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Second and third order forward difference operator: what is in between?

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Abstract
In this article, we present a new geometrical notion for a real-valued function defined in a discrete domain that depends on a parameter $\alpha \geq 2$. We give examples to illustrate connections between convexity and this new concept. We then prove two criteria based on the sign of the discrete fractional operator of a function u, $\Delta^\alpha u$ with $2 \leq \alpha < 4$. Two examples show that the given criteria are optimal with respect to the established geometrical notion.

Keywords Jerk dynamics · Discrete fractional operator · Convexity · Transference principle · Positive α-jerk

Mathematics Subject Classification 26A51 · 39A12 · 34A08 · 26A33

1 Introduction

Consider the third-order differential equation

$$x'''(t) = J(t, x(t), x'(t), x''(t)), \quad x(0) = x_0, \quad x'(0) = x_1, \quad x''(0) = x_2, \quad t \geq 0. \tag{1.1}$$

Nonlinear equations involving the third temporal derivative have been widely studied [8, 24,25,32,34,37,42]. The jerk term $x'''(t)$ with $x(t)$ being the displacement appear in the Abraham–Lorentz equation [28], describing the motion of a radiating charged particle. On the other hand, it is known that the requirement for the occurrence of chaos of a nonlinear autonomous system is at least the third-order temporal derivative being involved, like the nonlinear jerk equations [23,41]. In the last time, third-order differential equations appeared

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in a variety of dissimilar areas such as elastically deformable matter, in the geometric design of roads and tracks, in motion control and in manufacturing processes. Therefore, the study of nonlinear jerk equations is an interesting issue that deserves to be investigated.

Recently, in the papers [15,16] the authors show that the jerk dynamics are naturally obtained for electrical circuits using the fractional calculus approach, i.e. replacing the third order derivative x''' by a fractional order operator D_t^α with order $2 < \alpha < 4$. The electrical circuits studied in such papers and their respective analogue mechanical system can be used to analyze the vibration levels of machinery, serial mechanisms, robotics, oscillating circuits modeling, and instability of electrical and mechanical circuits, to evaluate reconfigurable machines or to make mobility analysis or algebraic formulations of motion equations [15].

The discrete analogue is important for the numerical analysis of nonautonomous and nonlinear fractional evolution problems. Using Euler's method, the third order derivative $x'''(t)$ may be approximated by $\frac{1}{h^3}\Delta^3 u(n)$ where h is the step size of the approximation and $u(n) := x(nh)$. Here, $\Delta u(n) := u(n+1) - u(n)$ is the forward difference operator and, for $m \in \mathbb{N}$

$$\Delta^m u(n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} u(n+j), \quad n \in \mathbb{N}_0.$$

From a numerical point of view, the discrete fractional operator Δ^α—as defined in [33]—approximates the Riemann–Liouville fractional order operator D_t^α and coincides with the generalized forward Grünwald–Letnikov derivative [36], defined by

$$\Delta^\alpha u(n) := \Delta^m \sum_{j=0}^{n} k^{m-\alpha}(j) u(j), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m$, $m \in \mathbb{N}$ and $k^\alpha(n) := \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}$, $n \in \mathbb{N}_0$. Here, Γ denotes the Gamma function.

In this paper we focus on the fractional cases $2 < \alpha < 3$ and $3 < \alpha < 4$.

Fractional powers of the forward difference operator Δ have been defined since many years ago. In 1956, Kutter [31] mentioned for the first time differences of fractional order. In 1974, Diaz and Osler [12] introduced a discrete fractional operator defined as an infinite series. In 1988, Grey and Zhang [26] developed a fractional calculus for the discrete nabla (backward) operator. Miller and Ross [35] defined a fractional sum via the solution of a linear difference equation. Their definition is the discrete analogue of the Riemann-Liouville fractional integral, which can be obtained via the solution of a linear differential equation. In 2007, Atici and Eloe [2–4] introduced the Riemann–Liouville like fractional difference by using the definition of a fractional sum of Miller and Ross, and developed some of its properties that allow one to obtain solutions of certain fractional difference equations. In 2010, Anastassiou [1] defined the Caputo like fractional difference by using also the notion of a fractional sum from Miller and Ross. At the same year, Ferreira [14] introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations. See also Sengul [38] for related work. In 2011, Holm [27] further developed and applied the tools of discrete fractional calculus to the arena of fractional difference equations [44,45].

We point out that the discrete fractional operator Δ^α coincides up to translation with the more studied definition of discrete fractional operator Δ_a^ν given by Atici and Eloe in 2007, namely

$$\left(\Delta_a^\nu f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s),\, t \in \mathbb{N}_{a+N-\nu}, \tag{1.2}$$

where $f \in s(\mathbb{N}_a; \mathbb{R})$, $N \in \mathbb{N}_1$ is the unique integer satisfying $N-1 < \nu < N$, and the map $t \mapsto t^{\underline{\nu}}$ is defined by $t^{\underline{\nu}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. This property, called transference principle, has been proved recently by Goodrich and Lizama [21].

In 2014, the following question was addressed by Dahal and Goodrich [9,10]:

(P) Is there a connection between the sign of the discrete fractional operator $\Delta^\alpha u$, and the positivity, monotonicity and convexity of the sequence u on which it acts?

While it is trivial to prove that $\Delta u(n) \geq 0$ for $n \in \mathbb{N}_0$ implies that u is increasing on $\mathbb{N}_0$, it is very nontrivial to decide how monotonicity is connected to the positivity or negativity of the discrete fractional operator. Similarly, while it is equally trivial to prove that $\Delta^2 u(n) \geq 0$ for $n \in \mathbb{N}_0$ implies that u is increasing on $\mathbb{N}_0$ and thus that u satisfies a convexity-type property, the analogue of this sort of result in the discrete fractional setting is much more difficult to obtain. This is a very nontrivial program due to the inherent nonlocal nature of the fractional operator, a fact that causes great difficulty when trying to equip the operator with some reasonable geometrical meaning.

In case $2 \leq \alpha < 3$ convexity results for discrete fractional operators as well applications to fractional boundary value problems were studied in [17] and then reviewed in the monograph [18, Section 7.3] by Goodrich and Peterson. We point out that due to a flurry of recent work in the area, the basic convexity and concavity results presented in [18] have been substantively extended in a variety of directions. See, for instance [5,11,13,19–22,40].

Recently, in the Ref. [21] the authors studied the problem (P) in case $2 \leq \alpha < 3$ and found that if $\Delta^\alpha u(n) \geq 0$ for all $n \in \mathbb{N}_0$ and $u(0) \geq 0$, $u(1) \geq \alpha u(0)$, $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$ then u is convex. See [21, Theorem 7.1]. This result has been recently refined in [6] where it was proved that u should be positive and nondecreasing, too.

To understand the behavior between monotonicity and convexity, the concept of α-convex sequence was defined in [6, Definition 6.1]. It refers to the continuous behavior (with respect to the parameter α) between convex and non-decreasing sequences. Roughly speaking, an α-convex sequence should be placed geometrically above sequences of the form $\beta a^n + b$ where $\beta, b \in \mathbb{R}$, $a > 0$, see [6]. By means of this concept, the connection between the sign of the operator $\Delta^\alpha u$ and the convexity of the sequence u has been studied.

However, even though many authors have studied various generalizations of convexity, see e.g. [30] and its references, as far as we know none of these address the problem of what is between the second and third order powers of the forward difference operator Δ and thus remains an open problem.

