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Strongly L^p well-posedness for abstract time-fractional Moore-Gibson-Thompson type equations

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Abstract

We obtain necessary and sufficient conditions for the strongly L^p well-posedness of three abstract evolution equations, arising from fractional Moore-Gibson-Thompson type equations which have recently appeared in the literature. We use Fourier multiplier techniques to derive new characterizations in terms of the R-boundedness of the operator-valued symbol associated to each abstract model, when endowed with the time-fractional Liouville-Grünwald derivative. As a consequence of our characterization, we give new insights into the differences between the models based on the structure of the respective operator-valued symbols and show novel applications by including several classes of operators other than the Laplacian.

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1. Introduction

Let A be a closed linear operator defined on a complex Banach space X. In this article, we are concerned with time-fractional abstract linear models that admit the form

$$(1+\tau^{\alpha}D_t^{\alpha})(u''(t)-c^2Au(t))-\delta D_t^{2-\beta}Au(t)=f(t), \quad t\in[0,2\pi], \tag{1.1}$$

where $\tau,\delta>0$ and D_t^{α} denotes the α^{th} Liouville-Grünwald derivative [16,29, Definition 2.1] (also called Weyl derivative [22, Section 6.2], [10,34]). The above model in case $A=\Delta$, the Laplacian operator, and considering Caputo-Djrbashian fractional derivative has been recently introduced by Kaltenbacher and Nikolic [25]. It arises from using fractional temperature laws proposed by Compte and Metzler instead of the standard heat flux law within the governing equations to the classical Moore-Gibson-Thompson (MGT) equation. Due to their interest in applied fields such as viscoelasticity theory [20], where they have gained increasing importance, in [1,2,23], the authors have proposed similar approaches for constructing a fractional order version of the MGT equation. They used diverse time-fractional order operators to create fractional Moore-Gibson-Thompson (fMGT) equations. For instance, in the reference [1] the authors used the fractional Atangana-Baleanu operator, whereas in the reference [2] the more standard Riemann-Liouville definition of fractional derivative is used. On the other hand, in [2] the authors considered the Caputo-Djrbashian definition of fractional operator. Using the same definition, in [25, Section 7] the authors introduce the following cases: Type I: $\alpha\in(0,1]$ and $\beta=\alpha$; Type II: $\alpha\in(0,1]$ and $\beta=2-\alpha$; and Type III: $\alpha=1$ and $\beta\in(0,1]$ and they name them as fMGT equations. In the same work, it was shown the well-posedness for these linear time-fractional models in the Hilbert space $X=L^2(\Omega)$.

We point out that a recent and extensive analysis has been done on the singular limit of the integer model (1.1) when $\alpha=\beta=1$ and the relaxation parameter τ vanishes. See [5,9] and references therein. In the new and interesting articles of Meliani [35] and Katenbacher and Nikolic [26,37], the authors consider abstract memory kernel convolution terms instead of fractional derivatives, analyzing the singular limit and generalizing [25].

Well-posedness in vector-valued Lebesgue-Bochner spaces for the abstract model (1.1) in case $\tau=0$ has been provided in [29]. In case $\tau\neq 0,\alpha=1,\beta=1$, the well-posedness of equation (1.1) named as the abstract Moore-Gibson-Thompson (aMGT) equation, has been studied by Poblete and Pozo in [38] in Lebesgue-Bochner spaces and, more generally, by Cai and Bu [18] in the scales of vector-valued Besov and Triebel-Lizorkin spaces. However, the analysis of well-posedness for the fractional abstract MGT equation (afMGT) introduced in this article is still an open problem in most vector-valued spaces of interest. See also [7,11,12,14,15] for more research on this topic.

In this article we provide for the first time a complete characterization of strong well-posedness for (1.1) on the scale of vector-valued Lebesgue spaces $L^p(0,2\pi;X)$, showing new and interesting relationships and differences between equations of Type I, II and III within the framework of the abstract model (1.1). We consider equation (1.1) when endowed with time-fractional Liouville-Grünwald derivatives. This definition arises to preserve periodicity and is used e.g., in physics for the study of fractional fields at a positive temperature where fractional oscillators are replaced by fractional thermal oscillators [33, Section 3], and in approximation theory of periodic functions by trigonometric polynomials [32,39]. Several properties of the Liouville-Grunwald derivative are given in [17, Section 3].

It should be noted that our results are general enough not to require additional assumptions on the operator A, such as being the generator of a semigroup or cosine family of operators. Our methods are based on the use of Fourier multipliers theorems with operator-valued symbols obtained by Arendt and Bu [8], R-bounded operators, UMD spaces, and an original technique introduced in the recent reference [19].

