mathematics

MDPI

Article

The Abstract Cauchy Problem with Caputo–Fabrizio Fractional Derivative

Jennifer Bravo [1] and Carlos Lizama [2,*]

[1] Departamento de Matemática y Estadística, Facultad de Ciencias de la Educación, Universidad San Sebastian, Bellavista 7, Santiago 8420524, Chile
[2] Departamento de Matemática y Ciencia de la Computación, Facultad de Ciencias, Universidad de Santiago de Chile, Las Sophoras 173, Estación Central, Santiago 9170124, Chile
* Correspondence: carlos.lizama@usach.cl

Abstract: Given an injective closed linear operator A defined in a Banach space X, and writing ${}_{CF}D_t^{\alpha}$ the Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$, we show that the unique solution of the abstract Cauchy problem $(*)$ ${}_{CF}D_t^{\alpha}u(t) = Au(t) + f(t), \;\; t \geq 0$, where f is continuously differentiable, is given by the unique solution of the first order abstract Cauchy problem $u'(t) = B_{\alpha}u(t) + F_{\alpha}(t), \;\; t \geq 0; u(0) = -A^{-1}f(0)$, where the family of bounded linear operators B_{α} constitutes a Yosida approximation of A and $F_{\alpha}(t) \to f(t)$ as $\alpha \to 1$. Moreover, if $\frac{1}{1-\alpha} \in \rho(A)$ and the spectrum of A is contained outside the closed disk of center and radius equal to $\frac{1}{2(1-\alpha)}$ then the solution of $(*)$ converges to zero as $t \to \infty$, in the norm of X, provided f and f' have exponential decay. Finally, assuming a Lipchitz-type condition on $f = f(t,x)$ (and its time-derivative) that depends on α, we prove the existence and uniqueness of mild solutions for the respective semilinear problem, for all initial conditions in the set $\mathcal{S} := \{x \in D(A) \, : \, x = A^{-1}f(0,x)\}$.

Keywords: Caputo–Fabrizio fractional derivative; Yosida approximation; stability; linear and semi-linear abstract Cauchy problem; one-parameter semigroups of operators

MSC: 47D06; 35R11; 35B35

check for updates

Citation: Bravo, J.; Lizama, C. The Abstract Cauchy Problem with Caputo–Fabrizio Fractional Derivative. *Mathematics* **2022**, *10*, 3540. https://doi.org/10.3390/math10193540

Academic Editor: Dongfang Li

Received: 14 August 2022
Accepted: 19 September 2022
Published: 28 September 2022

1. Introduction

In 2015, the authors Caputo and Fabrizio proposed a new concept of fractional derivative with a regular kernel [1]. This concept has proven to have valuable properties that make it very useful in various areas of science and engineering (see [2–13]).

For example, in [2], Abbas, Benchohra and Nieto provided sufficient conditions to ensure the existence of solutions for functional fractional differential equations with instantaneous impulses involving the Caputo–Fabrizio derivative. As methods, they used fixed point theory and measure of noncompactness. In [4], Baleanu, Jajarmi, Mohammadi and Rezapour proposed a new fractional model for the human liver involving the Caputo–Fabrizio derivative. In the paper, comparative results with real clinical data indicated the superiority of the new fractional model over the preexisting integer order model with ordinary time derivatives. A similar study carried out by the aforementioned authors, but for the Rubella disease model, was performed in reference [5], while in [6] the analysis was performed in terms of a differential equation model for COVID-19. In the paper [14], Baleanu, Sajjadi, Jajarmi and Defterli analyzed the complicated behaviors of a nonlinear suspension system in the framework of the Caputo–Fabrizio derivative. They showed that both the chaotic and nonchaotic behaviors of the considered system can be identified by the fractional order mathematical model. Very recently, in the reference [15], Kumar, Das and Ong analyzed tumor cells in the absence and presence of chemotherapeutic treatment by use of the Caputo–Fabrizio derivative. This is one of the few studies, together with the references [13,16], where the presence of partial differential equations with the Caputo–Fabrizio derivative over time was considered.

Although this notion of fractional derivative appears to be very auspicious in a variety of concrete applications, so far an unified analysis in the context of abstract partial differential equations, where there is a wider range of mathematical models, remains undeveloped. In this context, one of the basic problems to be studied corresponds to the so-called abstract Cauchy problem.

In this article, our concern is the study of existence, uniqueness and qualitative properties for the solutions of the abstract Cauchy problem

$$ {}_{CF}D_t^{\alpha} u(t) = Au(t) + f(t), \;\; t \geq 0, \tag{1}$$

and semilinear versions of it, i.e., where the term $f(\cdot)$ is replaced by $f(\cdot, u(\cdot))$. In Equation (1), A is a closed linear operator with domain $D(A)$ defined in a Banach space X and ${}_{CF}D_t^{\alpha}$ denotes the Caputo–Fabrizio fractional derivative of order $\alpha \in (0,1)$.

One of the motivations for this study is that, to our knowledge, similar work has not been done before in abstract spaces with Caputo-Fabrizio or other fractional derivatives that have *non-singular* kernels. Our goal is to clarify to what extent this type of fractional derivative offers advantages/disadvantages in this abstract scenario.

In the existing literature, the problem (1) has been studied when A is scalar or even a matrix, but when A is simply a closed linear operator, e.g., partial differential operators such as the Laplacian, the problem (1) remains unsolved.

In the border case $\alpha = 1$, it is well-known that solving the linear problem

$$ u'(t) = Au(t) + f(t), \;\; t \geq 0, \tag{2}$$

requires A as the generator of a C_0-semigroup.

In contrast, in this article we will show that this requirement is not necessary for the study of (1). Such an advantageous property occurs because we realize in this article that solving the problem (1) is equivalent to solving the following problem:

$$ \begin{cases} u'(t) = B_{\alpha} u(t) + F_{\alpha}(t), \;\; t \geq 0; \\ u(0) = -A^{-1} f(0), \end{cases} \tag{3}$$

where B_{α} are bounded linear operators that behave like a *Yosida approximation* of A, being $B_{\alpha} \to A$ and $F_{\alpha}(t) \to f(t)$ as $\alpha \to 1$, in an appropriate sense. In this way, some qualitative properties for (1) could be directly deduced from the corresponding ones of (3) with due care, given the special initial condition $u(0) = -A^{-1} f(0)$ that appears in our new context.

Once this key result is established, we study the important issue of stability. We show that under a simple condition, which depends on α, about the location of the spectrum of the operator A, and a decay condition on f, we can conclude that the unique solution u of the *nonhomogeneous* Equation (1) satisfies $\|u(t)\| \to 0$ as $t \to \infty$. A concrete example is shown that illustrates this asymptotic behavior and how the connection between (1) and (3) works.

Finally, if A is a closed linear operator, we show existence and uniqueness of mild solutions for the nonlinear equation

$$ {}_{CF}D_t^{\alpha} u(t) = Au(t) + f(t, u(t)), \;\; t \in [0, T], \quad T > 0, \tag{4}$$

under a Lipschitz type condition on f that also depends on α. In particular, assuming that A is densely defined, we realize that as $\alpha \to 1$, our result matches a classical result for Equation (2) stated in ([17], Theorem 6.1.2), where the condition for A to be the generator of a C_0 semigroup appears. Our studies reveal that this condition turns out to be natural thanks to the property $B_{\alpha} \to A$ as a Yosida approximation, mentioned before.

It should be noted that one of the keys that was taken into account to carry out this work is that the Caputo-Fabrizio fractional derivative has a non-singular kernel. Therefore, it is natural to ask—and we leave it as an open problem—in what extent the results of this article could be reproduced if the Caputo-Fabrizio derivative is replaced by another type of fractional derivatives with non-singular kernel. For example, there are fractional time derivatives by the use of Gaussian kernels [18] (Section 8), or Mittag-Leffler kernels [19], the last also known as the Atangana-Baleanu-Caputo derivative.

2. Preliminaries

In this section, we recall some preliminary results and definitions that will be used throughout the paper. Let X be a Banach space; by $\mathcal{B}(X)$ we denote the space of all bounded linear operators from X to X. If A is a closed linear operator in X, we denote by $D(A)$ the domain of A.

Definition 1 ([11], Definition 2). *Let $0 < \alpha < 1$ and $u : \mathbb{R}_+ \to X$ be a continuously differentiable function. The Caputo–Fabrizio fractional derivative of u of order α is given by:*

$$ {}_{CF}D_t^{\alpha}u(t) := \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)u'(s)ds, \quad t \geq 0. $$

We recall two important properties (see [1], Section 2):

(i) For $\alpha \to 1$ we have that

$$ \lim_{\alpha\to 1} {}_{CF}D_t^{\alpha}u(t) = u'(t). \tag{5} $$

(ii) We denote by $\mathscr{L}[u]$ the Laplace Transform of a function u. The Laplace Transform of the fractional operator ${}_{CF}D_t^{\alpha}$ with $0 < \alpha < 1$ is:

$$ \mathscr{L}[{}_{CF}D_t^{\alpha}u](\lambda) = \frac{\lambda\mathscr{L}[u](\lambda) - u(0)}{\lambda(1-\alpha)+\alpha}, \quad \lambda > 0. $$

Remark 1. *Note that the Caputo–Fabrizio fractional derivative has a non-singular kernel, namely, $\exp\left(-\frac{\alpha t}{1-\alpha}\right)$. This special feature, when compared with the classical Caputo or Riemann–Liouville fractional derivative that instead has the singular kernel $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $0 < \alpha < 1$, allows us to obtain distinguished properties of the non-local operator ${}_{CF}D_t^{\alpha}$. One of these properties, which is obvious but important in our analysis, is the following:*

$$ {}_{CF}D_t^{\alpha}u(0) = 0, \tag{6} $$

whenever $0 < \alpha < 1$. This behavior has been remarked by Diethelm, Garrapa, Giusti and Stynes [20], where the general issue of the use of regular kernels in the theory of fractional calculus is discussed.

