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Well-posedness for the abstract Blackstock–Crighton–Westervelt equation

LAURA R. GAMBERA, CARLOS LIZAMA AND ANDRÉA PROKOPCZYK

Abstract. In this paper, an abstract degenerate hyperbolic equation is considered that includes the semilinear Blackstock–Crighton–Westervelt equation. By proposing a new approach based on strongly continuous semigroups and resolvent families of operators, we prove an explicit representation of the strong and mild solutions for the linearized model by means of a kind of variation of parameters formula. Moreover, we show that under nonlocal initial conditions, the existence of a mild solution of the semilinear equation can be established.

1. Introduction

The classical models in nonlinear acoustics are partial differential equations of second order in time and characterized by the presence of a viscoelastic damping. The most general of these popular models is Kuznetsov's equation

$$u_{tt} - c^2\Delta u - b\Delta u_t = \left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_t, \tag{1}$$

where u denotes the acoustic velocity potential, $c > 0$ is the speed of sound, $b \geq 0$ is the diffusivity of sound and B/A is the parameter of nonlinearity. Neglecting local nonlinear effects one arrives at the Westervelt equation

$$u_{tt} - b\Delta u_t - c^2\Delta u = \left(\frac{1}{c^2}\left(1 + \frac{B}{2A}\right)(u_t)^2\right)_t. \tag{2}$$

The Kuznetsov equation can be regarded in some sense as a simplification of the following higher-order model

$$(a\Delta - \partial_t)(u_{tt} - c^2\Delta u - b\Delta u_t) = \left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)_{tt}, \quad a > 0, \tag{3}$$

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which is called the Blackstock–Crighton–Kuznetsov equation. The constant a is the heat conductivity of the fluid. Neglecting local nonlinear effects as it is done when reducing the Kuznetsov to the Westervelt equation, one arrives at the Blackstock–Crighton–Westervelt equation

$$(a\Delta - \partial_t)(u_{tt} - c^2\Delta u - b\Delta u_t) = \left(\frac{1}{c^2}\left(1 + \frac{B}{2A}\right)(u_t)^2\right)_{tt}. \tag{4}$$

The analysis of partial differential equations describing nonlinear sound propagation has been a fruitful source of research in the last decade. For an overview, we refer to the monograph [15]. The main emphasis lies on well-posedness and decay results. The seminal contributions in this area of research are due to Kaltenbacher and Lasiecka [16,17]. Her research on the analysis of the Kuznetsov and Westervelt equation are an inspiration source of many additional studies, for instance the Jordan–Moore–Gibson–Thompson equation [19,20].

A further line of research is the study of linearized abstract models, because this constitutes an important preliminary step for the nonlinear analysis. For example, in reference [18] the authors consider an abstract Moore–Gibson–Thompson equation which is driven by a self-adjoint positive operator defined on a Hilbert space, in [1] the Laplacian is replaced by an arbitrary closed linear operator defined in a Banach space and, in [10], two closed linear operators are considered.

The seminal mathematical study of Eq. (4) was initiated in 2014 by Brunnhuber and Kaltenbacher [6]. These authors used the theory of C_0-semigroups in order to investigate the linearization of the model, proving that the underlying semigroup is analytic. That leads to exponential decay results for the linear homogeneous equation. Moreover, it was proved local in time well-posedness of the model under the assumption that initial data are sufficiently small and a fixed-point argument. Global in time well-posedness was also obtained, by performing energy estimates and using the classical barrier method, again for sufficiently small initial data. Additionally, Brunnhuber and Kaltenbacher provided results concerning exponential decay of solutions of the nonlinear equation.

Later, Eq. (3) was studied by Brunnhuber and Meyer in Ref. [7] to show optimal regularity and exponential stability in L_p-spaces with Dirichlet and Neumann boundary conditions. In such reference, it was also proved long-time well-posedness and exponential stability for sufficiently small data.

More recently, in Ref. [9], Celik and Kyed have considered the Blackstock–Crighton–Westervelt equation in a three-dimensional bounded domain with both nonhomogeneous Dirichlet and Neumann boundary values. Existence of a solution was obtained via a fixed-point argument based on appropriate a priori estimates for the linearized equations.

However, up to date, there is no research on the abstract modeling of Eq. (4), i.e., replacing the Laplace operator $-\Delta$ by a general closed linear operator A defined on a Banach space.

In this work, we will study the linearized Blackstock–Crighton–Westervelt equation in the form

$$(-aA - D_t)(u''(t) + c^2Au(t) + bAu'(t)) = f(t),\ t \geq 0, \tag{5}$$

defined in a Banach space X with initial conditions $x = u(0)$, $y = u'(0)$, $z = u''(0)$ and $A\colon D(A) \subset X \to X$ a closed linear densely defined operator that satisfy appropriate conditions which we will describe later. Moreover, we denote in (5) by D_t the differentiation operator of order 1 with respect to the temporal variable t.

As we have intimate before, this approach in the setting of Banach spaces is completely new and has been not studied until now. The main advantage is that the abstract model can serve as prototype for other common operators A, like, e.g., the fractional Laplacian, among others. As we said before, the linear problem constitutes an important preparation for the nonlinear one. It should be noted that when $a = 0$ Eq. (5) reduces to the Moore–Gibson–Thompson equation.

One of our main and surprising results that we obtain using this abstract approach use the theory of C_0-semigroups of operators $\{T(t)\}_{t\geq 0}$, combined with the theory of resolvent families $\{S(t)\}_{t\geq 0}$, see [25], to solve *explicitly* the linearized Eq. (5) which provides new insights even in case that $A = -\Delta$, the negative Laplacian. Namely, we will prove that the solution of the linearized equation can be represented as:

$$\begin{aligned} u(t) = {} & e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T(s)x\mathrm{d}s + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)x\mathrm{d}s \\ & + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)x\mathrm{d}s + \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)x\mathrm{d}s \\ & + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)x\mathrm{d}\tau\mathrm{d}s + R(t)y - \frac{a+b}{a-b}\int_0^t T(s)y\mathrm{d}s \\ & + \frac{a+b}{a-b}\int_0^t S(s)y\mathrm{d}s - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)y\mathrm{d}s + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)y\mathrm{d}\tau\mathrm{d}s \\ & + \int_0^t R(s)z\mathrm{d}s + \int_0^t\int_0^s R(s-\tau)f(\tau)\mathrm{d}\tau\mathrm{d}s, \end{aligned} \tag{6}$$

where $R(t) = (S * T)(t)$ is the finite convolution of a resolvent family $\{S(t)\}_{t\geq 0}$ generated by $-A$ with kernel $\mathrm{a}(t) = b + c^2t$, and $\{T(t)\}_{t\geq 0}$ a C_0-semigroup generated by $-aA$.

Our second contribution in this paper is that, assuming the existence of the above representation, and certain hypothesis on a semilinear source $f(\cdot, u(\cdot))$, the existence of at least one mild solution of the corresponding semilinear model (5) can be guaranteed.

This paper is organized as follows: The second section is concerned with the preliminaries necessary for this work. In the third section, we show an explicit representation of the solution presentingx conditions for a mild solution to be strong. A local mild solution for the semilinear Blackstock–Crighton–Westervelt equation is proved in Sect. 4. And Sect. 5 is concerned with the mild solution of the semilinear version of

Eq. (5) with nonlocal initial conditions. We consider nonlocal initial conditions motivated by the observation that this type of conditions is more practical than classical conditions when treating physical problems. For instance, the sum

$$u(x,0)+\sum_{k=1}^{n}\beta_k(x)u(x,T_k) \tag{7}$$

is more accurate to measurement of a state than $u(x,0)$ alone. This approach was used by Deng [11] to describe the diffusion phenomenon of a small amount of gas in a tube. If there is too little gas at the initial time, the measurement (7) of the sum of the amounts of the gas is more reliable than the measurement $u(x,0)$ of the amount of the gas at the instant $t=0$. For more information, we refer the reader to the articles [2,8,26,27] and references therein.

2. Preliminaries

Most of the notations used throughout this paper are standard. X and Y always are complex Banach spaces with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$; the subscript will be dropped when there is no possibility of confusion. We denote the space of all bounded linear operators from X to Y by $\mathcal{L}(X,Y)$, and in the case $X=Y$, we will write briefly $\mathcal{L}(X)$.

Let $I\subset\mathbb{R}$ be a compact interval. We denote by $C(I;X)$ the space of all continuous functions from I to X. This space endowed with the norm

$$\|f\|_\infty=\sup_{t\in I}\|f(t)\|_X$$

is a Banach space.

Let A be a closed linear operator defined in X. We will denote its domain by $D(A)$, its resolvent set by $\rho(A)$ and its spectrum by $\sigma(A)$. We also denote

$$\Sigma(\omega,\theta):=\{\lambda\in\mathbb{C};\,|\arg(\lambda-\omega)|<\theta\},\qquad \omega\in\mathbb{R},\quad 0<\theta\le\pi.$$

We remind the following definition of resolvent family due to Pruss [25].