1.1. Main results

Next, we provide an overview of our main findings. We consider the complex variable functions

$$d^j(z) = \frac{\tau^\alpha z^{\alpha+2} + z^2}{c^2 + c^2\tau^\alpha z^\alpha + \delta z^{2-\beta}} \quad \text{with } z = ik, \quad k \in \mathbb{Z}, \tag{1.2}$$

where $j \in \{I, II, III\}$. We show that for a fixed j, and under the assumption that X is a UMD space (e.g. $X = L^q(\Omega)$, $1 < q < \infty$), a necessary and sufficient condition to have for every $f \in L^p(0, 2\pi; X)$, $1 < p < \infty$, a unique function u that satisfies equation (1.1) and belongs to the respective maximal regularity space, is that $\{d^j(ik)\}_{k\in\mathbb{Z}} \subseteq \rho(A)$, the resolvent set of A, and the set

$$\{d^j(ik)(d^j(ik) - A)^{-1}\}_{k\in\mathbb{Z}},$$

is R-bounded. See Theorems 3.15, 4.21 and 4.24 below. As a consequence, important a priori estimates for the solutions can be established. We also observe that Theorem 3.15 extends [38, Corollary 3.12] from the (aMGT) to the (afMGT) equation.

We show in all cases that $Im(d_k^j) \neq 0$ for all $k \neq 0$, but that the real part may vary as follows:

Type II: $Re(d_k^{II}) < 0$ for all $k \neq 0$ and $Im(d_k^{II})/Re(d_k^{II})$ has order $1/|k|^\alpha$ ($|k| \to \infty$);

Type I: $Re(d_k^I) < 0$ for all $k \neq 0$ if the condition

$$1/2 < \alpha \leq 1 \quad \text{and} \quad \cos(\alpha\pi/2) + \tau^\alpha \cos(\alpha\pi) < 0, \tag{1.3}$$

holds. Moreover $Im(d_k^I)/Re(d_k^I) \to \pm\tan(\alpha\pi)$ as $k \to \pm\infty$;

Type III: $Re(d_k^{III}) < 0$ for all $k \neq 0$ if the condition

$$0 < \beta \leq 1 \quad \text{and} \quad \cos\left(\frac{\beta\pi}{2}\right) - \tau \sin\left(\frac{\beta\pi}{2}\right) < 0 \tag{1.4}$$

is imposed. Moreover, $Im(d_k^{III})/Re(d_k^{III}) \to \mp\cot(\beta\pi/2)$ as $k \to \pm\infty$.

This implies several interesting consequences. First of all, Type II is the prototypical model for the fMGT equation in case $A = \Delta$ since d_k^{II} follows the same dynamical behavior as the case $\alpha = 1$ for the MGT equation. Our abstract results confirm the same behavior for this fractional model on an arbitrary Hilbert space for $0 < \alpha \leq 1$, see Theorem 5.26 below. We observe that this result is in agreement with [25, Proposition 7.1] in the sense that well-posedness is allowed without any restriction (except, of course, $\delta > 0$). As an important advantage, our abstract result allows considering any selfadjoint operator A in this model, such as the negative bilaplacian operator $A = -\Delta^2$ in an appropriate domain.

According to (1.3), the Type I model has two restrictions to have a similar behavior to the Type II model. The first analytical restriction is on the fractional order $\alpha \in (1/2, 1]$ which, surprisingly, is in line with the physical interpretation of model I, namely, for $0 < \alpha \leq 1/2$ the temperature which is represented by the solution $u(t)$ of the model, could be negative, see [25, Section 2]. On the other hand, the second restriction gives new insights into the dependence between the parameter τ, that accounts for relaxation, and the fractional order α. According to this dependence, and taking into account that τ must be small in several, though not all, practical situations (for a discussion and examples, see [9, p.150]) we conclude from (1.3) that in such situations α must be close to the MGT case $\alpha = 1$. Note that obeying the behavior of $Im(d_k^I)/Re(d_k^I)$ as $k \to \pm\infty$ in the model I, sectorial operators are admitted in contrast to model II. Hence, the same behavior as for the MGT case is valid for several sectorial operators A defined on a UMD space, see Corollary 5.32 below. In particular, our results for the Type I model are valid in Lebesgue spaces $L^q(\Omega)$ where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain. As a concrete example, we prove new results on strongly L^p well-posedness for the fMGT equation when $A = \Delta$ is defined on a cylindrical domain. See Theorem 5.33.

Finally and taking into account now (1.4), we observe that, in contrast with model II, there is no analytical restriction on the values of $\beta \in (0, 1]$ for model III, but a constraint on the values of τ is needed. Again this dependence shows that in practical situations of interest, the fractional order β must be close to $\beta = 1$. In addition, as in the case of model II, sectorial operators could also be admitted. We finish this article with Theorem 5.36 showing that operators like $A = \Delta$ but also $A = -(-\Delta)^{1/2}$ and $A = -\Delta^2$ are eventually admissible for this model.

2. Preliminaries

Let X and Y be complex Banach spaces. We denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$ then $\mathcal{B}(X, Y)$ will be denoted as $\mathcal{B}(X)$. We denote as $L^p(0, 2\pi; X)$, $1 \leq p \leq \infty$ the space of all 2π-periodic Bochner measurable X-valued functions f such that f restricted to $[0, 2\pi]$ is p-integrable (essentially bounded if $p = \infty$).