In this paper, we propose a new concept that we have called α-jerk sequence. This notion interpolate between the concepts of convex and positive jerk sequence, where the latter refers to sequences that verify $\Delta^3 u(n) \geq 0$. This concept allows us to answer the problem (P) about the connection between the sign of the operator $\Delta^\alpha u$, when $2 \leq \alpha < 4$ and the property of the sequence u on which it acts. It is worthwhile to mention that this new concept not only allows a continuous transition between the geometry of the sequence u as α moves from 2 to 4 but also a continuous transition between the previous results existing in the literature and ours. In this way, our main results in this article are the following:

Theorem 1.1 *Let $2 \leq \alpha \leq 3$ and $u \in s(\mathbb{N}_0; \mathbb{R})$ be given and assume that*

1. $\Delta^\alpha u(n) \geq 0$, *for all $n \in \mathbb{N}_0$;*

2. $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$;
3. $u(1) \geq \alpha u(0)$;
4. $u(0) \geq 0$.

Then u is positive, increasing, convex and has positive α-jerk on $\mathbb{N}_0$.

and

Theorem 1.2 *Let $3 \leq \alpha < 4$ and $u \in s(\mathbb{N}_0; \mathbb{R})$ be given and assume that*

1. $\Delta^\alpha u(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
2. $u(3) \geq \alpha u(2) - \frac{\alpha(\alpha-1)}{2} u(1) + \frac{\alpha(\alpha-1)(\alpha-2)}{6} u(0)$;
3. $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$;
4. $u(1) \geq \alpha u(0)$;
5. $u(0) \geq 0$.

Then u is positive, increasing, convex and has positive jerk on $\mathbb{N}_0$.

This paper is organized as follows: In Sect. 2, we provided the necessary preliminaries on the definition of fractional discrete operator that we will use. We also recall the fundamental transference principle. Section 3 is dedicated to our main results. After providing our definition of positive α-jerk sequence, we realize that its graph must be placed above sequences of the form $\beta a^n + bn + c$. This is shown in Proposition 3.2. Next, our main results are proved and, by means of the transference principle, the analogous results corresponding to the operator Δ_a^ν are established. Two additional examples are given. One example refers to the optimality of condition (2) in Theorem 1.1 to guarantee the convexity of the sequence u. See the Example 3.5. The second refers to the optimality of the condition (2) in Theorem 1.2 to ensure that the sequence u has positive jerk. All these results provide new insights and propose original concepts to better understand the qualitative behavior of discrete fractional operators in this challenging area of study.

2 Preliminaries

In this section, we provide the necessary preliminaries on discrete fractional operators that will be used throughout the paper. In what follows, we denote $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$, for some $a \in \mathbb{R}$, and $\mathbb{N} \equiv \mathbb{N}_1$ as usual. We denote by $s(\mathbb{N}_a; \mathbb{R})$ the vectorial space that consists of all sequences $f : \mathbb{N}_a \to \mathbb{R}$. Recall that given a sequence $f \in s(\mathbb{N}_a; \mathbb{R})$ the first-order forward difference operator, denoted by Δ_a, is defined by

$$\big(\Delta_a f\big)(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

Then one may define iteratively the higher order differences Δ_a^n, for $n \in \mathbb{N}_1$, by writing

$$\big(\Delta_a^n f\big)(t) := \big(\Delta_a \circ \Delta_a^{n-1} f\big)(t).$$

We also denote $\Delta_a^0 \equiv I_a$, where $I_a : s(\mathbb{N}_a; \mathbb{R}) \to s(\mathbb{N}_a; \mathbb{R})$ is the identity operator, $\Delta_a^1 \equiv \Delta_a$, and $\Delta^n \equiv \Delta_0^n$.

Remark 2.1 For any $f \in s(\mathbb{N}_0; \mathbb{R})$, $l \in \mathbb{N}_0$ we have

$$\Delta^l f(t) = \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} f(t+j), \quad t \in \mathbb{N}_0.$$

We define for $n \in \mathbb{N}_0$, the following sequence

$$k^{\alpha}(n) := \begin{cases} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} & \text{if } \alpha \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}, \\ \delta_0(n) & \text{if } \alpha = 0, \end{cases}$$

where $\delta_0(n)$ is the delta function,

$$\delta_0(n) := \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

The sequence k^{α} has been introduced in [33]. This special sequence has a number of distinguished properties that are fundamental to understand the behavior or discrete fractional operators. Their importance has been recognized in several papers. For an overview, we refer to the reference [21]. Among others, that will be useful in this paper, we notice the semigroup property:

$$(k^{\alpha} * k^{\beta})(n) = k^{\alpha+\beta}(n), \qquad n \in \mathbb{N}_0, \quad \alpha, \beta > 0, \tag{2.1}$$

which will be frequently used. We recall from [21, Lemma 3.2] the following result.

Lemma 2.2 *For any $\alpha > 0$ and $n \in \mathbb{N}_0$, the following identities hold:*

1. $\Delta k^{\alpha}(n) = (\alpha - 1)\dfrac{k^{\alpha}(n)}{n+1}$.
2. $\Delta^2 k^{\alpha}(n) = (\alpha - 2)(\alpha - 1)\dfrac{k^{\alpha}(n)}{(n+1)(n+2)}$.
3. $\Delta^3 k^{\alpha}(n) = (\alpha - 3)(\alpha - 2)(\alpha - 1)\dfrac{k^{\alpha}(n)}{(n+1)(n+2)(n+3)}$.

Now we recall from [33] the definition of α-th fractional sum operator on the set $\mathbb{N}_0$:

Definition 2.3 For each $\alpha > 0$ and $f \in s(\mathbb{N}_0; \mathbb{R})$, we define the fractional sum of order α as follows:

$$\Delta^{-\alpha} f(n) := \sum_{j=0}^{n} k^{\alpha}(n-j) f(j), \quad n \in \mathbb{N}_0.$$

The next concept, originally proposed in [33], is analogous to the definition of a fractional derivative in the sense of Riemann–Liouville [35].

Definition 2.4 Let $\alpha > 0$ be given and $f \in s(\mathbb{N}_0; \mathbb{R})$. The $\alpha-$th fractional discrete operator is defined by

$$\Delta^{\alpha} f(n) := \Delta^{m} \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha \leq m, \ m \in \mathbb{N}$.

We recall that the finite convolution $*$ of two sequences u, v where $u \in s(\mathbb{N}_0; \mathbb{C})$ and $v \in s(\mathbb{N}_0; \mathbb{R})$ is defined by

$$(u * v)(n) = \sum_{j=0}^{n} u(n-j)v(j), \quad n \in \mathbb{N}_0.$$

Given $a, b \in \mathbb{R}$, we define the translation (by $a \in \mathbb{R}$) operator $\tau_a : s(\mathbb{N}_a; \mathbb{R}) \to s(\mathbb{N}_0; \mathbb{R})$ by

$$\tau_a f(n) := f(a+n), \quad n \in \mathbb{N}_0.$$

Note that $\tau_a^{-1} = \tau_{-a}$ and $\tau_{a+b} = \tau_a \circ \tau_b = \tau_b \circ \tau_a$.