Definition 2.1. [25, *Definition 1.3 p. 32]* A family $\{S(t)\}_{t\ge0}\subset\mathcal{L}(X)$ of bounded linear operators in X is called a resolvent family if the following conditions are satisfied:

(S1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0)=I$;

(S2) $S(t)$ commutes with A, which means that $S(t)D(A)\subset D(A)$ and $AS(t)x=S(t)Ax$ for all $x\in D(A)$ and $t\ge0$;

(S3) the resolvent equation holds

$$S(t)x=x+\int_0^t \mathrm{a}(t-s)AS(s)x\mathrm{d}s,\ \text{for all } x\in D(A),\ t\ge0,$$

where $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ is a scalar kernel $\not\equiv 0$. In this case, A is called the generator of the resolvent family $\{S(t)\}_{t\geq 0}$.

The following definition was introduced by Pruss [25] and has ultimate importance in the development of our main results.

Definition 2.2. [25, *Definition 3.2 p. 68]* Let $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. The kernel $a(t)$ is called k-regular if there is a constant $C > 0$ such that

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \leq C|\hat{a}(\lambda)|, \text{ for all } Re(\lambda) > 0, \ 0 \leq n \leq k. \tag{8}$$

Here, the symbol hat denotes Laplace transform. We illustrate the above definition with the following examples that will be useful later.

Example 2.3. Define $a(t) = b + c^2 t$, $c \in \mathbb{R}$, $b > 0$. Then we have that $a(t)$ is k-regular for all $k \in \mathbb{N}$. Indeed, given $k \in \mathbb{N}$ notice that for each $n \in \{0, 1, \dots, k\}$, we have

$$\lambda^n \hat{a}^{(n)}(\lambda) = (-1)^n n! b\lambda^{-1} + (-1)^n (n+1)! c^2 \lambda^{-2};$$

then,

$$\begin{aligned}\frac{\lambda^n \hat{a}^{(n)}(\lambda)}{\hat{a}(\lambda)} =& \frac{(-1)^n n![b\lambda^{-1} + c^2\lambda^{-1}] + (-1)^n n(n!) c^2 \lambda^{-2}}{b\lambda^{-1} + c^2\lambda^{-2}} \\ =& (-1)^n n! + (-1)^n n(n!) \frac{c^2}{b\lambda + c^2},\end{aligned}$$

which is bounded, for all $Re(\lambda) > 0$ and $0 \leq n \leq k$. Therefore, $a(t)$ is k-regular, for all $k \in \mathbb{N}$.

The following result will be very useful in this paper. And its proof is very similar to the proof of [25, Theorem 3.1, p. 73]

Theorem 2.4. *Let* $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ *be* k*-regular and suppose that* A *is closed and densely defined operator on a Banach space* X *such that* $\hat{a}(\lambda) \neq 0$ *and* $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ *for all* $Re(\lambda) > \omega$*. Suppose that*

$$\|\lambda^{-1}(I - \hat{a}(\lambda)A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \text{ for all } Re(\lambda) > \omega. \tag{9}$$

Then there exists a resolvent family $\{S(t)\}_{t\geq 0}$ *such that* $S \in C^{k-1}((0,\infty), \mathcal{L}(X))$ *and* $\|S(t)\| \leq Me^{wt}$.

Remark 2.5. It is possible to obtain spatial regularity of the resolvent $\{S(t)\}_{t\geq 0}$, provided $a(t)$ is k-regular for some large enough k. We have the estimate

$$\|t^k AS(t)\| \leq Me^{wt}, \ t \geq 0, \tag{10}$$

i.e., $S(t)X \subset D(A)$ for all $t > 0$ (see [25, Comment (f) p. 82]).

Example 2.6. The following is an example of a kernel $\mathrm{a} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ which is k-regular and such that $|\arg(\hat{\mathrm{a}}(\lambda))| \leq \theta_0$, for all $Re(\lambda) > \omega := \frac{c^2}{b} \in \mathbb{R}$ for some $\theta_0 < \pi$, and where $c \in \mathbb{R}$, $b > 0$.

Consider $\mathrm{a}(t) = -b - c^2 t$. We will prove that there exists $\theta_0 < \pi$ such that $|\arg(\hat{\mathrm{a}}(\lambda))| < \theta_0$ for all $Re(\lambda) > \frac{c^2}{b}$. Indeed, for $\lambda = re^{i\theta}$ with $Re(\lambda) > \frac{c^2}{b}$ and $\theta < \pi$ we have

$$\begin{aligned}
\arg(\hat{\mathrm{a}}(re^{i\theta})) =& \operatorname{Im}\log(\hat{\mathrm{a}}(re^{i\theta})) = \operatorname{Im}\int_0^\theta \frac{\mathrm{d}}{\mathrm{d}t}\log(\hat{\mathrm{a}}(re^{it}))\mathrm{d}t \\
=& \operatorname{Im}\int_0^\theta \frac{\hat{\mathrm{a}}'(re^{it})ire^{it}}{\hat{\mathrm{a}}(re^{it})}\mathrm{d}t \\
=& -\operatorname{Im}\int_0^\theta 1i + \frac{c^2 i}{br(\cos t + i\sin t) + c^2}\mathrm{d}t \\
=& -\operatorname{Im}\theta i - \operatorname{Im}\int_0^\theta \frac{c^2 i(c^2 + br\cos t - ibr\sin t)}{(c^2 + br\cos t)^2 + (br\sin t)^2}\mathrm{d}t \\
=& -\theta - c^2\int_0^\theta \frac{c^2 + br\cos t}{2brc^2\cos t + c^4 + b^2r^2}\mathrm{d}t.
\end{aligned}$$

Using [14, 2554 2, p. 173], we obtain

$$\begin{aligned}
\arg(\hat{\mathrm{a}}(re^{i\theta})) =& -\theta - c^2\left[\frac{brt}{2brc^2}\right]_0^\theta - c^2\frac{2brc^4 - (b^3r^3 + brc^4)}{2brc^2}\int_0^\theta \frac{\mathrm{d}t}{b^2r^2 + c^4 + 2brc^2\cos t} \\
=& -\theta - \frac{\theta}{2} + \frac{b^2r^2 - c^4}{2}\int_0^\theta \frac{\mathrm{d}t}{b^2r^2 + c^4 + 2brc^2\cos t}.
\end{aligned}$$

From [14, 2553 3, p. 172], we conclude that

$$\begin{aligned}
\arg(\hat{\mathrm{a}}(re^{i\theta})) =& -\frac{3\theta}{2} + \frac{b^2r^2 - c^4}{2}\left[\frac{2}{\sqrt{(b^2r^2 - c^4)^2}}\arctan\left(\frac{(br - c^2)^2}{\sqrt{(b^2r^2 - c^4)^2}}\tan\left(\frac{\theta}{2}\right)\right)\right] \\
=& -\frac{3\theta}{2} + \arctan\left(\frac{br - c^2}{br + c^2}\tan\left(\frac{\theta}{2}\right)\right).
\end{aligned}$$

Since $Re(\lambda) > \frac{c^2}{b}$, we have $|\theta| < \frac{\pi}{2}$. Let $\theta_1 > 0$ be such that $|\theta| < \frac{\theta_1}{2} < \frac{\pi}{2}$. Then $|\tan\left(\frac{\theta}{2}\right)| < 1$. Notice that

$$\left|\frac{br - c^2}{br + c^2}\right| \leq 1$$

then

$$\tan\left(-\frac{\pi}{4}\right) = -1 < \frac{br - c^2}{br + c^2}\tan\left(\frac{\theta}{2}\right) < 1 = \tan\left(\frac{\pi}{4}\right),$$

that is,

$$-\frac{\pi}{4} < \arctan\left(\frac{br - c^2}{br + c^2}\tan\left(\frac{\theta}{2}\right)\right) < \frac{\pi}{4};$$

therefore,

$$-\frac{3\theta_1}{4}-\frac{\pi}{4}<-\frac{3\theta}{2}+\arctan\left(\frac{br-c^2}{br+c^2}\tan\left(\frac{\theta}{2}\right)\right)<\frac{3\theta_1}{4}+\frac{\pi}{4}=:\theta_0<\pi.$$

We conclude that $|\arg\hat{a}(\lambda)|<\pi$ for all $Re(\lambda)>\frac{c^2}{b}$.

Example 2.7. From Example 2.6, we note that $\frac{1}{\hat{a}(\lambda)}\in\Sigma(0,\pi)$ for all $Re(\lambda)>\frac{c^2}{b}$, and where $c\in\mathbb{R}$, $b>0$.

Let $-A$ be a closed and densely defined operator on X. If $\rho(-A)\supset\Sigma(0,\pi)$ and

$$\|\lambda(\lambda^2I+(b\lambda+c^2)A)^{-1}\|\leq\frac{M}{\left|\lambda-\frac{c^2}{b}\right|},\quad\text{for all }\ Re(\lambda)>\frac{c^2}{b},\tag{11}$$

then $-A$ satisfies the hypothesis of Theorem 2.4 and there exists a resolvent family $\{S(t)\}_{t\geq 0}$ such that $S\in C^\infty((0,\infty),\mathcal{L}(X))$ and $\|S(t)\|\leq Me^{\frac{c^2}{b}t}$. Moreover, from Remark 2.5, $S(t)X\subset D(-A)$ for all $t>0$.

Now we will present some definitions and results that will be useful in Sect. 5.

