ and $\nu \in (0,1)$ the situation was more complicated—see [31, Theorem 2.9]. In this second case, a monotonicity-type result was only obtained on a proper subset of the parameter space $(\mu,\nu) \in (1,2) \times (0,1)$. So, we conclude that

(1) Theorem 6.17 demonstrates that by using the nonzero lower bound condition

$$(\Delta^{\beta}\circ\Delta^{\alpha}u)(n) \geq \frac{\beta}{2}(1-\beta)\Delta^{\alpha-1}u(0)$$

we can instead get a clean result holds on the entire parameter space $(\alpha,\beta) \in [1,2) \times [0,1)$; and

(2) by subsequently using the transference property in Section 5, we can obtain Corollary 6.19, which connects Theorem 6.17 back to the case studied in [31].

Corollary 6.19: *Let $a \in \mathbb{R}$ be given. Suppose that $1 < \alpha < 2$, $0 \leq \beta < 1$, and $1 \leq \alpha + \beta < 2$. In addition, suppose that*

(i) $(\Delta_{a+2-\alpha}^{\beta} \circ \Delta_a^{\alpha} v)(t) \geq \frac{\beta}{2}(1-\beta)\Delta_a^{\alpha-1} v(a+2-\alpha)$, *for each* $t \in \mathbb{N}_{a+3-\alpha-\beta}$;

(ii) $v(a+1) \geq (\alpha+\beta)v(a)$; *and*

(iii) $v(a) \geq 0$.

Then v is monotone increasing and $(\alpha+\beta)$-convex on $\mathbb{N}_a$.

Proof. Define $u := \tau_a v$. Then, using the transference theorems we have

$$\begin{aligned}\Delta^{\beta} \circ \Delta^{\alpha} u(n) &= \tau_{a+1-\beta} \circ \Delta_a^{\beta} \circ \tau_{-a} \circ \tau_{a+2-\alpha} \circ \Delta_a^{\alpha} \circ \tau_{-a} u(n) \\ &= \tau_{a+1-\beta} \circ (\Delta_a^{\beta} \circ \tau_{2-\alpha}) \circ \Delta_a^{\alpha} \circ \tau_{-a} u(n) \\ &= \tau_{a+1-\beta} \circ (\tau_{2-\alpha} \circ \Delta_{a+2-\alpha}^{\beta}) \circ \Delta_a^{\alpha} v(n) \\ &= \tau_{a+3-\beta-\alpha} \circ \Delta_{a+2-\alpha}^{\beta} \circ \Delta_a^{\alpha} v(n),\end{aligned}$$

for each $n \in \mathbb{N}_0$. Therefore,

$$\Delta^{\beta} \circ \Delta^{\alpha} u(n) = \Delta_{a+2-\alpha}^{\beta} \circ \Delta_a^{\alpha} v(t),$$

where $t := n+a+3-\alpha-\beta \in \mathbb{N}_{a+3-\alpha-\beta}$. Moreover, since $0 < \alpha - 1 < 1$ we have

$$\Delta^{\alpha-1} u(n) = \tau_{a+2-\alpha} \circ \Delta_a^{\alpha-1} \circ \tau_{-a} u(n) = \Delta_a^{\alpha-1} v(n+a+2-\alpha),$$

which implies $\Delta^{\alpha-1}u(0) = \Delta_a^{\alpha-1} v(a+2-\alpha)$. The result follows from Theorem 6.17. ∎

6.3. Case (c): $0 < \alpha < 1$ and $1 < \beta < 2$. The general result in the parameter space $[0,1] \times [1,2]$ is the following theorem.

Theorem 6.20: *Suppose that $1 \leq \alpha < 2$, $0 \leq \beta < 1$. In addition, suppose that*

(i) $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \geq 0$, *for each* $n \in \mathbb{N}_0$;

(ii) $\Delta^{\alpha} u(0) \geq 0$;

(iii) $\Delta^{\alpha} u(1) \geq \beta \Delta^{\alpha} u(0)$; *and*

(iv) $u(0) \geq 0$.

Then u is α-monotone increasing and positive on $\mathbb{N}_0$.

Proof. Let $v = \Delta^{\alpha} u$. By hypotheses (i) and (iii) we obtain that the assumptions of Theorem 6.3 are satisfied and, consequently, we obtain that v is positive and monotone increasing on $\mathbb{N}_1$. Since (ii) holds, we deduce that $\Delta^{\alpha} u$

is positive on $\mathbb{N}_0$. This, together with hypothesis (iv), implies that the conditions of Theorem 5.4 are satisfied. We conclude that u is α-monotone increasing and positive on $\mathbb{N}_0$. ∎

Our pivotal result in this subsection is the following Theorem 6.21.

THEOREM 6.21: *Suppose that* $0 < \alpha < 1$, $1 < \beta < 2$, *and* $1 < \alpha + \beta < 2$. *Then*

$$(\Delta^\beta \circ \Delta^\alpha u)(n) = (\Delta^{\alpha+\beta} u)(n+1) - \Delta^2 k^{2-\beta}(n+1)u(0). \tag{6.9}$$

Proof. Since $0 < \alpha < 1$, $1 < \beta < 2$ and $1 < \alpha + \beta < 2$ we obviously have $0 < \alpha < 1$, $0 < \beta - 1 < 1$ and $0 < \alpha + (\beta - 1) < 1$. Therefore, applying Theorem 5.7 we have the identity:

$$\begin{aligned}(\Delta^{\beta-1} \circ \Delta^\alpha u)(n) &= (\Delta^{\alpha+\beta-1} u)(n+1) - (\Delta k^{1-(\beta-1)})(n+1)u(0)\\ &= (\Delta^{\alpha+\beta-1} u)(n+1) - (\Delta k^{2-\beta})(n+1)u(0).\end{aligned}$$

Hence,

$$\Delta \circ (\Delta^{\beta-1} \circ \Delta^\alpha u)(n) = \Delta \circ (\Delta^{\alpha+\beta-1} u)(n+1) - \Delta \circ (\Delta k^{2-\beta})(n+1)u(0).$$

