

## POSITIVITY, MONOTONICITY, AND CONVEXITY FOR CONVOLUTION OPERATORS

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ABSTRACT. We consider the convolution inequality

$$a * u \geq v$$

for given functions  $a$  and  $v$ , and we then investigate conditions on  $a$  and  $v$  that force the unknown function  $u$  to be positive or monotone or convex. We demonstrate that these results for abstract convolution equations can be specialized to yield new insights into the qualitative properties of fractional difference and differential operators. Finally, we apply our results to finite difference methods for fractional differential equations, and we show that our results yield insights into the qualitative behavior of these types of numerical approximations.

1. **Introduction.** The division problem is the following: Let  $a$  and  $v$  be functions defined on a time scale  $\mathbb{T} \subset \mathbb{R}$ . Can we find a function  $u$  which satisfies

$$a * u = v? \tag{1}$$

This problem was posed and solved in complete generality by B. Malgrange and L. Ehrenpreis in the fifties and sixties. However, more precise questions, e.g., the regularity of the function  $u$ , were left open. This was called the *Problem B* by Ehrenpreis [14]. In this paper, we address following form of the *Problem B*: Under which conditions on  $a$  and, eventually, on initial values of  $u$ , can we find a positive, monotone or convex function  $u$  satisfying (1) – more precisely, we replace (1) with the somewhat more general convolution inequality  $a * u \geq v$ ? As we will explain momentarily, it turns out that this question has many applications in the theory of fractional differential and difference operators as well as finite difference methods for fractional differential equations.

Convolution is a widely used technique in mathematical analysis, statistics and approximation theory, and has many applications, e.g., image and signal processing.

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Convolution is also used in connection with univariate splines over uniform knots and B-splines [35].

Various convolution-type equations have been studied in a number of settings. For example, Schumacher [34] and Xu and Wu [38] studied wave propagation dynamics in the case of convolution-type equations; the qualitative properties of such solutions was studied. Similarly, Diekmann and Kaper [11] analyzed the existence and uniqueness of solution to nonlinear convolution equations similar to (1). Similarly, Constantin and Hargraves [8] considered the solvability of the convolution-type equation

$$\Phi(u(t)) = L(t) + \int_0^t P(t-s)u(s) ds,$$

and specifically considered the existence of monotone solutions; a similar sort of problem was earlier considered by Lipovan [28]. In a similar manner, Darwish [10] considered the existence of monotone solutions to a convolution-type integral equation. A different example of convolution equations can be found in the paper by Bright [6], in which the author uses a convolution-type equation to approximate solutions of a given initial value problem in the context of ODEs. Other studies involving convolution equations have addressed some of the regularity properties of such solutions – see, for example, Bonet, Fernández, and Meise [5], Gómez-Callado and Jordá [16], and Lv, Gao, Wei, and Wu [32], and the references therein. Consideration of the injectivity or surjectivity of convolution operators has been considered in other papers such as Choi [7], Dieudonné [13], and Fernández, Galbis, and Jornet [15]. So, we see that there are many studies involving convolution equations, particularly as applied to various differential equations, integral equations, numerical approximations, regularity properties, and arithmetical properties.

But in spite of the extensive literature treating convolution-type problems in general, there seems to be no discernible progress toward answering *Problem B* in the specific setting of the qualitative properties of  $u$  – e.g., positivity, monotonicity, and convexity. While some authors, such as Constantin and Hargraves [8] and Darwish [10], do consider, for example, the existence of certain types of solutions (e.g., monotone) to specific convolution-type equations, there does not seem to be any general results – only very specific results for very specific types of problems.

It is quite natural to ask, therefore, the following more general question: given specific properties imposed on the functions  $a$  and  $v$ , does this confer any particular positivity, monotonicity, and convexity on  $u$ ? This sort of connection is particularly natural when one recalls (as we do below) that fractional differences and derivatives can be realized as convolutions. Yet it does not seem to have been a question considered in the literature to date.

So, it is this question that we attempt to begin to address in this present work. More precisely, our results are typically phrased in the following way. We begin (in the continuous case) by assuming that  $a$  and  $u$  satisfy a convolution inequality of the form

$$\frac{d^n}{dt^n}(a * u)(t) \geq 0, \tag{2}$$

for some nonnegative integer  $n \in \mathbb{N}_0$  and  $t \geq 0$ , say; the discrete case is very similar in that we simply replace the derivative with a forward difference of order  $n$ . We then impose, perhaps, some additional auxiliary conditions on the functions  $a$  and  $u$  – specifically, on their “initial” values. For example, we might require that  $u(0) \geq 0$ . In addition, we will typically assume that  $a$  belongs to the class  $\mathcal{PC}$ , which essentially

means that there exists a nonnegative function  $b$  such that  $(b * a)(t) \equiv 1$ , for  $t > 0$  – see Section 2 for additional details. Then the convolution inequality condition (2) coupled with the auxiliary conditions is sufficient to derive a variety of results about the qualitative behavior of  $u$  itself. For example, we are able to deduce that  $u$  must be variously positive, monotone, or convex on its domain, depending on the value of  $n$  in (2) and the specific character of the auxiliary conditions on either  $a$  or  $u$ . These results, then, make some serious progress toward answering *Problem B* as posed in [14].

Furthermore, as suggested earlier a significant application of our results are to the theory of fractional difference and differential operators, as well as numerical methods for fractional differential equations. To understand why this is true, we would like to recall that both the fractional derivative and the fractional difference can be realized in the form of a particular convolution. It is perhaps worth mentioning at this juncture that the class  $\mathcal{PC}$ , which was described in the preceding paragraph, is not at all restrictive when it comes to these various examples. In fact, as we show in this paper the kernels  $a$  that naturally arise in the study of various fractional differential and difference equations as well as numerical methods for the former can be written in a convolution-type form where the relevant kernel  $a$  satisfies precisely the condition necessary for it to belong to the  $\mathcal{PC}$  class.

So, for example (see Section 2 for additional details), if, for  $0 < \alpha < 1$ , we put  $a(t) := \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ , then the Riemann-Liouville fractional derivative of  $u$  at  $t$ , denoted  $(D_t^\alpha u)(t)$ , can be defined as the convolution

$$(D_t^\alpha u)(t) := \frac{d}{dt}(a * u)(t).$$

This means that a general theorem regarding the derivative of a convolution can be related back to the specific setting of Riemann-Liouville fractional derivatives. As a specific example of what we have in mind, we show in Theorem 2.1 that if  $\frac{d}{dt}(a * u)(t) \geq 0$ , for  $t \geq 0$ , and  $a$  is of class  $\mathcal{PC}$ , then  $u(t) \geq 0$  for  $t > 0$ . As an application of this abstract result, see Example 1, we argue that any solution of the fractional differential equation  $D_t^\alpha u(t) = f(t, u(t))$ , where  $t \geq 0$  and  $0 < \alpha < 1$ , must be nonnegative whenever  $f$  is a nonnegative function. Thus, using the result for a general convolution equation we are able to deduce information about a specific class of fractional differential equations. We are then able to repeat this sort of analysis several times for a variety of different differential equations – see, for instance, Examples 2, 3, 4, 5, and 6. So, we show that a wide range of qualitative properties of solutions of fractional differential equations can be deduced as direct consequences of our more general convolution results.

Similarly, for example, if we consider the function  $a(n) = k^\alpha(n) := \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)n!}$ , for  $n \in \mathbb{N}_0$ , then for  $1 < \alpha < 2$  the  $\alpha$ -th discrete fractional difference is defined by (see either Goodrich and Lizama [19] or Goodrich and Peterson [20])

$$(\Delta^\alpha u)(n) := \Delta^2(k^{2-\alpha} * u)(n),$$

for each  $n \in \mathbb{N}_0$ . Due to the convolution form in the definition of  $\Delta^\alpha$  it is, once again, the case that questions regarding the qualitative properties of  $u$  can be related back to the more general *Problem B*. It therefore follows (see Sections 3 and 4) that the qualitative properties of the discrete fractional difference  $\Delta^\alpha$  can be realized

as a specific application of the more general theory of the qualitative properties of solutions of the convolution equation (1).

Finally, as a third application of our abstract convolution results we demonstrate in Section 5 an application to finite difference methods for fractional differential equations. In particular, following a recent paper by Jin, Li, and Zhou [23] we consider approximation methods for fractional differential equations of the form  $\partial_t^\alpha u(t) = f(t, u(t))$ , for  $t > 0$ . As an approximation to  $\partial_t^\alpha u(t_n)$ , where  $t_n := n\tau$  with  $\tau$  the constant step size of the approximation, the convolution  $\tau^{-\alpha}(a * u)(n)$  is used. Consequently, because the numerical method can be reduced to the realization of a particular convolution, it follows that our general theory from Sections 3 and 4 can be brought to bear on this numerical problem in order to deduce certain qualitative properties of the numerical approximation.