In what follows, we use the notation

$$(ik)^\alpha := \begin{cases} |k|^\alpha e^{i\,\mathrm{sgn(k)}\alpha\pi/2} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

Moreover, we define $a_k := 1/(ik)^\alpha$ for $k \neq 0$ and $a_0 = 1$.

The Fourier coefficients of $f \in L^p(0, 2\pi; X)(1 \leq p < \infty)$ will be denoted by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t) dt, \quad k \in \mathbb{Z},$$

where $e_k(t) := e^{ikt}$, $t \in [0, 2\pi]$.

Let $\alpha > 0$. We define the α^{th} Liouville-Grünwald fractional derivative operator D^α in $L^p(0, 2\pi; X)$ by

$$D^\alpha u := \sum_{k \in \mathbb{Z}} (ik)^\alpha \hat{u}(k) e_k, \quad u \in H^{\alpha,p}(0, 2\pi; X), \tag{2.1}$$

where we denote by $H^{\alpha,p}(0,2\pi;X)$ the vector-valued fractional Sobolev space

$$H^{\alpha,p}(0,2\pi;X) := \{u \in L^p(0,2\pi;X) : \exists v \in L^p(0,2\pi;X) \ : \ \hat{v}(k) = (ik)^\alpha \hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

In case $\alpha = 0$ we denote $H^{0,p}(0,2\pi;X) = L^p(0,2\pi;X)$.

The expression given in (2.1) is also known as the α^{th} Weyl derivative [34]. The present formulation can be found e.g. in the references [10,13]. This derivative usually appears in problems that involve periodicity, see for example, Section 9 in [16] in which the authors investigated a fractional diffusion-type equation, or [29] where periodic solutions for time-fractional differential equations are analyzed.

Remark 2.1. Let $p > 1$ be given. It is shown in [10, p. 203-204] that for each $n \in \mathbb{N} \cup \{0\}$ if $n + \frac{1}{p} < \alpha \le n + 1 + \frac{1}{p}$ and $u \in H^{\alpha,p}(0,2\pi;X)$ then u is n-times continuously differentiable and $u^{(k)}(0) = u^{(k)}(2\pi)$ for all $0 \le k \le n$. In particular, if $u \in H^{3,p}(0,2\pi;X)$ then $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$.

The vector-valued Sobolev space $H^{\alpha,p}(0,2\pi;X)$ is equipped with the norm

$$\|u\|_{H^{\alpha,p}} := \|u\|_{L^p} + \|D^\alpha u\|_{L^p}$$

so that it becomes a Banach space. If $\alpha = 1$ we denote $D^1 = D$. By [16, Theorem 4.1] we have the following characterization

$$H^{\alpha,p}(0,2\pi;X) = \{u \in L^p(0,2\pi;X) : D^\alpha u \in L^p(0,2\pi;X)\}.$$

For further use we state the following result.

Proposition 2.2. *Let* $\alpha, \beta > 0$. *Then*

$$H^{\alpha+\beta,p}(0,2\pi;X) = \{u \in H^{\beta,p}(0,2\pi;X) : \ D^\beta u \in H^{\alpha,p}(0,2\pi;X)\}.$$

Proof. Let $u \in H^{\alpha+\beta,p}(0,2\pi;X)$. Then $u \in L^p(0,2\pi;X)$ and $D^{\alpha+\beta}u \in L^p(0,2\pi;X)$. By [29, Proposition 2.3 (ii)] we have that $D^\beta u$ exists and $D^{\alpha+\beta}u = D^\alpha(D^\beta u)$. This means that $D^\alpha(D^\beta u) \in L^p(0,2\pi;X)$. Since $\beta < \alpha + \beta$, part (i) of the same Proposition gives $D^\beta u \in L^p(0,2\pi;X)$. Hence $u \in H^{\beta,p}(0,2\pi;X)$ and $D^\beta u \in H^{\alpha,p}(0,2\pi;X)$.

Conversely, assume that $u \in H^{\beta,p}(0,2\pi;X)$ and $D^\beta u \in H^{\alpha,p}(0,2\pi;X)$. Then [29, Proposition 2.3 (ii)] guarantees that $D^{\alpha+\beta}u$ exists and $D^{\alpha+\beta}u = D^\alpha(D^\beta u)$. This implies $u \in H^{\alpha+\beta,p}(0,2\pi;X)$. □

Remark 2.3. [10] As a consequence of Proposition 2.2, if $0 < \xi_1 \le \xi_2$, then

$$H^{\xi_2,p}(0,2\pi;X) \subset H^{\xi_1,p}(0,2\pi;X).$$

We recall the notion of operator-valued Fourier multiplier [8,10,30].