For further use, we recall the following definition.

Definition 2 ([21], Definition II.4.1, [17], Section 2.2.5). *A closed linear operator A with dense domain $D(A)$ in a Banach space X is called sectorial (of angle δ) if there exists $0 < \delta < \pi/2$ such that the sector*

$$ \Sigma := \{\lambda \in \mathbb{C} : |\arg\lambda| < \pi/2 + \delta\} \cup \{0\} $$

is contained in the resolvent set $\rho(A)$, and if there exists $M \geq 1$ such that

$$ ||(\lambda - A)^{-1}|| \leq M/|\lambda| \ \ \textit{for} \ \ \lambda \in \Sigma, \lambda \neq 0. $$

3. Well-Posedness

Let X be a complex Banach space and $A : D(A) \subset X \to X$ be a closed linear operator on X. Given $0 < \alpha < 1$ and $f : [0, \infty) \to X$ be a function.In this section, we are concerned with the problem of existence of solutions to the equation

$$ {}_{CF}D_t^\alpha u(t) = Au(t) + f(t), \;\; t \geq 0. \tag{7} $$

Definition 3. *A function $u : [0, \infty) \to X$ is said to be a strong solution of (7) if u is continuously differentiable with $u(t) \in D(A)$, $t \geq 0$, and satisfies (7).*

Remark 2. *Observe that in Equation (7) when $t = 0$ we have ${}_{CF}D_t^\alpha u(0) = Au(0) + f(0)$, i.e., $Au(0) = -f(0)$. Therefore, the value $u(0)$ is implicitly prescribed although it is not given as an initial condition. This condition will be important to show that the solution is unique in the classical sense depending on the properties of the operator A.*

Let $0 < \alpha < 1$ be fixed. Assuming that $\frac{1}{1-\alpha} \in \rho(A)$, we define

$$ N_\alpha := (I - (1-\alpha)A)^{-1} \in \mathcal{B}(X). $$

Since $B_\alpha := \frac{\alpha}{1-\alpha}(N_\alpha - I)$ is a bounded operator, it defines the uniformly continuous group $(T_\alpha(t))_{t \in \mathbb{R}}$ on X, given by (see [21], Theorem I.3.7):

$$ T_\alpha(t) = \exp(tB_\alpha) = \sum_{k=0}^{\infty} \frac{t^k B_\alpha^k}{k!}, \; t \in \mathbb{R}. \tag{8} $$

Let $f : [0, \infty) \to X$ be continuously differentiable and define

$$ F_\alpha(t) := (1-\alpha) N_\alpha f'(t) + \alpha N_\alpha f(t). $$

Note that $F_\alpha(t) \in D(A)$ for all $t \in [0, \infty)$. For each $x_0 \in X$, we define:

$$ u(t) = T_\alpha(t)x_0 + \int_0^t T_\alpha(t-\tau) F_\alpha(\tau) d\tau, \; t \geq 0. \tag{9} $$

Then, it is well-known ([17], Section 4.2) that u is a strong solution of

$$ u'(t) = B_\alpha u(t) + F_\alpha(t), \;\; t \geq 0. \tag{10} $$

Observe from (9) that $x_0 = u(0)$. If $x_0 \in D(A)$ is such that $Ax_0 = -f(0)$ then u solves the following particular initial value problem

$$ \begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), \;\; t \geq 0; \\ Au(0) = -f(0). \end{cases} \tag{11} $$

Note that in the initial condition we are not yet assuming any conditions on the invertibility of A.

We recall the following definitions, applied to (11).

Definition 4. *Let $x_0 \in X$ be given. The function u defined by (9) is called a mild solution of the initial value problem (11).*

Definition 5. *A function $u : [0, \infty) \to X$ is a strong solution of (11) if u is continuously differentiable with $u(0) \in D(A)$ and satisfies (11).*

The following result is well-known, except for the new necessary condition imposed on the operator A.

Proposition 1. *Every strong solution of* (11) *is also a mild solution, and, if A is injective and $u(0) \in D(A)$ then every mild solution of* (11) *is also a strong solution.*

Proof. Suppose u is a strong solution of (11), then it is well-known that u is defined by

$$u(t) = T_\alpha(t)u(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,\ t \geq 0,$$

and is a mild solution (see [17], Section 4.2). Now, if A is injective, then u defined by

$$u(t) = -T_\alpha(t)A^{-1}f(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,,\ t \geq 0 \tag{12}$$

verifies (10) and, since $u(0) \in D(A)$, we have $Au(0) = -AT_\alpha(0)A^{-1}f(0) = -f(0)$. Hence, (12) is a strong solution of (11). □

Next, we collect some important properties of the operators previously defined.

Proposition 2. *Let $0 < \alpha < 1$, $\beta := \frac{\alpha}{1-\alpha}$, A be a closed linear operator on X with domain $D(A)$, $\frac{1}{1-\alpha} \in \rho(A)$ and $T_\alpha(t) = \exp(tB_\alpha)$, $t \in \mathbb{R}$, where $B_\alpha = \beta(N_\alpha - I)$ and $N_\alpha = (I-(1-\alpha)A)^{-1}$. The following statements hold:*

(i) $B_\alpha x = \alpha N_\alpha Ax$ and $AN_\alpha x = N_\alpha Ax$, for $x \in D(A)$;
(ii) $B_\alpha^k x \in D(A)$ and $B_\alpha^k Ax = AB_\alpha^k x$, for $x \in D(A)$ and $k \in \mathbb{N}_0$;
(iii) $T_\alpha(t)x \in D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$, for $t \in \mathbb{R}$ and $x \in D(A)$;
(iv) $B_\alpha^k N_\alpha x = N_\alpha B_\alpha^k x$, for $x \in X$ and $k \in \mathbb{N}_0$;
(v) $N_\alpha T_\alpha(t)x = T_\alpha(t)N_\alpha x$, for $t \in \mathbb{R}$ and $x \in X$.

Proof. (i) For $x \in D(A)$, we have $Ix = N_\alpha N_\alpha^{-1}x = N_\alpha x - (1-\alpha)N_\alpha Ax$ then $N_\alpha x - Ix = (1-\alpha)N_\alpha Ax$, obtaining the claim.

(ii) First, by proceeding by induction on k, we prove that $B_\alpha^k x \in D(A)$, for $x \in D(A)$. In fact, for $k = 0$ is trivial and for $k = 1$, by property (i), we have $B_\alpha x = \alpha N_\alpha Ax \in D(A)$, for all $x \in D(A)$. Suppose that for $k \in \mathbb{N}_0$, we have $B_\alpha^k x \in D(A)$, for all $x \in D(A)$. Then, for $k+1$ and $x \in D(A)$, we obtain $B_\alpha^{k+1}x = B_\alpha^k(B_\alpha x) \in D(A)$.
Now, again by proceeding by induction on k, we prove that $B_\alpha^k Ax = AB_\alpha^k x$, for $x \in D(A)$. Indeed, for $k = 0$ is trivial and for $k = 1$, by property (i) and the above case, we have for $x \in D(A)$

$$B_\alpha Ax = \beta(N_\alpha - I)Ax = \beta[N_\alpha Ax - Ax] = \beta[AN_\alpha x - Ax] = A[\beta(N_\alpha - I)]x = AB_\alpha x.$$

Suppose that for $k \in \mathbb{N}_0$, we have $B_\alpha^k Ax = AB_\alpha^k x$, for all $x \in D(A)$. Then, by property (i), we obtain for $k+1$ and $x \in D(A)$:

$$B_\alpha^{k+1}Ax = B_\alpha^k(B_\alpha A)x = B_\alpha^k(AB_\alpha)x = (B_\alpha^k A)B_\alpha x = (AB_\alpha^k)B_\alpha x = AB_\alpha^{k+1}x.$$

(iii) Let $n \in \mathbb{N}_0$ and $x \in D(A)$, define $T_{\alpha,n}(t)x := \sum_{k=0}^n \frac{t^k B_\alpha^k}{k!}x$, $t \in \mathbb{R}$. By property (ii), we have $T_{\alpha,n}(t)x \in D(A)$, for $n \in \mathbb{N}_0$, and

$$\lim_{n\to\infty} T_{\alpha,n}(t)x = T_\alpha(t)x,\ t \in \mathbb{R}.$$

Note that by property (ii), we obtain for $x \in D(A)$

$$AT_{\alpha,n}(t)x = \sum_{k=0}^n \frac{t^k AB_\alpha^k}{k!}x = \sum_{k=0}^n \frac{t^k B_\alpha^k A}{k!}x = T_{\alpha,n}(t)Ax,\ t \in \mathbb{R}.$$

Hence, by the above identities, we have for $x \in D(A)$

$$\lim_{n\to\infty} AT_{\alpha,n}(t)x = \lim_{n\to\infty} T_{\alpha,n}(t)Ax = T_\alpha(t)Ax,\ t\in\mathbb{R}.$$