Using item (ii) of Proposition 2.9 we have

$$\Delta \circ \Delta^{\beta-1} = \Delta^\beta \quad \text{and} \quad \Delta \circ \Delta^{\alpha+\beta-1} = \Delta^{\alpha+\beta}.$$

Consequently, we obtain (6.9) from the above equality. ∎

The main result of this subsection is the following Theorem 6.22. As suggested in the previous subsection, we note that this result can be compared to [31, Theorem 2.4]. The result there, in addition to supposing an analogue of condition (i) below, in [31, Theorem 2.4] it was also assumed that $f(0) \geq 0$, $\Delta f(0) \geq 0$, and $\Delta f(1) \geq 0$. By contrast, we see that Theorem 6.22 below does not involve a condition on $u(2)$, and so, in this particular sense, is an improvement of the corresponding result deduced in [31].

THEOREM 6.22: *Suppose that* $0 < \alpha < 1$, $1 \leq \beta < 2$, *and* $1 < \alpha + \beta < 2$. *In addition, suppose that*

(i) $(\Delta^\beta \circ \Delta^\alpha u)(n) \geq 0$ *for each* $n \in \mathbb{N}_0$;
(ii) $u(1) \geq (\alpha + \beta)u(0)$; *and*
(iii) $u(0) \geq 0$.

Then u is monotone increasing and $(\alpha + \beta)$-convex on $\mathbb{N}_0$.

Proof. The case $\beta = 1$ is exactly Theorem 6.12. Suppose $1 < \beta < 2$. By Theorem 6.21 we have

$$\begin{aligned}(\Delta^{\alpha+\beta}u)(n+1) &= (\Delta^{\beta} \circ \Delta^{\alpha}u)(n) + \Delta^2 k^{2-\beta}(n+1)u(0)\\ &= (\Delta^{\beta} \circ \Delta^{\alpha}u)(n) + \beta(\beta-1)\frac{k^{-\beta}(n+1)}{(n+2)(n+3)}u(0)\\ &\geq (\Delta^{\beta} \circ \Delta^{\alpha}u)(n)u(0) \geq 0.\end{aligned}$$

We also observe that (ii) implies

$$\Delta^{\alpha+\beta}u(0) \geq 0.$$

The conclusion follows from Theorem 6.3. ∎

Remark 6.23: Similar to Remark 6.6, condition (ii) can be replaced by:
(ii)′ $\Delta u(0) \geq u(0)$.

Corollary 6.24: *Suppose that $0 < \alpha < 1$, $1 \leq \beta < 2$, and $1 < \alpha + \beta < 2$. In addition, suppose that*

(i) $(\Delta^{\beta}_{a+1-\alpha} \circ \Delta^{\alpha}_{a} v)(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+3-\alpha-\beta}$;
(ii) $v(a+1) \geq (\alpha+\beta)v(a)$; *and*
(iii) $v(a) \geq 0$.

Then v is monotone increasing and $(\alpha+\beta)$-convex on $\mathbb{N}_a$.

Proof. Set $u = \tau_a v$. Then, using the transference theorems, we have

$$\begin{aligned}\Delta^{\beta} \circ \Delta^{\alpha}u(n) &= (\tau_{a+2-\beta} \circ \Delta^{\beta}_{a} \circ \tau_{-a}) \circ (\tau_{a+1-\alpha} \circ \Delta^{\alpha}_{a} \circ \tau_{-a})u(n)\\ &= \tau_{a+2-\beta} \circ (\Delta^{\beta}_{a} \circ \tau_{1-\alpha}) \circ \Delta^{\alpha}_{a} v(n)\\ &= \tau_{a+2-\beta} \circ (\tau_{1-\alpha} \circ \Delta^{\beta}_{a+1-\alpha}) \circ \Delta^{\alpha}_{a} v(n)\\ &= \tau_{a+3-\alpha-\beta} \circ \Delta^{\beta}_{a+1-\alpha} \circ \Delta^{\alpha}_{a} v(n)\\ &= \Delta^{\beta}_{a+1-\alpha} \circ \Delta^{\alpha}_{a} v(t),\end{aligned}$$

where

$$t = n + a + 3 - \alpha - \beta \in \mathbb{N}_{a+3-\alpha-\beta}.$$

Since (ii) and (iii) transfer to $u(1) \geq (\alpha+\beta)u(0)$ and $u(0) \geq 0$, the result follows from Theorem 6.22. ∎

7. Convexity

In this section we will consider the following cases.

(a) $0 < \alpha < 1$ and $2 < \beta < 3$
(b) $0 < \alpha < 1$ and $1 < \beta < 2$
(c) $1 < \alpha < 2$ and $1 < \beta < 2$
(d) $1 < \alpha < 2$ and $0 < \beta < 1$
(e) $2 < \alpha < 3$ and $0 < \beta < 1$

The following new convexity result (in the non-sequential setting) is central. We note that this result can be compared to earlier results in the delta discrete fractional setting using definition (1.1)—for example, see Goodrich [26, 28] and Jia et al. [39]. In these papers, typical conditions, in addition to the nonnegativity of the discrete fractional difference $\Delta_a^\nu f(t)$, where $2 < \nu < 3$, was that $f(0) \leq 0$, $\Delta f(0) \geq 0$, and $\Delta^2 f(0) \geq 0$. Moreover, Goodrich [28] replaced these conditions with somewhat more flexible conditions that required $\Delta_a^\nu f(t) \geq 0$ together with an inequality involving a linear combination of $f(0)$, $f(1)$, and $f(2)$—see [28, Theorem 2]. So, we see that Theorem 7.1 utilizes different conditions that do not seem to have appeared exactly in the existing literature.

THEOREM 7.1: *Let $2 \leq \alpha < 3$ be given and assume that*

(i) $(\Delta^\alpha u)(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
(ii) $\Delta^2 u(0) + \frac{(\alpha-2)(\alpha-1)}{2} u(0) \geq (\alpha - 2)\Delta u(0)$;
(iii) $u(1) \geq \alpha u(0)$;
(iv) $u(0) \geq 0$.