Definition 2.4. Let X and Y be Banach spaces. For $1 \leq p \leq \infty$, $\alpha \geq 0$ we say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an $(L^p, H^{\alpha,p})$-multiplier, if for each $f \in L^p(0, 2\pi; X)$ there exists $u \in H^{\alpha,p}(0, 2\pi; Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

In particular, in the border case $\alpha = 0$ the definition coincides with the one contained in [8, Proposition 1.1]. The next important lemma can be found in [29, Lemma 2.6].

Lemma 2.5. *Let $1 \leq p < \infty$, $\alpha \geq 0$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$. The following assertions are equivalent*
(i) $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha,p})$-multiplier;
(ii) $((ik)^\alpha M_k)_{k\in\mathbb{Z}}$ is an (L^p, L^p)- multiplier.

We recall from [30] the following definition.

Definition 2.6. A sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C} \setminus \{0\}$ is called 1-regular if it is bounded as well as the set $\{\frac{k(c_{k+1}-c_k)}{c_k}\}_{k\in\mathbb{Z}}$.

The next lemma is a direct consequence of [31, Corollary 3.10 and Remark 2.2].

Lemma 2.7. *Let X be a UMD space. Let $\{M_k\}_{k\in\mathbb{Z}}$ be an (L^p, L^p)-multiplier and $\{b_k\}_{k\in\mathbb{Z}}$ a bounded sequence satisfying*

$$\sup_{k\in\mathbb{Z}} |b_k| + \sup_{k\in\mathbb{Z}} |k(b_{k+1} - b_k)| < \infty. \tag{2.2}$$

Then $\{b_k M_k\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

We now recall the notion of an R-bounded set of operators. For a summary about the main properties about R-bounded sets we refer to [21].

Definition 2.8. Let X and Y be Banach spaces. A set $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded if there is a constant $c \geq 0$ such that

$$\|(T_1x_1, ..., T_nx_n)\|_R \leq c\|(x_1, ..., x_n)\|_R, \tag{2.3}$$

for all $T_1, ..., T_n \in \mathcal{T},\ x_1, ..., x_n \in X,\ n \in \mathbb{N}$ where

$$\|(x_1, ..., x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j\in\{-1,1\}^n} \Big\| \sum_{j=1}^{n} \epsilon_j x_j \Big\|.$$

Let $\Sigma_\psi \subset \mathbb{C}$ be the open sector

$$\Sigma_\psi = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \psi\}.$$

Definition 2.9. [29] A closed densely defined operator A is said to be sectorial of angle θ if it satisfies the following conditions

(i) $\sigma(A) \subseteq \mathbb{C} \setminus \Sigma_\theta$;

(ii) The set $\{z(z-A)^{-1} : z \in \Sigma_\theta\}$ is bounded in $\mathcal{B}(X)$.

The operator A is called R-sectorial of angle θ if the set $\{z(z-A)^{-1} : z \in \Sigma_\theta\}$ is R-bounded.

3. A characterization of well-posedness for the afMGT equation of Type II

Let $\tau, c, \delta > 0$ and $0 < \alpha \le 1$ be given. In this section we study well-posedness in Lebesgue spaces $L^p(0, 2\pi; X)$ for the Type II afMGT given by:

$$\tau^\alpha D^\alpha u''(t) + u''(t) - c^2 Au(t) - (\tau^\alpha c^2 + \delta) D^\alpha Au(t) = f(t), \quad t \in [0, 2\pi], \tag{3.1}$$

where A is a closed linear operator defined on a Banach space X.

Next, we introduce the definition of a strong L^p-solution of the fractional evolution equation (3.1) and the associated concept of well-posedness.

Definition 3.10. Let $1 \le p < \infty$ and $0 < \alpha \le 1$. A function u is called a strong L^p-solution of (3.1) if

$$u \in H^{\alpha+2,p}(0, 2\pi; X) \cap H^{\alpha,p}(0, 2\pi; D(A)) =: MR(\alpha, X),$$

and equation (3.1) holds for almost all $t \in [0, 2\pi]$.

Remark 3.11. It is not difficult to see that

$$\begin{aligned} MR(\alpha, X) = \{u \in H^{\alpha,p}(0, 2\pi; X) \cap L^p(0, 2\pi; D(A)) : D^\alpha u \in L^p(0, 2\pi; D(A)), \\ u \in H^{\alpha+2,p}(0, 2\pi; X),\ u \in H^{2,p}(0, 2\pi; X)\}. \end{aligned} \tag{3.2}$$

The space $MR(1, X)$ was considered by Bu in [10] and it is called the maximal regularity space.