Thus, since A is a closed operator, for $x\in D(A)$ we obtain $T_\alpha(t)x\in D(A)$ and

$$AT_\alpha(t)x = T_\alpha(t)Ax,\ t\in\mathbb{R}.$$

(iv) By proceeding by induction on k. For $k=0$ is trivial and for $k=1$, we have for $x\in X$: $B_\alpha N_\alpha x = \beta(N_\alpha - I)N_\alpha x = \beta[N_\alpha^2 x - N_\alpha x] = N_\alpha[\beta(N_\alpha - I)]x = N_\alpha B_\alpha x$.
Suppose that for $k\in\mathbb{N}_0$, we have $B_\alpha^k N_\alpha x = N_\alpha B_\alpha^k x$, for all $x\in X$. Then, we obtain for $k+1$ and $x\in X$:

$$B_\alpha^{k+1}N_\alpha x = B_\alpha^k(B_\alpha N_\alpha)x = B_\alpha^k(N_\alpha B_\alpha)x = (B_\alpha^k N_\alpha)B_\alpha x = (N_\alpha B_\alpha^k)B_\alpha x = N_\alpha B_\alpha^{k+1}x.$$

(v) We define $T_{\alpha,n}(t)x := \sum_{k=0}^{n}\frac{t^k B_\alpha^k}{k!}x,\ t\in\mathbb{R}$, for $n\in\mathbb{N}_0$ and $x\in X$. Since N_α is a bounded operator, we have for $x\in X$

$$N_\alpha T_\alpha(t)x = N_\alpha \lim_{n\to\infty} T_{\alpha,n}(t)x = \lim_{n\to\infty} N_\alpha T_{\alpha,n}(t)x,\ t\in\mathbb{R}.$$

By property (iv), we have for $x\in X$

$$N_\alpha T_{\alpha,n}(t)x = \sum_{k=0}^{n}\frac{t^k N_\alpha B_\alpha^k}{k!}x = \sum_{k=0}^{n}\frac{t^k B_\alpha^k N_\alpha}{k!}x = T_{\alpha,n}(t)N_\alpha x,\ t\in\mathbb{R}.$$

Thus, $N_\alpha T_\alpha(t)x = T_\alpha(t)N_\alpha x$, $t\in\mathbb{R}$, for all $x\in X$. □

Remark 3. *Let $0<\alpha<1$ and A be a closed linear operator on X with $\frac{1}{1-\alpha}\in\rho(A)$. First note that by Proposition 2 part (i), we have*

$$B_\alpha x = \alpha A N_\alpha x = \alpha A(I-(1-\alpha)A)^{-1}x = \frac{\alpha}{1-\alpha}A(\frac{1}{1-\alpha}I - A)^{-1}x,\ \textit{for all}\ \ x\in D(A).$$

Thus, by the above identity, we obtain

$$\lim_{\alpha\to 1} B_\alpha x = Ax,\ \ \textit{for all}\ \ x\in D(A). \tag{13}$$

Now, let $s := \frac{1}{1-\alpha}$, we define for every $s>1$

$$B_s := (s-1)A(sI-A)^{-1} = (1-s)[s(sI-A)^{-1} - I].$$

Note that if A is a densely defined operator on X then we deduce the following: B_s is a Yosida approximation of A ([17], Theorem 1.3.1). Moreover, since each B_s is bounded, it generates a uniformly continuous semigroup $(T_s(t))_{t>0}$ on X. Then, there exists $M\geq 1$ such that

$$||T_s(t)||_{\mathcal{B}(X)} \leq M\ \ \textit{for all}\ \ t\geq 0. \tag{14}$$

Observe that for $\alpha\to 1$ we have $s\to\infty$. Thus, by (13), we obtain

$$\lim_{s\to\infty} B_s x = Ax,\ \ \textit{for all}\ \ x\in D(A). \tag{15}$$

Since $\frac{1}{1-\alpha}\in\rho(A)$, we have for $\alpha\to 1$ that

$$(\omega,\infty)\subset\rho(A),\ \ \omega>1. \tag{16}$$

Therefore, by ([22], Corollary 3.6.3), (14)–(16), we have that A generates a C_0-semigroup T and for all $x\in X$,

$$\lim_{s\to\infty} T_s(t)x = T(t)x$$

uniformly for $t \in [0, \tau]$ *for all* $\tau > 0$.

The following is the main result of this section, and one of the main theorems of this paper: We show that (7) is well-posed if and only if (11) is well-posed.

Theorem 1. *Let* $0 < \alpha < 1$*,* A *be a closed linear operator on* X *with domain* $D(A)$ *and* $f : [0, \infty) \to X$ *continuously differentiable. Assume that* $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$*. Then, the problem given by*

$$ {}_{CF}D_t^{\alpha} v(t) = Av(t) + f(t), \ \ t \geq 0, \tag{17} $$

has a unique strong solution if and only if the initial value problem given by

$$ \begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), \ \ t \geq 0; \\ u(0) = -A^{-1} f(0), \end{cases} \tag{18} $$

has a unique strong solution, where $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ *and* $F_\alpha(t) = (1-\alpha)N_\alpha f'(t) + \alpha N_\alpha f(t)$ *with* $N_\alpha = (I - (1-\alpha)A)^{-1}$.

Proof. Suppose that v is the unique strong solution of (18). Then, v is continuously differentiable with $v(0) \in D(A)$ and satisfies (18). In particular, v is a mild solution of (18). Thus, by Definition 4

$$ v(t) = T_\alpha(t)v(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \ t \geq 0, \tag{19} $$

where $(T_\alpha(t))_{t\in\mathbb{R}}$ is the uniformly continuous group generated by B_α.

We first observe that $v(t) \in D(A)$, $t \geq 0$. Indeed, by hypothesis, $v(0) \in D(A)$ and by Proposition 2, part (iii), we have $T_\alpha(t)v(0) \in D(A)$. Again by Proposition 2, part (iii), we obtain that $\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau \in D(A)$, because $F_\alpha(t) \in D(A)$ and A is a closed operator. This proves that

$$ v(t) \in D(A), \ \ t \geq 0. $$

Next, we observe some identities that v verifies. Since v is a strong solution, we have $v(0) \in D(A)$ and $v(0) = -A^{-1}f(0)$, i.e., $Av(0) = -f(0)$. Then, by Proposition 2, part (iii), we have

$$ AT_\alpha(t)v(0) = T_\alpha(t)Av(0) = -T_\alpha(t)f(0). $$

Thus, applying A to the identity (19) and using the above identity, we obtain that v verifies

$$ Av(t) = -T_\alpha(t)f(0) + A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \ \ t \geq 0. \tag{20} $$

Note that by Proposition 2, part (i), we have

$$ N_\alpha A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau = AN_\alpha\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \ t \geq 0, \tag{21} $$

since that $\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau \in D(A)$ for all $t \geq 0$. Then, operating by αN_α the identity (20) and using the identity (21), we have

$$ \alpha N_\alpha Av(t) = -\alpha N_\alpha T_\alpha(t)f(0) + \alpha AN_\alpha\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \ t \geq 0. $$

By Proposition 2, part (i), and the property of group, we have that the previous identity is equivalent to

$$ B_\alpha v(t) = -\alpha N_\alpha T_\alpha(t)f(0) + \alpha AN_\alpha\int_0^t T_\alpha(t)T_\alpha(-\tau)F_\alpha(\tau)d\tau, \ t \geq 0. \tag{22} $$

After these preliminaries, we will show that v satisfies (17). By definition of the Caputo–Fabrizio fractional derivative and the fact that v satisfies Equation (18), we have

$$
\begin{aligned}
{}_{CF}D_t^{\alpha}v(t) &= \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)v'(s)ds \\
&= \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)[B_\alpha v(s)+F_\alpha(s)]ds \\
&= \frac{\exp(-\frac{\alpha t}{1-\alpha})}{1-\alpha}\left[\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)B_\alpha v(s)ds + \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)F_\alpha(s)ds\right] \\
&= \frac{\exp(-\frac{\alpha t}{1-\alpha})}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)B_\alpha v(s)ds \\
&\quad + \frac{\exp(-\frac{\alpha t}{1-\alpha})}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)F_\alpha(s)ds.
\end{aligned} \tag{23}
$$

In the above identity, we will find an equivalent representation of the following expression:

$$
\frac{\exp(-\frac{\alpha t}{1-\alpha})}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)B_\alpha v(s)ds. \tag{24}
$$

Replacing the identity (22) in $\int_0^t \exp(\frac{\alpha s}{1-\alpha})B_\alpha v(s)ds$, we obtain

$$
\begin{aligned}
\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)B_\alpha v(s)ds = {}&-\alpha N_\alpha \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)T_\alpha(s)f(0)ds \\
&+ \alpha A N_\alpha \int_0^t\int_0^s \exp\left(\frac{\alpha s}{1-\alpha}\right)T_\alpha(s)T_\alpha(-\tau)F_\alpha(\tau)d\tau ds.
\end{aligned} \tag{25}
$$

Observe that $(\exp(\frac{\alpha t}{1-\alpha})T_\alpha(t))_{t\in\mathbb{R}}$ is a group whose generator is $\frac{\alpha}{1-\alpha}I + B_\alpha = \frac{\alpha}{1-\alpha}N_\alpha$. Hence, by ([21], Chapter II), we have

$$
\alpha N_\alpha \int_\tau^t \exp\left(\frac{\alpha s}{1-\alpha}\right)T_\alpha(s)f(0)ds = (1-\alpha)[\exp\left(\frac{\alpha t}{1-\alpha}\right)T_\alpha(t) - \exp\left(\frac{\alpha\tau}{1-\alpha}\right)T_\alpha(\tau)]f(0). \tag{26}
$$