Lemma 2.5 [21, Lemma 2.3] *Let $f, g \in s(\mathbb{N}_0; \mathbb{R})$ be sequences, then for each $p \in \mathbb{N}$ we have*

$$(f * \tau_p g)(n) = \tau_p(f * g)(n) - \sum_{j=0}^{p-1} \tau_p f(n-j)g(j).$$

We recall the most commonly used fractional difference operator of order $\nu > 0$ as defined by Atici and Eloe [2–4]

$$\left(\Delta_a^{\nu} f\right)(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-s-1)^{\underline{-\nu-1}} f(s), \quad t \in \mathbb{N}_{a+N-\nu}, \tag{2.2}$$

where $f \in s(\mathbb{N}_a; \mathbb{R})$, $N \in \mathbb{N}_1$ is the unique integer satisfying $N-1 < \nu < N$, and the map $t \mapsto t^{\underline{\nu}}$ is defined by $t^{\underline{\nu}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$. In the integer cases of $\nu = N$ we have

$$\Delta_a^N f(t) = \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} f(t+j), \quad t \in \mathbb{N}_a. \tag{2.3}$$

In [21, Theorem 4.3] the authors related the operator Δ_a^{ν} in (2.2) to the operator Δ^{α} in Definition 2.4 by means of the operator of translation, which allowed to transfer the properties between both definitions. This is called a *transference principle*. In the following, we have extended the formulation of the transference principle in order to include the integer cases $\alpha = N \in \mathbb{N}$, being the proof immediate taking into account (2.3).

Theorem 2.6 (Transference Principle) *Let $N-1 < \alpha \le N$, $N \in \mathbb{N}$ and $a, \beta \in \mathbb{R}$. For each sequence $f \in s(\mathbb{N}_a; \mathbb{R})$ we have*

$$\tau_{a+N-\alpha} \circ \Delta_a^{\alpha} f = \Delta^{\alpha} \circ \tau_a f,$$

and for each $f \in s(\mathbb{N}_{a+N-\beta}; \mathbb{R})$,

$$\tau_{N-\beta} \circ \Delta_{a+N-\beta}^{\alpha} f = \Delta_a^{\alpha} \circ \tau_{N-\beta} f.$$

The next results generalize [21, Proposition 2.9, part (v)]

Proposition 2.7 *For any $a, b \in s(\mathbb{N}_0; \mathbb{R})$ and $l \in \mathbb{N}$ we have*

$$\Delta^l (a * b)(n) = (\Delta^l a * b)(n) + \sum_{j=1}^{l} \sum_{i=0}^{j-1} \binom{l}{j} (-1)^{l-j} a(i) b(n+j-i).$$

Proof Note that, by Remark 2.1 and Lemma 2.5 we get for $l \in \mathbb{N}$

$$\begin{aligned}
(\Delta^l a * b)(n) &= \sum_{i=0}^{n} b(i) \Delta^l a(n-i) = \sum_{i=0}^{n} b(i) \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} a(n+j-i) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} a(n+j-i) b(i) \\
&= \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \sum_{i=0}^{n} \tau_j a(n-i) b(i) \\
&= \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} (\tau_j a * b)(n), \quad n \in \mathbb{N}_0.
\end{aligned}$$

Thus, we have that for $n \in \mathbb{N}_0$,

$$
\begin{aligned}
\Delta^l (a*b)(n) &= \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} (a*b)(n+j) = \sum_{j=1}^{l} \binom{l}{j} (-1)^{l-j} \tau_j (a*b)(n) \\
&\quad + \binom{l}{0} (-1)^{l-0} (a*b)(n) \\
&= \sum_{j=1}^{l} \binom{l}{j} (-1)^{l-j} \left[(\tau_j a * b)(n) + \sum_{i=0}^{j-1} a(i) \tau_j b(n-i) \right] + (-1)^l (a*b)(n) \\
&= \sum_{j=1}^{l} \binom{l}{j} (-1)^{l-j} (\tau_j a * b)(n) + (-1)^l (a*b)(n) \\
&\quad + \sum_{j=1}^{l} \binom{l}{j} (-1)^{l-j} \sum_{i=0}^{j-1} a(i) \tau_j b(n-i) \\
&= \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} (\tau_j a * b)(n) + \sum_{j=1}^{l} \sum_{i=0}^{j-1} \binom{l}{j} (-1)^{l-j} a(i) b(n-i+j) \\
&= (\Delta^l a * b)(n) + \sum_{j=1}^{l} \sum_{i=0}^{j-1} \binom{l}{j} (-1)^{l-j} a(i) b(n+j-i).
\end{aligned}
$$

which proves the result. □

Remark 2.8 In particular, by Proposition 2.7, we have for $l = 1$: $\Delta(a*b)(n) = (\Delta a*b)(n) + a(0)b(n+1)$ and $\Delta(a*b)(n) = (a*\Delta b)(n) + a(n+1)b(0)$. Thus,

$$
(a * \Delta b)(n) = (\Delta a * b)(n) - a(n+1)b(0) + a(0)b(n+1). \tag{2.4}
$$

3 Main results

From a physical point of view, jerk is the rate at which the acceleration of an object changes with respect to time. In a discrete setting (following Euler method of discretization) it corresponds to the third order power of the forward difference operator, namely: Δ^3. The connection between positive jerk, i.e. $\Delta^3 u \geq 0$ and u has been an object of recent interest [29,30] and can be characterized as follows: A sequence u has positive jerk if and only if

$$
u(n) = \frac{1}{2} \sum_{k=0}^{n} (n-k+2)(n-k+1) b(k)
$$

with $b(k) \geq 0$ for all $k \in \mathbb{N}_3$. See e.g. [30, Lemma 1.1] and references therein. In order to analyze the intermediate cases between Δ^2 and Δ^3 we propose the following definition.

Definition 3.1 Let $\alpha \geq 2$. We say that a sequence $u \in s(\mathbb{N}_0, \mathbb{R})$ has positive α-jerk if

$$
u(n+3) - \alpha u(n+2) + (2\alpha - 3) u(n+1) - (\alpha - 2) u(n) \geq 0, \quad n \in \mathbb{N}_0, \tag{3.1}
$$

When $\alpha = 3$ we recover the notion of positive jerk. Note that when $\alpha = 2$ we retrieve the concept of convex sequence on the set $\mathbb{N}$.

Our first result tell us about the geometrical meaning of a positive α-jerk sequence. Compared with a convex sequence, whose graph lies above sequences of the form: $\beta a^n + b$, in the case of positive α-jerk sequences, the graph is placed above sequences of the form: $\beta a^n + bn + c$. This is the content of the following result.

Proposition 3.2 *If $u \in s(\mathbb{N}_0, \mathbb{R})$ has positive α-jerk then we have*

$$u(n) \geq \frac{1}{(3-\alpha)^2}[n(3-\alpha) + (\alpha-2)^n - 1]\Delta^2 u(0) + n\Delta u(0) + u(0)$$

for $\alpha \geq 2, \alpha \neq 3$ and

$$u(n) \geq \frac{n(n-1)}{2}\Delta^2 u(0) + n\Delta u(0) + u(0),$$

in case $\alpha = 3$.

Proof From the definition, we note that u has positive α-jerk if and only if $\Delta^2 u(n+1) \geq (\alpha-2)\Delta^2 u(n),\ n \in \mathbb{N}_0$. Iterating, we obtain

$$\Delta^2 u(n) \geq (\alpha-2)^n \Delta^2 u(0). \tag{3.2}$$

Thus, $\Delta u(n+1) - \Delta u(n) \geq (\alpha-2)^n \Delta^2 u(0)$ and hence $\Delta u(n+1) \geq (\alpha-2)^n \Delta^2 u(0) + \Delta u(n)$. Iterating again this last inequality, we obtain

$$\Delta u(n) \geq \sum_{j=0}^{n-1} (\alpha-2)^j \Delta^2 u(0) + \Delta u(0) = \frac{1-(\alpha-2)^n}{3-\alpha}\Delta^2 u(0) + \Delta u(0).$$

in case $\alpha \neq 3$ and

$$\Delta u(n) \geq n\Delta^2 u(0) + \Delta u(0),$$

in case $\alpha = 3$.