Note that $MR(\alpha, X)$ is a Banach space under the norm

$$\|u\|_{MR} := \|Au\|_{L^p(0,2\pi;X)} + \|D^\alpha Au\|_{L^p(0,2\pi;X)} + \|u''\|_{L^p(0,2\pi;X)} + \|D^\alpha u''\|_{L^p(0,2\pi;X)}.$$

Remark 3.12. Note that if $u \in H^{\alpha+2,p}(0, 2\pi; X)$ then by Remark 2.3 we have that $u \in H^{2,p}(0, 2\pi; X)$ and hence, by Remark 2.1, we have $u(0) = u(2\pi), u'(0) = u'(2\pi)$. On the other hand, by Proposition 2.2 we have $D^{\alpha+1}u \in H^{1,p}(0, 2\pi; X)$, and by [41, Chapter XII, (9.1)], we get that $D^{\alpha+1}u(0) = D^{\alpha+1}u(2\pi)$. Therefore, Definition 3.10 implicitly implies that the equation (3.1) possesses the initial conditions $u(0) = u(2\pi), u'(0) = u'(2\pi)$ and $D^{\alpha+1}u(0) = D^{\alpha+1}u(2\pi)$. Note that in the case of $\alpha = 1$ these initial conditions coincide with those considered in the reference [38].

Definition 3.13. Let $1 \le p < \infty$. We say that the problem (3.1) is strongly L^p well-posed if for every $f \in L^p(0, 2\pi; X)$ there exists a unique strong L^p-solution of (3.1).

Note that if the problem (3.1) is strongly L^p well-posed then $M : MR(\alpha, X) \to L^p(0, 2\pi; X)$ such that $u \to u_f$, where u_f is the unique strong L^p-solution of (3.1), is a closed mapping. Then, by the closed graph theorem, we deduce that there exists a constant $C > 0$ independent of $f \in L^p(0, 2\pi; X)$ such that the estimate

$$\|Au\|_{L^p(0,2\pi;X)} + \|D^\alpha Au\|_{L^p(0,2\pi;X)} + \|u''\|_{L^p(0,2\pi;X)} + \|D^\alpha u''\|_{L^p(0,2\pi;X)} \le C\|f\|_{L^p(0,2\pi;X)}$$

holds.

The following result characterizes L^p well-posedness for the problem (3.1) in terms of certain operator-valued Fourier multipliers symbols. In what follows, we will denote

$$\gamma_\alpha := \tau^\alpha c^2 + \delta.$$

Theorem 3.14. *Let X be a UMD space, $\tau, c, \delta > 0$ and let $A : D(A) \subset X \to X$ be a closed linear operator. The following assertions are equivalent for $p \in [1, \infty)$:*

(a) The problem (3.1) is strongly L^p well-posed;
(b) $S(ik) := (\tau^\alpha (ik)^{\alpha+2} + (ik)^2 - (c^2 + \gamma_\alpha (ik)^\alpha)A)^{-1}$ exists in $\mathcal{B}(X)$ for all $k \in \mathbb{Z}$ and the sequence

$$\{(ik)^{\alpha+2} S(ik)\}_{k\in\mathbb{Z}}$$

is an (L^p, L^p)-multiplier.

Proof. First, we show that (a) implies (b). Let $k \in \mathbb{Z}$ and $y \in X$. We define $f(t) = e^{ikt}y$. Then $f \in L^p(0, 2\pi; X)$, $\widehat{f}(k) = y$ and $\widehat{f}(j) = 0$ for $j \neq k$. Since the problem (3.1) is well-posed there exists a unique $u \in H^{\alpha+2,p}(0, 2\pi; X) \cap H^{\alpha,p}(0, 2\pi; D(A))$ such that

$$\tau^\alpha D^\alpha u''(t) + u''(t) - c^2 Au(t) - \gamma_\alpha D^\alpha Au(t) = f(t) \quad \text{a.a.} \quad t \in [0, 2\pi]. \tag{3.3}$$

Observe that $u \in H^{2,p}(0, 2\pi; X)$ by Remark 3.11 and hence $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$ (see the second part of Remark 2.1). We also have $\widehat{u''}(k) = (ik)^2 \hat{u}(k)$. Multiplying equation (3.3) by $e_{-k}(t), k \in \mathbb{Z}$ and integrating from 0 to 2π in both sides of the equation, we get using the definition of D^α and [8, Lemma 3.1] that $\widehat{u}(k) \in D(A)$ and

$$(\tau^\alpha (ik)^{\alpha+2} + (ik)^2 - (c^2 + \gamma_\alpha (ik)^\alpha)A)\widehat{u}(k) = \widehat{f}(k) = y, \quad k \in \mathbb{Z}.$$

It follows that the operators $(\tau^\alpha (ik)^{\alpha+2} + (ik)^2 - (c^2 + \gamma_\alpha (ik)^\alpha)A$ are surjective for all $k \in \mathbb{Z}$.