Definition 2.8. Let W be a bounded subset of a normed space Y. The Hausdorff measure of noncompactness of W is defined by

$$\eta(W)=\inf\{\varepsilon>0\colon Z\text{ has a finite cover by balls of radius }\varepsilon\}.$$

Lemma 2.9. [3, Lemma 5.1, p. 222] *Let X be a real Banach space and B_1, B_2 be bounded subsets of X. Then*

(i) $\eta(B_1)=0$ *if and only if B_1 is totally bounded*
(ii) $\eta(B_1)\leq\eta(B_2)$ *if* $B_1\subseteq B_2$;
(iii) $\eta(B_1)=\eta(\overline{B_1})=\eta(\overline{co}(B_1))$, *where $\overline{B_1}$ denotes the closure of B_1 and $\overline{co}(B_1)$ is the closed convex hull of B_1*;
(iv) $\eta(B_1\bigcup B_2)=\max\{\eta(B_1),\eta(B_2)\}$;
(v) $\eta(\lambda B_1)=|\lambda|\eta(B_1)$, *with* $\lambda\in\mathbb{R}$;
(vi) $\eta(B_1+B_2)\leq\eta(B_1)+\eta(B_2)$, *where* $B_1+B_2=\{b_1+b_2;\,b_1\in B_1,b_2\in B_2\}$.

We will use ξ to denote the Hausdorff measure of noncompactness defined in X and γ to denote the Hausdorff measure of noncompactness on $C(I;X)$. Moreover, we will use η for the Hausdorff measure of noncompactness for general Banach spaces Y. For more general information of measure of noncompactness, we refer the reader to the monograph of Banás and Goebel [4].

Lemma 2.10. [28, Property 1.1, p. 10] *Let $W\subseteq C(I;X)$ be a subset of continuous functions. If W is bounded and equicontinuous, then the set $\overline{co}(W)$ is also bounded and equicontinuous.*

Let W be a set of functions from I to X and $t\in I$ fixed, and we denote $W(t)=\{w(t)\colon w\in W\}$. The proof of the following lemma can be found in [4, Theorem 11.3].

Lemma 2.11. *Let $W \subseteq C(I; X)$ be a bounded set. Then $\xi(W(t)) \leq \gamma(W)$ for all $t \in I$. Furthermore, if W is equicontinuous on I, then $\xi(W(t))$ is continuous on I, and*

$$\gamma(W) = \sup\{\xi(W(t)) : t \in I\}.$$

A set of functions $W \subseteq L^1(I; X)$ is said to be uniformly integrable if there exists a positive function $\kappa \in L^1(I; X)$ such that $\|w(t)\| \leq \kappa(t)$ a.e. for all $w \in W$.

Lemma 2.12. [23, Proposition 1.6] *If $\{u_n\}_{n=1}^{\infty} \subseteq L^1(I; X)$ is uniformly integrable, then for each $n \in \mathbb{N}$ the function $t \mapsto \xi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\xi\left(\left\{\int_0^t u_n(s)\mathrm{d}s\right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \xi(\{u_n(t)\}_{n=1}^{\infty}).$$

The next result is crucial for our work. The reader can see its proof in [5, Theorem 2].

Lemma 2.13. *Let Y be a Banach space. If $W \subseteq Y$ is a bounded subset, then for each $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq W$ such that*

$$\eta(W) \leq 2\eta(\{u_n\}_{n=1}^{\infty}) + \varepsilon.$$

The following lemmata will be useful for the proof of our results. For details, see [21, Theorem 3.1] and [21, Lemma 2.4].

Lemma 2.14. *For all $0 \leq m \leq n$, denote $C_m^n = \binom{n}{m}$. If $0 < \epsilon < 1$, $h > 0$ and let*

$$S_n = \epsilon^n + C_1^n \epsilon^{n-1} h + C_2^n \epsilon^{n-2} \frac{h^2}{2!} + \cdots + \frac{h^n}{n!}, \; n \in \mathbb{N}, \tag{12}$$

then $\lim_{n\to\infty} S_n = 0$.

Lemma 2.15. *Let W be a closed and convex subset of a complex Banach space Y, and let $F : W \to W$ be a continuous operator such that $F(W)$ is a bounded set. Define*

$$F^1(W) = F(W), \; F^n(W) = F(\overline{co}(F^{n-1}(W))), \; n = 2, 3, \ldots.$$

If there exist a constant $0 \leq r < 1$ and $n_0 \in \mathbb{N}$ such that

$$\eta(F^{n_0}(W)) \leq r\eta(W), \tag{13}$$

then F has a fixed point in the set W.

3. Well-posedness

Let X be a Banach space. In this section, we will study the well-posedness for the abstract Eq. (5) that we rewrite including their initial conditions as follows:

$$\begin{cases} u'''(t) + (a+b)Au''(t) + (abA + c^2)Au'(t) + ac^2A^2u(t) = f(t), \ t \geq 0 \\ u(0) = x, \ u'(0) = y \ u''(0) = z, \end{cases} \tag{14}$$

where $x, y, z \in X$, $c \in \mathbb{R}, b > 0, c > 0$ and $A: D(A) \subset X \to X$ is a operator that satisfy appropriate conditions which we will describe later. We first introduce our notion of solution.

Definition 3.1. A function $u: \mathbb{R}_+ \to X$ is called a strong solution of (14) if it satisfies

(i) $u \in C(\mathbb{R}_+; D(A^2)) \cap C^3(\mathbb{R}_+; X)$;
(ii) $u' \in C(\mathbb{R}_+; D(A^2))$;
(iii) $u'' \in C(\mathbb{R}_+; D(A))$;
(iv) (14) holds on $\mathbb{R}_+$.

We say that a closed linear densely defined operator A satisfies the **Hypothesis** (H) if

(i) $-A$ is the generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ uniformly bounded, that is, $\|T(t)\| \leq M, \ \forall t \geq 0$.
(ii) $-A$ generates a resolvent family $\{S(t)\}_{t\geq 0}$ with kernel $\mathrm{a}(t) = b + c^2t$ and satisfying $\|S(t)\| \leq Me^{\omega t}$ $\forall t \geq 0$ and $S(t)X \subset D(A)$ $\forall t \geq 0$.

In such case, we denote by $T_a(t)$ the semigroup generated by $-aA$, and

$$R(t) := (S * T_a)(t) := \int_0^t S(t-s)T_a(s)\mathrm{d}s, \quad t \geq 0, \tag{15}$$

the finite convolution. Here, the integral is understood in the Bochner sense. Note that we have $\|R(t)\| \leq Ke^{\omega t}$ for some $K > 0$ and $\omega \in \mathbb{R}$.

Remark 3.2. If $-A$ generates an analytic semigroup uniformly bounded such that $\Sigma(0, \pi) \subset \rho(-A)$, then A satisfies hypothesis (H). Indeed, item (i) is clear and it follows that

$$\|(\lambda I + A)^{-1}\| \leq \frac{M}{|\lambda|}, \ \forall \, Re(\lambda) > 0,$$

and then, since $Re(\lambda^2(b\lambda + c^2)^{-1}) > 0$,

$$\begin{aligned} \|\lambda(\lambda^2 I + (b\lambda + c^2)A)^{-1}\| &= \|\lambda(b\lambda + c^2)^{-1}(\lambda^2(b\lambda + c^2)^{-1}I + A)^{-1}\| \\ &\leq \frac{|\lambda(b\lambda + c^2)|M}{|\lambda^2(b\lambda + c^2)|} = \frac{M}{|\lambda|} \\ &\leq \frac{M}{\left|\lambda - \frac{c^2}{b}\right|}, \qquad Re(\lambda) > \frac{c^2}{b}. \end{aligned}$$

From Example 2.7, we conclude that $-A$ generates a resolvent family $\{S(t)\}_{t\geq 0}$ of type $\left(M, \frac{c^2}{b}\right)$ with kernel $\mathrm{a}(t) = b + c^2t, \ b > 0, c \in \mathbb{R}$, and such that $S(t)X \subset D(-A)$.

Remark 3.3. By (ii) in hypothesis (H), and (S3) in Definition (2.1), we have

$$S(t)x = x - b\int_0^t S(s)Ax\mathrm{d}s - c^2\int_0^t (t-s)S(s)Ax\mathrm{d}s \quad x \in D(A), \quad t \geq 0,$$

and since A is closed

$$S(t)x = x - bA\int_0^t S(s)x\mathrm{d}s - c^2 A\int_0^t (t-s)S(s)x\mathrm{d}s \quad x \in X, \quad t \geq 0,$$

(see [25, Proposition 1.1]). In particular, we have that

$$S'(t)x = -bS(t)Ax - c^2\int_0^t S(s)Ax\mathrm{d}s \quad x \in D(A), \quad t \geq 0,$$

and since $S(t)X \subset D(A)$,

$$S'(t)x = -bAS(t)x - c^2 A\int_0^t S(s)x\mathrm{d}s \quad x \in X, \quad t \geq 0.$$

Moreover,

$$S''(t)x = -bS'(t)Ax - c^2 S(t)Ax, \quad x \in D(A^2), \quad t \geq 0,$$

and

$$S'''(t)x = -bS''(t)Ax - c^2 S'(t)Ax, \quad x \in D(A^3), \quad t \geq 0.$$

Example 3.4. If $1 < p < \infty$, the Laplacian operator Δ in $L^p(\mathbb{R}^n)$, i.e., the operator Δ_p with domain $D(\Delta_p) = \{f \in L^p(\mathbb{R}^n);\ \Delta f \in L^p(\mathbb{R}^n)\}$ satisfies hypothesis (H). Indeed, $-\Delta_p$ generates an analytic semigroup uniformly bounded and $\Sigma(0,\pi) \subset \rho(-\Delta_p)$ (see [22, Theorem 2.3.3 p. 40 and A.7.6 p. 329]).