Thus, using the identity (26) in (25), we obtain

$$
\begin{aligned}
\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)B_\alpha v(s)ds = {}&(1-\alpha)f(0) - (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right)T_\alpha(t)f(0) \\
&+ \alpha A N_\alpha \int_0^t\int_0^s \exp\left(\frac{\alpha s}{1-\alpha}\right)T_\alpha(s)T_\alpha(-\tau)F_\alpha(\tau)d\tau ds.
\end{aligned} \tag{27}
$$

Note that, using Fubini's theorem, Proposition 2, part (v), and the identity (26), we obtain that

$$
\begin{aligned}
&\alpha A N_\alpha \int_0^t\int_0^s \exp\left(\frac{\alpha s}{1-\alpha}\right)T_\alpha(s)T_\alpha(-\tau)F_\alpha(\tau)d\tau ds \\
&= \alpha A N_\alpha \int_0^t\int_\tau^t \exp\left(\frac{\alpha s}{1-\alpha}\right)T_\alpha(s)T_\alpha(-\tau)F_\alpha(\tau)ds d\tau \\
&= A\int_0^t T_\alpha(-\tau)\alpha N_\alpha \int_\tau^t \exp\left(\frac{\alpha s}{1-\alpha}\right)T_\alpha(s)F_\alpha(\tau)ds d\tau \\
&= A\int_0^t T_\alpha(-\tau)(1-\alpha)\left[\exp\left(\frac{\alpha t}{1-\alpha}\right)T_\alpha(t) - \exp\left(\frac{\alpha\tau}{1-\alpha}\right)T_\alpha(\tau)\right]F_\alpha(\tau)d\tau \\
&= (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right)A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau - (1-\alpha)A\int_0^t \exp\left(\frac{\alpha\tau}{1-\alpha}\right)F_\alpha(\tau)d\tau.
\end{aligned} \tag{28}
$$

Thus, replacing (28) in (27) and using the identity (20), we obtain

$$
\begin{aligned}
&\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) B_\alpha v(s)ds \\
&= (1-\alpha)f(0) - (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) T_\alpha(t) f(0) \\
&\quad + (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) A \int_0^t T_\alpha(t-\tau) F_\alpha(\tau) d\tau - (1-\alpha) A \int_0^t \exp\left(\frac{\alpha \tau}{1-\alpha}\right) F_\alpha(\tau) d\tau \\
&= (1-\alpha)f(0) + (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right)\left[-T_\alpha(t)f(0) + A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau\right] \\
&\quad - (1-\alpha) A \int_0^t \exp\left(\frac{\alpha \tau}{1-\alpha}\right) F_\alpha(\tau) d\tau \\
&= (1-\alpha)f(0) + (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) Av(t) - (1-\alpha) A \int_0^t \exp\left(\frac{\alpha \tau}{1-\alpha}\right) F_\alpha(\tau) d\tau.
\end{aligned}
\tag{29}
$$

Therefore, multiplying by $\frac{1}{1-\alpha}\exp\left(-\frac{\alpha t}{1-\alpha}\right)$ the identity (29), we finally obtain (24):

$$
\begin{aligned}
&\frac{\exp\left(-\frac{\alpha t}{1-\alpha}\right)}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) B_\alpha v(s)ds \\
&= \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) + Av(t) - \exp\left(\frac{-\alpha t}{1-\alpha}\right) A \int_0^t \exp\left(\frac{\alpha \tau}{1-\alpha}\right) F_\alpha(\tau) d\tau.
\end{aligned}
\tag{30}
$$

This gives us the desired representation.

We return to the identity (23). Replacing (30) in (23), we obtain

$$
\begin{aligned}
{}_{CF}D_t^\alpha v(t) &= \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) + Av(t) - \exp\left(\frac{-\alpha t}{1-\alpha}\right) A \int_0^t \exp\left(\frac{\alpha \tau}{1-\alpha}\right) F_\alpha(\tau) d\tau \\
&\quad + \frac{\exp\left(\frac{-\alpha t}{1-\alpha}\right)}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) F_\alpha(s) ds \\
&= Av(t) + \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) \\
&\quad + \left[-\exp\left(\frac{-\alpha t}{1-\alpha}\right) A + \frac{\exp\left(\frac{-\alpha t}{1-\alpha}\right)}{1-\alpha} I\right] \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) F_\alpha(s) ds.
\end{aligned}
\tag{31}
$$

Now, we calculate $\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) F_\alpha(s)ds$. By definition of F_α and integration by parts, we have

$$
\begin{aligned}
&\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) F_\alpha(s) ds \\
&= \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right)[(1-\alpha)N_\alpha f'(s) + \alpha N_\alpha f(s)]ds \\
&= (1-\alpha)N_\alpha \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) f'(s)ds + \alpha N_\alpha \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) f(s)ds \\
&= (1-\alpha)N_\alpha\left[\exp\left(\frac{\alpha t}{1-\alpha}\right) f(t) - f(0) - \frac{\alpha}{1-\alpha}\int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) f(s)ds\right] \\
&\quad + \alpha N_\alpha \int_0^t \exp\left(\frac{\alpha s}{1-\alpha}\right) f(s)ds \\
&= (1-\alpha)N_\alpha\left[\exp\left(\frac{\alpha t}{1-\alpha}\right) f(t) - f(0)\right] \\
&= (1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) N_\alpha f(t) - (1-\alpha)N_\alpha f(0).
\end{aligned}
\tag{32}
$$

Thus, replacing (32) in (31) and using the identity $I = N_\alpha - (1-\alpha)AN_\alpha$, we obtain

$$
\begin{aligned}
&{}_{CF}D_t^\alpha v(t) \\
&= Av(t) + \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) \\
&\quad + \left[-\exp\left(\frac{-\alpha t}{1-\alpha}\right) A + \frac{\exp\left(\frac{-\alpha t}{1-\alpha}\right)}{1-\alpha} I \right] \left[(1-\alpha)\exp\left(\frac{\alpha t}{1-\alpha}\right) N_\alpha f(t) - (1-\alpha) N_\alpha f(0) \right] \\
&= Av(t) + \exp\left(\frac{-\alpha t}{1-\alpha}\right) f(0) - (1-\alpha) A N_\alpha f(t) \\
&\quad + (1-\alpha)\exp\left(\frac{-\alpha t}{1-\alpha}\right) A N_\alpha f(0) + N_\alpha f(t) - \exp\left(\frac{-\alpha t}{1-\alpha}\right) N_\alpha f(0) \\
&= Av(t) + [N_\alpha - (1-\alpha)AN_\alpha] f(t) + \exp\left(\frac{-\alpha t}{1-\alpha}\right) [I + (1-\alpha)AN_\alpha - N_\alpha] f(0) \\
&= Av(t) + f(t).
\end{aligned}
$$

The above shows that v is a strong solution of (17) and, by (19):

$$v(t) = T_\alpha(t)v(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau. \tag{33}$$

Finally, we show uniqueness. Assume that w is a strong solution of (17) and set $s := v - w$. Then, by linearity of the operator ${}_{CF}D_t^\alpha$, we have that s is a strong solution of the equation

$${}_{CF}D_t^\alpha s(t) = As(t),\ t \geq 0. \tag{34}$$

Since ${}_{CF}D_t^\alpha s(0) = 0$, then $As(0) = 0$. Thus, $s(0) = 0$ because A is injective. Using the identity (33) for the problem (34) ($f \equiv 0$), we have $s(t) = T_\alpha(t)s(0) = 0$. Hence, $v(t) = w(t)$, $t \geq 0$. This proves the first part of the theorem.

Conversely, assume that v is the unique strong solution of (17). Then, v is continuously differentiable with $v(t) \in D(A)$, $t \geq 0$, and satisfies (17), i.e.,

$${}_{CF}D_t^\alpha v(t) = Av(t) + f(t),\ \ t \geq 0. \tag{35}$$

Note that since B_α is a bounded operator and f is continuously differentiable, then we can define the function u by

$$u(t) := T_\alpha(t)v(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,\ \ t \geq 0, \tag{36}$$

where $(T_\alpha(t))_{t\in\mathbb{R}}$ is the group generated by B_α. By identities (35) and (36), we have that

$$u(0) = v(0) \in D(A). \tag{37}$$

We claim that the function u defined by (36) is continuously differentiable and satisfies (18). In fact, it is clear that it is continuously differentiable because $(T_\alpha(t))_{t\in\mathbb{R}}$ is a uniformly continuous group. We will check that it satisfies (18). First, note that by ([21], Section VI.7) it clearly satisfies the identity

$$u'(t) = B_\alpha u(t) + F_\alpha(t),\ \ t\ \geq 0.$$

It remains to check that it satisfies the initial condition. In fact, using (35) and (37), we have $Au(0) = Av(0) = {}_{CF}D_t^\alpha v(0) - f(0) = -f(0)$ since ${}_{CF}D_t^\alpha v(0) = 0$ by Remark 1. Thus, $u(0) = -A^{-1}f(0)$ because A is injective. This proves the claim. Therefore, u is a strong solution of (18).