Therefore, if $\alpha \neq 3$ we have

$$u(n+1) \geq \frac{1-(\alpha-2)^n}{3-\alpha}\Delta^2 u(0) + \Delta u(0) + u(n). \tag{3.3}$$

Thus, iterating again we arrive at

$$\begin{aligned} u(n) &\geq \sum_{j=0}^{n-1} \frac{1-(\alpha-2)^j}{3-\alpha}\Delta^2 u(0) + n\Delta u(0) + u(0) \\ &= \frac{1}{(3-\alpha)^2}[n(3-\alpha) + (\alpha-2)^n - 1]\Delta^2 u(0) + n\Delta u(0) + u(0) \end{aligned}$$

which finish the proof in this case. In contrast, if $\alpha = 3$, we obtain

$$u(n) \geq \frac{n(n-1)}{2}\Delta^2 u(0) + n\Delta u(0) + u(0)$$

which finishes the proof. □

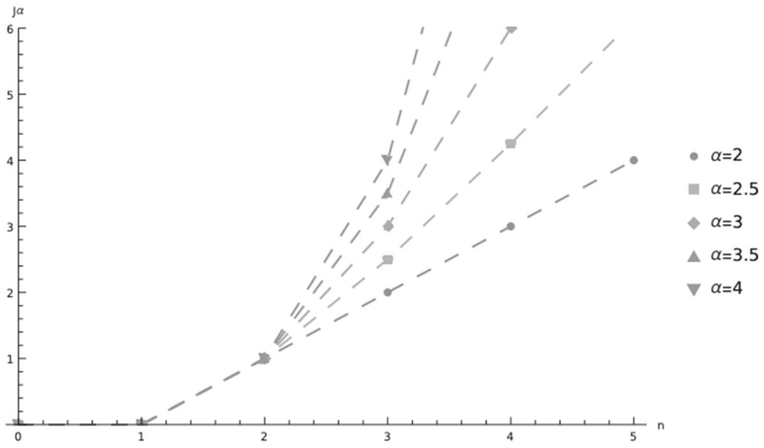

Fig. 1 J_α with $u(0) = u(1) = 0$ and $u(2) = 1$

Remark 3.3 If a sequence u has positive $\alpha-$jerk and we assume that $u(0) = u(1) = 0$, $u(2) = 1$, then their graph lies above the graph of the sequence $J_\alpha(n) := \frac{1}{(3-\alpha)^2}[n(3-\alpha) + (\alpha-2)^n - 1]$. The behavior of the sequence $J_\alpha(n)$ for different values of $\alpha \neq 3$ is drawn in Fig. 1. In case $\alpha = 3$ the graph of a positive jerk sequence lies above the graph of the sequence $J_3(n) = \frac{n(n-1)}{2}$.

The following is our main result in case $2 \leq \alpha \leq 3$.

Theorem 3.4 *Let $2 \leq \alpha \leq 3$ and $u \in s(\mathbb{N}_0; \mathbb{R})$ be given and assume that*

1. $\Delta^\alpha u(n) \geq 0$, *for all $n \in \mathbb{N}_0$;*
2. $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$;
3. $u(1) \geq \alpha u(0)$;
4. $u(0) \geq 0$.

Then u is positive, increasing, convex and has positive α-jerk on $\mathbb{N}_0$.

Proof First, we study the borderline cases. If $\alpha = 2$ then by hypothesis (1) we have $\Delta^2 u(n) \geq 0$, for all $n \in \mathbb{N}_0$, i.e. u is convex (positive 2-jerk) on $\mathbb{N}_0$. Now, using the fact that u is convex on $\mathbb{N}_0$, we get $\Delta u(n+1) \geq \Delta u(n)$. By hypotheses (3) and (4) we also have $u(1) \geq u(0)$, then $\Delta u(0) \geq 0$ and $\Delta u(n) \geq \cdots \geq \Delta u(0) \geq 0$. Hence, u is monotone increasing and positive on $\mathbb{N}_0$.

On the other hand, if $\alpha = 3$, by hypothesis (1), we have $\Delta^3 u(n) \geq 0$ on $\mathbb{N}_0$, i.e., u has positive jerk. Moreover, by hypotheses (2), (3) and (4) we obtain $\Delta^2 u(0) = u(2) - 2u(1) + u(0) \geq u(1) - 2u(0) \geq 0$, since that positive jerk is equivalent to $\Delta^2 u(n+1) \geq \Delta^2 u(n)$, then $\Delta^2 u(n) \geq ... \geq \Delta^2 u(0) \geq 0$. Thus, u is convex on $\mathbb{N}_0$ and by hypothesis (3) and (4), u is monotone increasing and positive.

Now, we assume $2 < \alpha < 3$. By Proposition 2.7, with $a := u$, $b := k^{3-\alpha}$ and $l = 3$, we obtain

$$(k^{3-\alpha} * \Delta^3 u)(n) = \Delta^\alpha u(n) - \sum_{j=1}^{3} \sum_{i=0}^{j-1} \binom{3}{j} (-1)^{3-j} u(i) k^{3-\alpha}(n+j-i). \tag{3.4}$$

Since $k^{3-\alpha}(n+j-i)=\tau_{j-i}k^{3-\alpha}(n)$, then convolving (3.4) with $k^{\alpha-2}$ we obtain

$$(k^{\alpha-2}*k^{3-\alpha}*\Delta^3 u)(n)=(k^{\alpha-2}*\Delta^\alpha u)(n)\\ -\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)(k^{\alpha-2}*\tau_{j-i}k^{3-\alpha})(n). \quad (3.5)$$

Observe that $(k^{\alpha-2}*k^{3-\alpha}*\Delta^3 u)(n)=(k*\Delta^3 u)(n)=\Delta^2 u(n+1)-\Delta^2 u(0)$. Moreover, by Lemma 2.5 and the semigroup property of the kernel k^γ, we get

$$(k^{\alpha-2}*\tau_{j-i}k^{3-\alpha})(n)=(k^{\alpha-2}*k^{3-\alpha})(n+j-i)-\sum_{l=0}^{j-i-1}k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l)\\ =1-\sum_{l=0}^{j-i-1}k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l).$$

Therefore, replacing the above identity in (3.5), we have

$$\Delta^2 u(n+1)-\Delta^2 u(0)=(k^{\alpha-2}*\Delta^\alpha u)(n)-\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\\ +\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\sum_{l=0}^{j-i-1}k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l). \quad (3.6)$$

Note that,

$$\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)=3u(0)-3u(0)\\ -3u(1)+u(0)+u(1)+u(2)=\Delta^2 u(0). \quad (3.7)$$

Also, since for any $\gamma>0$, $k^\gamma(0)=1$, $k^\gamma(1)=\gamma$ and $k^\gamma(2)=\frac{\gamma(\gamma+1)}{2}$, we have

$$\sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}u(i)\sum_{l=0}^{j-i-1}k^{\alpha-2}(n-l+j-i)k^{3-\alpha}(l)\\ =3u(0)k^{\alpha-2}(n+1)-3[u(0)(k^{\alpha-2}(n+2)\\ +k^{\alpha-2}(n+1)(3-\alpha))+u(1)k^{\alpha-2}(n+1)]\\ +[u(0)(k^{\alpha-2}(n+3)+k^{\alpha-2}(n+2)(3-\alpha)+k^{\alpha-2}(n+1)\frac{1}{2}(3-\alpha)(4-\alpha))\\ +u(1)(k^{\alpha-2}(n+2)+k^{\alpha-2}(n+1)(3-\alpha))+u(2)k^{\alpha-2}(n+1)]. \quad (3.8)$$