Then u is convex on $\mathbb{N}_0$.

Proof. If $\alpha = 2$ then the result is clear. We assume $2 < \alpha < 3$. By definition and Lemma 2.3 we have

$$\begin{aligned}
&\Delta^\alpha u(n)\\
&=(k^{3-\alpha} * u)(n+3) - 3(k^{3-\alpha} * u)(n+2) + 3(k^{3-\alpha} * u)(n+1) - (k^{3-\alpha} * u)(n)\\
&=[(k^{3-\alpha} * \tau_3 u)(n) + k^{3-\alpha}(n+3)u(0) + k^{3-\alpha}(n+2)u(1) + k^{3-\alpha}(n+1)u(2)]\\
&\quad - 3[(k^{3-\alpha} * \tau_2 u)(n) + k^{3-\alpha}(n+2)u(0) + k^{3-\alpha}(n+1)u(1)]\\
&\quad + 3[(k^{3-\alpha} * \tau_1 u)(n) + k^{3-\alpha}(n+1)u(0)] - (k^{3-\alpha} * u)(n)\\
&=(k^{3-\alpha} * [\tau_3 u - 3\tau_2 u + 3\tau_1 u - u])(n) + k^{3-\alpha}(n+3)u(0)\\
&\quad + k^{3-\alpha}(n+2)[u(1) - 3u(0)] + k^{3-\alpha}(n+1)[u(2) - 3u(1) + 3u(0)].
\end{aligned}$$

That is,

$$\begin{aligned}
k^{3-\alpha} * \Delta^3 u(n) \\
&= \Delta^\alpha u(n) - k^{3-\alpha}(n+1)x_1 - k^{3-\alpha}(n+2)x_2 - k^{3-\alpha}(n+3)x_3 \\
&= \Delta^\alpha u(n) - \tau_1 k^{3-\alpha}(n)x_1 - \tau_2 k^{3-\alpha}(n)x_2 - \tau_3 k^{3-\alpha}(n)x_3,
\end{aligned}$$

where

$$\begin{aligned}
x_1 &:= \Delta^2 u(0) - \Delta u(0) + u(0); \\
x_2 &:= \Delta u(0) - 2u(0); \\
x_3 &:= u(0).
\end{aligned}$$

We have

$$\begin{aligned}
k^{\alpha-2} * k^{3-\alpha} * \Delta^3 u(n) \\
&= k^{\alpha-2} * \Delta^\alpha u(n) - k^{\alpha-2} * \tau_1 k^{3-\alpha}(n)x_1 \\
&\quad - k^{\alpha-2} * \tau_2 k^{3-\alpha}(n)x_2 - k^{\alpha-2} * \tau_3 k^{3-\alpha}(n)x_3 \\
&= k^{\alpha-2} * \Delta^\alpha u(n) - [k^{\alpha-2} * k^{3-\alpha}(n+1) - k^{\alpha-2}(n+1)k^{3-\alpha}(0)]x_1 \\
&\quad - [k^{\alpha-2} * k^{3-\alpha}(n+2) - k^{\alpha-2}(n+2)k^{3-\alpha}(0) - k^{\alpha-2}(n+1)k^{3-\alpha}(1)]x_2 \\
&\quad - [k^{\alpha-2} * k^{3-\alpha}(n+3) - k^{\alpha-2}(n+3)k^{3-\alpha}(0) \\
&\qquad - k^{\alpha-2}(n+2)k^{3-\alpha}(1) - k^{\alpha-2}(n+1)k^{3-\alpha}(2)]x_3.
\end{aligned}$$

Since for any $\gamma > 0$, $k^\gamma(0) = 1$,

$$k^\gamma(1) = \gamma \quad \text{and} \quad k^\gamma(2) = (\gamma+1)\gamma/2,$$

we obtain

$$\begin{aligned}
\Delta^2 u(n+1) - \Delta^2 u(0) \\
&= k^{\alpha-2} * \Delta^\alpha u(n) - [1 - k^{\alpha-2}(n+1)]x_1 \\
&\quad - [1 - k^{\alpha-2}(n+2) - k^{\alpha-2}(n+1)(3-\alpha)]x_2 \\
&\quad - \left[1 - k^{\alpha-2}(n+3) - k^{\alpha-2}(n+2)(3-\alpha) - k^{\alpha-2}(n+1)\frac{(4-\alpha)(3-\alpha)}{2}\right]x_3 \qquad (7.1) \\
&= k^{\alpha-2} * \Delta^\alpha u(n) - (x_1 + x_2 + x_3) + k^{\alpha-2}(n+3)x_3 \\
&\quad + k^{\alpha-2}(n+2)[x_2 + (3-\alpha)x_3] \\
&\quad + k^{\alpha-2}(n+1)\left[x_1 + (3-\alpha)x_2 + \frac{(4-\alpha)(3-\alpha)}{2}x_3\right],
\end{aligned}$$

where $x_1 + x_2 + x_3 = \Delta^2 u(0)$, $x_3 = u(0)$, $x_2 + (3-\alpha)x_2 = u(1) - \alpha u(0)$ and

$$\begin{aligned} x_1 + (3-\alpha)x_2 + &\frac{(4-\alpha)(3-\alpha)}{2} x_3 \\ &= \Delta^2 u(0) + \frac{(\alpha-2)(\alpha-1)}{2} u(0) - (\alpha-2)\Delta u(0) \\ &= \Delta^2 u(0) - (\alpha-2)u(1) + \frac{(\alpha-2)(\alpha+1)}{2} u(0). \end{aligned} \tag{7.2}$$