We show uniqueness. Assume that w is strong solution of (18) and set $s := u - w$. Then, $s(0) \in D(A)$ and is a strong solution of the problem

$$s'(t) = B_\alpha s(t),\ t \geq 0.$$

It is well-known that $s(t) = T_\alpha(t)s(0)$, $t \geq 0$, with $T_\alpha(t)$ the group generated by B_α. Note that $s(0) = u(0) - w(0) = 0$, then $s(t) \equiv 0$, $t \geq 0$. Hence, $u(t) = w(t)$, $t \geq 0$. □

Remark 4. *Examining the previous proof, we deduce that if A is a non-injective operator, then $\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau \in D(A)$ and that a strong solution v of (17) (and of (18)) verifies $Av(0) = -f(0)$ and*

$$Av(t) = T_\alpha(t)Av(0) + A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,\ \ t \geq 0,$$

where $(T_\alpha(t))_{t\in\mathbb{R}}$ is a uniformly continuous group generated by $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ and $F_\alpha(t) = (1-\alpha)N_\alpha f'(t) + \alpha N_\alpha f(t)$.

Remark 5. *Note that Theorem 1 does not assume A to be the generator of any one-parameter family of operators, or A to be densely defined, in contrast with the limit case $\alpha = 1$ that requires A to be the generator of a C_0-semigroup. This reveals an important advantage of the fractional abstract Cauchy problem (1) when compared with the abstract Cauchy problem (2). However, we have a restriction over the spectrum, namely: $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$. In conclusion, although the Cauchy problem with the Caputo–Fabrizio fractional derivative can always be theoretically reduced to a first order abstract Cauchy problem, the first could be much more flexible when dealing with applications. This is probably the reason why problems with the Caputo–Fabrizio derivative find many applications in the real world.*

Remark 6. *If $f \equiv 0$ in Theorem 1 since $u(0) = -A^{-1}f(0) = 0$, then the unique strong solution u of (17) (and of (18)) is $u \equiv 0$.*

Remark 7. *If f is not zero in Theorem 1, then the unique strong solution u of (17) (and of (18)) is*

$$u(t) = T_\alpha(t)A^{-1}f(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau,\ \ t \geq 0.$$

In order to avoid the hypothesis of injectivity, we now introduce the following definitions.

Definition 6. *Let A be a closed linear operator. A function $u : [0,\infty) \to X$ is called an A-unique strong solution of (7)–(11) if and only if any strong solution v of (7)–(11) satisfies that $Au(t) = Av(t)$, $t \geq 0$.*

Remark 8. *Observe that if A is an injective operator, then an A-unique strong solution is unique in the classical sense.*

With the above preliminaries, we show the following corollary of Theorem 1.

Corollary 1. *Let $0 < \alpha < 1$, A be a closed linear operator on X with domain $D(A)$ and $f : [0,\infty) \to X$ continuously differentiable. Assume that $\frac{1}{1-\alpha} \in \rho(A)$. Then, the problem given by*

$$_{CF}D_t^\alpha v(t) = Av(t) + f(t),\ \ t \geq 0, \tag{38}$$

has an A-unique strong solution if and only if the initial value problem given by

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), & t \geq 0; \\ Au(0) = -f(0), \end{cases} \tag{39}$$

has an A-unique strong solution, where $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ *and* $F_\alpha(t) = (1-\alpha)N_\alpha f'(t) + \alpha N_\alpha f(t)$ *with* $N_\alpha = (I-(1-\alpha)A)^{-1} \in \mathcal{B}(X)$.

Proof. The proof of the existence of strong solutions for problems (38) and (39) is the same as Theorem 1. We show A-uniqueness. Let v be a strong solution of (38). By Remark 4 we have that v verifies

$$Av(t) = T_\alpha(t)Av(0) + A\int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \ \ t \geq 0. \tag{40}$$

Assume that w is another strong solution of (38) and set $s := v - w$, then we have that s is a strong solution of the following

$${}_{CF}D_t^\alpha s(t) = As(t), \ t \geq 0. \tag{41}$$

Since ${}_{CF}D_t^\alpha s(0) = 0$, then $As(0) = 0$. Using identity (40) for problem (41) ($f \equiv 0$), we have $As(t) = T_\alpha(t)As(0) = 0$. Hence, $Av(t) = Aw(t)$, $t \geq 0$.

Now, let u be a strong solution of (39) and assume that w is another strong solution of (39). We set $s := u - w$, then $s(0) \in D(A)$ and is a strong solution of the following

$$s'(t) = B_\alpha s(t), \ t \geq 0, \ \text{ and } \ As(0) = 0.$$

Therefore, we obtain$s(t) = T_\alpha(t)s(0)$, $t \geq 0$, with $T_\alpha(t)$ the group generated by B_α. By Proposition 2, part (iii), we obtain $s(t) = T_\alpha(t)s(0) \in D(A)$, $t \geq 0$, and $As(t) = AT_\alpha(t)s(0) = T_\alpha(t)As(0) = 0$, since $s(0) \in D(A)$ and $As(0) = 0$. Hence, $Au(t) = Aw(t)$, $t \geq 0$. □

Remark 9. *If $f \equiv 0$ in Corollary 1, then the A-unique strong solution u of* (17) *(and of* (18)*) is not zero for an initial condition in the kernel of A.*

Next, we present an immediate consequence of the above results.

Corollary 2. *Let $0 < \alpha < 1$, A be a sectorial operator on X of angle $\delta \in (0, \pi/2)$ with domain $D(A)$ and $f : [0, \infty) \to X$ continuously differentiable. Then, the problem given by*

$${}_{CF}D_t^\alpha v(t) = Av(t) + f(t), \ \ t \geq 0, \tag{42}$$

has a unique strong solution if and only if the initial value problem given by

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t), \ \ t \geq 0; \\ u(0) = -A^{-1}f(0), \end{cases} \tag{43}$$

has a unique strong solution.

Remark 10. *By ([17], Corollary II.4.7), if A is a normal operator on a Hilbert space H satisfying*

$$\sigma(A) \subset \{z \in \mathbb{C} : \arg(-z) < \delta\}$$

for some $\delta \in [0, \pi/2)$, then A generates a bounded analytic semigroup, and hence A is sectorial. Therefore, Corollary 2 applies.

4. Stability

The stability of the fractional order linear systems has been studied for many years, and powerful criteria have been proposed. The best known one is Matignon's stability theorem [23], and it is the starting point for several useful and important results in the field. The stability of the linear fractional order systems described by the Caputo–Fabrizio derivative has recently been studied in reference [10], where the authors gave necessary and sufficient conditions for the stability of the solutions of the problem

$$_{CF}D_t^{\alpha}u(t) = Au(t), \ \ t \geq 0,$$

where A *is a matrix*. In what follows, we will extend the results of [10] to the case of closed linear operators A.

After recalling some spectral properties, we study the asymptotic behavior of the solutions for the problem

$$_{CF}D_t^{\alpha}u(t) = Au(t) + f(t), \ \ t \geq 0, \tag{44}$$

where A is a closed injective linear operator.

Remark 11. *We recall that the Spectral Mapping Theorem for the resolvent operator ([21], Theorem IV.1.13) and for polynomials ([24], Proposition A.6.2, [25], Theorem VII.9.10) says that given $A : D(A) \subset X \to X$, a closed operator with nonempty resolvent set $\rho(A)$, we have*

(i) $\sigma((\beta - A)^{-1}) \setminus \{0\} = (\beta - \sigma(A))^{-1}$ *for each* $\beta \in \rho(A)$.
(ii) $\sigma(q(A)) = q(\sigma(A))$ *for each polynomial* $q \in \mathbb{C}[z]$.

Using the previous spectral properties, we can prove the following result.

Proposition 3. *Let $0 < \alpha < 1$ and A be a closed operator on X. Assume that $\frac{1}{1-\alpha} \in \rho(A)$. Let $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$, where $N_\alpha = (I - (1-\alpha)A)^{-1}$, then the following identity holds*

$$\sigma(B_\alpha) = \frac{\alpha}{1-\alpha}\left[\frac{1}{1-(1-\alpha)\sigma(A)} - 1\right].$$

Proof. By Remark 11, we have

$$\sigma(B_\alpha) = \sigma\left(\frac{\alpha}{1-\alpha}(N_\alpha - I)\right) = \frac{\alpha}{1-\alpha}\sigma(N_\alpha - I) = \frac{\alpha}{1-\alpha}[\sigma(N_\alpha) - 1].$$

Thus, by definition of N_α, we obtain

$$\sigma(B_\alpha) = \frac{\alpha}{1-\alpha}[\sigma((I-(1-\alpha)A)^{-1}) - 1] = \frac{\alpha}{1-\alpha}\left[\frac{1}{1-(1-\alpha)\sigma(A)} - 1\right],$$

and we obtain the claim. □

Remark 12. *Note that by Proposition 3, we obtain*

$$\sigma(A) = \frac{1}{1-\alpha}I - \frac{1}{1-\alpha}\frac{\alpha}{(1-\alpha)\sigma(B_\alpha) + \alpha}.$$

We recall that a semigroup $(T(t))_{t>0}$ on a Banach space X is called uniformly exponentially stable if there exist constants $\omega > 0$, $M \geq 1$ such that

$$||T(t)||_{\mathcal{B}(X)} \leq Me^{-\omega t} \ \text{ for all } \ t \geq 0.$$

Remark 13. *Let $0 < \alpha < 1$. Since $(T_\alpha(t))_{t>0}$ is the uniformly continuous semigroup generated by B_α, by ([21], Proposition I.3.12 and Theorem I.3.14), the following assertions are equivalent*

(i) $(T_\alpha(t))_{t>0}$ *is uniformly exponentially stable.*
(ii) $\lim_{t\to\infty} ||T_\alpha(t)||_{\mathcal{B}(X)} = 0$.
(iii) $\mathrm{Re}(\lambda) < 0$ *for all* $\lambda \in \sigma(B_\alpha)$.