The next lemma concerns the definition of $R(t)$ given in (15).

Lemma 3.5. *Let A be a closed linear operator satisfying hypothesis (H). Suppose $a \neq b$, $a > 0, b > 0, c \in \mathbb{R}$, then, for all $x \in X$ we have $R(t)x \in D(A)$ and satisfy*

$$AR(t)x = \frac{1}{a-b}\left[S(t)x - T_a(t)x + \frac{c^2}{a}\int_0^t S(s)x\mathrm{d}s - \frac{c^2}{a}R(t)x\right], \quad x \in X. \tag{16}$$

Moreover, $AR(t)x \in D(A)$ for all $x \in X$.

Proof. Since A commutes with $T_a(t)$ and $S(t)$, and A is closed, we have that A commutes with $R(t)$ on $D(A)$.

Since $T_a(t)$ is uniformly bounded and $S(t)$ and $R(t)$ are exponentially bounded, we can apply the Laplace transform, and we have for all $Re(\lambda) > \omega$ and $x \in D(A)$

$$\hat{R}(\lambda)Ax = \hat{S}(\lambda)\hat{T}_a(\lambda)Ax = \hat{S}(\lambda)\left[-\frac{\lambda}{a}\hat{T}_a(\lambda) + \frac{1}{a}I\right]x$$
$$= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)\lambda\hat{S}(\lambda)x.$$

By $(S3)$ of Definition 2.1 with $\mathrm{a}(t) = b + c^2 t$, we obtain after application of the Laplace transform

$$\lambda\hat{S}(\lambda)x = Ix + b\hat{S}(\lambda)Ax + \frac{c^2}{\lambda}\hat{S}(\lambda)Ax, \tag{17}$$

and therefore, we have

$$\begin{aligned}
\hat{R}(\lambda)Ax &= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)\left[I - b\hat{S}(\lambda)A - \frac{c^2}{\lambda}\hat{S}(\lambda)A\right]x \\
&= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}\hat{R}(\lambda)Ax + \frac{c^2}{a\lambda}\hat{T}_a(\lambda)\hat{S}(\lambda)Ax \\
&= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}\hat{R}(\lambda)Ax + \frac{c^2}{a\lambda}\left[-\frac{\lambda}{a}\hat{T}_a(\lambda) + \frac{1}{a}I\right]\hat{S}(\lambda)x \\
&= \frac{1}{a}\hat{S}(\lambda)x - \frac{1}{a}\hat{T}_a(\lambda)x + \frac{b}{a}\hat{R}(\lambda)Ax - \frac{c^2}{a^2}\hat{R}(\lambda)x + \frac{c^2}{a^2\lambda}\hat{S}(\lambda)x.
\end{aligned}$$

Then

$$\frac{a-b}{a}\hat{R}(\lambda)Ax = \frac{1}{a}\left[\hat{S}(\lambda) - \hat{T}_a(\lambda) - \frac{c^2}{a}\hat{R}(\lambda) + \frac{c^2}{a\lambda}\hat{S}(\lambda)\right]x.$$

So, applying the inversion of the Laplace transform, and by the uniqueness theorem, we have for all $x \in D(A)$

$$AR(t)x = R(t)Ax = \frac{1}{a-b}\left[S(t)x - T_a(t)x + \frac{c^2}{a}\int_0^t S(s)x\mathrm{d}s - \frac{c^2}{a}R(t)x\right]. \tag{18}$$

Because A is closed and $D(A)$ is dense in X, we deduce from the above identity that $R(t)x \in D(A)$ for all $x \in X$ and (16) holds.

For the last conclusion of the Lemma, we first notice that by hypothesis, $S(t)X \subset D(A)$, and $T_a(t)X \subset D(A)$ for any $t > 0$, where $\{T_a(t)\}_{t\geq 0}$ is an analytic semigroup generated by $-aA$ (see [12, Theorem 4.6 (c) p. 101]). Next, we observe that from (17) and defining $e(t) := e^{-\frac{c^2}{b}t}$ we have for all $x \in D(A)$

$$\begin{aligned}
A\frac{1}{\lambda}\hat{S}(\lambda)x &= \frac{1}{b}\frac{\lambda}{\lambda + c^2/b}\hat{S}(\lambda)x - \frac{1}{b}\frac{1}{\lambda + c^2/b}x = \frac{1}{b}\lambda\hat{e}(\lambda)\hat{S}(\lambda)x - \frac{1}{b}\hat{e}(\lambda)x \\
&= \frac{1}{b}\widehat{e'}(\lambda)\hat{S}(\lambda)x + \frac{1}{b}\hat{S}(\lambda)x - \frac{1}{b}\hat{e}(\lambda)x.
\end{aligned}$$

Therefore,

$$A\int_0^t S(s)x\mathrm{d}s = -\frac{c^2}{b^2}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)x\mathrm{d}s + \frac{1}{b}S(t)x - \frac{1}{b}e^{-\frac{c^2}{b}t}x, \quad x \in D(A). \tag{19}$$

Using that $D(A)$ is dense in X and A is closed, we deduce from the above identity that $\int_0^t S(s)x\mathrm{d}s \in D(A)$ for all $x \in X$ and (19) holds on X. It follows that, for each $x \in X$, all the terms in the right-hand side of (18) belong to the domain of A, proving the lemma.

□

The following lemma gives an additional property.

Lemma 3.6. *Let A be a closed linear operator satisfying hypothesis (H). Suppose $a > 0$, $a \neq b$, $b > 0$, $c \in \mathbb{R}$. Then, for all $x \in X$ we have*

$$\begin{aligned}
A^2\int_0^t R(s)x\mathrm{d}s &= \frac{1}{a(a-b)}T_a(t)x - \frac{1}{b(a-b)}S(t)x - \frac{c^2}{a(a-b)^2}\int_0^t S(s)x\mathrm{d}s \\
&\quad + \frac{c^2}{a(a-b)^2}\int_0^t T_a(s)x\mathrm{d}s + \frac{c^4}{a^2(a-b)^2}\int_0^t R(s)x\mathrm{d}s \\
&\quad - \frac{c^4}{a^2(a-b)^2}\int_0^t\int_0^s S(\tau)x\mathrm{d}\tau\mathrm{d}s \\
&\quad + \frac{1}{ab}e^{-\frac{c^2}{b}t}x + \frac{c^2}{ab^2}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)x\mathrm{d}s.
\end{aligned} \tag{20}$$

Proof. Given $x \in X$, define by $G(t)x$ the right-hand side of (20). Notice that $G(t)x$ is well defined for all $t \geq 0$. Moreover, we have the identity

$$-\frac{1}{b(a-b)}S(t)x = -\left[\frac{1}{a(a-b)} + \frac{1}{ab}\right]S(t)x. \tag{21}$$

Then, replacing (21) in $G(t)x$ and applying the Laplace transform, which is possible since $T_a(t)$ is uniformly bounded, and $S(t)$ and $R(t)$ are exponentially bounded, we have for all $Re(\lambda) > \omega$

$$\begin{aligned}
\hat{G}(\lambda)x &= \frac{1}{a(a-b)}\hat{T}_a(\lambda)x - \frac{1}{a(a-b)}\hat{S}(\lambda)x + \frac{c^2}{a(a-b)^2\lambda}\hat{T}_a(\lambda)x - \frac{c^2}{a(a-b)^2\lambda}\hat{S}(\lambda)x \\
&\quad + \frac{c^4}{a^2(a-b)^2\lambda}\hat{R}(\lambda)x - \frac{c^4}{a^2(a-b)^2\lambda^2}\hat{S}(\lambda)x + \frac{1}{ab}\frac{1}{\lambda+\frac{c^2}{b}}x \\
&\quad + \frac{c^2}{ab^2}\frac{1}{\lambda+\frac{c^2}{b}}\hat{S}(\lambda)x - \frac{1}{ab}\hat{S}(\lambda)x \\
&= \frac{1}{a(a-b)}[\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{c^2}{a(a-b)^2\lambda}\hat{T}_a(\lambda)x - \frac{c^2}{a(a-b)^2\lambda}\hat{S}(\lambda)x \\
&\quad + \frac{c^4}{a^2(a-b)^2\lambda}\hat{R}(\lambda)x - \frac{c^4}{a^2(a-b)^2\lambda^2}\hat{S}(\lambda)x + \frac{1}{a(b\lambda+c^2)}[I - \lambda\hat{S}(\lambda)]x
\end{aligned}$$

$$= \frac{1}{a(a-b)}[\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{1}{a(b\lambda + c^2)}[I - \lambda\hat{S}(\lambda)]x$$
$$+ \frac{c^2}{a(a-b)^2\lambda}\left[\hat{T}_a(\lambda) - \hat{S}(\lambda) + \frac{c^2}{a}\hat{R}(\lambda) - \frac{c^2}{a\lambda}\hat{S}(\lambda)\right]x.$$

By Lemma 3.5 and since $\hat{a}(\lambda) = \frac{b}{\lambda} + \frac{c^2}{\lambda^2}$, we have

$$\hat{G}(\lambda)x = \frac{1}{a(a-b)}[\hat{T}_a(\lambda) - \hat{S}(\lambda)]x + \frac{1}{a\hat{a}(\lambda)}\left[\frac{1}{\lambda^2}I - \frac{1}{\lambda}\hat{S}(\lambda)\right]x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x.$$