Replacing (3.7) and (3.8) in (3.6) we obtain that for $n\in\mathbb{N}_0$,

$$\Delta^2 u(n+1)=(k^{\alpha-2}*\Delta^\alpha u)(n)+k^{\alpha-2}(n+3)u(0)+k^{\alpha-2}(n+2)[u(1)-\alpha u(0)]\\ +k^{\alpha-2}(n+1)\left[u(2)-\alpha u(1)+\frac{\alpha(\alpha-1)}{2}u(0)\right]. \quad (3.9)$$

Using the hypotheses (1)–(4) we conclude from (3.9) that $\Delta^2 u(n) \geq 0$, for all $n \in \mathbb{N}$. On the other hand, using hypothesis (2), we have

$$u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2} u(0) = \Delta^2 u(0) - (\alpha - 2)u(1) + \frac{(\alpha-2)(\alpha+1)}{2} u(0) \geq 0.$$

Hence, hypotheses (3) and (4) show that $\Delta^2 u(0) \geq 0$. Indeed,

$$\begin{aligned}\Delta^2 u(0) &\geq (\alpha-2)u(1) - \frac{(\alpha-2)(\alpha+1)}{2} u(0) \geq \left[(\alpha-2)\alpha - \frac{(\alpha-2)(\alpha+1)}{2}\right] u(0) \\ &= \frac{(\alpha-2)(\alpha-1)}{2} u(0).\end{aligned}$$

This proves that $\Delta^2 u(n) \geq 0$ for all $n \in \mathbb{N}_0$ – i.e., u is convex on the set $\mathbb{N}_0$ as claimed. It follows that u is positive and increasing (because it corresponds to the case $\alpha = 2$ proved at the beginning of the proof).

Next, we prove that u has positive α-jerk. Indeed, from (3.4) we obtain

$$\begin{aligned}\Delta^\alpha u(n) &= (k^{3-\alpha} * \Delta^3 u)(n) + \sum_{j=1}^{3} \sum_{i=0}^{j-1} \binom{3}{j} (-1)^{3-j} u(i) k^{3-\alpha}(n+j-i) \\ &= (k^{3-\alpha} * \Delta^3 u)(n) + k^{3-\alpha}(n+3)u(0) + k^{3-\alpha}(n+2)[u(1) - 3u(0)] \\ &\quad + k^{3-\alpha}(n+1)[u(2) - 3u(1) + 3u(0)].\end{aligned} \tag{3.10}$$

By Eq. (2.4), with $a := k^{3-\alpha}$ and $b := \Delta^2 u$, we get

$$(k^{3-\alpha} * \Delta^3 u)(n) = (\Delta k^{3-\alpha} * \Delta^2 u)(n) - k^{3-\alpha}(n+1)\Delta^2 u(0) + k^{3-\alpha}(0)\Delta^2 u(n+1).$$

Thus, replacing the above identity in (3.10) and by hypothesis (1) we have

$$\begin{aligned}0 \leq \Delta^\alpha u(n) &= (\Delta k^{3-\alpha} * \Delta^2 u)(n) - k^{3-\alpha}(n+1)\Delta^2 u(0) + k^{3-\alpha}(0)\Delta^2 u(n+1) \\ &\quad + k^{3-\alpha}(n+3)u(0) + k^{3-\alpha}(n+2)[u(1) - 3u(0)] \\ &\quad + k^{3-\alpha}(n+1)[u(2) - 3u(1) + 3u(0)] \\ &= \sum_{j=1}^{n} \Delta k^{3-\alpha}(j)\Delta^2 u(n-j) + \Delta k^{3-\alpha}(0)\Delta^2 u(n) + \Delta^2 u(n+1) \\ &\quad + k^{3-\alpha}(n+3)u(0) + k^{3-\alpha}(n+2)[u(1) - 3u(0)] \\ &\quad - k^{3-\alpha}(n+1)[u(1) - 2u(0)] \\ &= \sum_{j=1}^{n} \Delta k^{3-\alpha}(j)\Delta^2 u(n-j) + (2-\alpha)\Delta^2 u(n) + \Delta^2 u(n+1) \\ &\quad + \Delta k^{3-\alpha}(n+2)u(0) + \Delta k^{3-\alpha}(n+1)[u(1) - 2u(0)],\end{aligned}$$

where we have used $\Delta k^{3-\alpha}(0) = 2 - \alpha$ and $k^{3-\alpha}(0) = 1$. By Lemma 2.2, part (1), we have $\Delta k^{3-\alpha}(m) \leq 0$. Recalling that $\Delta^2 u(m) \geq 0$ we obtain by hypotheses and the above inequality

$$\begin{aligned}(2-\alpha)\Delta^2 u(n) + \Delta^2 u(n+1) &\geq -\sum_{j=1}^{n} \Delta k^{3-\alpha}(j)\Delta^2 u(n-j) - \Delta k^{3-\alpha}(n+2)u(0) \\ &\quad - \Delta k^{3-\alpha}(n+1)[u(1) - 2u(0)] \geq 0.\end{aligned}$$

for all $n \in \mathbb{N}_0$ – i.e., u has positive α-jerk on $\mathbb{N}_0$. □

The following example show that the condition (2) in Theorem 3.4 is necessary in order to guarantee the convexity of the sequence u.

Example 3.5 Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = \frac{2^n - 1}{2^{n-1}}$, and assume that $\frac{4+\sqrt{2}}{2} < \alpha < 3$. The following statements holds.

(i) $\Delta^\alpha u(n) \geq 0$, for all $n \in \mathbb{N}_0$;
(ii) $u(1) \geq \alpha u(0)$;
(iii) $u(0) \geq 0$;
(iv) u is positive and increasing.
(v) u has positive α-jerk and is concave.

Indeed, first observe that $u(0) = 0$ and $u(1) = 1$, therefore (ii) and (iii) holds. By their own definition u is positive and it is clear that $\Delta u(n) = u(n+1) - u(n) = \frac{1}{2^n} \geq 0$, i.e., u is increasing on $\mathbb{N}_0$. It proves (iv).