Let $x \in D(A)$. If we assume that $[\tau^\alpha (ik)^{\alpha+2} + (ik)^2 - (c^2 + \gamma_\alpha (ik)^\alpha)A]x = 0$ for all $k \in \mathbb{Z}$, then $u(t) := e^{ikt}x$, $k \in \mathbb{Z}$, defines a periodic solution of equation (3.1) with $f \equiv 0$. Indeed,

$$\begin{aligned} &\tau^\alpha D^\alpha u''(t) + u''(t) - c^2 Au(t) - \gamma_\alpha (ik)^\alpha D^\alpha Au(t) \\ &\quad = (ik)^2 \tau^\alpha D^\alpha e^{ikt}x + (ik)^2 e^{ikt}x - c^2 Ae^{ikt}x - \gamma_\alpha (ik)^\alpha D^\alpha Ae^{ikt}x \\ &\quad = e^{ikt}[\tau^\alpha (ik)^{\alpha+2} + (ik)^2 - (c^2 + \gamma_\alpha (ik)^\alpha)A]x = 0, \end{aligned}$$

proving the claim. From the uniqueness of the strong solution, we conclude that $x = 0$.

Next, we show that the sequence $\{(ik)^{\alpha+2}S(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. Indeed, let $f \in L^p(0, 2\pi; X)$ be given. By hypothesis, we have that $\widehat{u}(k) \in D(A)$ for all $k \in \mathbb{Z}$ and there exists a unique $u \in H^{\alpha+2,p}(0, 2\pi; X) \cap H^{\alpha,p}(0, 2\pi; D(A))$ such that

$$(\tau^\alpha (ik)^{\alpha+2} + (ik)^2 - (c^2 + \gamma_\alpha (ik)^\alpha)A)\widehat{u}(k) = \widehat{f}(k), \quad k \in \mathbb{Z}.$$

Let $v := D^\alpha u''$. Then $v \in L^p(0, 2\pi; X)$. Since $S(ik)$ exists, we have

$$\widehat{v}(k) = (ik)^{\alpha+2}\widehat{u}(k) = (ik)^{\alpha+2}S(ik)\widehat{f}(k), \quad k \in \mathbb{Z}.$$

Consequently, we deduce that $\{(ik)^{\alpha+2}S(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier, proving (b).

Next, we prove that (b) implies (a). Let $f \in L^p(0, 2\pi; X)$ be fixed. By hypothesis, there exists $v \in L^p(0, 2\pi; X)$ such that

$$\widehat{v}(k) = (ik)^{\alpha+2}S(ik)\widehat{f}(k), \quad k \in \mathbb{Z}. \tag{3.4}$$

By Lemma 2.5 we have that $\{S(ik)\}_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha+2,p})$-multiplier and thus there exists

$$u \in H^{\alpha+2,p}(0, 2\pi; X) \subset H^{\alpha,p}(0, 2\pi; X) \tag{3.5}$$

such that

$$\widehat{u}(k) = S(ik)\widehat{f}(k), \quad k \in \mathbb{Z} \tag{3.6}$$

and from Remark 3.12 it follows that $D^{\alpha+1}u(0) = D^{\alpha+1}u(2\pi)$. This implies that $\widehat{u}(k) \in D(A)$. Combining (3.4) with (3.6), we obtain

$$\widehat{v}(k) = (ik)^{\alpha+2}\widehat{u}(k), \quad k \in \mathbb{Z}.$$

From here we deduce that $D^{\alpha+2}u = v$. Proposition 2.2 gives that $u'' \in H^{\alpha,p}(0, 2\pi; X)$ and $D^{\alpha+2}u = D^\alpha u'' = v$.

Next, since the sequence $\{(ik)^{-\alpha}\}_{k\in\mathbb{Z}}$ is 1-regular and by hypothesis $\{(ik)^{\alpha+2}S(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier, then, the hypothesis of UMD allows to use Lemma 2.7 and hence the identity

$$(ik)^2 S(ik) = \frac{1}{(ik)^2}(ik)^{\alpha+2}S(ik), \quad k \in \mathbb{Z}$$

implies that $\{(ik)^2 S(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. It follows that there exists $v_1 \in L^p(0, 2\pi; X)$ such that

$$\widehat{v_1}(k) = (ik)^2 S(ik)\widehat{f}(k) = (ik)^2\widehat{u}(k), \quad k \in \mathbb{Z}. \tag{3.7}$$

Consequently, $u \in H^{2,p}(0, 2\pi, X)$ and $v_1 = u''$. From the identity

$$I = \tau^\alpha (ik)^{\alpha+2}S(ik) + (ik)^2 S(ik) - (c^2 + \gamma_\alpha (ik)^\alpha)AS(ik), \quad k \in \mathbb{Z}, \tag{3.8}$$

it follows that

$$\widehat{f}(k) = \tau^\alpha (ik)^{\alpha+2} S(ik)\widehat{f}(k) + (ik)^2 S(ik)\widehat{f}(k) - (c^2 + \gamma_\alpha (ik)^\alpha) AS(ik)\widehat{f}(k), \quad k \in \mathbb{Z}. \tag{3.9}$$

The above identity and the fact that $\{(ik)^2 S(ik)\}_{k\in\mathbb{Z}}$ and $\{(ik)^{\alpha+2} S(ik)\}_{k\in\mathbb{Z}}$ are (L^p, L^p)-multipliers, imply that the sequence $\{(c^2 + \gamma_\alpha (ik)^\alpha) AS(ik)\}_{k\in\mathbb{Z}}$ is also an (L^p, L^p)-multiplier.