Moreover, by definition of $S(t)$, we have

$$A\hat{S}(\lambda)x = \frac{1}{\lambda\hat{a}(\lambda)}x - \frac{\hat{S}(\lambda)}{\hat{a}(\lambda)}x.$$

Then,

$$\hat{G}(\lambda)x = \frac{1}{a(a-b)\lambda}[\lambda\hat{T}_a(\lambda) - \lambda\hat{S}(\lambda)]x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x$$
$$= \frac{1}{a(a-b)\lambda}[\widehat{T_a'}(\lambda) - \widehat{S'}(\lambda)]x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x$$
$$= \frac{1}{a(a-b)\lambda}\left[\widehat{T_a'}(\lambda) + \frac{c^2}{\lambda}A\hat{S}(\lambda) + bA\hat{S}(\lambda)\right]x + \frac{1}{a\lambda}A\hat{S}(\lambda)x - \frac{c^2}{a(a-b)\lambda}A\hat{R}(\lambda)x$$
$$= \frac{1}{(a-b)\lambda}\left[\frac{1}{a}\widehat{T_a'}(\lambda) + \frac{c^2}{a\lambda}A\hat{S}(\lambda) + A\hat{S}(\lambda) - \frac{c^2}{a}A\hat{R}(\lambda)\right]x.$$

Using that $-aA$ is the generator of $T_a(t)$ and Lemma 3.5, we obtain

$$\hat{G}(\lambda)x = \frac{1}{(a-b)\lambda}\left[A\hat{S}(\lambda) - A\hat{T}_a(\lambda) + \frac{c^2}{a\lambda}A\hat{S}(\lambda) - \frac{c^2}{a}A\hat{R}(\lambda)\right]x$$
$$= A^2\frac{1}{\lambda}\hat{R}(\lambda)x.$$

Notice that from Lemma 3.5, $AR(t)x \in D(A)$ for $x \in X$, and then, $A^2R(t)x$ is well defined for $x \in X$. So, applying the inversion of the Laplace transform and by the uniqueness theorem, we obtain

$$G(t)x = A^2\int_0^t R(s)x\mathrm{d}s,$$

for all $x \in X$. This finishes the proof. □

Remark 3.7. If we consider

$$R(t)x = (T_a * S)(t)x = \int_0^t T_a(t-s)S(s)x\mathrm{d}s, \ x \in X,$$

then

$$R'(t)x = S(t)x - aAR(t)x, \quad x \in X.$$

In particular, under the hypothesis that $S(t)X \subset D(A)$ and in view of Lemma 3.5 we obtain that $R'(t)X \subset D(A)$ for all $t \geq 0$. Moreover,

$$\begin{aligned} R''(t)x &= S'(t)x - aR'(t)Ax = S'(t)x - aS(t)Ax + a^2AR(t)Ax \\ &= -bAS(t)x - c^2A\int_0^t S(s)x\mathrm{d}s - aAS(t)x + a^2AR(t)Ax \\ &= -(a+b)AS(t)x - c^2A\int_0^t S(s)x\mathrm{d}s + a^2AR(t)Ax \end{aligned}$$

for all $x \in D(A)$ (see Remark 3.3), and

$$\begin{aligned} R'''(t)x &= -(a+b)S'(t)Ax - c^2S(t)Ax + a^2R'(t)A^2x \\ &= -(a+b)S'(t)Ax - c^2AS(t)x + a^2S(t)A^2x - a^3AR(t)A^2x \\ &= -(a+b)S'(t)Ax + (a^2A - c^2)AS(t)x - a^3AR(t)A^2x, \end{aligned}$$

for all $x \in D(A^2)$. Moreover,

$$\begin{aligned} \|R(t)\| &\leq \int_0^t \|T_a(t-s)\|\|S(s)\|\mathrm{d}s \leq M^2\int_0^t e^{\omega s}\mathrm{d}s \\ &= \frac{M^2}{\omega}[e^{\omega t} - 1] \leq \frac{M^2}{\omega}e^{\omega t}. \end{aligned}$$

Considering $K = \dfrac{M^2}{\omega}$, we obtain

$$\|R(t)\| \leq Ke^{\omega t}.$$

With this preliminaries, we arrive at the following main result in this paper.

Theorem 3.8. *Let A be a closed linear operator satisfying the hypothesis (H), $a > 0$, $a \neq b$, $b > 0$, $c \in \mathbb{R}$. If $f \in C(\mathbb{R}_+, D(A))$, $x \in D(A^3)$, $y \in D(A^2)$ and $z \in D(A)$, then $u(t)$ given by*

$$\begin{aligned} u(t) = &\left[S(t) + bAR(t) + (abA + c^2)A\int_0^t R(s)\mathrm{d}s\right]x + \left[R(t) + (a+b)A\int_0^t R(s)\mathrm{d}s\right]y \\ &+ \int_0^t R(s)z\mathrm{d}s + \int_0^t (R * f)(s)\mathrm{d}s \end{aligned} \tag{22}$$

is the unique strong solution of the initial value problem (14).

Proof. For $x, y, z \in X$ and $f \in L^1_{loc}(\mathbb{R}_+; X)$, we have

$$u(t) = \left[S(t) + bAR(t) + (abA + c^2)A\int_0^t R(s)\mathrm{d}s\right]x + \left[R(t) + (a+b)A\int_0^t R(s)\mathrm{d}s\right]y$$
$$+ \int_0^t R(s)z\mathrm{d}s + \int_0^t (R * f)(s)\mathrm{d}s.$$

We will show that $u(t)$ is a strong solution of (14). Indeed, since $f \in L^1_{loc}(\mathbb{R}_+, D(A))$, $x \in D(A^2)$, $y \in D(A^2)$ and $z \in D(A)$ then recalling that $S(t)$ commutes with A on $D(A)$ (see (S2) in Definition 2.1), as well as $R(t)$, and using the second part of Lemma (3.5), we obtain that $u \in C(\mathbb{R}_+; D(A^2))$. By Remark 3.3, we also note that $u \in C^1(\mathbb{R}_+; X)$. Moreover,

$$u(0) = [S(0) + bAR(0)]x + R(0)y + (R * f)(0) = x.$$

Since $AR(t)x \in D(A)$ for all $x \in X$ (Lemma 3.5) and taking into account Remarks 3.3 and 3.7, we have, for $x \in D(A)$, $y \in D(A)$ and $z \in X$,

$$\begin{aligned} u'(t) &= [S'(t) + bR'(t)A + (abA + c^2)R(t)A]x + [R'(t) + (a+b)AR(t)]y \\ &\quad + R(t)z + (R * f)(t) \\ &= [S'(t) + b[S(t) - aAR(t)]A + (abA + c^2)R(t)A]x \\ &\quad + [S(t) - aAR(t) + (a+b)AR(t)]y + R(t)z + (R * f)(t) \\ &= [S'(t) + bAS(t) + c^2AR(t)]x \\ &\quad + [S(t) + bAR(t)]y + R(t)z + (R * f)(t). \end{aligned}$$

From the above identity, and using that $S(t)X \subset D(A)$ and $AR(t)X \subset D(A)$, we deduce that $u' \in C(\mathbb{R}_+; D(A^2))$. Moreover, by Remarks 3.3 and 3.7 we obtain $u' \in C^1(\mathbb{R}_+; X)$ and we have

$$u'(0) = [S'(0) + bAS(0) + c^2AR(0)]x + [S(0) + bAR(0)]y + R(0)z + (R * f)(0) = y.$$

Further, for $x \in D(A^2), y \in D(A^2), z \in D(A)$ and using again Remark 3.3 and Remark 3.7 we have

$$\begin{aligned} u''(t) &= [S''(t) + bS'(t)A + c^2R'(t)A]x + [S'(t) + bR'(t)A]y + R'(t)z + (R' * f)(t) \\ &= [-c^2S(t)A + c^2R'(t)A]x + [S'(t) + bR'(t)A]y + R'(t)z + (R' * f)(t). \end{aligned}$$

Since $R'(t)X \subset D(A)$ (cf. Remark 3.7), we deduce from the above identity that $u'' \in C(\mathbb{R}_+; D(A))$, and since $y \in D(A^2)$ and $z \in D(A)$, $u'' \in C^1(\mathbb{R}_+; X)$. Therefore, $u \in C^3(\mathbb{R}_+; X)$. In addition, we have

$$u''(0) = [-c^2AS(0) + c^2AR'(0)]x + [S'(0) + bAR'(0)]y + R'(0)z + (R' * f)(0) = z.$$

At last, for $x \in D(A^2)$, $y \in D(A^2)$ and $z \in D(A)$ we have

$$\begin{aligned} u'''(t) &= [-c^2 S'(t)A + c^2 R''(t)A]x + [S''(t) + bR''(t)A]y + R''(t)z + (R'' * f)(t) + f(t) \\ &= [-c^2 aAR'(t)A]x + [-c^2 S(t)A - abAR'(t)A]y + R''(t)z + (R'' * f)(t) + f(t). \end{aligned}$$

Then by hypothesis, we conclude that conditions (i), (ii) and (iii) of Definition 3.1 are satisfied.