We now verify (i): By Proposition 2.7, with $a := k^{3-\alpha}$, $b := u$ and $l = 3$, we obtain for each $n \in \mathbb{N}_0$

$$\begin{aligned}\Delta^\alpha u(n) &= (\Delta^3 k^{3-\alpha} * u)(n) + \sum_{j=1}^{3}\sum_{i=0}^{j-1}\binom{3}{j}(-1)^{3-j}k^{3-\alpha}(i)u(n+j-i)\\ &= (\Delta^3 k^{3-\alpha} * u)(n) + u(n+3) - \alpha u(n+2) + \frac{\alpha(\alpha-1)}{2}u(n+1).\end{aligned} \tag{3.11}$$

Using Eq. (2.4), with $a := u$ and $b := \Delta^2 k^{3-\alpha}$, we obtain

$$(\Delta^3 k^{3-\alpha} * u)(n) = (\Delta^2 k^{3-\alpha} * \Delta u)(n) + \Delta^2 k^{3-\alpha}(n+1)u(0) - \Delta^2 k^{3-\alpha}(0)u(n+1).$$

Now, replacing the above identity in (3.11) we have

$$\begin{aligned}\Delta^\alpha u(n) &= (\Delta^2 k^{3-\alpha} * \Delta u)(n) + \Delta^2 k^{3-\alpha}(n+1)u(0)\\ &\quad - \Delta^2 k^{3-\alpha}(0)u(n+1) + u(n+3) - \alpha u(n+2)\\ &\quad + \frac{\alpha(\alpha-1)}{2}u(n+1)\\ &= (\Delta^2 k^{3-\alpha} * \Delta u)(n) - \frac{(\alpha-1)(\alpha-2)}{2}u(n+1) + u(n+3)\\ &\quad - \alpha u(n+2) + \frac{\alpha(\alpha-1)}{2}u(n+1)\\ &= (\Delta^2 k^{3-\alpha} * \Delta u)(n) + u(n+3)\\ &\quad - \alpha u(n+2) + (\alpha-1)u(n+1)\\ &= \sum_{j=1}^{n}\Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \Delta^2 k^{3-\alpha}(0)\Delta u(n) + u(n+3)\\ &\quad - \alpha u(n+2) + (\alpha-1)u(n+1)\\ &= \sum_{j=1}^{n}\Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \frac{(\alpha-1)(\alpha-2)}{2}\\ &\quad \times [u(n+1) - u(n)] + u(n+3) - \alpha u(n+2) + (\alpha-1)u(n+1)\end{aligned}$$

$$= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + u(n+3) - \alpha u(n+2)$$
$$+ \left[\frac{(\alpha-1)(\alpha-2)}{2} + (\alpha-1) \right] u(n+1)$$
$$- \frac{(\alpha-1)(\alpha-2)}{2} u(n)$$
$$= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + u(n+3) - \alpha u(n+2) + \frac{\alpha(\alpha-1)}{2} u(n+1)$$
$$- \frac{(\alpha-1)(\alpha-2)}{2} u(n)$$
$$= \sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) + \frac{2^{n+3}-1}{2^{n+2}} - \alpha \frac{2^{n+2}-1}{2^{n+1}} + \frac{\alpha(\alpha-1)}{2} \frac{2^{n+1}-1}{2^n}$$
$$- \frac{(\alpha-1)(\alpha-2)}{2} \frac{2^n-1}{2^{n-1}}.$$

Note that by Lemma 2.2, part (2), and using the fact that $\Delta u(n) \geq 0$, we obtain $\sum_{j=1}^{n} \Delta^2 k^{3-\alpha}(j)\Delta u(n-j) \geq 0$. Thus, since $\alpha \in \left(\frac{4+\sqrt{2}}{2}, 3\right)$, from the above identity we obtain

$$\Delta^\alpha u(n) \geq \frac{2^{n+3}-1}{2^{n+2}} - \alpha \frac{2^{n+2}-1}{2^{n+1}} + \frac{\alpha(\alpha-1)}{2} \frac{2^{n+1}-1}{2^n} - \frac{(\alpha-1)(\alpha-2)}{2} \frac{2^n-1}{2^{n-1}}$$
$$= \frac{2\alpha^2 - 8\alpha + 7}{2^{n+2}} \geq 0.$$

This proves (i) as claimed.

Finally, we prove (v). Note that

$$\Delta^2 u(n) = \Delta u(n+1) - \Delta u(n) = -\frac{1}{2^{n+1}} \leq 0.$$

Therefore u is concave. Since u has positive α-jerk if and only if $(2-\alpha)\Delta^2 u(n) + \Delta^2 u(n+1) \geq 0$, we obtain after a computation

$$(2-\alpha)\Delta^2 u(n) + \Delta^2 u(n+1) = \frac{1}{2^{n+1}} \left[\alpha - \frac{3}{2} \right] \geq 0,$$

since $\alpha > \frac{4+\sqrt{2}}{2} > \frac{3}{2}$. This proves (v).

However, note that $u(2) = \frac{3}{2} < \alpha = \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$. It follows that the condition (2) in Theorem 3.4 is necessary in order to ensure the convexity of the sequence u.

From Theorem 3.4 and the transference principle (Theorem 2.6) we deduce the following corollary.

Corollary 3.6 *Let* $2 \leq \alpha \leq 3$, $a \in \mathbb{R}$ *and* $v \in s(\mathbb{N}_a; \mathbb{R})$ *be given and assume that*

1. $\Delta_a^\alpha v(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+3-\alpha}$;
2. $v(a+2) \geq \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2} v(a)$;
3. $v(a+1) \geq \alpha v(a)$;
4. $v(a) \geq 0$.

Then v is positive, increasing, convex and has positive α-jerk on $\mathbb{N}_a$.

Proof In case $\alpha = 2$ the conclusion is clear from the hypothesis. Define $u := \tau_a v$. Using the transference principle, we have,

$$\Delta^\alpha u(n) = \tau_{a+3-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u(n) = \tau_{a+3-\alpha} \circ \Delta_a^\alpha \circ v(n) = \Delta_a^\alpha v(t) \geq 0,$$

for each $t := n+a+3-\alpha \in \mathbb{N}_{a+3-\alpha}$. Moreover, $u(2) = v(a+2) \geq \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2} v(a) = \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$, $u(1) = v(a+1) \geq \alpha v(a) = \alpha u(0)$, and $u(0) = v(a) \geq 0$. The conclusion follows from Theorem 3.4. □

The following is our main result in case $3 \leq \alpha < 4$.

Theorem 3.7 *Let $3 \leq \alpha < 4$, $u \in s(\mathbb{N}_0; \mathbb{R})$ be given and assume that*

1. $\Delta^\alpha u(n) \geq 0$, *for all* $n \in \mathbb{N}_0$;
2. $u(3) \geq \alpha u(2) - \frac{\alpha(\alpha-1)}{2} u(1) + \frac{\alpha(\alpha-1)(\alpha-2)}{6} u(0)$;
3. $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2} u(0)$;
4. $u(1) \geq \alpha u(0)$;
5. $u(0) \geq 0$.

Then u is positive, increasing, convex and has positive jerk on $\mathbb{N}_0$.

Proof Observe that in the limit case $\alpha = 3$ all the given hypothesis and conclusions coincides with those of Theorem 3.4 (note that (1) equals (2) in such case) and therefore the proof follows as in such theorem. Suppose that $3 < \alpha < 4$. By Proposition 2.7, with $a := u$, $b := k^{4-\alpha}$ and $l = 4$, we obtain

$$(k^{4-\alpha} * \Delta^4 u)(n) = \Delta^\alpha u(n) - \sum_{j=1}^{4} \sum_{i=0}^{j-1} \binom{4}{j} (-1)^{4-j} u(i) k^{4-\alpha}(n+j-i). \tag{3.12}$$

Since $k^{4-\alpha}(n+j-i) = \tau_{j-i} k^{4-\alpha}(n)$, then convolving (3.12) with $k^{\alpha-3}$ we obtain

$$\begin{aligned}(k^{\alpha-3} * k^{4-\alpha} * \Delta^4 u)(n) = &(k^{\alpha-3} * \Delta^\alpha u)(n) \\ &- \sum_{j=1}^{4} \sum_{i=0}^{j-1} \binom{4}{j} (-1)^{4-j} u(i) (k^{\alpha-3} * \tau_{j-i} k^{4-\alpha})(n).\end{aligned} \tag{3.13}$$