Next, let

$$b_k := \frac{1}{c^2 + \gamma_\alpha (ik)^\alpha}, \quad k \in \mathbb{Z}.$$

It is clear that $\{b_k\}_{k\in\mathbb{Z}}$ is a bounded sequence. Let $\xi > 0$, we get from the mean value theorem and the definition of $(ik)^\xi$ that

$$|(i(k+1))^\xi - (ik)^\xi| \le \xi |k|^{\xi-1}, \quad k \in \mathbb{Z}. \tag{3.10}$$

Then, for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} |k(b_{k+1} - b_k)| &= \frac{\gamma_\alpha |k((i(k+1))^\alpha - (ik)^\alpha)|}{|(c^2 + \gamma_\alpha (i(k+1))^\alpha)(c^2 + \gamma_\alpha (ik)^\alpha)|} \\ &\le \frac{\alpha\gamma_\alpha |k|^\alpha}{|(c^2 + \gamma_\alpha (i(k+1))^\alpha)(c^2 + \gamma_\alpha (ik)^\alpha)|} = O(1/|k|^\alpha) \quad \text{as } |k| \to \infty. \end{aligned} \tag{3.11}$$

From (3.11) we immediately get

$$\sup_{k\in\mathbb{Z}} |b_k| + \sup_{k\in\mathbb{Z}} |k(b_{k+1} - b_k)| < \infty.$$

It follows from Lemma 2.7 that the sequence

$$\{b_k(c^2 + \gamma_\alpha (ik)^\alpha) AS(ik)\}_{k\in\mathbb{Z}} = \{AS(ik)\}_{k\in\mathbb{Z}}$$

is an (L^p, L^p)-multiplier. Therefore, there exists $v_2 \in L^p(0, 2\pi; X)$ such that

$$\widehat{v_2}(k) = AS(ik)\widehat{f}(k) = A\widehat{u}(k), \quad k \in \mathbb{Z}. \tag{3.12}$$

Since $0 \in \rho(A)$, we define $w_2 := A^{-1} v_2$. Then, by (3.12), we obtain

$$u = w_2 \in L^p(0, 2\pi; D(A)). \tag{3.13}$$

Since $\{AS(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier, the identity (3.9) implies that $\{(ik)^\alpha AS(ik)\}_{k\in\mathbb{Z}}$ is also an (L^p, L^p)-multiplier. Therefore there exists $v_3 \in L^p(0, 2\pi; X)$ such that

$$\widehat{v_3}(k) = (ik)^\alpha AS(ik)\widehat{f}(k) = (ik)^\alpha A\widehat{u}(k), \tag{3.14}$$

for all $k \in \mathbb{Z}$. As before, define $w_3 := A^{-1} v_3$. Then by (3.14) we obtain

$$D^\alpha u = A^{-1}v_3 = w_3 \in L^p(0,2\pi; D(A)). \tag{3.15}$$

Thus, inserting (3.4), (3.7) and (3.14) in (3.9) we obtain that

$$\widehat{f}(k) = \tau^\alpha \widehat{v}(k) + \widehat{v}_1(k) - c^2\widehat{v}_2(k) - \gamma_\alpha \widehat{v}_3(k), \quad k \in \mathbb{Z}.$$

Hence, (3.5), (3.13) and (3.15) together with Remark 3.11 imply that $u \in MR(\alpha, X)$ and

$$\tau^\alpha D^\alpha u''(t) + u''(t) - c^2 Au(t) - \gamma_\alpha D^\alpha Au(t) = f(t),$$

for almost all $t \in [0, 2\pi]$. Hence u is a strong L^p-solution of (3.1).

Let us see the uniqueness. If $u \in MR(\alpha, X)$ is such that

$$\tau^\alpha D^\alpha u''(t) + u''(t) - c^2 Au(t) - \gamma_\alpha D^\alpha Au(t) = 0,$$

then for the Fourier coefficients we get

$$(\tau^\alpha (ik)^{\alpha+2} + (ik)^\alpha) - (c^2 + \gamma_\alpha (ik)^\alpha)A)\widehat{u}(k) = 0.$$

Since $S(ik)$ exists, we deduce that $\widehat{u}(k) = 0$ for all $k \in \mathbb{Z}$, which means that the equation (3.1) has the null solution $u \equiv 0$. □

The following is the main result of this section.

Theorem 3.15. *Let A be a closed linear operator defined on a UMD space X and let $\tau, c, \delta > 0$. The following conditions are equivalent:*

(i) Equation (3.1) is strongly L^p well-posed;

(ii) $\left\{\dfrac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2)}{c^2 + \gamma_\alpha (ik)^\alpha}\right\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and the set*

$$\left\{\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha}\left(\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha} - A\right)^{-1}\right\}_{k\in\mathbb{Z}} \tag{3.16}$$

is R-bounded.