All in all, then, in this paper we provide a relatively comprehensive answer to *Problem B* as posed by Ehrenpreis. And we show that this general treatment provides a number of interesting applications to the qualitative and numerical theory of fractional differential and difference equations. Since the qualitative properties of discrete and continuous fractional operators remains a challenging and relatively new area of study, we believe that the results here demonstrate a new methodology, via convolution, that can be helpful in refining our understanding of these operators.

**2. Positivity and monotonicity on  $\mathbb{R}_+$ .** Recall that by  $*$  we denote the finite convolution – i.e.,

$$(f * g)(t) := \int_0^t f(t-s)g(s) ds, \quad t \geq 0,$$

for  $f, g : [0, \infty) \rightarrow \mathbb{R}$ . We say that a kernel  $a \in L^1_{loc}(\mathbb{R}_+)$  is of type  $\mathcal{PC}$ , if the following condition is satisfied

( $\mathcal{PC}$ ) : There exists a nonnegative kernel  $b$  such that  $b * a = \mathbf{1}$  on  $(0, \infty)$ .

Here, by  $\mathbf{1}$  we denote the constant function whose value is 1 on  $[0, \infty)$ . The condition  $\mathcal{PC}$  originated from a paper of N. Sonine, published in 1884. The pairs  $(a, b)$  appearing in  $\mathcal{PC}$  are called the Sonine pairs. See a survey in [33] and the papers [27], [21]. The subject is connected with the theory of complete Bernstein functions. A subclass of the Sonine pairs was recently used by Vergara and Zacher in the paper [36]. However, note that our definition is weaker than those used in [36] because, in contrast with [36], we have omitted the condition of nonincreasing to the kernel  $b$ . Examples of kernels of type  $\mathcal{PC}$  are the standard kernel

$$a(t) = g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0, \quad \alpha > 0;$$

and the kernels

$$a(t) = \int_0^1 g_\beta(t) d\beta, \quad a(t) = \int_0^\infty \frac{e^{-st}}{1+s} ds, \quad t > 0.$$

For other examples, see [36, Section 6] and [33]. In what follows, for  $n \in \mathbb{N}$ , we denote by  $C^n(\mathbb{R}_+)$  the space of  $n$ -times continuously differentiable functions on  $\mathbb{R}_+$ . We begin with the following result.

**Theorem 2.1.** *Let  $a \in L^1_{loc}(\mathbb{R}_+)$  and  $u \in C^1(\mathbb{R}_+)$  be given. Suppose that*

$$\frac{d}{dt}(a * u)(t) \geq 0 \quad \text{for each } t \geq 0.$$

Assume that  $a$  is of type  $\mathcal{PC}$ . Then  $u(t) \geq 0$  for all  $t \geq 0$ .

*Proof.* From the hypothesis, the Leibniz Rule, and the commutativity of  $*$  we obtain

$$\frac{d}{dt}(a * u)(t) = a(t)u(0) + (a * u')(t), \quad t \geq 0. \tag{3}$$

Convolving the above identity with  $b$  we have

$$\begin{aligned} 0 \leq b * \left( \frac{d}{dt}(a * u) \right) (t) &= (b * a)(t)u(0) + b * (a * u')(t) \\ &= u(0) + ((b * a) * u')(t) = u(t), \quad t \geq 0, \end{aligned}$$

where we have used the associativity of  $*$  and that  $a$  is of type  $\mathcal{PC}$ . □

**Remark 1.** When  $u(0) = 0$ , the expression  $\frac{d}{dt}(a * u)(t)$  is known as distributed order derivative, see Kochubei [26, formula (2.5)]. It appear in mathematical physics in the context of anomalous diffusion processes. An account of their main properties can be found in [26]. The investigation of optimal decay estimates for non-local subdiffusion equations involving the distributed order derivative can be found in the reference [36] by Vergara and Zacher.

**Example 1.** Let  $a(t) = g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$  where  $0 < \alpha < 1$ . Then, thanks to the semigroup property of  $g_\alpha$  (i.e.  $g_\alpha * g_\beta = g_{\alpha+\beta}$  for all  $\alpha, \beta > 0$ ) there exists  $b(t) = g_\alpha(t)$  such that  $a * b = \mathbf{1}$  on  $(0, \infty)$ . Therefore  $g_{1-\alpha}$  is of type  $\mathcal{PC}$  for all  $0 < \alpha < 1$ . This example produces the following consequence: Suppose that  $u$  is a solution of the fractional differential equation

$$D_t^\alpha u(t) = f(t, u(t)), \quad t \geq 0, \quad 0 < \alpha < 1,$$

where  $D^\alpha$  denotes the fractional derivative of order  $\alpha$  in the sense of Riemann-Liouville, i.e.  $D_t^\alpha u(t) := \frac{d}{dt}(g_{1-\alpha} * u)(t)$ . If  $f(t, x) \geq 0$  for all  $x \in \mathbb{R}, t \geq 0$  and  $u$  is differentiable then, necessarily,  $u(t) \geq 0$ .

We have the following corollary.

**Corollary 1.** Let  $a \in L^1_{loc}(\mathbb{R}_+)$  of type  $\mathcal{PC}$  be given. Assume that  $u \in C^1(\mathbb{R}_+)$  and satisfies

$$(a * u')(t) \geq 0 \quad \text{for each } t \geq 0. \tag{4}$$

If  $a$  is nonnegative on  $(0, \infty)$  and  $u(0) \geq 0$  then  $u(t) \geq 0$  for all  $t \geq 0$ .

*Proof.* From the given hypothesis and the identity (3) we deduce that  $\frac{d}{dt}(a * u)(t) \geq 0$  for all  $t \geq 0$ . The result is now an immediate consequence of Theorem 2.1. □

**Remark 2.** It should be noted that alone under the condition (4) of the above corollary, we cannot expect *monotonicity* of  $u$ . To see an example, consider  $a(t) = g_{1-\alpha}(t)$  where  $0 < \alpha < 1$ . It is easy to verify that  $a(t)$  satisfies all the conditions of the corollary but, however, the condition  $(g_{1-\alpha} * u')(t) \geq 0$  alone does not imply that  $u$  is monotone, see [12, Theorem 2.2].

**Example 2.** In order to illustrate the power of the apparently simple Corollary 1, we consider the following problem: To find the existence of positive solutions for the problem

$$\begin{cases} {}_C D_t^\alpha u(t) + {}_C D_t^\beta u(t) &= f(t, u(t)), \quad t \geq 0, & 0 < \beta < \alpha < 1; \\ u(0) &= u_0, \end{cases} \tag{5}$$

where  ${}_C D_t^\alpha$  denotes the derivative in the sense of Caputo, i.e.  ${}_C D_t^\alpha u(t) := (g_{1-\alpha} * u')(t)$ . We assume that  $f(t, x) \geq 0$  and  $u_0 \geq 0$ . Next, we will see that positivity is a consequence of Corollary 1. In fact, thanks to the properties of the finite convolution, we note that the equation in (5) can be rewritten as

$$\left( (g_{1-\alpha} + g_{1-\beta}) * u' \right)(t) = f(t, u(t)), \quad t \geq 0.$$

Define  $a(t) := g_{1-\alpha}(t) + g_{1-\beta}(t)$ . Then, using, for example, Laplace transform, it is easy to check, using the kernel

$$b(t) := t^{\alpha-1} E_{\alpha-\beta, \alpha}(-t^{\alpha-\beta})(t), \quad t > 0,$$

and the property

$$\int_0^\infty e^{-\lambda t} t^{\nu-1} E_{\mu, \nu}(\pm \omega t^\mu) dt = \frac{\lambda^{\mu-\nu}}{(\lambda^\mu \mp \omega)}, \quad \operatorname{Re}(\lambda) > |\omega|^{1/\mu},$$

where  $E_{\mu, \nu}$  denotes the Mittag-Leffler function, that  $a$  is of type  $\mathcal{PC}$ . Since clearly  $a$  is nonnegative, the result about positivity of solutions of equation (5) follows immediately from Corollary 1, as claimed.

**Example 3.** Let  $0 < \mu < 1, 0 < \nu < 1$  and  $u$  be given such that the Hilfer fractional derivative [24] of  $u$  exists and satisfy

$$D_t^{\nu, \mu} u(t) \geq 0, \quad \text{for all } t \geq 0.$$

Then, by definition,  $g_{\nu(1-\mu)} * \frac{d}{dt}(g_{(1-\nu)(1-\mu)} * u)(t) \geq 0$  for all  $t \geq 0$ . See [24] and references therein.

Assume that  $u \in C^1(\mathbb{R}_+)$  and  $u(0) = 0$ . Then we have

$$\begin{aligned} 0 \leq g_{\nu(1-\mu)} * \frac{d}{dt}(g_{(1-\nu)(1-\mu)} * u)(t) &= (g_{\nu(1-\mu)} * g_{(1-\nu)(1-\mu)})(t)u(0) \\ &\quad + \left( g_{\nu(1-\mu)} * (g_{(1-\nu)(1-\mu)} * u') \right)(t) \\ &= g_{1-\mu}(t)u(0) + (g_{1-\mu} * u')(t) \\ &= (g_{1-\mu} * u')(t). \end{aligned}$$

By Example 1 with  $\alpha = \mu$  we obtain that  $a := g_{1-\mu}$  is of type  $\mathcal{PC}$  and nonnegative on  $(0, \infty)$ . Corollary 1 shows that  $u(t) \geq 0$  for all  $t \geq 0$ .