We conclude that

$$\begin{aligned} \Delta^2 u(n+1) = &k^{\alpha-2} * \Delta^\alpha u(n) + k^{\alpha-2}(n+3)u(0) + k^{\alpha-2}(n+2)[u(1) - \alpha u(0)] \\ &+ k^{\alpha-2}(n+1)\Big[\Delta^2 u(0) + \frac{(\alpha-2)(\alpha-1)}{2} u(0) - (\alpha-2)\Delta u(0)\Big], \end{aligned}$$

for all $n \in \mathbb{N}_0$. Finally, note that (ii) (using the identity (7.2)), (iii) and (iv) show that

$$\begin{aligned} \Delta^2 u(0) \geq &(\alpha-2)u(1) - \frac{(\alpha-2)(\alpha-1)}{2} u(0) \\ \geq &\Big[(\alpha-2)\alpha - \frac{(\alpha-2)(\alpha-1)}{2}\Big] u(0) = \frac{(\alpha-2)(\alpha+1)}{2} u(0) \geq 0. \end{aligned}$$

This proves that $\Delta^2 u(m) \geq 0$ for all $m \in \mathbb{N}_0$—i.e., u is convex. ∎

Remark 7.2: According to the proof (see (7.1)) we can replace conditions (ii)–(iv) by the following:

(ii)′ $\Delta^2 u(0) \geq \Delta u(0)$.
(iii)′ $\Delta u(0) \geq 2u(0)$;
(iv)′ $u(0) \geq 0$.

Note that (ii)′ can be rewriten as $\Delta u(1) \geq 2\Delta u(0)$. As already noted, these conditions appear to be somewhat different than any that have appeared in the existing literature.

Remark 7.3: Note that for $1 < \gamma < 2$ an easy calculation using the definition shows that

$$\Delta^\gamma u(0) = \Delta^2(k^{2-\gamma} * u)(0) = \frac{\gamma(\gamma-1)}{2} u(0) - \gamma u(1) + u(2).$$

Note that from this we cannot directly replace condition (ii) by

$$\Delta^\alpha u(0) \geq 0.$$

In the parameter space $[1,2) \times [1,2)$ we give the following general result.

Theorem 7.4: *Let $1 \le \alpha < 2$ and $1 \le \beta < 2$ be given and assume that*

(i) $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \ge 0$, *for all* $n \in \mathbb{N}_0$;
(ii) $\Delta^{\alpha} u(1) \ge \beta \Delta^{\alpha} u(0)$;
(iii) $\Delta^{\alpha} u(0) \ge 0$; *and*
(iv) $u(0) \ge 0$.

Then u is monotone increasing on $\mathbb{N}_1$ and positive on $\mathbb{N}_0$. Moreover, u is α-convex, inclusive of case $\alpha = 2$.

Proof. Let $v = \Delta^{\alpha} u$. By hypothesis $\Delta^{\beta} v \ge 0$. Moreover,

$$v(1) = \Delta^{\alpha} u(1) \ge (\beta - 1)\Delta^{\alpha} u(0) = (\beta - 1)v(0)$$

and

$$v(0) \ge 0.$$

By Theorem 6.3 we obtain that $\Delta^{\alpha} u(n) = v(n) \ge 0$ for all $n \in \mathbb{N}_0$. Note that by (iii) and (iv) we have

$$u(1) \ge \alpha u(0) \ge (\alpha - 1)u(0).$$

Therefore, again by Theorem 6.3, we deduce the assertion. ∎

In the parameter space $[2, 3) \times [0, 1]$ we have

Theorem 7.5: *Let $2 \le \alpha < 3$ and $0 \le \beta \le 1$ be given and assume that*

(i) $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \ge 0$, *for all* $n \in \mathbb{N}_0$;
(ii) $\Delta^{\alpha} u(0) \ge 0$;
(iii) $\Delta^{2} u(0) + \frac{(\alpha-2)(\alpha-1)}{2} u(0) \ge (\alpha - 2)\Delta u(0)$;
(iv) $u(1) \ge \alpha u(0)$; *and*
(v) $u(0) \ge 0$.

Then u is convex on $\mathbb{N}_0$.

Proof. Let $v = \Delta^{\alpha} u$. By Theorem 5.4 and (ii) of the hypothesis we obtain that v is positive on $\mathbb{N}_0$. Then, by Theorem 7.1 and hypotheses (iii)–(v), we have the conclusion. ∎

Remark 7.6: We note that condition (ii) is equivalent to

$$\frac{\alpha(\alpha - 1)}{2} u(1) + u(3) \ge \alpha u(2) + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6} u(0).$$

In case of the parameter space $[2, 3) \times [1, 2)$ we have

THEOREM 7.7: *Let $2 \leq \alpha < 3$ and $1 \leq \beta < 2$ be given and assume that*

(i) $(\Delta^{\beta} \circ \Delta^{\alpha} u)(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
(ii) $\Delta^{\alpha} u(0) \geq 0$;
(iii) $\Delta^{\alpha} u(1) \geq \beta \Delta^{\alpha} u(0)$
(iv) $\Delta^2 u(0) + \frac{(\alpha-2)(\alpha-1)}{2} u(0) \geq (\alpha - 2)\Delta u(0)$;
(v) $u(1) \geq \alpha u(0)$; *and*
(vi) $u(0) \geq 0$.

Then u is convex on $\mathbb{N}_0$.

Proof. Let $v = \Delta^{\alpha} u$. By Theorem 6.3 and (ii)–(iii) of the hypotheses we obtain that v is positive on $\mathbb{N}_0$. Then, by Theorem 7.1 and hypotheses (iv)–(vi), we have the conclusion. ∎

The key result for remaining sequential results on convexity is the following theorem.

THEOREM 7.8: *Assume* $2 < \alpha + \beta < 3$.