On the other hand, by Proposition 3, we obtain the following result that will be important for our main result on stability. In what follows, we denote by $\overline{\mathbb{D}}(z,r) := \{w \in \mathbb{C} : |w-z| \le r\}$ the closed disk of center z and radius $r > 0$.

Proposition 4. *Let $0 < \alpha < 1$ and A be a closed operator on X. Assume that $\frac{1}{1-\alpha} \in \rho(A)$. Let $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$, where $N_\alpha = (I-(1-\alpha)A)^{-1}$, then we have*

$$\sigma(A) \subset \mathbb{C} \setminus \overline{\mathbb{D}}\Big(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)}\Big) \iff \mathrm{Re}(\lambda) < 0 \ \textit{for all} \ \lambda \in \sigma(B_\alpha).$$

Proof. Suppose $\mu \in \sigma(A)$ such that $\mu = a + ib \in \mathbb{C}$. Since $\sigma(A) \subset \mathbb{C} \setminus \overline{\mathbb{D}}(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)})$, we have that μ verifies $\left(a - \frac{1}{2(1-\alpha)}\right)^2 + b^2 > \left(\frac{1}{2(1-\alpha)}\right)^2$ which, after a computation, is equivalent to

$$a - (1-\alpha)(a^2+b^2) < 0. \tag{45}$$

By Proposition 3, we have

$$\lambda = \frac{\alpha}{1-\alpha}\left[\frac{1}{1-(1-\alpha)\mu} - 1\right] \in \sigma(B_\alpha),$$

where after some computations, we obtain the equivalent representation

$$\lambda = \alpha \frac{[a-(1-\alpha)(a^2+b^2)] + ib}{(1-(1-\alpha)a)^2 + ((1-\alpha)b)^2}. \tag{46}$$

Hence, by identity (46), we have $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(B_\alpha)$ if and only if $a - (1-\alpha)(a^2 + b^2) < 0$. Thus, by equivalence (45), we conclude the claim. □

Remark 14. *Observe that the condition $\sigma(A) \subset \mathbb{C} \setminus \overline{\mathbb{D}}(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)})$ implies that the operator A is injective.*

Next, we apply Theorem 1 to study the stability of the solution to the problem (44). The following is our main result in this section.

Theorem 2. *Let $0 < \alpha < 1$, A be a closed operator on X with domain $D(A)$ and $f : [0,\infty) \to X$ continuously differentiable. Assume that*

(i) $\frac{1}{1-\alpha} \in \rho(A)$,
(ii) $\sigma(A) \subset \mathbb{C} \setminus \overline{\mathbb{D}}(\frac{1}{2(1-\alpha)}, \frac{1}{2(1-\alpha)})$,
(iii) there exist constants $\beta, M > 0$ such that

$$||f(t)||_X + ||f'(t)||_X \le Me^{-\beta t}, \ \ t \ge 0.$$

Then, the problem given by

$${}_{CF}D_t^\alpha u(t) = Au(t) + f(t), \ \ t \ge 0, \tag{47}$$

has a unique strong solution u such that

$$\lim_{t\to\infty} ||u(t)||_X = 0.$$

Proof. Suppose $f \equiv 0$, then we have that the problem (47) has a unique strong solution $u \equiv 0$, by Remark 6. Thus, the theorem holds.

Now, suppose f is not zero. By Remark 7, we have that problem (47) has a unique strong solution given by

$$u(t) = T_\alpha(t)A^{-1}f(0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau)d\tau, \ \ t \geq 0,$$

where $(T_\alpha(t))_{t\in\mathbb{R}}$ is a uniformly continuous group generated by $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$ and $F_\alpha(t) = (1-\alpha)N_\alpha f'(t) + \alpha N_\alpha f(t)$ with $N_\alpha = (I-(1-\alpha)A)^{-1} \in \mathcal{B}(X)$. Thus, we have

$$||u(t)||_X \leq ||T_\alpha(t)A^{-1}f(0)||_X + \int_0^t ||T_\alpha(t-\tau)||_{\mathcal{B}(X)}||F_\alpha(\tau)||_X d\tau, \ \ t \geq 0. \tag{48}$$

Note that by Proposition 4 and Remark 13, we have that there exist constants $\omega_\alpha > 0$, $M_\alpha \geq 1$ such that $||T_\alpha(t)||_{\mathcal{B}(X)} \leq M_\alpha e^{-\omega_\alpha t}$ for all $t \geq 0$. Hence, from inequality (48), we obtain

$$||u(t)||_X \leq ||T_\alpha(t)A^{-1}f(0)||_X + \int_0^t M_\alpha e^{-\omega_\alpha(t-\tau)}||F_\alpha(\tau)||_X d\tau. \tag{49}$$

On the other hand, by hypothesis (iii) we have

$$\begin{aligned}
||F_\alpha(t)||_X &\leq (1-\alpha)||N_\alpha f'(t)||_X + \alpha||N_\alpha f(t)||_X \\
&\leq (1-\alpha)||N_\alpha||_{\mathcal{B}(X)}||f'(t)||_X + \alpha||N_\alpha||_{\mathcal{B}(X)}||f(t)||_X \\
&\leq (1-\alpha)||N_\alpha||_{\mathcal{B}(X)}Me^{-\beta t} + \alpha||N_\alpha||_{\mathcal{B}(X)}Me^{-\beta t} \\
&= ||N_\alpha||_{\mathcal{B}(X)}Me^{-\beta t},
\end{aligned}$$

where $C_\alpha := ||N_\alpha||_{\mathcal{B}(X)}M$ is a positive constant. Let $\gamma_\alpha := \min\{\beta, \omega_\alpha/2\}$, then we obtain that

$$||F_\alpha(t)||_X \leq C_\alpha e^{-\gamma_\alpha t}, \ \ t \geq 0.$$

Therefore, by (49) we have

$$\begin{aligned}
||u(t)||_X &\leq ||T_\alpha(t)A^{-1}f(0)||_X + \int_0^t M_\alpha e^{-\omega_\alpha(t-\tau)}C_\alpha e^{-\gamma_\alpha\tau}d\tau \\
&= ||T_\alpha(t)A^{-1}f(0)||_X + M_\alpha C_\alpha e^{-\omega_\alpha t}\int_0^t e^{(\omega_\alpha-\gamma_\alpha)\tau}d\tau \\
&= ||T_\alpha(t)A^{-1}f(0)||_X + \frac{M_\alpha C_\alpha}{\omega_\alpha - \gamma_\alpha}e^{-\omega_\alpha t}[e^{(\omega_\alpha-\gamma_\alpha)t} - 1] \\
&= ||T_\alpha(t)A^{-1}f(0)||_X + \frac{M_\alpha C_\alpha}{\omega_\alpha - \gamma_\alpha}[e^{-\gamma_\alpha t} - e^{-\omega_\alpha t}].
\end{aligned} \tag{50}$$

Observe that by Proposition 4 and Remark 13, we obtain

$$\lim_{t\to\infty} ||T_\alpha(t)A^{-1}f(0)||_X = 0.$$

Finally, by the above identity and (48), we conclude

$$\lim_{t\to\infty} ||u(t)||_X \leq \lim_{t\to\infty} \frac{M_\alpha C_\alpha}{\omega_\alpha - \gamma_\alpha}[e^{-\gamma_\alpha t} - e^{-\omega_\alpha t}] = 0.$$

This proves the claim. □

The following example illustrates how Theorems 1 and 2 can be applied to obtain solutions of ${}_{CF}D_t^\alpha u(t) = Au(t) + f(t)$, $t \geq 0$, and know about its behavior.

Example 1. *Fix $0 < \alpha < 1$ and consider in $X := C([0,1])$ the operator $Au(x) = u''(x)$, $x \in [0,1]$, with $D(A) = \{u \in C^2([0,1]) : u(0) = u(1) = 0\}$. Since $\sigma(A) = \{-\pi^2 k^2 : k \in \mathbb{N}\}$, we have that A is an injective operator and*

$$A^{-1}g(x) = -x\int_0^1 (1-s)g(s)ds + \int_0^x (x-s)g(s)ds, \quad x \in [0,1].$$

Let us now consider the problem

$${}_{CF}D_t^{\alpha} u(t,x) = u_{xx}(t,x) + \gamma e^{-\beta t}\sin(x),$$

where $\gamma, \beta > 0$, $x \in [0,1]$ and $t > 0$. Since the equation forces the initial condition, we obtain

$$u(0,x) = A^{-1}[\gamma \sin(x)] = \gamma[x\sin(1) - \sin(x)].$$

Our results show that the solution is given by

$$u(t,x) = T_{\alpha}(t)\gamma[x\sin(1) - \sin(x)] + \int_0^t T_{\alpha}(t-\tau)F_{\alpha}(\tau,x)d\tau, \tag{51}$$

where $(T_{\alpha}(t))_{t\geq 0}$ is the uniformly continuous semigroup generated by $B_{\alpha} = \frac{\alpha}{1-\alpha}(N_{\alpha} - I)$ and $F_{\alpha}(t,x) = \gamma[\alpha(1+\beta) - \beta]e^{-\beta t}N_{\alpha}\sin(x)$ with $N_{\alpha} = (I - (1-\alpha)A)^{-1} \in \mathcal{B}(X)$.