Concerning monotonicity, we give the following theorem.

**Theorem 2.2.** *Let  $a \in C^1(\mathbb{R}_+)$  be of type  $\mathcal{PC}$  and nonincreasing. Assume that  $u \in C^2(\mathbb{R}_+)$  and*

$$\frac{d^2}{dt^2}(a * u)(t) \geq 0 \quad \text{for each } t \geq 0.$$

*If  $u(0) \geq 0$ , then  $u$  is nondecreasing.*

*Proof.* A simple computation shows the following identity

$$\frac{d^2}{dt^2}(a * u)(t) = a'(t)u(0) + a(t)u'(0) + (a * u'')(t), \quad t \geq 0. \tag{6}$$

Since  $a$  is of type  $\mathcal{PC}$ , there exists  $b$  such that  $(b * a)(t) = 1$  for all  $t > 0$ , and we obtain

$$b * (a * u)''(t) = (b * a')(t)u(0) + u'(0) + (g_1 * u'')(t), \quad t \geq 0.$$

Thus,

$$u'(t) = b * (a * u)''(t) - (b * a')(t)u(0), \quad t \geq 0. \tag{7}$$

Since  $b$  is nonnegative and  $a$  is nonincreasing, the conclusion follows from hypothesis and the identity (7).  $\square$

An interesting application of the last theorem is the following.

**Example 4.** For given  $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}_+$  and  $\lambda, T > 0$ , we consider the existence of positive and nonincreasing solutions for the boundary value problem

$$\begin{cases} {}_C D_t^\alpha u(t) + \lambda f(t, u(t)) = 0, & t \in (0, 1), \quad 1 < \alpha < 2; \\ u'(0) = 0, \quad u(1) = T. \end{cases} \tag{8}$$

Convolving the equation with  $g_\alpha$ , using the definition of the Caputo fractional derivative, and the initial conditions, it is easy to see that a solution of (11) is a fixed point of the equation

$$u(t) = T + \lambda \int_0^1 g_\alpha(1-s)f(s, u(s))ds - \lambda \int_0^t g_\alpha(t-s)f(s, u(s))ds. \tag{9}$$

We claim that assuming existence of a solution for (9) then both, positivity and monotonicity of such solution, follows immediately from Theorem 2.2. Indeed, define momentarily  $h(t) := \lambda f(t, u(t))$  and let  $v(t) := (g_\alpha * h)(t)$ . Then  $(g_{2-\alpha} * v)(t) = (g_2 * h)(t)$  and hence by hypothesis

$$\frac{d^2}{dt^2}(g_{2-\alpha} * v)(t) = h(t) \geq 0 \quad \text{for each } t \in [0, 1].$$

and therefore we could take  $a(t) := g_{2-\alpha}(t)$  in Theorem 2.2. Observe that  $a(t)$  satisfies all the hypothesis of such theorem. Indeed, choosing  $b(t) = g_{\alpha-1}(t)$  we see that  $a$  is of type  $\mathcal{PC}$  and is clearly nonincreasing since  $0 < 2 - \alpha < 1$  (see e.g. [19, Proposition 3.1 (iii)]). Moreover,  $v(0) = 0$ . Therefore, we conclude by Theorem 2.2 that  $v$  is nondecreasing, which implies:

$$\int_0^t g_\alpha(t-s)\lambda f(s, u(s))ds = (g_\alpha * h)(t) \leq (g_\alpha * h)(\tau) = \int_0^\tau g_\alpha(\tau-s)\lambda f(s, u(s))ds, \tag{10}$$

for all  $0 \leq t < \tau \leq 1$ . Taking  $\tau = 1$  in (10) we obtain from (9) that  $u$  is positive. Moreover,

$$u(t) - u(\tau) = (g_\alpha * h)(\tau) - (g_\alpha * h)(t) \geq 0$$

i.e. the solution  $u$  is nonincreasing.

The method just developed allows to explain in an easy way some of the arguments used for the main results in [37]. In such paper, the authors analyzed the existence of positive solutions for the boundary value problem

$$\begin{cases} D_t^\alpha u(t) + \lambda f(t, u(t)) = 0, & t \in (0, 1), \quad 2 < \alpha < 3; \\ u(0) = u'(0) = u(1) = 0. \end{cases} \tag{11}$$

where  $D_t^\alpha$  stands for Riemann-Liouville fractional derivative. By [37, Lemma 2.2], and using our notation of the standard kernel  $g_\gamma$ , we have that any solution of (11) is a fixed point of the equation

$$u(t) = t^{\alpha-1}(g_\alpha * h)(1) - (g_\alpha * h)(t), \quad t \in (0, 1),$$

where  $h(t) := \lambda f(t, u(t))$  as before. We will see that any solution of (11), if it exists, is always positive. Indeed, we have for  $t \in (0, 1)$

$$\begin{aligned} (g_\alpha * h)(t) &= \int_0^t g_\alpha(t-s)h(s)ds = t^{\alpha-1} \int_0^t g_\alpha(1-s/t)h(s)ds \\ &\leq t^{\alpha-1} \int_0^t g_\alpha(1-s)h(s)ds \leq t^{\alpha-1} \int_0^1 g_\alpha(1-s)h(s)ds = t^{\alpha-1}(g_\alpha * h)(1), \end{aligned}$$

where we have used the facts that the function  $g_\alpha$  is homogeneous of degree  $\alpha - 1$ , which means  $g_\alpha(t\tau) = t^{\alpha-1}g_\alpha(\tau)$  for all  $t, \tau > 0$ ; and nondecreasing.

A consequence of Theorem (2.2) is the following analog to Corollary 1.

**Corollary 2.** *Let  $a \in C^1(\mathbb{R}_+)$  be of type  $\mathcal{PC}$  nonnegative and nonincreasing. Assume that  $u \in C^2(\mathbb{R}_+)$  and*

$$(a * u'')(t) \geq 0 \quad \text{for each } t \geq 0.$$

*If  $u(0) = 0$  and  $u'(0) \geq 0$  then  $u$  is nondecreasing.*

*Proof.* From the hypothesis and the identity (6) we obtain that  $\frac{d^2}{dt^2}(a * u)(t) \geq 0$  for all  $t \geq 0$ . Then, the conclusion follows from Theorem 2.2. □

The following is a simple application of Corollary 2 which, in turn, provides new insights on the qualitative behavior of fractional differential equations.

**Example 5.** Let  $1 < \alpha < 2$  and consider  $a(t) = g_{2-\alpha}(t)$ . Then, it is easy to see that  $a$  satisfies the hypotheses of Corollary 2. Therefore, the equation

$${}_C D_t^\alpha u(t) = f(t, u(t)), \quad t \geq 0,$$

with initial conditions  $u(0) = 0$  and  $u'(0) \geq 0$  admits nondecreasing solutions whenever  $f(t, x) \geq 0$  for all  $t \geq 0$  and  $x \in \mathbb{R}$ .

Finally, we consider convexity.

**Theorem 2.3.** *Let  $a \in C^2(\mathbb{R}_+)$  be of type  $\mathcal{PC}$  nonincreasing and convex. Assume that  $u \in C^3(\mathbb{R}_+)$  and*

$$\frac{d^3}{dt^3}(a * u)(t) \geq 0 \quad \text{for each } t \geq 0.$$

*If  $u(0) \leq 0$  and  $u'(0) \geq 0$ , then  $u$  is convex.*

*Proof.* Differentiating one more time the identity (6) we obtain

$$(a * u''')(t) = (a * u)'''(t) - a''(t)u(0) - a'(t)u'(0) - a(t)u''(0), \quad t \in I. \tag{12}$$

Since  $a$  is of type  $\mathcal{PC}$  there exists a positive kernel  $b$  such that  $(b * a)(t) = 1$  for all  $t \in I$ . Therefore we obtain

$$\begin{aligned} u''(t) - u''(0) &= (g_1 * u'')(t) \\ &= ((b * a) * u''')(t) \\ &= b * (a * u''')(t) \\ &= (b * (a * u'''))(t) - (b * a'')(t)u(0) - (b * a')(t)u'(0) - (b * a)(t)u''(0) \\ &= (b * (a * u'''))(t) - (b * a'')(t)u(0) - (b * a')(t)u'(0) - u''(0). \end{aligned}$$

Thus

$$u''(t) = (b * (a * u'''))(t) - (b * a'')(t)u(0) - (b * a')(t)u'(0).$$

This last identity together with the hypothesis imply the convexity of  $u$ . □



We prove the following corollary.