By Lemma 2.5 and the semigroup property of the kernel k^γ, we get

$$\begin{aligned}(k^{\alpha-3} * \tau_{j-i} k^{4-\alpha})(n) &= (k^{\alpha-3} * k^{4-\alpha})(n+j-i) - \sum_{l=0}^{j-i-1} k^{\alpha-3}(n-l+j-i) k^{4-\alpha}(l) \\ &= 1 - \sum_{l=0}^{j-i-1} k^{\alpha-3}(n-l+j-i) k^{4-\alpha}(l).\end{aligned}$$

Therefore, replacing the above identity in (3.13), and since $(k^{\alpha-3} * k^{4-\alpha} * \Delta^4 u)(n) = (k * \Delta^4 u)(n) = \Delta^3 u(n+1) - \Delta^3 u(0)$, we obtain

$$\begin{aligned}\Delta^3 u(n+1) - \Delta^3 u(0) &= (k^{\alpha-3} * \Delta^\alpha u)(n) - \sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)\\ &\quad + \sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)\sum_{l=0}^{j-i-1} k^{\alpha-3}(n-l+j-i)k^{4-\alpha}(l).\end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned}\sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i) &= -4u(0) + 6[u(0)+u(1)] - 4[u(0)+u(1)+u(2)]\\ &\quad + [u(0)+u(1)+u(2)+u(3)]\\ &= \Delta^3 u(0).\end{aligned} \tag{3.15}$$

Also, since for any $\gamma > 0, k^\gamma(0) = 1, k^\gamma(1) = \gamma, k^\gamma(2) = \frac{\gamma(\gamma+1)}{2}$ and $k^\gamma(3) = \frac{\gamma(\gamma+1)(\gamma+2)}{6}$ we have

$$\begin{aligned}&\sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)\sum_{l=0}^{j-i-1} k^{\alpha-3}(n-l+j-i)k^{4-\alpha}(l)\\ &= -4u(0)k^{\alpha-3}(n+1) + 6\Big[u(0)\Big(k^{\alpha-3}(n+2) + k^{\alpha-3}(n+1)(4-\alpha)\Big)\\ &\quad + u(1)k^{\alpha-3}(n+1)\Big]\\ &\quad - 4\Big[u(0)\Big(k^{\alpha-3}(n+3) + k^{\alpha-3}(n+2)(4-\alpha) + k^{\alpha-3}(n+1)\frac{1}{2}(4-\alpha)(5-\alpha)\Big)\\ &\quad + u(1)\Big(k^{\alpha-3}(n+2) + k^{\alpha-3}(n+1)(4-\alpha)\Big) + u(2)k^{\alpha-3}(n+1)\Big]\\ &\quad + \Big[u(0)\Big(k^{\alpha-3}(n+4) + k^{\alpha-3}(n+3)(4-\alpha) + k^{\alpha-3}(n+2)\frac{1}{2}(4-\alpha)(5-\alpha)\\ &\quad + k^{\alpha-3}(n+1)\frac{1}{6}(4-\alpha)(5-\alpha)(6-\alpha)\Big) + u(1)\Big(k^{\alpha-3}(n+3) + k^{\alpha-3}(n+2)(4-\alpha)\\ &\quad + k^{\alpha-3}(n+1)\frac{1}{2}(4-\alpha)(5-\alpha)\Big) + u(2)\Big(k^{\alpha-3}(n+2) + k^{\alpha-3}(n+1)(4-\alpha)\Big)\\ &\quad + u(3)k^{\alpha-3}(n+1)\Big].\end{aligned} \tag{3.16}$$

Replacing (3.15) and (3.16) in (3.14) we obtain that for $n \in \mathbb{N}_0$,

$$\Delta^3 u(n+1) = (k^{\alpha-3} * \Delta^\alpha u)(n) + k^{\alpha-3}(n+4)u(0) + k^{\alpha-3}(n+3)[u(1) - \alpha u(0)]$$
$$+ k^{\alpha-3}(n+2)\left[u(2) - \alpha u(1) + \frac{\alpha(\alpha-1)}{2}u(0)\right]$$
$$+ k^{\alpha-3}(n+1)\left[u(3) - \alpha u(2) + \frac{\alpha(\alpha-1)}{2}u(1) - \frac{\alpha(\alpha-1)(\alpha-2)}{6}u(0)\right]. \tag{3.17}$$

Using the hypotheses (2), (3), (4) and (5) we conclude from (3.17) that $\Delta^3 u(n) \geq 0$, for all $n \in \mathbb{N}$.

We claim that $\Delta^3 u(0) \geq 0$. In fact, using (2), we have

$$u(3) - \alpha u(2) + \frac{\alpha(\alpha-1)}{2}u(1) - \frac{\alpha(\alpha-1)(\alpha-2)}{6}u(0)$$
$$= \Delta^3 u(0) - (\alpha-3)u(2) + \frac{(\alpha-3)(\alpha+2)}{2}u(1)$$
$$- \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0) \geq 0.$$

Note that, since $3 < \alpha < 4$, then $\alpha^2 - 6\alpha + 11 > 0$. Hence, hypotheses (3), (4) and (5) show that

$$\Delta^3 u(0) \geq (\alpha-3)u(2) - \frac{(\alpha-3)(\alpha+2)}{2}u(1) + \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0)$$
$$\geq (\alpha-3)\left[\alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)\right] - \frac{(\alpha-3)(\alpha+2)}{2}u(1)$$
$$+ \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0)$$
$$= \frac{(\alpha-2)(\alpha-3)}{2}u(1) - \frac{\alpha(\alpha-1)(\alpha-3)}{2}u(0) + \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0)$$
$$\geq \frac{(\alpha-2)(\alpha-3)}{2}\alpha u(0) - \frac{\alpha(\alpha-1)(\alpha-3)}{2}u(0) + \frac{(\alpha+1)(\alpha^2-4\alpha+6)}{6}u(0)$$
$$= \frac{\alpha^3 - 6\alpha^2 + 11\alpha + 6}{6}u(0) = \frac{\alpha(\alpha^2 - 6\alpha + 11) + 6}{6}u(0) \geq 0.$$

This proves the claim and that $\Delta^3 u(n) \geq 0$ for all $n \in \mathbb{N}_0$ – i.e., the sequence u has positive jerk.

Finally, by hypotheses we obtain $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0) \geq \alpha u(1) - \frac{(\alpha-1)}{2}u(1) = \frac{\alpha+1}{2}u(1) \geq 2u(1)$, then $\Delta^2 u(0) = u(2) - 2u(1) + u(0) \geq 0$. Moreover, $\Delta^2 u(n+1) \geq \Delta^2 u(n) \geq ... \geq \Delta^2 u(0)$, therefore u is convex on $\mathbb{N}_0$. Now, using the fact that u is convex on $\mathbb{N}_0$, we get $\Delta u(n+1) \geq \Delta u(n)$. By hypotheses (4) and (5) we also have $u(1) \geq u(0) \geq 0$, then $\Delta u(0) \geq 0$ and $\Delta u(n) \geq ... \geq \Delta u(0) \geq 0$. Hence, u is monotone increasing and positive on $\mathbb{N}_0$. □

The following example show that the condition (2) in Theorem 3.7 is necessary in order to ensure that the sequence u has positive jerk.

Example 3.8 Define the sequence $u : \mathbb{N}_0 \to \mathbb{R}$ by $u(n) = 2n + 2^{2-n} - 4$, and assume that $\frac{6+\sqrt{2}}{2} < \alpha < 4$. We have that the following collection of statements are true.