By Theorem 1, we have that (51) is also the solution of the problem

$$u_t(t,x) = B_{\alpha}u(t,x) + F_{\alpha}(t,x), \quad t \geq 0, \quad x \in [0,1].$$

A computation shows that for $f : [0,\infty) \times X \to X$ we have

$$\begin{aligned} N_{\alpha}f(x) = {} & \frac{\sqrt{1-\alpha}}{1-\alpha}\frac{\sinh\left(\frac{x}{\sqrt{1-\alpha}}\right)}{\sin\left(\frac{1}{\sqrt{1-\alpha}}\right)}\int_0^1 \sinh\left(\frac{1-y}{\sqrt{1-\alpha}}\right)f(y)dy \\ & - \frac{\sqrt{1-\alpha}}{1-\alpha}\int_0^x \sinh\left(\frac{x-y}{\sqrt{1-\alpha}}\right)f(y)dy. \end{aligned}$$

In particular, using [26] we have

$$N_{\alpha}\sin(x) = \frac{\sin(x)}{2-\alpha} - \frac{\sin(1)}{(2-\alpha)\sinh\left(\frac{1}{\sqrt{1-\alpha}}\right)}\sinh\left(\frac{x}{\sqrt{1-\alpha}}\right)$$

and hence we conclude that

$$F_{\alpha}(t,x) = \gamma e^{-\beta t}[\alpha(1+\beta) - \beta]\left[\frac{\sin(x)}{2-\alpha} - \frac{\sin(1)}{(2-\alpha)\sinh\left(\frac{1}{\sqrt{1-\alpha}}\right)}\sinh\left(\frac{x}{\sqrt{1-\alpha}}\right)\right].$$

On the other hand, by Theorem 2, we conclude that the solution (51) satisfies

$$\lim_{t\to\infty}\sup_{x\in[0,1]}|u(t,x)| = 0.$$

Remark 15. *Let $f \equiv 0$ in Example 1, i.e., $\gamma = 0$, then from (51) we see that the solution is $u(t,x) \equiv 0$.*

5. The Semilinear Problem

Let $0 < \alpha < 1$ and assume that $A : D(A) \subset X \to X$ is a closed linear operator defined on a complex Banach space X and $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$. We recall that $B_{\alpha} := \frac{\alpha}{1-\alpha}(N_{\alpha} - I)$, where $N_{\alpha} := (I - (1-\alpha)A)^{-1} \in \mathcal{B}(X)$, defines the uniformly continuous group $(T_{\alpha}(t))_{t\in\mathbb{R}}$ on X, given by (8).

Let $f : [0, \infty) \times X \to X$ be continuously differentiable in $t \in [0, \infty)$. We define

$$F_\alpha(t, x) := (1-\alpha)N_\alpha f_t(t, x) + \alpha N_\alpha f(t, x).$$

For each $u_0 \in X$, we consider the integral equation:

$$u(t) = T_\alpha(t)u_0 + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau, u(\tau))d\tau,\ t \geq 0. \tag{52}$$

Let $u_0 \in X$ be given. It is well-known (see [17], Chapter 6) that a continuous solution u of the integral Equation (52) is called a *mild solution* of the following semilinear initial value problem:

$$\begin{cases} u'(t) = B_\alpha u(t) + F_\alpha(t, u(t)), & t \geq 0; \\ u(0) = u_0. \end{cases}$$

For our purposes, we use the equivalence given by Theorem 1 to extend the previous terminology as follows.

Definition 7. *Assume that $0 \in \rho(A)$ and let $\mathcal{S} := \{x \in D(A) : x = A^{-1}f(0, x)\}$. A continuous solution u of the integral equation*

$$u(t) = -T_\alpha(t)A^{-1}f(0, u_0) + \int_0^t T_\alpha(t-\tau)F_\alpha(\tau, u(\tau))d\tau,\ t \geq 0 \tag{53}$$

is called a mild solution of the initial value problem

$$\begin{cases} {}_{CF}D_t^\alpha u(t) = Au(t) + f(t, u(t)), & t \geq 0; \\ u(0) = u_0 \in \mathcal{S}. \end{cases} \tag{54}$$

We finish with the following result that assures the existence and uniqueness of mild solutions of (54) for Lipschitz continuous functions f and A an injective operator. The proof is relatively standard, but we give it for completeness.

Theorem 3. *Let $0 < \alpha < 1$ and A be a closed linear operator on a complex Banach space X. Suppose $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$ and $f : [0, T] \times X \to X$ is continuously differentiable in t on $[0, T]$ and satisfies the following Lipschitz type condition with constant L on X:*

$$(1-\alpha)||f_t(t, x) - f_t(t, y)||_X + \alpha||f(t, x) - f(t, y)||_X \leq L||x - y||_X$$

for all $x, y \in X$. Then, the problem given by

$$\begin{cases} {}_{CF}D_t^\alpha u(t) = Au(t) + f(t, u(t)), & 0 \leq t \leq T; \\ u(0) = u_0 \in \mathcal{S}, \end{cases} \tag{55}$$

has a unique mild solution in $C([0, T] : X)$.

Proof. By hypothesis, we define $N_\alpha = (I - (1-\alpha)A)^{-1} \in \mathcal{B}(X)$. Then, we obtain

$$\begin{aligned} &||F_\alpha(t, x) - F_\alpha(t, y)||_X = ||N_\alpha[(1-\alpha)f_t(t, x) + \alpha f(t, x)] - N_\alpha[(1-\alpha)f_t(t, y) + \alpha f(t, y)]||_X \\ &\leq ||N_\alpha||_{\mathcal{B}(X)}||[(1-\alpha)f_t(t, x) + \alpha f(t, x)] - [(1-\alpha)f_t(t, y) + \alpha f(t, y)]||_X \\ &\leq ||N_\alpha||_{\mathcal{B}(X)}[(1-\alpha)||f_t(t, x) - f_t(t, y)|| + \alpha||f(t, x) - f(t, y)||_X] \\ &\leq ||N_\alpha||_{\mathcal{B}(X)}L||x - y||_X. \end{aligned} \tag{56}$$

Hence, F_α is uniformly Lipschitz continuous with constant $L||N_\alpha||_{\mathcal{B}(X)}$. Moreover, we recall that $(T_\alpha(t))_{t\geq 0}$ is the uniformly continuous semigroup generated by $B_\alpha = \frac{\alpha}{1-\alpha}(N_\alpha - I)$. In particular, there exists $M = M(\alpha, T) \geq 1$ such that

$$||T_\alpha(t)||_{\mathcal{B}(X)} \leq M \text{ for all } t \in [0, T]. \tag{57}$$

For a given $u_0 \in \mathcal{S}$, we define a mapping $G : C([0, T] : X) \to C([0, T] : X)$ by

$$(G_\alpha u)(t) := -T_\alpha(t)A^{-1}f(0, u_0) + \int_0^t T_\alpha(t-s)F_\alpha(s, u(s))ds, \quad t \in [0, T]. \tag{58}$$

Denoting by $||u||_\infty$ the norm of u as an element of $C([0, T] : X)$ it follows readily from the definition of G and (56) and (57) that

$$\begin{aligned} ||(G_\alpha u)(t) - (G_\alpha v)(t)|| &= ||\int_0^t T_\alpha(t-s)[F_\alpha(s, u(s)) - F_\alpha(s, v(s))]ds|| \\ &\leq \int_0^t ||T_\alpha(t-s)||_{\mathcal{B}(X)}||F_\alpha(s, u(s)) - F_\alpha(s, v(s))||_X ds \\ &\leq \int_0^t M||N_\alpha||_{\mathcal{B}(X)}L||u(s) - v(s)||_X ds \\ &\leq M||N_\alpha||_{\mathcal{B}(X)}Lt||u - v||_\infty. \end{aligned} \tag{59}$$

In general, we obtain using (58) and (59) and induction on n that

$$||(G_\alpha^n u)(t) - (G_\alpha^n v)(t)|| \leq \frac{(M||N_\alpha||_{\mathcal{B}(X)}Lt)^n}{n!}||u - v||_\infty$$

whence

$$||G_\alpha^n u - G_\alpha^n v|| \leq \frac{(M||N_\alpha||_{\mathcal{B}(X)}LT)^n}{n!}||u - v||_\infty.$$

Since $\frac{(M||N_\alpha||_{\mathcal{B}(X)}LT)^n}{n!} < 1$ for n is sufficiently large, applying the contraction principle we conclude that G_α has a unique fixed point u in $C([0, T] : X)$. This fixed point is the desired solution of the integral Equation (54). Thus, by Definition 7, we have that (55) has a mild solution.