(a) *If* $0 < \alpha < 1$ *and* $2 < \beta < 3$, *then*

$$\Delta^{\beta} \circ \Delta^{\alpha} u(n) = \Delta^{\alpha+\beta} u(n+1) - \Delta^3 k^{3-\beta}(n+1)u(0).$$

(b) *If* $0 < \alpha < 1$ *and* $1 < \beta < 2$, *then*

$$\Delta^{\beta} \circ \Delta^{\alpha} u(n) = \Delta^{\alpha+\beta} u(n) - \Delta^2 k^{2-\beta}(n+1)u(0).$$

(c) *If* $1 < \alpha < 2$ *and* $1 < \beta < 2$, *then*

$$\begin{aligned}\Delta^{\beta} &\circ \Delta^{\alpha} u(n) \\ &= \Delta^{\alpha+\beta} u(n+1) - \Delta^2 k^{2-\beta}(n+1)\Delta^{\alpha-1} u(0) - \Delta^3 k^{2-\beta}(n+1)u(0).\end{aligned}$$

(d) *If* $1 < \alpha < 2$ *and* $0 < \beta < 1$, *then*

$$\begin{aligned}\Delta^{\beta} &\circ \Delta^{\alpha} u(n) \\ &= \Delta^{\alpha+\beta} u(n) - \Delta^2 k^{1-\beta}(n+1)u(0) - \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1} u(0).\end{aligned}$$

(e) *If* $2 < \alpha < 3$ *and* $0 < \beta < 1$, *then*

$$\begin{aligned}\Delta^{\beta} \circ \Delta^{\alpha} u(n) =& \Delta^{\alpha+\beta} u(n+1) - \Delta^3 k^{1-\beta}(n+1)u(0) \\ &- \Delta^2 k^{1-\beta}(n+1)\Delta^{\alpha-2} u(0) - \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1} u(0).\end{aligned}$$

Proof. (a) We have $0<\alpha<1$, $1<\beta-1<2$ and $1<\alpha+(\beta-1)<2$. Applying Theorem 6.21 we obtain

$$\Delta^{\beta-1}\circ\Delta^{\alpha}u(n)=\Delta^{\alpha+\beta-1}u(n+1)-\Delta^{2}k^{2-(\beta-1)}(n+1)u(0).$$

Hence,

$$\Delta\circ\Delta^{\beta-1}\circ\Delta^{\alpha}u(n)=\Delta\circ\Delta^{\alpha+\beta-1}u(n+1)-\Delta\circ\Delta^{2}k^{3-\beta}(n+1)u(0).$$

Therefore, item (iii) of Proposition 2.9 implies

$$\Delta^{\beta}\circ\Delta^{\alpha}u(n)=\Delta^{\alpha+\beta}u(n+1)-\Delta^{3}k^{3-\beta}(n+1)u(0),$$

proving (a).

(b) We have $0<\alpha<1$, $0<\beta-1<1$ and $1<\alpha+(\beta-1)<2$. Applying Theorem 6.11 we obtain

$$(\Delta^{\beta-1}\circ\Delta^{\alpha}u)(n)=(\Delta^{\alpha+\beta-1}u)(n)-(\Delta k^{1-(\beta-1)})(n+1)u(0).$$

Hence,

$$\Delta\circ\Delta^{\beta-1}\circ\Delta^{\alpha}u(n)=\Delta\circ\Delta^{\alpha+\beta-1}u(n)-\Delta\circ\Delta k^{2-\beta}(n+1)u(0).$$

Therefore, item (ii) of Proposition 2.9 implies

$$\Delta^{\beta}\circ\Delta^{\alpha}u(n)=\Delta^{\alpha+\beta}u(n)-\Delta^{2}k^{2-\beta}(n+1)u(0),$$

proving (b).

(c) We have $1<\alpha<2$, $0<\beta-1<1$ and $1<\alpha+\beta-1<2$. Applying Theorem 6.16 we obtain

$$\begin{aligned}&\Delta^{\beta-1}\circ\Delta^{\alpha}u(n)\\&\quad=\Delta^{\alpha+\beta-1}u(n+1)-\Delta k^{1-(\beta-1)}(n+1)\Delta^{\alpha-1}u(0)-\Delta^{2}k^{1-(\beta-1)}(n+1)u(0).\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta\circ\Delta^{\beta-1}\circ\Delta^{\alpha}u(n)\\&\quad=\Delta\circ\Delta^{\alpha+\beta-1}u(n+1)-\Delta^{2}k^{2-\beta}(n+1)\Delta^{\alpha-1}u(0)-\Delta^{3}k^{2-\beta}(n+1)u(0).\end{aligned}$$

Therefore, item (ii) of Proposition 2.9 implies

$$\Delta^{\beta}\circ\Delta^{\alpha}u(n)=\Delta^{\alpha+\beta}u(n+1)-\Delta^{2}k^{2-\beta}(n+1)\Delta^{\alpha-1}u(0)-\Delta^{3}k^{2-\beta}(n+1)u(0).$$

proving (c).

(d) We have $0 < \alpha - 1 < 1$ and $0 < \beta < 1$ and $1 < \alpha - 1 + \beta < 2$. Applying Theorem 6.11 we obtain

$$\Delta^{\beta} \circ \Delta^{\alpha-1} u(n) = \Delta^{\alpha-1+\beta} u(n) - \Delta k^{1-\beta}(n+1) u(0).$$

Hence,

$$\Delta \circ \Delta^{\beta} \circ \Delta^{\alpha-1} u(n) = \Delta \circ \Delta^{\alpha-1+\beta} u(n) - \Delta^{2} k^{1-\beta}(n+1) u(0).$$

Therefore, item (iv) of Proposition 2.9 implies

$$\begin{aligned}\Delta^{\beta} \circ \Delta \circ \Delta^{\alpha-1} u(n) &+ \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0) \\ &= \Delta \circ \Delta^{\alpha-1+\beta} u(n) - \Delta^{2} k^{1-\beta}(n+1) u(0).\end{aligned}$$

Now, by items (ii) and (iii) of Proposition 2.9 we obtain

$$\begin{aligned}&\Delta^{\beta} \circ \Delta^{\alpha} u(n) \\ &\quad = \Delta^{\alpha+\beta} u(n) - \Delta^{2} k^{1-\beta}(n+1) u(0) - \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0),\end{aligned}$$

proving (d).