To verify that (14) holds, first notice that

$$\begin{aligned} & S'''(t)x + (a+b)S''(t)Ax + (abA + c^2)S'(t)Ax + ac^2 S(t)A^2 x \\ &= \{S'(t)(b^2 A - c^2)A + (a+b)[-bS'(t)A - c^2 S(t)A]A + ac^2 S(t)A^2 \\ &\quad + S'(t)(abA + c^2)A + c^2 bS(t)A^2\}x \\ &= 0, \end{aligned}$$

for all $x \in D(A^3)$. Moreover, we have

$$\begin{aligned} & R'''(t)x + (a+b)R''(t)Ax + R'(t)(abA + c^2)Ax + ac^2 R(t)A^2 x \\ &= -(a+b)S'(t)Ax + S(t)(a^2 A - c^2)Ax + (a+b)[S'(t) - aS(t)A \\ &\quad + a^2 AR(t)A]Ax - a^3 AR(t)A^2 x + [S(t) - aAR(t)](abA + c^2)Ax \\ &\quad + ac^2 R(t)A^2 x = 0, \end{aligned}$$

for all $x \in D(A^2)$. Now, if we define

$$h(t)x := R''(t)x + (a+b)AR'(t)x + (abA + c^2)AR(t)x + ac^2 A^2 \int_0^t R(s)x\mathrm{d}s, \;\; x \in X,$$

then we have $h'(t)x = 0$ as we have seen above. But, $h(0)x = 0$, then $h \equiv 0$. Then $u(t)$ given by (22) satisfies (14) and is the unique strong solution. This concludes the proof. □

Remark 3.9. By Lemmas 3.5 and 3.6, an equivalent representation of (22) is

$$\begin{aligned} u(t) &= e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)x\mathrm{d}s + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)x\mathrm{d}s \\ &\quad + \frac{c^2(a^2 - ab - b^2)}{a(a-b)^2}\int_0^t S(s)x\mathrm{d}s - \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)x\mathrm{d}s \\ &\quad - \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)x\mathrm{d}\tau\mathrm{d}s + R(t)y - \frac{a+b}{a-b}\int_0^t T_a(s)y\mathrm{d}s \\ &\quad + \frac{a+b}{a-b}\int_0^t S(s)y\mathrm{d}s - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)y\mathrm{d}s + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)y\mathrm{d}\tau\mathrm{d}s \\ &\quad + \int_0^t R(s)z\mathrm{d}s + \int_0^t\int_0^s R(s-\tau)f(\tau)\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{23}$$

4. The semilinear problem

Let A be a closed linear operator satisfying hypothesis (H). For $(x, y, z) \in X \times X \times X$ and $a \neq b$, $a > 0$, $b > 0$, $c \in \mathbb{R}$, we will consider the semilinear problem

$$\begin{cases} u'''(t) + (a+b)Au''(t) + (abA + c^2)Au'(t) + ac^2A^2u(t) = f(t, u(t)), \\ u(0) = x, \ u'(0) = y, \ u''(0) = z. \end{cases} \tag{24}$$

We note that the problem (24) exclude the full nonlinear term in (4), because we are only interested in a semilinear form of the Blackstock–Crighton–Westervelt equation. We first introduce the following definition.

Definition 4.1. Given $(x, y, z) \in X \times X \times X$, a continuous function $u(t, x, y, z)$ that satisfies

$$\begin{aligned} u(t) &= e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)x\mathrm{d}s + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)x\mathrm{d}s \\ &\quad + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)x\mathrm{d}s + \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)x\mathrm{d}s \\ &\quad - \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)x\mathrm{d}\tau\mathrm{d}s + R(t)y - \frac{a+b}{a-b}\int_0^t T_a(s)y\mathrm{d}s \\ &\quad + \frac{a+b}{a-b}\int_0^t S(s)y\mathrm{d}s - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)y\mathrm{d}s + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)y\mathrm{d}\tau\mathrm{d}s \\ &\quad + \int_0^t R(s)z\mathrm{d}s + \int_0^t\int_0^s R(s-\tau)f(\tau, u(\tau))\mathrm{d}\tau\mathrm{d}s \end{aligned} \tag{25}$$

is called mild solution of the problem (24).

Our next main result shows that there exists a mild solution of the problem (24).

Theorem 4.2. *Let A be a closed linear operator satisfying hypothesis (H). If $f\colon [0, +\infty) \times X \to X$ satisfies a Lipschitz condition in x uniformly in $t \in \mathbb{R}$, with Lipschitz constant $L > 0$, then there is a unique mild solution of* (24).

Proof. Consider $R(t) = (S * T_a)(t)$. We know that there exist constants $K > 0$ and $\omega > 0$ such that

$$\|R(t)\| \leq Ke^{\omega t}, \ \forall t \in [0, +\infty).$$

Let $T > 0$ be given and consider the space $C([0, T], X)$ of the continuous functions from $[0, T]$ to X, with the norm

$$\|u\|_{\tilde{L}} = \max_{t\in[0,T]}\{e^{-\left(\tilde{L}+\omega\right)t}\|u(t)\|\},$$

where $u \in C([0, T], X)$ and $\tilde{L} > \dfrac{-\omega + \sqrt{\omega^2 + 4KL}}{2} > 0$ is arbitrary but fixed.

First we will show that the norm $\|\cdot\|_{\tilde{L}}$ is equivalent with the standard norm $\|\cdot\|_\infty$, which is defined by

$$\|u\|_\infty = \max_{t\in[0,T]} \|u(t)\|.$$

Indeed, since the function $e^{-\left(\tilde{L}+\omega\right)t}$ is decreasing in $[0, T]$, for all $t \in [0, T]$ we have

$$e^{-\left(\tilde{L}+\omega\right)T}\|u(t)\| \leq e^{-(\tilde{L}+\omega)t}\|u(t)\| \leq \|u(t)\|.$$

Then,

$$e^{-\left(\tilde{L}+\omega\right)T}\|u\|_\infty \leq \|u\|_{\tilde{L}} \leq \|u\|_\infty.$$

Therefore, the norms are equivalent and $C([0, T], X)$ is a Banach space with the norm $\|\cdot\|_{\tilde{L}}$. Let $x, y, z \in X$ be fixed and define the operator $\Gamma\colon C([0, T], X) \to C([0, T], X)$ by

$$\begin{aligned}\Gamma u(t) &= e^{-\frac{c^2}{b}t}x - \frac{bc^2}{a(a-b)}R(t)x + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)x\mathrm{d}s + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)x\mathrm{d}s \\ &\quad + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)x\mathrm{d}s + \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)x\mathrm{d}s \\ &\quad - \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)x\mathrm{d}\tau\mathrm{d}s + R(t)y - \frac{a+b}{a-b}\int_0^t T_a(s)y\mathrm{d}s \\ &\quad + \frac{a+b}{a-b}\int_0^t S(s)y\mathrm{d}s - \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)y\mathrm{d}s + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)y\mathrm{d}\tau\mathrm{d}s \\ &\quad + \int_0^t R(s)z\mathrm{d}s + \int_0^t\int_0^s R(s-\tau)f(\tau, u(\tau))\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{26}$$

Notice that given $u \in C([0, T], X)$, the function $s \mapsto f(s, u(s))$ is continuous in $[0, T]$ and therefore integrable in $[0, t]$ for all $t \in [0, T]$. Then Γu is a continuous function from $[0, T]$ to X, implying that Γ is well defined.

Consider $u, v \in C([0, T], X)$ and $t \in [0, T]$. We have

$$\begin{aligned}\|\Gamma u(t) - \Gamma v(t)\| &= \left\|\int_0^t\int_0^s R(s-\tau)[f(\tau, u(\tau)) - f(\tau, v(\tau))]\mathrm{d}\tau\mathrm{d}s\right\| \\ &\leq \int_0^t\int_0^s \|R(s-\tau)\|\|f(\tau, u(\tau)) - f(\tau, v(\tau))\|\mathrm{d}\tau\mathrm{d}s;\end{aligned}$$

then,

$$\begin{aligned}e^{-(\tilde{L}+\omega)t}\|\Gamma u(t) - \Gamma v(t)\| &\leq e^{-(\tilde{L}+\omega)t}\int_0^t\int_0^s \|R(s-\tau)\|\|f(\tau, u(\tau)) - f(\tau, v(\tau))\|\mathrm{d}\tau\mathrm{d}s \\ &\leq e^{-(\tilde{L}+\omega)t}\int_0^t\int_0^s Ke^{\omega(s-\tau)}L\|u(\tau) - v(\tau)\|d\tau\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= e^{-(\tilde{L}+\omega)t} KL \int_0^t e^{\omega s} \int_0^s e^{\tilde{L}\tau} e^{-\omega\tau} e^{-\tilde{L}\tau} \|u(\tau) - v(\tau)\| \mathrm{d}\tau \mathrm{d}s \\
&\leq KLe^{-(\tilde{L}+\omega)t} \int_0^t e^{\omega s} \int_0^s e^{\tilde{L}\tau} \|u - v\|_{\tilde{L}} \mathrm{d}\tau \mathrm{d}s \\
&\leq KLe^{-(\tilde{L}+\omega)t} \|u - v\|_{\tilde{L}} \int_0^t e^{\omega s} \int_0^s e^{\tilde{L}\tau} \mathrm{d}\tau \mathrm{d}s \\
&= \frac{KL}{\tilde{L}} \|u - v\|_{\tilde{L}} e^{-(\tilde{L}+\omega)t} \int_0^t e^{\omega s} [e^{\tilde{L}s} - 1] \mathrm{d}s \\
&\leq \frac{KL}{\tilde{L}} \|u - v\|_{\tilde{L}} e^{-(\tilde{L}+\omega)t} \int_0^t e^{\left(\omega+\tilde{L}\right)s} \mathrm{d}s \\
&= \frac{KL}{\tilde{L}(\omega + \tilde{L})} \|u - v\|_{\tilde{L}} e^{-(\tilde{L}+\omega)t} [e^{(\omega+\tilde{L})t} - 1] \\
&\leq \frac{KL}{\tilde{L}(\omega + \tilde{L})} \|u - v\|_{\tilde{L}}.
\end{aligned}
$$