Now, we show the uniqueness. Assume that v is a mild solution of (55) on $[0, T]$ with the initial value $v_0 \in \mathcal{S}$. Then

$$\begin{aligned} ||u(t) - v(t)||_X &\leq || - T_\alpha(t)A^{-1}f(0, u_0) + T_\alpha(t)A^{-1}f(0, v_0)||_X \\ &\quad + \int_0^t ||T_\alpha(t-s)[F_\alpha(s, u(s)) - F_\alpha(s, v(s))]||_X ds \\ &\leq M||A^{-1}f(0, u_0) - A^{-1}f(0, v_0)||_X \\ &\quad + M||N_\alpha||_{\mathcal{B}(X)}L\int_0^t ||u(s) - v(s)||_X ds. \end{aligned}$$

which implies, by Gronwall's inequality, that

$$||u(t) - v(t)||_X \leq M||A^{-1}f(0, u_0) - A^{-1}f(0, v_0)||_X e^{M||N_\alpha||_{\mathcal{B}(X)}LT},$$

and therefore

$$||u - v||_\infty \leq M||A^{-1}f(0, u_0) - A^{-1}f(0, v_0)||_X e^{M||N_\alpha||_{\mathcal{B}(X)}LT}$$

which yield the uniqueness of u (with $v_0 = u_0$). □

Remark 16. *Theorem 3 does not assume that A is the generator of a C_0-semigroup in contrast to the case of the first order abstract Cauchy problem ([17], Theorem 6.1.2).*

Remark 17. *Note that if $\alpha \to 1$ in Theorem 3, then we obtain that the Lipschitz condition with respect to f and f_t simplifies to a Lipschitz condition with respect to f only. On the other hand, by identity (5) and Remark 3, and assuming that A is densely defined, we obtain by Theorem 3 that, when $\alpha \to 1$, A is the infinitesimal generator of a C_0-semigroup. It shows that Theorem 3 extends the case $\alpha = 1$ proved in ([17], Theorem 6.1.2) to the case $0 < \alpha < 1$.*

Example 2. *We consider the following semilinear problem:*

$$\begin{cases} {}_{CF}D_t^{\alpha} u(t,x) &=& u_{xx}(t,x) - e^{-\beta t}\sin(u(t,x)), \qquad 0 \le t \le T; \\ \\ u(0,x) &=& 0. \end{cases} \tag{60}$$

where $\beta > 0$, $x \in [0,1]$ and $0 < \alpha < 1$.

Let us consider X and $(A, D(A))$ as in Example 1. Then, it is clear that $\{0, \frac{1}{1-\alpha}\} \subset \rho(A)$. We define $f(t,u) = -e^{-\beta t}\sin(u)$ in $[0,T] \times X$, where $\sin(u)(x) := \sin(u(x))$, $x \in [0,1]$. Further, we observe that

$$0 \in \mathcal{S} = \{u \in C^2([0,1]) \,:\, u''(x) = -\sin(u(x)),\ u(0) = u(1) = 0\,\}.$$

Moreover, by the mean value theorem, we obtain $\|f(t,u) - f(t,v)\| \le \|u - v\|$ as well as $\|f_t(t,u) - f_t(t,v)\| \le \beta\|u - v\|$ for any $u, v \in X$. Therefore,

$$(1-\alpha)\|f_t(t,u) - f_t(t,v)\| + \alpha\|f(t,u) - f(t,v)\| \le L||u - v||,$$

with Lipschitz constant $L = (1-\alpha)\beta + \alpha$. By Theorem 3 we conclude that (60) has a unique mild solution in $C([0,T];X)$.

6. Conclusions

In this article we study the abstract Cauchy problem with the fractional derivative of order $\alpha \in (0,1)$ of Caputo-Fabrizio and compare its performance from a mathematical point of view. As advantage, and in contrast to the finite dimensional case, i.e., A being a matrix, we observe that being A an unbounded closed linear operator (e.g. a differential operator like the Laplacian), the abstract Cauchy problem with operator A turns out to be equivalent to a first order abstract Cauchy problem with a family of *bounded* operators B_α—that behave like a Yosida's approximation of A—and that makes unnecessary any previous assumptions about A, to solve it, such as a generator of a C_0-semigroup or cosine family of operators, for example. As disadvantage, the non-singular character of the kernel that defines the Caputo-Fabrizio derivative, forces an initial condition (somewhat artificial) that involves the operator A itself, condition that can be overcome if we assume certain conditions of invertibility of the operator A that hold for certain classes of differential operators, for example, the Dirichlet Laplacian operator on a smooth bounded domain. We leave similar studies for other classes of fractional derivatives with non-singular (or regular) kernels as an open problem.

Author Contributions: Conceptualization, C.L.; Formal analysis, J.B. and C.L.; Investigation, J.B. and C.L.; Methodology, J.B. and C.L.; Validation, C.L.; Writing—original draft, J.B. and C.L. All authors have read and agreed to the published version of the manuscript.

Funding: C. Lizama is partially supported by ANID—FONDECYT/1220036. J. Bravo is partially supported by ANID-PFCHA/Doctorado Nacional/2019-21190764.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank DICYT-VRIDEI of the University of Santiago de Chile (USACH) for the financial support for the open access publication of this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
2. Abbas, S.; Benchohra, M.; Nieto, J.J. Caputo–Fabrizio fractional differential equations with instantaneous impulses. *Mathematics* **2021**, *6*, 2932–2946. [CrossRef]
3. Alizadeh, S.; Baleanu, D.; Rezapour, S. Analyzing transient response of the parallel RCL circuit by using the Caputo–Fabrizio fractional derivative. *Adv. Differ. Equ.* **2020**, *2020*, 55. [CrossRef]
4. Baleanu, D.; Jajarmi, A.; Mohammadi, H.; Rezapour, S. A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. *Chaos Solitons Fractals* **2020**, *134*, 109705. [CrossRef]
5. Baleanu, D.; Mohammadi, H.; Rezapour, S. A mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the Rubella disease model. *Adv. Differ. Equ.* **2020**, *2020*, 184. [CrossRef]
6. Baleanu, D.; Mohammadi, H.; Rezapour, S. A fractional differential equation model for the COVID-19 transmission by using the Caputo–Fabrizio derivative. *Adv. Differ. Equ.* **2020**, *2020*, 299. [CrossRef] [PubMed]
7. Baleanu, D.; Aydogn, S.M.; Mohammadi, H.; Rezapour, S. On modelling of epidemic childhood diseases with the Caputo–Fabrizio derivative by using the Laplace Adomian decomposition method. *Alex. Eng. J.* **2020**, *59*, 3029–3039. [CrossRef]
8. Shah, K.; Sarwar, M.; Baleanu, D. Study on Krasnoselskii's fixed point theorem for Caputo–Fabrizio fractional differential operators. *Adv. Differ. Equ.* **2020**, *2020*, 178.
9. El-Dessoky, M.M.; Khan, M.A. Application of Caputo–Fabrizio derivative to a cancer model with unknown parameters. *Discr. Cont. Dyn. Syst. Ser. S* **2021**, *14*, 3557–3575. [CrossRef]
10. Li, H.; Cheng, J.; Li, H.-B.; Zhong, S.-M. Stability analysis of a fractional-order linear system described by the Caputo–Fabrizio derivative. *Mathematics* **2019**, *7*, 200. [CrossRef]
11. Losada, J.; Nieto, J.J. Properties of a new fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 87–92.
12. Momani, S.; Djeddi, N.; Al-Smadi, M.; Al-Omari, S. Numerical investigation for Caputo–Fabrizio fractional Ricatti and Bernoulli equations using iterative reproducing kernel method. *Appl. Numer. Math.* **2021**, *170*, 418–434. [CrossRef]
13. Sur, A.; Mondal, S. The Caputo–Fabrizio heat transport law in vibration analysis of a microscale beam induced by laser. *Z. Angew. Math. Mech.* **2021**, *101*, e202000215. [CrossRef]
14. Baleanu, D.; Sajjadi, S.S.; Jajarmi, A.; Defterli, O. On a nonlinear dynamical system with both chaotic and non-chaotic behaviors: A new fractional analysis and control. *Adv. Differ. Equ.* **2021**, *2021*, *234*. [CrossRef]
15. Kumar, S.; Das, S.; Ong, S.H. Analysis of tumor cells in the absence and presence of chemotherapeutic treatment: The case of Caputo–Fabrizio time fractional derivative. *Math. Comp. Simul.* **2021**, *190*, 1–14. [CrossRef]
16. Caputo, M.; Fabrizio, M. On the singular kernels for fractional derivatives. Some applications to partial differential equations. *Progr. Fract. Differ. Appl.* **2021**, *7*, 79–82.
17. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983.
18. Caputo, M.; Fabrizio, M. Applications of new time and spatial fractional derivatives with exponential kernels. *Prog. Fract. Differ. Appl.* **2016**, *1*, 1–11.
19. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769.
20. Diethelm, K.; Garrapa, R.; Giusti, A.; Stynes, M. Why singular derivatives with nonsingular kernels should not be used. *Fract. Calc. Appl. Anal.* **2020**, *23*, 610–634. [CrossRef]
21. Engel, K.J.; Nagel, R. *One-Parameter Semigroups for Linear Evolution Equations*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2000. [CrossRef]
22. Arendt, W.; Batty, C.J.K.; Hieber, M.; Neubrander, F. *Vector-Valued Laplace Transforms and Cauchy Problems*, 2nd ed.; Monographs in Mathematics, 96; Birkhäuser/Springer Basel AG: Basel, Switzerland, 2011. [CrossRef]
23. Matignon, D. Stability results on fractional differential equations with applications to control processing. In *Proceedings of the Computational Engineering in Systems and Application Multiconference*; IMACS IEEE-SMC: Lille, France, 1996; Volume 2, pp. 963–968.
24. Haase, M. The Functional Calculus for Sectorial Operators. In *The Functional Calculus for Sectorial Operators. Operator Theory: Advances and Applications*; Birkhäuser: Basel, Switzerland, 2006; Volume 169.
25. Dunford, N.; Schwartz, J.T. *Linear Operators, Part I, General Theory*; Wiley-Interscience: New York, NY, USA, 1958.
26. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*, 7th ed.; Elsevier/Academic Press: Amsterdam, The Netherlands, 2007.