(e) We have $1 < \alpha - 1 < 2$ and $0 < \beta < 1$ and $1 < \alpha - 1 + \beta < 2$. Applying Theorem 6.16 we obtain

$$\begin{aligned}&\Delta^{\beta} \circ \Delta^{\alpha-1} u(n) \\ &\quad = \Delta^{\alpha-1+\beta} u(n+1) - \Delta^{2} k^{1-\beta}(n+1) u(0) - \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1-1} u(0).\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta \circ \Delta^{\beta} \circ \Delta^{\alpha-1} u(n) \\ &\quad = \Delta \circ \Delta^{\alpha-1+\beta} u(n+1) - \Delta^{3} k^{1-\beta}(n+1) u(0) - \Delta^{2} k^{1-\beta}(n+1) \Delta^{\alpha-2} u(0).\end{aligned}$$

Therefore, item (iv) of Proposition 2.9 implies

$$\begin{aligned}&\Delta^{\beta} \circ \Delta \circ \Delta^{\alpha-1} u(n) + \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0) \\ &\quad = \Delta \circ \Delta^{\alpha-1+\beta} u(n+1) - \Delta^{3} k^{1-\beta}(n+1) u(0) - \Delta^{2} k^{1-\beta}(n+1) \Delta^{\alpha-2} u(0).\end{aligned}$$

Consequently, by item (ii) of Proposition 2.9 we obtain

$$\begin{aligned}\Delta^{\beta} \circ \Delta^{\alpha} u(n) = &\Delta^{\alpha+\beta} u(n+1) - \Delta^{3} k^{1-\beta}(n+1) u(0) \\ &- \Delta^{2} k^{1-\beta}(n+1) \Delta^{\alpha-2} u(0) - \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0),\end{aligned}$$

proving the theorem. ∎

We now arrive at our main results concerning convexity, which are the following sequence of theorems. In particular, these treat the different parameter spaces for (α,β) enumerated at the beginning of this section.

Theorem 7.9: *Suppose that* $0<\alpha<1$, $2<\beta<3$, *and* $2<\alpha+\beta<3$. *In addition, suppose that*

(i) $\Delta^{\beta}\circ\Delta^{\alpha}u(n)\geq\beta(\beta-1)(\beta-2)\frac{(3-\beta)}{24}u(0)$, *for all* $n\in\mathbb{N}_0$;
(ii) $u(2)\geq(\alpha+\beta)u(1)-\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(0)$;
(iii) $u(1)\geq(\alpha+\beta)u(0)$;
(iv) $u(0)\geq 0$.

Then u *is convex on* $\mathbb{N}_0$.

Proof. We have

$$\Delta^{\alpha+\beta}u(n+1)=\Delta^{\beta}\circ\Delta^{\alpha}u(n)+\Delta^{3}k^{3-\beta}(n+1)u(0),$$

where by item (iii) of Lemma 3.2 and the fact that $0<3-\beta<1$

$$\begin{aligned}\Delta^{3}k^{3-\beta}(n+1)&=-\beta(\beta-1)(\beta-2)\frac{k^{3-\beta}(n+1)}{(n+2)(n+3)(n+4)}\\&\geq-\beta(\beta-1)(\beta-2)\frac{k^{3-\beta}(1)}{24}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta^{\alpha+\beta}u(n+1)=&\Delta^{\beta}\circ\Delta^{\alpha}u(n)+\Delta^{3}k^{3-\beta}(n+1)u(0)\\\geq&\Delta^{\beta}\circ\Delta^{\alpha}u(n)-\beta(\beta-1)(\beta-2)\frac{(3-\beta)}{24}u(0),\end{aligned}$$

and the conclusion follows from Theorem 7.1. ∎

Remark 7.10: According to Remark 7.2 we can replace conditions (ii)–(iv) by the following:

(ii)′ $\Delta^{2}u(0)\geq\Delta u(0)$;
(iii)′ $\Delta u(0)\geq 2u(0)$;
(iv)′ $u(0)\geq 0$.

Theorem 7.11: *Suppose that* $0<\alpha<1$, $1<\beta<2$, *and* $2<\alpha+\beta<3$. *In addition, suppose that*

(i) $\Delta^{\beta}\circ\Delta^{\alpha}u(n)\geq 0$, *for all* $n\in\mathbb{N}_0$;
(ii) $u(2)\geq(\alpha+\beta)u(1)-\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(0)$;

(iii) $u(1) \geq (\alpha+\beta)u(0)$;
(iv) $u(0) \geq 0$.

Then u is convex on $\mathbb{N}_0$.

Proof. We have

$$\Delta^\beta \circ \Delta^\alpha u(n) = \Delta^{\alpha+\beta} u(n) - \Delta^2 k^{2-\beta}(n+1)u(0),$$

where by item (ii) of Lemma 3.2

$$\Delta^2 k^{2-\beta}(n+1) = \beta(\beta-1)\frac{k^{2-\beta}(n+1)}{(n+2)(n+3)} \geq 0.$$

Therefore,

$$\begin{aligned}\Delta^{\alpha+\beta}u(n) =& \Delta^\beta \circ \Delta^\alpha u(n) + \Delta^2 k^{2-\beta}(n+1)u(0)\\ \geq& \Delta^\beta \circ \Delta^\alpha u(n) \geq 0,\end{aligned}$$

and the conclusion follows from Theorem 7.1. ∎

Remark 7.12: According to Remark 7.2 we can replace conditions (ii)–(iv) by the following:

(ii)′ $\Delta^2 u(0) \geq \Delta u(0)$;
(iii)′ $\Delta u(0) \geq 2u(0)$;
(iv)′ $u(0) \geq 0$.