**Corollary 3.** *Let  $a \in C^1(\mathbb{R}_+)$  be of type  $\mathcal{PC}$ , nonnegative, nonincreasing and convex and such that  $a' \in C^1(\mathbb{R}_+)$ . Assume that  $u \in C^3(\mathbb{R}_+)$  and*

$$(a * u''')(t) \geq 0 \quad \text{for each } t \geq 0.$$

*If  $u(0) = u'(0) = 0$  and  $u''(0) \geq 0$ , then  $u$  is convex.*

*Proof.* The proof follows from the identity (12) and Theorem 2.3. □

We finish this section with the following example.

**Example 6.** Let  $a(t) = g_{2-\alpha}(t)$  where  $1 < \alpha < 2$ . Then,  $a(t)$  satisfies all the required hypothesis of Corollary 3. The equation

$${}_C D_t^\alpha u(t) = f(t, u(t)), \quad t \geq 0,$$

with initial conditions  $u(0) = u'(0) = 0$  and  $u''(0) \geq 0$  admits convex solutions whenever  $f(t, x) \geq 0$  for all  $t \geq 0$  and  $x \in \mathbb{R}$ .

**3. Positivity and monotonicity on  $\mathbb{N}_0$ .** Let  $a \in s(\mathbb{N}_0; \mathbb{R})$  be a real valued sequence. We say that  $a$  belongs to the class  $\mathcal{C}$  if there exists  $b \in s(\mathbb{N}_0; \mathbb{R}_+)$  such that  $b(n) \geq 0$  and  $(b * a)(n) = 1$  for all  $n \in \mathbb{N}_0$ . We define the set

$$\mathcal{C} := \{b \in s(\mathbb{N}_0; \mathbb{R}_+) : (b * a)(n) = 1 \text{ for all } n \in \mathbb{N}_0\}.$$

Recall that in this discrete context we define the finite convolution  $(u * v)(n)$  by

$$(u * v)(n) := \sum_{j=0}^n u(n-j)v(j),$$

for any  $u, v \in s(\mathbb{N}_0; \mathbb{R})$ .

**Example 7.** For any  $0 < \alpha < 1$  set  $a(n) = k^\alpha(n) := \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)n!}, n \in \mathbb{N}_0$ . We note that this sequence of numbers, known as Cesàro numbers, have ultimate importance in the recent theory of fractional difference operators. See Section 5, below. For an account of their main properties, we refer the reader to [19].

We observe that, because of the semigroup property  $k^{\gamma+\beta} = k^\gamma * k^\beta$  valid for any  $\gamma, \beta > 0$ , the sequence  $b(n) := k^{1-\alpha}(n)$  shows that  $a$  belongs to the  $\mathcal{C}$  class. In the border case  $\alpha = 1$  we have  $a(n) \equiv 1$  and then the sequence  $b(n) = \delta_0(n)$ , where  $\delta_j(n)$  denotes the Kronecker delta, that is

$$\delta_j(n) = \begin{cases} 1 & \text{if } n = j; \\ 0 & \text{if } n \neq j, \end{cases} \tag{13}$$

shows that still we have  $a \in \mathcal{C}$ .

**Example 8.** For any  $1 < \alpha < 2$  set  $a(n) = k^{2-\alpha}(n)$ . Then, there exists  $b(n) = k^{\alpha-1}(n)$  satisfying  $(a * b)(n) \equiv 1$ . Therefore  $a \in \mathcal{C}$ .

**Example 9.** For any  $n \in \mathbb{N}_0$  define  $a(n) = \frac{1}{n+1}$  and  $b(n) = \frac{1}{n!} \int_0^1 \frac{\Gamma(\beta+n)}{\Gamma(\beta)} d\beta$ .

In addition, for any  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we denote by  $\mathcal{P}(f)(n) := \int_0^\infty p_n(t)f(t) dt$  the Poisson transformation [2, Section 4], [29]. Here, the function  $p_n(t) := \frac{t^n e^{-t}}{n!}$  is the Poisson distribution.

We claim that

$$(a * b)(n) = \mathcal{P}(c * d)(n), \quad n \in \mathbb{N}_0, \tag{14}$$

where  $c(t) := \int_0^\infty \frac{e^{-st}}{1+s} ds$ ,  $d(t) := \int_0^1 g_\beta(t) d\beta$  with  $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}$ ,  $\beta > 0$ . We observe that  $(c * d)(t) = 1$  for all  $t > 0$  (see [36, section 6] for a proof). Therefore, according to [29, Theorem 3.4], we obtain

$$\mathbf{1} = \mathcal{P}(c * d) = \mathcal{P}(c) * \mathcal{P}(d), \tag{15}$$

where  $\mathbf{1}(n) \equiv 1$ . Therefore, in light of (14) and (15) to prove that  $(a * b) = \mathbf{1}$  it is sufficient to check that  $\mathcal{P}(c) = a$  and  $\mathcal{P}(d) = b$ . In fact, we have:

$$\begin{aligned} \mathcal{P}(c)(n) &= \int_0^\infty p_n(t) \int_0^\infty \frac{e^{-st}}{1+s} ds dt = \int_0^\infty \frac{1}{n!(1+s)} \int_0^\infty t^n e^{(1+s)t} dt ds \\ &= \int_0^\infty \frac{1}{(1+s)^{n+2}} ds = \frac{1}{n+1}, \end{aligned}$$

and, using [29, Example 3.3],

$$\mathcal{P}(d)(n) = \int_0^\infty p_n(t) \int_0^1 g_\beta(t) d\beta dt = \int_0^1 \int_0^\infty p_n(t)g_\beta(t) dt d\beta = \int_0^1 k^\beta(n) d\beta.$$

Thus,  $a * b \equiv \mathcal{P}(c) * \mathcal{P}(d) \equiv \mathbf{1}$ . This proves that  $a$  and  $b$  belong to the  $\mathcal{C}$  class.

Before presenting our positivity and monotonicity results for convolution operators on  $\mathbb{N}_0$  it will be useful to introduce some notation and basic results regarding the map  $n \mapsto k^\alpha(n)$ . We refer the interested reader either to Goodrich and Peterson [20] or Goodrich and Lizama [19].

**Definition 3.1.** Given a number  $a \in \mathbb{R}$  we define the translation operator  $\tau_a : s(\mathbb{N}_a; \mathbb{R}) \rightarrow s(\mathbb{N}_0; \mathbb{R})$  by

$$(\tau_a u)(n) := u(a+n), \quad n \in \mathbb{N}_0. \tag{16}$$

**Lemma 3.2.** Let  $f, g \in s(\mathbb{N}_0; \mathbb{R})$  be sequences. Then for each  $p = 1, 2, \dots$  we have

$$(f * \tau_p g)(n) = \tau_p(f * g)(n) - \sum_{j=0}^{p-1} \tau_p f(n-j)g(j).$$

In particular, for  $p = 1$  we have

$$(f * \tau_1 g)(n) = (f * g)(n+1) - f(n+1)g(0),$$

and in case  $p = 2$  we have

$$(f * \tau_2 g)(n) = (f * g)(n+2) - f(n+2)g(0) - f(n+1)g(1),$$

and in case  $p = 3$  we have

$$(f * \tau_3 g)(n) = (f * g)(n+3) - f(n+3)g(0) - f(n+2)g(1) - f(n+1)g(2).$$

We start with the following result.

**Theorem 3.3.** Let  $a \in s(\mathbb{N}_0; \mathbb{R})$  be given. Suppose that

$$\Delta(a * u)(n) \geq 0 \quad \text{for each } n \in \mathbb{N}_0.$$

Assume that  $a \in \mathcal{C}$ . If  $a(0)u(0) \geq 0$ , then  $u(n) \geq 0$  for all  $n \in \mathbb{N}_0$ .

*Proof.* A simple computation shows that

$$\Delta(a * u)(n) = u(0)a(n + 1) + (a * \Delta u)(n) = u(0)(\tau_1 a)(n) + (a * \Delta u)(n), \quad n \in \mathbb{N}_0,$$

where  $\tau_1$  is defined as in Definition 3.1. Since  $a \in \mathcal{C}$ , there exists a positive sequence  $b$  such that  $(b * a)(n) = 1$  for all  $n \in \mathbb{N}_0$ . Convoluting with  $b$  the above identity, we obtain

$$(b * \Delta(a * u))(n) = u(0)(b * \tau_1 a)(n) + (b * (a * \Delta u))(n), \quad n \in \mathbb{N}_0.$$

Using the associativity property of the convolution, we obtain

$$(b * \Delta(a * u))(n) = u(0)(b * \tau_1 a)(n) + (1 * \Delta u)(n), \quad n \in \mathbb{N}_0.$$

Using the identity  $(b * \tau_1 a)(n) = (b * a)(n + 1) - b(n + 1)a(0) = 1 - b(n + 1)a(0)$ , we deduce

$$\begin{aligned} (b * \Delta(a * u))(n) &= [1 - b(n + 1)a(0)]u(0) + [u(n + 1) - u(0)] \\ &= -b(n + 1)a(0)u(0) + u(n + 1), \quad n \in \mathbb{N}_0. \end{aligned}$$

Hence, from the positivity of  $b$  and the given hypothesis, we have

$$u(n + 1) = (b * \Delta(a * u))(n) + b(n + 1)a(0)u(0) \geq 0, \quad n \in \mathbb{N}_0,$$

proving the theorem. □

**Remark 3.** Note that the condition  $a(0) > 0$  is satisfied by all the examples given above.

We can improve the condition  $\Delta(a * u)(n) \geq 0$  by imposing a more restrictive class of admissible kernels. We define the set

$$\mathcal{C}_+ := \{b \in \mathcal{C} : \Delta b(n) \geq 0 \text{ for all } n \in \mathbb{N}_0\}.$$

Then we can prove the following theorem.