Since $\tilde{L} > \dfrac{-\omega + \sqrt{\omega^2 + 4KL}}{2} > 0$, we have $\dfrac{KL}{\tilde{L}(\omega + \tilde{L})} < 1$. So Γ is a contraction. By the Banach fixed-point theorem, Γ has a unique fixed point; that is, there exists $u \in C([0, T], X)$ such that $u(t) = \Gamma u(t)$, so

$$
\begin{aligned}
u(t) =& e^{-\frac{c^2}{b}t} x - \frac{bc^2}{a(a-b)} R(t)x + \frac{c^2(2b-a)}{(a-b)^2} \int_0^t T_a(s)x \mathrm{d}s + \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t R(s)x \mathrm{d}s \\
&+ \frac{c^2(a^2 - ab - b^2)}{a(a-b)^2} \int_0^t S(s)x \mathrm{d}s + \frac{c^2}{b} \int_0^t e^{-\frac{c^2}{b}(t-s)} S(s)x \mathrm{d}s \\
&- \frac{c^4(2b-a)}{a(a-b)^2} \int_0^t \int_0^s S(\tau)x \mathrm{d}\tau \mathrm{d}s + R(t)y - \frac{a+b}{a-b} \int_0^t T_a(s)y \mathrm{d}s \\
&+ \frac{a+b}{a-b} \int_0^t S(s)y \mathrm{d}s - \frac{c^2(a+b)}{a(a-b)} \int_0^t R(s)y \mathrm{d}s + \frac{c^2(a+b)}{a(a-b)} \int_0^t \int_0^s S(\tau)y \mathrm{d}\tau \mathrm{d}s \\
&+ \int_0^t R(s)z \mathrm{d}s + \int_0^t \int_0^s R(s-\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s
\end{aligned}
$$

for all $t \in [0, T]$. By the uniqueness of the fixed point, the problem (24) has a unique solution in $[0, T]$. Since $T > 0$ is arbitrary, the result follows. □

5. Mild solutions with nonlocal initial conditions

Given $a, b > 0$, $c \in \mathbb{R}$, we will consider the semilinear Blackstock–Crighton–Westervelt with nonlocal initial conditions:

$$
\begin{cases}
u'''(t) + (a+b)Au''(t) + (abA + c^2)Au'(t) + ac^2A^2u(t) = f(t, u(t)), \; t \in I \\
u(0) = g_1(u), \; u'(0) = g_2(u), \; u''(0) = g_3(u),
\end{cases}
\tag{27}
$$

where A is a closed linear operator satisfying hypothesis (H) and considering $I = [0, 1]$. The functions $f: I \times X \to X$ and $g_1, g_2, g_3: C(I; X) \to X$ are X-valued functions that satisfy appropriate conditions which we will describe later.

We will consider the function $U: I \to \mathcal{L}(X)$ given by $U(t) = (1 * R)(t) = \int_0^t R(s)\mathrm{d}s$. Notice that U is uniformly continuous in $\mathcal{L}(X)$.

We will assume that the following assertions hold:

(H1) The functions $g_1, g_2, g_3: C(I; X) \to X$ are compact maps.

(H2) The function $f: I \times X \to X$ satisfies the Carathéodory type conditions; that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in I$.

(H3) There exist a function $m \in L^1(I; \mathbb{R}^+)$ and a nondecreasing continuous function $\Phi: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, x)\| \leq m(t)\Phi(\|x\|)$$

for all $x \in X$ and almost all $t \in I$.

(H4) There exists a function $G \in L^1(I; \mathbb{R}^+)$ such that for any bounded $S \subseteq X$

$$\xi(f(t, S)) \leq G(t)\xi(S)$$

for almost all $t \in I$.

Remark 5.1. Assuming that a function g satisfies hypothesis $(H1)$, it is clear that g takes bounded sets into bounded sets. For this reason, for each $J \geq 0$ we will denote $g_J = \sup\{\|g(u)\|: \|u\|_\infty \leq J\}$.

For the following result, we will consider the following number

$$L = \max\left\{1, \sup_{t\in I}\{\|S(t)\|\}, \sup_{t\in I}\{\|R(t)\|\}, \sup_{t\in I}\{\|T_a(t)\|\}, \sup_{t\in I}\|U(t)\|\right\}. \tag{28}$$

Theorem 5.2. *Suppose* $0 < a < b$ *and* $c \in \mathbb{R}$. *If hypotheses (H1)-(H4) are satisfied and there exists a constant* $J \geq 0$ *such that*

$$J \geq \left(1+\frac{2c^2(b^2+ab-a^2)}{a(a-b)^2}+\frac{c^2}{b}+\frac{2c^4(2b-a)}{a(a-b)^2}\right)Lg_{1J} + \left(1+\frac{4b(a+c^2)}{a(a-b)}\right)Lg_{2J}$$
$$+ Lg_{3J} + L\Phi(J)\int_0^1 m(s)\mathrm{d}s,$$

where L is given by (28)*, then the problem* (27) *has at least one mild solution.*

Proof. Given $x, y, z \in X$, we define $F: C(I; X) \to C(I; X)$ by

$$(Fu)(t) = e^{-\frac{c^2}{b}t}g_1(u) - \frac{bc^2}{a(a-b)}R(t)g_1(u) + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t T_a(s)g_1(u)\mathrm{d}s$$
$$+ \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t R(s)g_1(u)\mathrm{d}s + \frac{c^2(a^2-ab-b^2)}{a(a-b)^2}\int_0^t S(s)g_1(u)\mathrm{d}s$$
$$+ \frac{c^2}{b}\int_0^t e^{-\frac{c^2}{b}(t-s)}S(s)g_1(u)\mathrm{d}s - \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s S(\tau)g_1(u)\mathrm{d}\tau\mathrm{d}s$$

$$+ R(t)g_2(u) - \frac{a+b}{a-b}\int_0^t T_a(s)g_2(u)\mathrm{d}s + \frac{a+b}{a-b}\int_0^t S(s)g_2(u)\mathrm{d}s$$
$$- \frac{c^2(a+b)}{a(a-b)}\int_0^t R(s)g_2(u)\mathrm{d}s + \frac{c^2(a+b)}{a(a-b)}\int_0^t\int_0^s S(\tau)g_2(u)\mathrm{d}\tau\mathrm{d}s$$
$$+ \int_0^t R(s)g_3(u)\mathrm{d}s + \int_0^t U(t-s)f(s,u(s))\mathrm{d}s,$$

for all $u \in C(I;X)$.

We begin showing that F is a continuous map. Let $\{u_n\}_{n=1}^{\infty} \subseteq C(I;X)$ such that $u_n \to u$ as $n \to \infty$ (in the norm of $C(I;X)$). Since $0 < a < b$, it follows that $a^2 - ab - b^2 < 0$, and then,

$$\begin{aligned}
\|F(u_n) - F(u)\| \le{}& \left(1 + \frac{2c^2(-a^2+ab+b^2)}{a(a-b)^2} + \frac{c^2}{b} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) L\|g_1(u_n) - g_1(u)\| \\
&+ \left(1 + \frac{2(a+b)(a+c^2)}{a(b-a)}\right) L\|g_2(u_n) - g_2(u)\| + L\|g_3(u_n) - g_3(u)\| \\
&+ L\int_0^1 \|f(s,u_n(s)) - f(s,u(s))\|\mathrm{d}s \\
\le{}& \left(1 + \frac{2c^2(b^2+ab-a^2)}{a(a-b)^2} + \frac{c^2}{b} + \frac{2c^4(2b-a)}{a(a-b)^2}\right) L\|g_1(u_n) - g_1(u)\| \\
&+ \left(1 + \frac{4b(a+c^2)}{a(a-b)}\right) L\|g_2(u_n) - g_2(u)\| + L\|g_3(u_n) - g_3(u)\| \\
&+ L\int_0^1 \|f(s,u_n(s)) - f(s,u(s))\|\mathrm{d}s.
\end{aligned}$$

By hypotheses (H1) and (H2) and by the dominated convergence theorem, we have that $\|F(u_n) - F(u)\| \to 0$ when $n \to \infty$.