THEOREM 7.13: *Suppose that $1 < \alpha < 2$ and $1 < \beta < 2$, and $2 < \alpha+\beta < 3$. In addition, suppose that*

(i) $\Delta^\beta \circ \Delta^\alpha u(n) \geq \beta(1+\beta)(\beta-1)\frac{(2-\beta)}{24}u(0)$, *for all $n \in \mathbb{N}_0$;*
(ii) $u(2) \geq (\alpha+\beta)u(1) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(0)$;
(iii) $u(1) \geq (\alpha+\beta)u(0)$;
(iv) $u(0) \geq 0$.

Then u is convex on $\mathbb{N}$.

Proof. We have

$$\begin{aligned}&\Delta^\beta \circ \Delta^\alpha u(n)\\ &\quad= \Delta^{\alpha+\beta}u(n+1) - \Delta^2 k^{2-\beta}(n+1)\Delta^{\alpha-1}u(0) - \Delta^3 k^{2-\beta}(n+1)u(0),\end{aligned}$$

where by item (ii) of Lemma 3.2 we have $\Delta^2 k^{2-\beta}(n+1) \geq 0$ and, since $0 < 2-\beta < 1$,

$$\begin{aligned}\Delta^3 k^{2-\beta}(n+1) &= -\beta(1+\beta)(\beta-1)\frac{k^{2-\beta}(n+1)}{(n+2)(n+3)(n+4)} \\ &\geq -\beta(1+\beta)(\beta-1)\frac{k^{2-\beta}(1)}{24}.\end{aligned}$$

Also, observe that by (ii) and (i) we have

$$\begin{aligned}\Delta^{\alpha-1}u(0) &= u(1)-(\alpha-1)u(0) \\ &\geq (\alpha+\beta)u(0)-(\alpha-1)u(0) \\ &= (\beta+1)u(0) \geq 0.\end{aligned} \tag{7.3}$$

Therefore,

$$\begin{aligned}\Delta^{\alpha+\beta}u(n+1) &= \Delta^{\beta}\circ\Delta^{\alpha}u(n) + \Delta^2 k^{2-\beta}(n+1)\Delta^{\alpha-1}u(0) + \Delta^3 k^{2-\beta}(n+1)u(0) \\ &\geq \Delta^{\beta}\circ\Delta^{\alpha}u(n) - \beta(1+\beta)(\beta-1)\frac{(2-\beta)}{24}u(0),\end{aligned}$$

and the conclusion follows from the hypothesis and Theorem 7.1. ∎

Remark 7.14: According to Remark 7.2 we can replace conditions (ii)–(iv) by the following:

(ii)′ $\Delta^2 u(0) \geq \Delta u(0)$;
(iii)′ $\Delta u(0) \geq 2u(0)$;
(iv)′ $u(0) \geq 0$, replacing the argument in (7.3) by the following:

$$\begin{aligned}\Delta^{\alpha-1}u(0) &= u(1)-(\alpha-1)u(0) = u(1)-\alpha u(0)+u(0) \\ &\geq u(1)-\alpha u(0) \\ &\geq 3u(0)-\alpha u(0) \\ &= (3-\alpha)u(0) \geq 0.\end{aligned}$$

THEOREM 7.15: *Suppose that $1 < \alpha < 2$ and $0 < \beta \leq 1$, and $2 < \alpha+\beta < 3$. In addition, suppose that*

(i) $\Delta^{\beta}\circ\Delta^{\alpha}u(n) \geq \frac{\beta}{2}(1-\beta)\Delta^{\alpha-1}u(0)$, *for all* $n \in \mathbb{N}_0$;
(ii) $u(2) \geq (\alpha+\beta)u(1) - \frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)u(0)$;
(iii) $u(1) \geq (\alpha+\beta)u(0)$;
(iv) $u(0) \geq 0$.

Then u is convex on $\mathbb{N}$.

Proof. The limit case $\beta = 1$ coincides with Theorem 7.13. We have

$$\begin{aligned}&\Delta^{\beta}\circ\Delta^{\alpha}u(n)\\ &\quad=\Delta^{\alpha+\beta}u(n) - \Delta^2 k^{1-\beta}(n+1)u(0) - \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0),\end{aligned}$$

where by item (ii) of Lemma 3.2 we have

$$\Delta^2 k^{1-\beta}(n+1) = \beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)} \geq 0.$$

And since $0 < 1-\beta < 1$, it follows that

$$\Delta k^{1-\beta}(n+1) = -\beta\frac{k^{1-\beta}(n+1)}{n+2} \geq -\beta\frac{k^{1-\beta}(1)}{2}.$$

Also, since $0 < \alpha - 1 < 1$, we observe that by (ii) and (i) we have

$$\begin{aligned}\Delta^{\alpha-1}u(0) =& u(1) - (\alpha-1)u(0)\\ \geq&(\alpha+\beta)u(0) - (\alpha-1)u(0)\\ =&(\beta+1)u(0) \geq 0.\end{aligned} \tag{7.4}$$

Therefore

$$\begin{aligned}\Delta^{\alpha+\beta}u(n+1) =&\Delta^{\beta}\circ\Delta^{\alpha}u(n) + \Delta^2 k^{1-\beta}(n+1)u(0) + \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0)\\ \geq&\Delta^{\beta}\circ\Delta^{\alpha}u(n) - \frac{\beta}{2}(1-\beta)\Delta^{\alpha-1}u(0),\end{aligned}$$

and the conclusion follows from the hypothesis and Theorem 7.1. ∎

Remark 7.16: According to Remark 7.2 we can replace conditions (ii)–(iv) by the following:

(ii)′ $\Delta^2 u(0) \geq \Delta u(0)$.
(iii)′ $\Delta u(0) \geq 2u(0)$;
(iv)′ $u(0) \geq 0$, replacing the argument in (7.4) by the following:

$$\begin{aligned}\Delta^{\alpha-1}u(0) =&u(1) - (\alpha-1)u(0)\\ =&u(1) - \alpha u(0) + u(0)\\ \geq&u(1) - \alpha u(0)\\ \geq&3u(0) - \alpha u(0)\\ =&(3-\alpha)u(0) \geq 0.\end{aligned}$$

We finally prove:

Theorem 7.17: *Suppose that $2 < \alpha < 3$ and $0 < \beta < 1$, and $2 < \alpha + \beta < 3$. In addition, suppose that*

(i) $\Delta^{\beta} \circ \Delta^{\alpha} u(n) \geq \beta(1+\beta)(2+\beta)\frac{(1-\beta)}{24} u(0) + \beta \frac{(1-\beta)}{2} \Delta^{\alpha-1} u(0)$, *for all* $n \in \mathbb{N}_0$*;*
(ii) $\Delta u(1) \geq 2\Delta u(0)$;
(iii) $\Delta u(0) \geq 2u(0)$;
(iv) $u(0) \geq 0$.