**Theorem 3.4.** *Let  $a \in s(\mathbb{N}_0; \mathbb{R})$  be given. Suppose that*

$$(a * u)(n) \geq 0 \quad \text{for each } n \in \mathbb{N}_0.$$

*Assume that  $a \in \mathcal{C}_+$ . If  $a(0) > 0$ , then  $u(n) \geq 0$  for all  $n \in \mathbb{N}$ .*

*Proof.* We first observe that  $a \in \mathcal{C}$  implies the identity

$$0 = \Delta(b * a)(n) = (\Delta b * a)(n) + a(n + 1)b(0). \tag{17}$$

Define  $w(n) := (a * u)(n)$ . Using the above identity, we obtain

$$(\Delta b * w)(n) = ((\Delta b * a) * u)(n) = -(\tau_1 a * u)(n)b(0).$$

Now, taking into account the identity  $(\tau_1 a * u)(n) = (a * u)(n + 1) - u(n + 1)a(0)$  we conclude that

$$u(n + 1)a(0)b(0) = (a * u)(n + 1)b(0) + (\Delta b * w)(n) = w(n + 1)b(0) + (\Delta b * w)(n). \tag{18}$$

Since  $w(n) \geq 0$  and  $a \in \mathcal{C}_+$  we get  $u(n)a(0)b(0) \geq 0$  for all  $n \in \mathbb{N}$ .

We now claim that  $b(0) \neq 0$ . Indeed, otherwise, we obtain from (17) that  $b(1)a(0) - b(0)a(0) = \Delta b(0)a(0) = 0$ . As  $a(0) > 0$  we conclude that  $b(1) = 0$ . It proves that  $\Delta b(0) = 0$ . Using again (17) we obtain

$$0 = \Delta^2(b * a)(n) = (\Delta^2 b * a)(n) + a(n + 1)\Delta b(0).$$

Consequently, we deduce  $(b(2) - 2b(1) + b(0))a(0) = \Delta^2 b(0)a(0) = 0$ . Hence  $b(2) = 0$ . Proceeding inductively we conclude that  $b(n) \equiv 0$  in contradiction with the fact that

$(a * b)(n) = 1$ . Finally, since  $b(0) > 0$ , it follows from (18) that  $u(n) \geq 0$  for each  $n \in \mathbb{N}$ , as claimed.  $\square$

The following is our first main result concerning monotonicity.

**Theorem 3.5.** *Let  $a \in s(\mathbb{N}_0; \mathbb{R})$  be given. Suppose that*

$$\Delta(a * u)(n) \geq 0 \quad \text{for each } n \in \mathbb{N}_0.$$

*Assume that  $a \in \mathcal{C}_+$ . If  $a(0) > 0$  and  $u(0) \geq 0$ , then  $\Delta u(n) \geq 0$  for all  $n \in \mathbb{N}_0$ .*

*Proof.* Define  $w(n) := (a * u)(n)$ . From (18) and the hypothesis  $a \in \mathcal{C}_+$  we have that there exists  $b$  such that  $\Delta b(n) \geq 0$  and

$$u(n+1)a(0)b(0) = w(n+1)b(0) + (\Delta b * w)(n).$$

Therefore

$$\begin{aligned} \Delta u(n+1)a(0)b(0) &= \Delta w(n+1)b(0) + \Delta(\Delta b * w)(n) \\ &= \Delta w(n+1)b(0) + (\Delta b * \Delta w)(n) + \Delta b(n+1)w(0) \\ &= \Delta w(n+1)b(0) + (\Delta b * \Delta w)(n) + \Delta b(n+1)a(0)u(0). \end{aligned}$$

We conclude that

$$\Delta u(n+1)a(0)b(0) = \Delta w(n+1)b(0) + (\Delta b * \Delta w)(n) + \Delta b(n+1)a(0)u(0). \quad (19)$$

Since  $b(0) \neq 0$  we deduce from the above identity the claim of the theorem. The proof is finished.  $\square$

A second result that imposes a weaker condition on the kernel  $a$  is the following theorem.

**Theorem 3.6.** *Let  $a \in s(\mathbb{N}_0; \mathbb{R})$  be given. Assume that each of the following conditions is satisfied.*

1.  $a(0) \geq 0$
2.  $u(0) \geq 0$
3.  $\Delta(a * u)(0) \geq (a * u)(0)$
4.  $\Delta^2(a * u)(n) \geq 0$ , for each  $n \in \mathbb{N}_0$

*If, in addition,  $a \in \mathcal{C}$ , then  $(\Delta u)(n) \geq 0$ , for all  $n \in \mathbb{N}_0$ .*

*Proof.* In light of Lemma 3.2 we begin by calculating

$$\begin{aligned} \Delta^2(a * u)(n) &= \tau_2(a * u)(n) - 2\tau_1(a * u)(n) + (a * u)(n) \\ &= (a * \tau_2 u)(n) + a(n+2)u(0) + a(n+1)u(1) \\ &\quad - 2\left[(a * \tau_1 u)(n) + a(n+1)u(0)\right] + (a * u)(n). \end{aligned} \quad (20)$$

Noticing that

$$(a * \tau_2 u)(n) - 2(a * \tau_1 u)(n) + (a * u)(n) = a * (\tau_2 u - 2\tau_1 u + u)(n) = (a * \Delta^2 u)(n),$$

we can recast (20) as

$$\Delta^2(a * u)(n) = (a * \Delta^2 u)(n) + (\tau_2 a)(n)u(0) + (\tau_1 a)(n)u(1) - 2(\tau_1 a)(n)u(0). \quad (21)$$

Now convolving  $b$  with equality (21) we arrive at

$$\begin{aligned}
 (b * \Delta^2(a * u))(n) &= (b * (a * \Delta^2 u))(n) + (b * \tau_2 a)(n)u(0) + (b * \tau_1 a)(n)u(1) \\
 &\quad + (b * (-2\tau_1 a))(n)u(0) \\
 &= (\mathbf{1} * \Delta^2 u)(n) + [(b * \tau_2 a)(n) - 2(b * \tau_1 a)(n)]u(0) \\
 &\quad + (b * \tau_1 a)u(1),
 \end{aligned}
 \tag{22}$$

where we have used in (22) that

$$(b * (a * \Delta^2 u))(n) = \left( \underbrace{(b * a)}_{\equiv \mathbf{1}} * \Delta^2 u \right) (n) = (\mathbf{1} * \Delta^2 u)(n).$$

Now, we rewrite the coefficients of both  $u(0)$  and  $u(1)$  on the right-hand side of (22). In particular, we note that

$$(b * \tau_2 a)(n) = (b * a)(n+2) - b(n+2)a(0) - b(n+1)a(1) = 1 - b(n+2)a(0) - b(n+1)a(1)
 \tag{23}$$

and that

$$(b * \tau_1 a)(n) = (b * a)(n+1) - b(n+1)a(0) = 1 - b(n+1)a(0).
 \tag{24}$$

Therefore, using (23)–(24) we see that we may rewrite equality (22) in the form

$$\begin{aligned}
 (b * \Delta^2(a * u))(n) &= (\mathbf{1} * \Delta^2 u)(n) \\
 &\quad + \left[ \underbrace{1 - b(n+2)a(0) - b(n+1)a(1)}_{=(b * \tau_2 a)(n)} \underbrace{-2 + 2b(n+1)a(0)}_{=-2(b * \tau_1 a)(n)} \right] u(0) \\
 &\quad + [1 - b(n+1)a(0)]u(1).
 \end{aligned}
 \tag{25}$$

At the same time we notice that

$$(\mathbf{1} * \Delta^2 u)(n) = \sum_{j=0}^n (\Delta^2 u)(j) = (\Delta u)(n+1) - (\Delta u)(0).
 \tag{26}$$

Therefore, putting (26) into (25) we obtain

$$\begin{aligned}
 (b * \Delta^2(a * u))(n) &= (\Delta u)(n+1) - (\Delta u)(0) \\
 &\quad + [-1 - b(n+2)a(0) + 2b(n+1)a(0) - b(n+1)a(1)]u(0) \\
 &\quad + [1 - b(n+1)a(0)]u(1) \\
 &= (\Delta u)(n+1) \\
 &\quad + [-b(n+2)a(0) + 2b(n+1)a(0) - b(n+1)a(1)]u(0) \\
 &\quad - b(n+1)a(0)u(1).
 \end{aligned}
 \tag{27}$$