Denote $B_J = \{u \in C(I;X) \colon \|u(t)\| \le J, \forall t \in I\}$ and note that for any $u \in B_J$ we have

$$\begin{aligned}
\|(Fu)(t)\| \le{}& \|e^{-\frac{c^2}{b}t}g_1(u)\| + \frac{bc^2}{a(b-a)}\|R(t)g_1(u)\| + \frac{c^2(2b-a)}{(a-b)^2}\int_0^t \|T_a(s)g_1(u)\|\mathrm{d}s \\
&+ \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t \|R(s)g_1(u)\|\mathrm{d}s + \frac{c^2(-a^2+ab+b^2)}{a(a-b)^2}\int_0^t \|S(s)g_1(u)\|\mathrm{d}s \\
&+ \frac{c^2}{b}\int_0^t |e^{-\frac{c^2}{b}(t-s)}|\|S(s)g_1(u)\|\mathrm{d}s + \frac{c^4(2b-a)}{a(a-b)^2}\int_0^t\int_0^s \|S(\tau)g_1(u)\|\mathrm{d}\tau\mathrm{d}s \\
&+ \|R(t)g_2(u)\| + \frac{a+b}{b-a}\int_0^t \|T_a(s)g_2(u)\|\mathrm{d} + \frac{a+b}{b-a}\int_0^t \|S(s)g_2(u)\|\mathrm{d}s \\
&+ \frac{c^2(a+b)}{a(b-a)}\int_0^t \|R(s)g_2(u)\|\mathrm{d}s + \frac{c^2(a+b)}{a(b-a)}\int_0^t\int_0^s \|S(\tau)g_2(u)\|\mathrm{d}\tau\mathrm{d}s \\
&+ \int_0^t \|R(s)g_3(u)\|\mathrm{d}s + \int_0^t\int_0^s \|R(s-\tau)f(\tau,u(\tau))\|\mathrm{d}\tau\mathrm{d}s
\end{aligned}$$

$$\le \left(1+\frac{2c^2(b^2+ab-a^2)}{a(a-b)^2}+\frac{c^2}{b}+\frac{2c^4(2b-a)}{a(a-b)^2}\right)Lg_{1J}+\left(1+\frac{4b(a+c^2)}{a(a-b)}\right)Lg_{2J}$$
$$+Lg_{3J}+L\Phi(J)\int_0^1 m(s)\mathrm{d}s \le J.$$

Therefore, F maps B_J into itself and $F(B_J)$ is a bounded set. Moreover, by continuity of the functions $t \to R(t)$, $t \to T_a(t)$, $t \to S(t)$ and $t \to U(t)$ on [0, 1], we have that the set $F(B_J)$ is an equicontinuous set of functions.

Define $\mathcal{B} = \overline{co}(F(B_J))$ the closed convex hull of the set $F(B_J)$. It follows from Lemma 2.10 that the set $\mathcal{B}$ is equicontinuous. In addition, the operator $F\colon \mathcal{B} \to \mathcal{B}$ is continuous and $F(\mathcal{B})$ is a bounded set of functions.

Let $\varepsilon > 0$ be given. For $t \in I$, we recall the notation $F(\mathcal{B})(t) = \{v(t); v \in F(\mathcal{B})\}$. By Lemma 2.13 (with $W = F(\mathcal{B})$) there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset F(\mathcal{B})$ such that

$$\xi(F(\mathcal{B})(t)) \le 2\xi(\{v_n(t)\}_{n=1}^{\infty}) + \varepsilon$$

and since the functions g_1, g_2, g_3 are compacts maps, it follows that

$$2\xi(\{v_n(t)\}_{n=1}^{\infty}) + \varepsilon \le 2\xi\left(\int_0^t \{U(t-s)f(s,u_n(s))\}_{n=1}^{\infty}\mathrm{d}s\right) + \varepsilon,$$

where $v_n = F(u_n)$ for some $\{u_n\}_{n=1}^{\infty} \subset \mathcal{B}$. By hypothesis (H3), for each $t \in I$ we have $\|U(t-s)f(s,u_n(s))\| \le L\Phi(J)m(s)$. And then, from Lemma 2.12

$$2\xi\left(\int_0^t \{U(t-s)f(s,u_n(s))\}_{n=1}^{\infty}\mathrm{d}s\right) \le 4L\int_0^t \xi(\{f(s,u_n(s))\}_{n=1}^{\infty})ds.$$

Therefore, by condition (H4) we have

$$\xi(F(\mathcal{B}(t))) \le 4L\int_0^t \xi(\{f(s,u_n(s))\}_{n=1}^{\infty})\mathrm{d}s + \varepsilon \le 4L\int_0^t G(s)\xi(\{u_n(s)\}_{n=1}^{\infty})\mathrm{d}s + \varepsilon$$
$$\le 4L\gamma(\mathcal{B})\int_0^t G(s)\mathrm{d}s + \varepsilon.$$

By hypothesis $(H4)$, $G \in L^1(I;\mathbb{R}^+)$. Then for $\kappa < \frac{1}{4L}$ there exists $\varphi \in C(I;\mathbb{R}^+)$ satisfying $\int_0^1 |G(s)-\varphi(s)|\mathrm{d}s < \kappa$, where $\alpha = 4\kappa L$. Hence,

$$\xi(F(\mathcal{B})(t)) \le 4L\gamma(\mathcal{B})\left[\int_0^t |G(s)-\varphi(s)|\mathrm{d}s + \int_0^t \varphi(s)\mathrm{d}s\right] + \varepsilon$$
$$\le 4L\gamma(\mathcal{B})[\kappa + Nt] + \varepsilon,$$

where $N = \|\varphi\|_\infty$. Since $\varepsilon > 0$ is arbitrary, we have

$$\xi(F(\mathcal{B})(t)) \le (\alpha + \beta t)\gamma(\mathcal{B}), \text{ where } \beta = 4LN. \tag{29}$$

Let $\varepsilon > 0$ be given. Since the functions g_1, g_2, g_3 are compact maps and applying Lemma 2.13 there exists a sequence $\{w_n\}_{n=1}^{\infty} \subseteq \overline{co}(F(\mathcal{B}))$ such that

$$\begin{aligned}\xi(F^2(\mathcal{B})(t)) \leq & 2\xi\left(\int_0^t \{U(t-s)f(s,w_n(s))\}_{n=1}^{\infty}\mathrm{d}s\right)+\varepsilon\\ \leq & 4L\int_0^t \xi(\{f(s,w_n(s))\}_{n=1}^{\infty})\mathrm{d}s+\varepsilon\\ \leq & 4L\int_0^t G(s)\xi(\overline{co}(F(\mathcal{B})(s)))ds+\varepsilon,\end{aligned}$$

and by item (iii) of Lemma 2.9,

$$\xi(F^2(\mathcal{B})(t)) \leq 4L\int_0^t G(s)\xi(F(\mathcal{B})(s))\mathrm{d}s+\varepsilon.$$

Using inequality (29), we have that

$$\begin{aligned}\xi(F^2(\mathcal{B})(t)) &\leq 4L\int_0^t G(s)(\alpha+\beta s)\gamma(\mathcal{B})\mathrm{d}s+\varepsilon\\ &\leq 4L\int_0^t [|G(s)-\varphi(s)|+|\varphi(s)|](\alpha+\beta s)\gamma(\mathcal{B})\mathrm{d}s+\varepsilon\\ &\leq 4L(\alpha+\beta t)\gamma(\mathcal{B})\int_0^t |G(s)-\varphi(s)|\mathrm{d}s+4LN\gamma(\mathcal{B})\left(\alpha t+\frac{\beta t^2}{2}\right)+\varepsilon\\ &\leq \left[\alpha(\alpha+\beta t)+\beta\left(\alpha t+\frac{bt^2}{2}\right)\right]\gamma(\mathcal{B})+\varepsilon\\ &\leq \left(\alpha^2+2\beta t+\frac{(\beta t)^2}{2}\right)\gamma(\mathcal{B})+\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\xi(F^2(\mathcal{B})(t)) \leq \left(\alpha^2+2\beta t+\frac{(\beta t)^2}{2}\right)\gamma(\mathcal{B}).$$

By an inductive process, for all $n \in \mathbb{N}$, it holds

$$\xi(F^n(\mathcal{B})(t)) \leq \left(\alpha^n+C_1^n\alpha^{n-1}\beta t+C_2^n\alpha^{n-2}\frac{(\beta t)^2}{2!}+\cdots+\frac{(\beta t)^n}{n!}\right)\gamma(\mathcal{B}),$$

where, for $0 \leq m \leq n$, the symbol C_m^n denotes the binomial coefficient $\binom{n}{m}$.

In addition, for all $n \in \mathbb{N}$ the set $F^n(\mathcal{B})$ is an equicontinuous set of functions. Therefore, using the Lemma 2.11 we conclude that

$$\gamma(F^n(\mathcal{B})) \leq \left(\alpha^n+C_1^n\alpha^{n-1}\beta+C_2^n\alpha^{n-2}\frac{\beta^2}{2!}+\cdots+\frac{\beta^n}{n!}\right)\gamma(\mathcal{B}).$$

Since $0 \leq \alpha < 1$ and $\beta > 0$, it follows from Lemma 2.14 that there exists $n_0 \in \mathbb{N}$ such that

$$\left(\alpha^{n_0}+C_1^{n_0}\alpha^{n_0-1}\beta+C_2^{n_0}\alpha^{n_0-2}\frac{\beta^2}{2!}+\cdots+\frac{\beta^{n_0}}{n_0!}\right)=r<1.$$

Consequently, $\gamma(F^{n_0}(\mathcal{B})) \leq r\gamma(\mathcal{B})$. It follows from Lemma 2.15 that F has a fixed point in $\mathcal{B}$, and this fixed point is a mild solution of (27).

□

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