(i) $\Delta^\alpha u(n) \geq 0$, for all $n \in \mathbb{N}_0$;

(ii) $u(2) \geq \alpha u(1) - \frac{\alpha(\alpha-1)}{2}u(0)$;
(iii) $u(1) \geq \alpha u(0)$;
(iv) $u(0) \geq 0$;
(v) u is positive, monotone increasing and convex.

Indeed, first observe that $u(0) = u(1) = 0$ and $u(2) = 1$. This proves (ii), (iii) and (iv). Also, we have that u is positive. Since $\Delta u(n) = u(n+1) - u(n) = 2 - \frac{1}{2^{n-1}} \geq 0$ and $\Delta^2 u(n) = \Delta u(n+1) - \Delta u(n) = \frac{1}{2^n} \geq 0$, then u is monotone increasing and convex on $\mathbb{N}_0$. This proves (v).

We will prove (i). In fact, by Proposition 2.7, with $a := u$, $b := k^{4-\alpha}$ and $l = 4$, we obtain for each $n \in \mathbb{N}_0$

$$\begin{aligned}\Delta^\alpha u(n) &= (k^{4-\alpha} * \Delta^4 u)(n) + \sum_{j=1}^{4}\sum_{i=0}^{j-1}\binom{4}{j}(-1)^{4-j}u(i)k^{4-\alpha}(n+j-i) \\ &= (k^{4-\alpha} * \Delta^4 u)(n) + k^{4-\alpha}(n+2) - \frac{3}{2}k^{4-\alpha}(n+1).\end{aligned} \tag{3.18}$$

By Eq. (2.4), with $a := k^{4-\alpha}$ and $b := \Delta^3 u$, we have

$$(k^{4-\alpha} * \Delta^4 u)(n) = (\Delta k^{4-\alpha} * \Delta^3 u)(n) - k^{4-\alpha}(n+1)\Delta^3 u(0) + k^{4-\alpha}(0)\Delta^3 u(n+1).$$

Replacing the above identity in (3.18), we obtain

$$\begin{aligned}\Delta^\alpha u(n) &= (\Delta k^{4-\alpha} * \Delta^3 u)(n) - k^{4-\alpha}(n+1)\Delta^3 u(0) + k^{4-\alpha}(0)\Delta^3 u(n+1) + k^{4-\alpha}(n+2) \\ &\quad - \frac{3}{2}k^{4-\alpha}(n+1) \\ &= (\Delta k^{4-\alpha} * \Delta^3 u)(n) + \Delta k^{4-\alpha}(n+1) + \Delta^3 u(n+1).\end{aligned} \tag{3.19}$$

On the other hand, using again Eq. (2.4), with $a := \Delta k^{4-\alpha}$ and $b := \Delta^2 u$, we have

$$(\Delta k^{4-\alpha} * \Delta^3 u)(n) = (\Delta^2 k^{4-\alpha} * \Delta^2 u)(n) - \Delta k^{4-\alpha}(n+1)\Delta^2 u(0) + \Delta k^{4-\alpha}(0)\Delta^2 u(n+1).$$

Thus, replacing the above identity in (3.19), we have

$$\begin{aligned}\Delta^\alpha u(n) &= (\Delta^2 k^{4-\alpha} * \Delta^2 u)(n) - \Delta k^{4-\alpha}(n+1)\Delta^2 u(0) + \Delta k^{4-\alpha}(0)\Delta^2 u(n+1) \\ &\quad + \Delta k^{4-\alpha}(n+1) + \Delta^3 u(n+1) \\ &= (\Delta^2 k^{4-\alpha} * \Delta^2 u)(n) + (3-\alpha)\Delta^2 u(n+1) + \Delta^3 u(n+1) \\ &= \sum_{j=1}^{n}\Delta^2 k^{4-\alpha}(j)\Delta^2 u(n-j) + \Delta^2 k^{4-\alpha}(0)\Delta^2 u(n) \\ &\quad + (3-\alpha)\Delta^2 u(n+1) + \Delta^3 u(n+1) \\ &= \sum_{j=1}^{n}\Delta^2 k^{4-\alpha}(j)\Delta^2 u(n-j) + \frac{(3-\alpha)(2-\alpha)}{2}\Delta^2 u(n) \\ &\quad + (3-\alpha)\Delta^2 u(n+1) + \Delta^3 u(n+1).\end{aligned}$$

By Lemma 2.2, part (2), and $\Delta^2 u(n) \geq 0$, we have $\sum_{j=1}^{n} \Delta^2 k^{4-\alpha}(j)\Delta^2 u(n-j) \geq 0$. Thus, since $\alpha \in (\frac{6+\sqrt{2}}{2}, 4)$, and from the above, we have

$$\begin{aligned}\Delta^\alpha u(n) &\geq \frac{(3-\alpha)(2-\alpha)}{2}\Delta^2 u(n) + (3-\alpha)\Delta^2 u(n+1) + \Delta^3 u(n+1)\\ &= \frac{(3-\alpha)(2-\alpha)}{2}\frac{1}{2^n} + (3-\alpha)\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}\\ &= \frac{2\alpha^2 - 12\alpha + 17}{2^{n+2}} \geq 0.\end{aligned}$$

This proves (i).

However, u has negative jerk. In fact, a simple calculation show

$$\Delta^3 u(n) = \Delta^2 u(n+1) - \Delta^2 u(n) = -\frac{1}{2^{n+1}} \leq 0.$$

Finally, notice that $u(3) = \frac{5}{2} < \alpha = \alpha u(2) - \frac{\alpha(\alpha-1)}{2}u(1) + \frac{\alpha(\alpha-1)(\alpha-2)}{6}u(0)$. It follows that the condition (2) in Theorem 3.7 is necessary in order to ensure that the sequence u has positive jerk.

From Theorem 3.7 and the transference principle we deduce the following result.

Corollary 3.9 *Let* $3 \leq \alpha < 4$, $a \in \mathbb{R}$ *and* $v \in s(\mathbb{N}_a; \mathbb{R})$ *be given and assume that*

1. $\Delta_a^\alpha v(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+4-\alpha}$*;*
2. $v(a+3) \geq \alpha v(a+2) - \frac{\alpha(\alpha-1)}{2}v(a+1) + \frac{\alpha(\alpha-1)(\alpha-2)}{6}v(a)$*;*
3. $v(a+2) \geq \alpha v(a+1) - \frac{\alpha(\alpha-1)}{2}v(a)$*;*
4. $v(a+1) \geq \alpha v(a)$*;*
5. $v(a) \geq 0$.

Then v *is positive, monotone increasing, convex and has positive jerk on* $\mathbb{N}_a$.

Proof In case $\alpha = 3$ the conclusion is clear from the hypothesis. Define $u := \tau_a v$. Using the transference principle Theorem 2.6, we have,

$$\Delta^\alpha u(n) = \tau_{a+4-\alpha} \circ \Delta_a^\alpha \circ \tau_{-a} u(n) = \tau_{a+4-\alpha} \circ \Delta_a^\alpha \circ v(n) = \Delta_a^\alpha v(t) \geq 0,$$

for $t := n + a + 4 - \alpha \in \mathbb{N}_{a+3-\alpha}$. The conclusion follows from Theorem 3.7. □

We end this work, noting that further applications of the results and methods developed in this article are possible in related areas such as, for instance, the study of positive solutions for discrete nonlinear fractional boundary value problems [7].

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Declarations

Conflict of interest The authors declare that there is no conflicts of interest/competing interests.

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