Then u is convex on $\mathbb{N}$.

Proof. We have

$$\begin{aligned}\Delta^{\beta} \circ \Delta^{\alpha} u(n) =& \Delta^{\alpha+\beta} u(n+1) - \Delta^3 k^{1-\beta}(n+1) u(0) \\ & - \Delta^2 k^{1-\beta}(n+1) \Delta^{\alpha-2} u(0) - \Delta k^{1-\beta}(n+1) \Delta^{\alpha-1} u(0),\end{aligned}$$

where by item (ii) of Lemma 3.2 we have

$$\Delta^2 k^{1-\beta}(n+1) = \beta(1+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)} \geq 0.$$

And since $0 < 1-\beta < 1$ it also follows that

$$\Delta k^{1-\beta}(n+1) = -\beta \frac{k^{1-\beta}(n+1)}{n+2} \geq -\beta \frac{k^{1-\beta}(1)}{2}.$$

Moreover,

$$\begin{aligned}\Delta^3 k^{1-\beta}(n+1) =& -\beta(1+\beta)(2+\beta)\frac{k^{1-\beta}(n+1)}{(n+2)(n+3)(n+4)} \\ \geq& -\beta(1+\beta)(2+\beta)\frac{k^{1-\beta}(1)}{24}.\end{aligned}$$

Also, since $0 < \alpha - 2 < 1$, we observe that by (iii) and (iv) we have

$$\begin{aligned}\Delta^{\alpha-2} u(0) =& u(1) - (\alpha-2) u(0) \\ =& u(1) - \alpha u(0) + 2u(0) \\ \geq& u(1) - \alpha u(0) \\ \geq& (3-\alpha) u(0) \geq 0,\end{aligned}$$

and since $1 < \alpha - 1 < 2$, we obtain from Remark 7.3 as well as (iv), (ii) and (iii) that

$$\begin{aligned}\Delta^{\alpha-1}u(0) =&\frac{1}{2}(\alpha-1)(\alpha-2)u(0) - (\alpha-1)u(1) + u(2)\\ \geq& -(\alpha-1)u(1) + u(2)\\ \geq& -(\alpha-1)u(1) + 3u(1) - 2u(0)\\ =&(3-\alpha)u(1) + u(1) - 2u(0)\\ \geq& u(1) - 2u(0)\\ \geq& 3u(0) - 2u(0)\\ =& u(0)\\ \geq& 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta^{\alpha+\beta}u(n+1) =&\Delta^{\beta}\circ\Delta^{\alpha}u(n) + \Delta^3 k^{1-\beta}(n+1)u(0)\\ &+ \Delta^2 k^{1-\beta}(n+1)\Delta^{\alpha-2}u(0) + \Delta k^{1-\beta}(n+1)\Delta^{\alpha-1}u(0)\\ \geq&\Delta^{\beta}\circ\Delta^{\alpha}u(n) - \beta(1+\beta)(2+\beta)\frac{(1-\beta)}{24}u(0)\\ &- \beta\frac{(1-\beta)}{2}\Delta^{\alpha-1}u(0),\end{aligned}$$

and the conclusion follows from the hypothesis and Theorem 7.1. ∎

We finish this paper with the following illustrative numerical example.

Example 7.18: As an application for the results established in the previous sections we consider time-stepping schemes for fractional differential equations given in the form

$$D_t^{\alpha}u(t) = f(t, u(t)), \quad t > 0,$$

where D_t^{α} denotes the Riemann–Liouville fractional derivative of order $\alpha > 0$. A number of schemes in time t with a constant time step size $\tau > 0$ has been considered by several authors; see [40] and references therein. Here, we consider the approximation $\overline{\partial}_{\tau}^{\alpha}u(n)$ to $D_t^{\alpha}u(t_n)$; $t_n = n\tau$ given by

$$\overline{\partial}_{\tau}^{\alpha}u(n) := \tau^{-\alpha}\Delta^{\alpha}u(n).$$

We observe that it amounts to a convolution quadrature generated by the kernel $z^{1-\alpha}(1-z)^{\alpha}$; see [40]. We now consider the following time-stepping scheme

$$\tau^{-\alpha}\Delta^{\alpha}u(n) = f(n, u(n)), \quad n \in \mathbb{N}_0. \tag{7.5}$$

As a consequence of Theorem 5.4, Theorem 6.3, Theorem 7.1 and Remark 7.3 we have the following result.

THEOREM 7.19: *Let $f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ be given and assume in* (7.5) *that $f(n,x) \geq 0$ for all $n \in \mathbb{N}_0$ and all $x \in \mathbb{R}$. Then:*

(i) *In case $0 < \alpha < 1$: If $u(0) \geq 0$ then $u(n+1) \geq \alpha u(n)$ for all $n \in \mathbb{N}_0$.*
(ii) *In case $1 < \alpha < 2$: If $u(1) \geq \alpha u(0) \geq 0$ then u is monotone increasing on $\mathbb{N}_1$ and $u(n+2) \geq \alpha u(n+1) - (\alpha-1)u(n)$ for all $n \in \mathbb{N}_0$.*
(iii) *In case $2 < \alpha < 3$: If $\Delta^2 u(0) \geq \Delta u(0) \geq 2u(0) \geq 0$ then $\Delta^2 u(n) \geq 0$ for all $n \in \mathbb{N}_0$—i.e., u is convex.*

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