From (27) together with condition (4) in the statement of the theorem it follows that

$$\begin{aligned} (\Delta u)(n+1) &\geq (b * \Delta^2(a * u))(n) \\ &\quad + [b(n+2)a(0) - 2b(n+1)a(0) + b(n+1)a(1)]u(0) \\ &\quad + b(n+1)a(0)u(1). \end{aligned} \quad (28)$$

Since  $a \in \mathcal{C}$  it follows that

$$(b * \Delta^2(a * u))(n) = \sum_{j=0}^n \underbrace{b(n-j)}_{\geq 0} \underbrace{\Delta^2(a * u)(j)}_{\geq 0} \geq 0,$$

for each  $n \in \mathbb{N}_0$ , using also the assumption that  $\Delta^2(a * u)(n) \geq 0$ . So, inequality (28) becomes

$$\begin{aligned} (\Delta u)(n+1) &\geq [b(n+2)a(0) - 2b(n+1)a(0) + b(n+1)a(1)]u(0) \\ &\quad + b(n+1)a(0)u(1). \end{aligned} \quad (29)$$

Finally, in light of estimate (29) and the other assumptions in the statement of the theorem we conclude that

$$\begin{aligned} (\Delta u)(n+1) &\geq [b(n+2)a(0) - 2b(n+1)a(0) + b(n+1)a(1)]u(0) + b(n+1)a(0)u(1) \\ &= b(n+2)a(0)u(0) + b(n+1)[a(1)u(0) - 2a(0)u(0) + a(0)u(1)] \\ &= b(n+2)a(0)u(0) + b(n+1)[(a * u)(1) - 2(a * u)(0)] \\ &= b(n+2)a(0)u(0) + b(n+1)[\Delta(a * u)(0) - (a * u)(0)] \\ &\geq 0, \end{aligned}$$

for each  $n \in \mathbb{N}_0$ , which proves that  $u$  is monotone increasing on  $\mathbb{N}_0$ , as claimed.  $\square$

We next give a couple examples of condition (3) in Theorem 3.6 in the case of specific choices for the function  $a$ .

**Example 10.** Consider the function  $a(n) := \frac{1}{n+1}$ . Then condition (3) in Theorem 3.6 becomes  $u(1) \geq \frac{3}{2}u(0)$ .

**Example 11.** Consider the function  $a(n) := k^\alpha(n)$ . Then condition (3) in Theorem 3.6 becomes  $u(1) \geq (2-\alpha)u(0)$ . And, similarly, with  $a(n) := k^{2-\alpha}(n)$  condition (3) becomes  $u(1) \geq \alpha u(0)$ . In fact, this case has an interesting application to fractional difference calculus. Using the fact that, for  $1 < \alpha < 2$ ,

$$(\Delta^\alpha u)(n) := (\Delta^2 \Delta^{\alpha-2} u)(n) = \Delta^2(k^{2-\alpha} * u)(n), \quad (30)$$

where  $\Delta^\alpha$  is the  $\alpha$ -th order fractional difference (see, for example, [20, 19]), we see that if the conditions

1.  $u(0) \geq 0$ ;
2.  $u(1) \geq \alpha u(0)$ ; and
3.  $(\Delta^\alpha u)(n) \geq 0$ , for each  $n \in \mathbb{N}_0$

hold, then  $u$  is monotone increasing on  $\mathbb{N}_0$ . Note that this recovers [19, Theorem 6.3] as a special case of Theorem 3.6. One may further compare to Dahal and Goodrich [9, Theorem 2.2] – see also Baoguo, Erbe, Goodrich, and Peterson [4] and Jia, Erbe, and Peterson [22].

We conclude this section with an application to the analysis of fractional difference equations.

**Example 12.** Consider, for  $1 < \alpha < 2$ , the fractional difference equation

$$(\Delta^\alpha u)(n) = f(n, u(n)),$$

together with the initial condition  $u(0) = 0$ . Then if  $f(n, u) \geq 0$  for each  $(n, u) \in \mathbb{N}_0 \times \mathbb{R}$  and  $u$  also satisfies  $u(1) \geq 0$ , then any solution of this problem must be monotone increasing.

4. **Convexity.** We define the set

$$\mathcal{C}_{++} := \{b \in \mathcal{C}_+ : \Delta^2 b(n) \geq 0 \text{ for all } n \in \mathbb{N}_0\}.$$

We next consider a first convexity-type result based on the set  $\mathcal{C}_{++}$ .

**Theorem 4.1.** *Let  $a \in s(\mathbb{N}_0; \mathbb{R})$  be given. Suppose that*

$$\Delta^2(a * u)(n) \geq 0 \quad \text{for each } n \in \mathbb{N}_0.$$

*Assume that  $a \in \mathcal{C}_{++}$ . If  $a(0) > 0$ ,  $u(0) \geq 0$  and  $a(1)u(0) + a(0)u(1) - a(0)u(0) \geq 0$ , then  $\Delta^2 u(n) \geq 0$  for all  $n \in \mathbb{N}$ .*

*Proof.* Define  $w(n) := (a * u)(n)$ . From (19) and the hypothesis  $a \in \mathcal{C}_{++}$  we have that there exists  $b$  such that  $\Delta b(n) \geq 0$ ,  $\Delta^2 b(n) \geq 0$  and

$$\Delta u(n+1)a(0)b(0) = \Delta w(n+1)b(0) + (\Delta b * \Delta w)(n) + \Delta b(n+1)a(0)u(0).$$

Therefore

$$\begin{aligned} \Delta^2 u(n+1)a(0)b(0) &= \Delta^2 w(n+1)b(0) + \Delta(\Delta b * \Delta w)(n) + \Delta^2 b(n+1)a(0)u(0) \\ &= \Delta^2 w(n+1)b(0) + (\Delta b * \Delta^2 w)(n) + \Delta b(n+1)\Delta w(0) \\ &\quad + \Delta^2 b(n+1)a(0)u(0), \end{aligned}$$

where  $\Delta w(0) = w(1) - w(0) = a(1)u(0) + a(0)u(1) - a(0)u(0)$ . The conclusion follows.  $\square$

**Remark 4.** We remark that the condition  $a(1)u(0) + a(0)u(1) - a(0)u(0) \geq 0$  can be rewritten in the form

$$\Delta(a * u)(0) \geq 0.$$

We continue with a different convexity result by reverting to the set  $\mathcal{C}$  rather than  $\mathcal{C}_{++}$ .

**Theorem 4.2.** *Let  $a \in s(\mathbb{N}_0; \mathbb{R})$  be given and assume that  $a \in \mathcal{C}$ . In addition, suppose that each of the following inequalities is satisfied.*

1.  $a(0)u(0) \geq 0$
2.  $a(1)u(0) - 3a(0)u(0) + a(0)u(1) \geq 0$
3.  $a(2)u(0) - 3a(1)u(0) + 3a(0)u(0) + a(1)u(1) - 3a(0)u(1) + a(0)u(2) \geq 0$
4.  $\Delta^3(a * u)(n) \geq 0$ , for each  $n \in \mathbb{N}_0$

*Then  $\Delta^2 u(n) \geq 0$  for each  $n \in \mathbb{N}$  - i.e.,  $u$  is convex on its domain.*

*Proof.* Note first that

$$\begin{aligned} \Delta^3(a * u)(n) &= \tau_3(a * u)(n) - 3\tau_2(a * u)(n) + 3\tau_1(a * u)(n) - (a * u)(n) \\ &= (a * \tau_3 u)(n) + a(n+3)u(0) + a(n+2)u(1) + a(n+1)u(2) \\ &\quad - 3\left[ (a * \tau_2 u)(n) + a(n+2)u(0) + a(n+1)u(1) \right] \\ &\quad + 3\left[ (a * \tau_1 u)(n) + a(n+1)u(0) \right] \\ &\quad - (a * u)(n). \end{aligned} \tag{31}$$

Now notice that

$$(a * \Delta^3 u)(n) = (a * \tau_3 u)(n) - 3(a * \tau_2 u)(n) + 3(a * \tau_1 u)(n) - (a * u)(n). \quad (32)$$

Therefore, combining (32) with (31) it follows that

$$\begin{aligned} \Delta^3(a * u)(n) &= (a * \Delta^3 u)(n) \\ &\quad + (\tau_3 a)(n)u(0) + (\tau_2 a)(n)u(1) + (\tau_1 a)(n)u(2) \\ &\quad - 3(\tau_2 a)(n)u(0) - 3(\tau_1 a)(n)u(1) + 3(\tau_1 a)(n)u(0). \end{aligned} \quad (33)$$

Recall that  $a \in \mathcal{C}$  by assumption, and so, there exists  $b$  such that

$$(a * b)(n) \equiv 1.$$

So, convolving equation (33) with  $b$  we obtain

$$\begin{aligned} (b * \Delta^3(a * u))(n) &= (\mathbf{1} * \Delta^3 u)(n) \\ &\quad + \left[ (b * \tau_3 a)(n) - 3(b * \tau_2 u)(n) + 3(b * \tau_1 u)(n) \right] u(0) \\ &\quad + \left[ (b * \tau_2 a)(n) - 3(b * \tau_1 a)(n) \right] u(1) \\ &\quad + (b * \tau_1 a)(n)u(2), \end{aligned} \quad (34)$$

using the associativity of the convolution operator.

Now combining Lemma 3.2 with (34) we obtain the identity

$$\begin{aligned} (b * \Delta^3(a * u))(n) &= (\mathbf{1} * \Delta^3 u)(n) \\ &\quad + [1 - b(n+3)a(0) - b(n+2)a(1) \\ &\quad \quad - b(n+1)a(2) + 3b(n+2)a(0) + 3b(n+1)a(1) \\ &\quad \quad - 3b(n+1)a(0)]u(0) \\ &\quad + [-b(n+2)a(0) - b(n+1)a(1) - 2 + 3b(n+1)a(0)]u(1) \\ &\quad + [1 - b(n+1)a(0)]u(2). \end{aligned} \quad (35)$$

In addition, we see that identity (38) can be rewritten in the form

$$(\mathbf{1} * \Delta^3 u)(n) = (\Delta^2 u)(n+1) - (\Delta^2 u)(0).$$

Therefore, it follows that

$$\begin{aligned} (b * \Delta^3(a * u))(n) &= (\Delta^2 u)(n+1) \\ &\quad + [-b(n+3)a(0) - b(n+2)a(1) + 3b(n+1)a(1) \\ &\quad \quad - b(n+1)a(2) + 3b(n+2)a(0) - 3b(n+1)a(0)]u(0) \\ &\quad + [-b(n+2)a(0) - b(n+1)a(1) + 3b(n+1)a(0)]u(1) \\ &\quad + [-b(n+1)a(0)]u(2). \end{aligned} \quad (36)$$



So, we conclude from (39) that

$$\begin{aligned} & (\Delta^2 u)(n+1) \\ &= (b * \Delta^3(a * u))(n) \\ &+ [b(n+3)a(0) + b(n+2)a(1) - 3b(n+1)a(1) + b(n+1)a(2) - 3b(n+2)a(0) \\ &+ 3b(n+1)a(0)]u(0) + [b(n+2)a(0) + b(n+1)a(1) - 3b(n+1)a(0)]u(1) \\ &+ b(n+1)a(0)u(2), \end{aligned}$$

or, equivalently, that

$$\begin{aligned} (\Delta^2 u)(n+1) &= (b * \Delta^3(a * u))(n) + [a(0)u(0)]b(n+3) \\ &+ [a(1)u(0) - 3a(0)u(0) + a(0)u(1)]b(n+2) \\ &+ [a(2)u(0) - 3a(1)u(0) + 3a(0)u(0) + a(1)u(1) - 3a(0)u(1) \\ &+ a(0)u(2)]b(n+1). \end{aligned} \tag{37}$$

Finally, let us recall the conditions (1)–(4) imposed in the statement of this theorem. Then we see that (37) implies that

$$\begin{aligned} (\Delta^2 u)(n+1) &= \underbrace{(b * \Delta^3(a * u))(n)}_{\geq 0} \\ &+ \underbrace{[a(0)u(0)]}_{\geq 0} b(n+3) \\ &+ \underbrace{[a(1)u(0) - 3a(0)u(0) + a(0)u(1)]}_{\geq 0} b(n+2) \\ &+ \underbrace{[a(2)u(0) - 3a(1)u(0) + 3a(0)u(0) + a(1)u(1) - 3a(0)u(1) + a(0)u(2)]}_{\geq 0} b(n+1) \\ &\geq 0, \end{aligned}$$

for each  $n \in \mathbb{N}$ , where we have used the fact that  $b(n) \geq 0$  for each  $n$ . Therefore, we conclude that

$$(\Delta^2 u)(n+1) \geq 0, \quad n \in \mathbb{N},$$

and so,  $n \mapsto u(n)$  is a convex map on its domain, which completes the proof of the theorem.  $\square$

It is somewhat difficult to see what precisely the conditions (2)–(3) in Theorem 4.2 imply. Therefore, we next provide some examples of the application of Theorem 4.2 to some of the types of kernels that we have already mentioned earlier in this paper.

**Example 13.** Consider the kernel

$$a(n) := k^\alpha(n).$$

Note that  $a(0) = 1$ ,  $a(1) = \alpha$ , and  $a(2) = \frac{1}{2}\alpha(\alpha+1)$ . Then condition (2) of Theorem 4.2 becomes

$$u(1) \geq (3 - \alpha)u(0).$$

On the other hand, condition (3) of Theorem 4.2 becomes

$$u(2) + (\alpha - 3)u(1) + \frac{1}{2}(\alpha - 3)(\alpha - 2)u(0) \geq 0.$$

It is instructive to observe that in case  $\alpha = 1$ , these conditions reduce, respectively, to

$$u(1) \geq 2u(0) \text{ and } (\Delta^2 u)(0) \geq 0,$$

whereas in case  $\alpha = 2$ , these conditions reduce, respectively, to

$$(\Delta u)(0) \geq 0 \text{ and } (\Delta u)(1) \geq 0.$$

**Example 14.** Consider the kernel

$$a(n) := k^{3-\alpha}(n).$$

Note that  $a(0) = 1$ ,  $a(1) = 3 - \alpha$ , and  $a(2) = \frac{1}{2}(3 - \alpha)(4 - \alpha)$ . Then condition (2) of Theorem 4.2 becomes

$$u(1) \geq \alpha u(0).$$

On the other hand, condition (3) of Theorem 4.2 becomes

$$u(2) - \alpha u(1) + \frac{1}{2}\alpha(\alpha - 1)u(0) \geq 0.$$

It is instructive to observe that in case  $\alpha = 2$ , these conditions reduce, respectively, to

$$u(1) \geq 2u(0) \text{ and } (\Delta^2 u)(0) \geq 0,$$

whereas in case  $\alpha = 3$ , these conditions reduce, respectively, to

$$u(1) \geq 3u(0) \text{ and } (\Delta^2 u)(0) - (\Delta u)(0) + u(0) \geq 0.$$

As an application of Example 14 we consider the following example, which connects some of the abstract result of Theorem 4.2 to the setting of fractional difference operators.

**Example 15.** For  $2 < \alpha < 3$  consider the fractional difference (again, see [20, 19])

$$(\Delta^\alpha u)(n) := (\Delta^3 \Delta^{\alpha-3} u)(n) = \Delta^3 (k^{3-\alpha} * u)(n). \quad (38)$$

Suppose that

$$(\Delta^\alpha u)(n) \geq 0, \quad (39)$$

for each  $n \in \mathbb{N}_0$ . Then due to (38), Theorem 4.2, and Example 14 we see that if (39) holds along with the conditions

1.  $u(0) \geq 0$ ;
2.  $u(1) \geq \alpha u(0)$ ; and
3.  $u(2) - \alpha u(1) + \frac{1}{2}\alpha(\alpha - 1)u(0) \geq 0$ ;

then  $u$  must be convex on  $\mathbb{N}_0$ . One may compare this result other recent convexity-type results for fractional differences such as [17, Theorem 2.6], [18, Theorem 2], and [19, Theorem 7.1]. Especially, the conditions

- $u(0) \geq 0$ ;
- $u(1) \geq \alpha u(0)$ ;
- $u(2) - \alpha u(1) + \frac{1}{2}\alpha(\alpha - 1)u(0) \geq 0$ ; and
- $(\Delta^\alpha u)(n) \geq 0$ , for each  $n \in \mathbb{N}_0$ ;

as deduced above recover [19, Theorem 7.1] as a special case of Theorem 4.2.

Finally, we consider the following application to a fractional difference equation.

**Example 16.** Consider, for  $2 < \alpha < 3$ , the fractional difference equation

$$(\Delta^\alpha u)(n) = f(n, u(n)),$$

where  $f(n, u) \geq 0$  for each  $(n, u) \in \mathbb{N}_0 \times \mathbb{R}$ . Then any function  $u$  satisfying conditions (1)–(3) in Example 15 must be convex if it also solves the fractional difference equation.

**5. Time-stepping schemes.** A rich source of application for the results established in the previous sections are time-stepping schemes for fractional differential equations [23] in the form

$$\partial_t^\alpha u(t) = f(t, u(t)), \quad t > 0.$$

We will consider a number of schemes in time  $t$  with a constant time step size  $\tau > 0$  where the approximation  $\bar{\partial}_\tau^\alpha u(n)$  to  $\partial_t^\alpha u(t_n)$ ;  $t_n = n\tau$  is given by

$$\bar{\partial}_\tau^\alpha u(n) := \tau^{-\alpha} (a * u)(n)$$

with

$$\sum_{j=0}^\infty a(j)\xi^j = \delta(\xi), \tag{40}$$

being  $\delta(\xi)$  the characteristic function of the time-stepping scheme [23, Section 3].

A key tool in our analysis will be the following particular sequence that has been proved to be a main ingredient in the theory of fractional difference operators and differential-difference equations [3, 1, 2, 29, 30, 31]

$$k^\alpha(n) := \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)}{n!} \text{ for } n \in \mathbb{N}; \quad k^\alpha(0) := 1, \quad \alpha \in \mathbb{R}.$$

Recall that the function  $n \mapsto k^\alpha(n)$  was previously introduced in Example 7 and used from time to time in the examples of Sections 3 and 4. Note that the sequence  $k^\alpha$  can be written as

$$k^\alpha(n) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)\Gamma(n + 1)}, \quad n \in \mathbb{N}_0; \quad \alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\},$$

where  $\Gamma$  is the Euler Gamma function. Also the kernel  $k^\alpha$  could be defined by the generating function, that is,

$$\sum_{j=0}^\infty k^\alpha(j)z^j = \frac{1}{(1 - z)^\alpha}, \quad |z| < 1. \tag{41}$$

Therefore, these kernels satisfy the group property,  $k^\alpha * k^\beta = k^{\alpha+\beta}$  for  $\alpha, \beta \in \mathbb{R}$ .

**5.1. Backward Euler scheme.** In this case, by [23, Subsection 3.1] we have  $\delta(\xi) = (1 - \xi)^\alpha$ ,  $\alpha > 0$ , and a simple observation comparing (40) and (41) gives  $a(n) = k^{-\alpha}(n)$ . By the group property, it follows that there exists  $b(n) := k^{\alpha+1}(n)$  such that  $(a * b)(n) = 1$ . This shows that  $a \in \mathcal{C}$  and hence, as consequence of Theorem 3.3 in Section 3, we obtain the following positivity result.

**Theorem 5.1.** *Let  $f : \mathbb{N}_0 \times \mathbb{R} \rightarrow \mathbb{R}$  be given and suppose that  $f$  is non-decreasing in the first variable, that is*

$$\Delta f(n, x) \geq 0 \quad \text{for all } x \in \mathbb{R}, \text{ and any } n \in \mathbb{N}_0.$$

Consider the discretization

$$\bar{\partial}_\tau^\alpha u(n) = f(n, u(n))$$

with backward Euler time-stepping scheme. Assume that  $u(0) \geq 0$ . Then  $u(n) \geq 0$  for all  $n \in \mathbb{N}_0$ .

Due to the fact that the kernel  $b(n) = k^{\alpha+1}(n)$  is increasing, which follows from the identity [19, Lemma 3.2]

$$\Delta k^{\alpha+1}(n) = \alpha \frac{k^{\alpha+1}(n)}{n+1},$$

and thanks to Theorem 3.4, we can relax the hypothesis of  $f$  to a more natural one, as the following result shows.

**Theorem 5.2.** *Let  $f : \mathbb{N}_0 \times \mathbb{R} \rightarrow \mathbb{R}$  be given and suppose that  $f$  is positive in the first variable, that is*

$$f(n, x) \geq 0 \quad \text{for all } x \in \mathbb{R}, \text{ and any } n \in \mathbb{N}_0.$$

Consider the discretization

$$\bar{\partial}_\tau^\alpha u(n) = f(n, u(n))$$

with backward Euler time-stepping scheme. Then  $u(n) \geq 0$  for all  $n \in \mathbb{N}$ .

For instance, the conclusion holds for nonlinear terms when they have the separated variables form  $f(n, x) = A(n)B(x)$ , where  $A(n)$  is non-increasing and  $B(x) \geq 0$  for all  $x \in \mathbb{R}$ .

A monotonicity result follows in analogous way. We only have to assume an extra condition on the initial terms of the discretization that are imposed by the backward Euler scheme. Note that to obtain this next result we appeal to Theorem 3.6.

**Theorem 5.3.** *Let  $f : \mathbb{N}_0 \times \mathbb{R} \rightarrow \mathbb{R}$  be given and suppose that  $f$  is convex in the first variable, that is*

$$\Delta^2 f(n, x) \geq 0 \quad \text{for all } x \in \mathbb{R}, \text{ and any } n \in \mathbb{N}_0.$$

Consider the discretization

$$\bar{\partial}_\tau^\alpha u(n) = f(n, u(n))$$

with backward Euler time-stepping scheme. Assume that  $u(0) \geq 0$  and  $u(1) \geq (\alpha + 2)u(0)$ . Then  $\Delta u(n) \geq 0$  for all  $n \in \mathbb{N}_0$ .

We continue with a convexity-type result. This makes use of Theorem 4.2.

**Theorem 5.4.** *Let  $f : \mathbb{N}_0 \times \mathbb{R} \rightarrow \mathbb{R}$  be given and suppose that  $f$  satisfies the inequality*

$$\Delta^3 f(n, x) \geq 0 \quad \text{for all } x \in \mathbb{R}, \text{ and any } n \in \mathbb{N}_0.$$

Consider the discretization

$$\bar{\partial}_\tau^\alpha u(n) = f(n, u(n))$$

with backward Euler time-stepping scheme, and assume that the function  $(n, x) \mapsto f(n, x)$  satisfies the inequality

$$\Delta^3 f(n, x) \geq 0,$$

for each  $x \in \mathbb{R}$  and  $n \in \mathbb{N}_0$ . Finally, assume that each of the following conditions holds.

1.  $u(0) \geq 0$
2.  $u(1) \geq (3 + \alpha)u(0)$
3.  $u(2) - (3 + \alpha)u(1) + \frac{1}{2}(\alpha + 2)(\alpha + 3)u(0) \geq 0$

Then  $(\Delta^2 u)(n) \geq 0$  for all  $n \in \mathbb{N}_0$  - i.e.,  $u$  is convex on its domain.

**5.2. Second-order backward difference scheme.** By the theory developed in [23, Subsection 3.2] we have  $\delta(\xi) = \left(\frac{3}{2} - 2\xi + \frac{1}{2}\xi^2\right)^\alpha$ . We note the identity

$$\left(\frac{3}{2} - 2\xi + \frac{1}{2}\xi^2\right)^\alpha = \left(\frac{3}{2}\right)^\alpha \left(1 - \frac{\xi}{3}\right)^\alpha (1 - \xi)^\alpha.$$

Comparing (40) and (41) we deduce

$$a(n) = \left(\frac{3}{2}\right)^\alpha \sum_{j=0}^n k^{-\alpha}(n-j) \frac{1}{3^j} k^{-\alpha}(j),$$

so that defining

$$b(n) := \left(\frac{2}{3}\right)^\alpha \sum_{j=0}^n k^{\alpha+1}(n-j) \frac{1}{3^j} k^\alpha(j),$$

we obtain  $(a * b)(n) = 1$ . Therefore,  $a \in \mathcal{C}$ . We also observe that  $a(0) = \left(\frac{3}{2}\right)^\alpha$  and

$$a(1) = -\alpha \left(\frac{3}{2}\right)^\alpha \frac{4}{3}.$$

We can then state the following result. Note that to obtain this next theorem we again use Theorem 3.6.

**Theorem 5.5.** *Let  $f : \mathbb{N}_0 \times \mathbb{R} \rightarrow \mathbb{R}$  be given. Consider the discretization*

$$\bar{\partial}_\tau^\alpha u(n) = f(n, u(n)), \quad \alpha > 0,$$

*with the second-order backward difference time-stepping scheme. Suppose that  $u(0) \geq 0$ .*

- (i) *If  $f$  is non-decreasing in the first variable then  $u$  is non-negative.*
- (ii) *If  $f$  is convex in the first variable and*

$$u(1) \geq \frac{4\alpha + 6}{3} u(0)$$

*holds, then  $u$  is non-decreasing.*

As in the previous subsection, we again continue with a convexity-type result. As before, the proof of this result makes use of Theorem 4.2. Note that to derive this result, we use the fact that  $a(0) = \left(\frac{3}{2}\right)^\alpha$ ,  $a(1) = -\alpha \left(\frac{3}{2}\right)^\alpha \frac{4}{3}$ , and  $a(2) = \left(\frac{3}{2}\right)^\alpha \left[\frac{5}{9}\alpha(\alpha - 1) + \frac{1}{3}\alpha^2\right]$ .

**Theorem 5.6.** *Let  $f : \mathbb{N}_0 \times \mathbb{R} \rightarrow \mathbb{R}$  be given. Consider the discretization*

$$\bar{\partial}_\tau^\alpha u(n) = f(n, u(n))$$

*with backward Euler time-stepping scheme, and assume that the function  $(n, x) \mapsto f(n, x)$  satisfies the inequality*

$$\Delta^3 f(n, x) \geq 0,$$

for each  $x \in \mathbb{R}$  and  $n \in \mathbb{N}_0$ . Finally, assume that each of the following conditions holds.

1.  $u(0) \geq 0$
2.  $u(1) \geq \left(\frac{4\alpha + 9}{3}\right) u(0)$
3.  $u(2) - \left(\frac{4\alpha + 9}{3}\right) u(1) + \frac{1}{9} (8\alpha^2 + 31\alpha + 27) u(0) \geq 0$

Then  $(\Delta^2 u)(n) \geq 0$  for each  $n \in \mathbb{N}$  – i.e.,  $u$  is convex on its domain.

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