Proof. Suppose (i). By Theorem 3.14 we obtain that $\left\{\dfrac{\tau^\alpha (ik)^{\alpha+1} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha}\right\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and the sequence $\{(ik)^{\alpha+2}S(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. By [8, Proposition 1.11] we conclude that $\{(ik)^{\alpha+2}S(ik)\}_{k\in\mathbb{Z}}$ is R-bounded. Since the sequence $\frac{\tau^\alpha (ik)^{2+\alpha} + (ik)^2}{(ik)^{2+\alpha}}$ is uniformly bounded, the identity

$$\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha}\left(\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha} - A\right)^{-1}$$

$$= \frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{(ik)^{\alpha+2}} \frac{(ik)^{\alpha+2}}{c^2 + \gamma_\alpha (ik)^\alpha} \left(\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha} - A \right)^{-1}$$

shows that (3.16) is R-bounded, too.

Conversely, suppose (ii). Since (3.16) is R-bounded and the sequence $\frac{(ik)^{2+\alpha}}{\tau^\alpha (ik)^{2+\alpha} + (ik)^2}$ is uniformly bounded, the identity

$$\frac{(ik)^{\alpha+2}}{c^2 + \gamma_\alpha (ik)^\alpha} \left(\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha} - A \right)^{-1}$$
$$= \frac{(ik)^{\alpha+2}}{\tau^\alpha (ik)^{\alpha+2} + (ik)^2} \frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha} \left(\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \gamma_\alpha (ik)^\alpha} - A \right)^{-1}$$

shows that the set $\{M_k := (ik)^{\alpha+2} S(ik)\}_{k\in\mathbb{Z}}$ is R-bounded. By the Marcinkiewicz operator-valued multiplier theorem [8, Theorem 1.3], it is enough to show that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. Then, the result follows from Theorem 3.14. In order to verify this property, we will use [19, Theorem 3.7].

For any $\xi > 0$ we define $r_k := (ik)^\xi$, $k \in \mathbb{Z}$. We prove that the set $\{r_k\}_{k\in\mathbb{Z}}$ is 1-regular, that is, the set $\{k\frac{r_{k+1} - r_k}{r_k}\}_{k\in\mathbb{Z}}$ is bounded. Indeed, by (3.10) we obtain

$$\left| k \frac{r_{k+1} - r_k}{r_k} \right| = \left| k \frac{(i(k+1))^\xi - (ik)^\xi}{(ik)^\xi} \right| \le \xi |k| \frac{|k|^{\xi-1}}{|k|^\xi} = \xi, \quad k \in \mathbb{Z} \setminus \{0\}, \tag{3.17}$$

for any $\xi > 0$, proving the claim.

Using (3.17) with $\xi = \alpha + 2$ it follows that the set $\{(ik)^{\alpha+2}\}_{k\in\mathbb{Z}}$ is 1-regular. It proves the first condition stated in [19, Theorem 3.7].

Next, in order to prove the second statement of [19, Theorem 3.7], we must consider

$$L_k := (S(ik)^{-1} - S(i(k+1))^{-1}) S(ik).$$

Observe that from the identity $[-k^2(\tau^\alpha (ik)^\alpha + 1) - (c^2 + \gamma_\alpha (ik)^\alpha)A]S(ik) = I$, we have

$$AS(ik) = (ik)^2 \frac{(\tau^\alpha (ik)^\alpha + 1)}{c^2 + \gamma_\alpha (ik)^\alpha} S(ik) - \frac{1}{c^2 + \gamma_\alpha (ik)^\alpha} I =: (ik)^2 c_k S(ik) - b_k I.$$

Since the sequences $\{c_k\}_{k\in\mathbb{Z}}$ and $\{(ik)^\alpha b_k\}_{k\in\mathbb{Z}}$ are clearly bounded, the identity

$$(ik)^\alpha AS(ik) = c_k (ik)^{\alpha+2} S(ik) - (ik)^\alpha b_k I = c_k M_k - (ik)^\alpha b_k I, \tag{3.18}$$

proves that the set $\{(ik)^\alpha AS(ik)\}_{k\in\mathbb{Z}}$ is R-bounded. Finally, we must verify that the set $\{kL_k\}_{k\in\mathbb{Z}}$ is R-bounded. In fact, for any $k \in \mathbb{Z} \setminus \{0\}$ we get

$$kL_k = k\Big[-k^2(\tau^\alpha (ik)^\alpha + 1) - (c^2 + \gamma_\alpha (ik)^\alpha)A + (k+1)^2(\tau^\alpha (i(k+1))^\alpha + 1)$$
$$+ (c^2 + \gamma_\alpha (i(k+1))^\alpha)A \Big] S(ik)$$
$$= \Big[\tau^\alpha k[(ik)^{\alpha+2} - (i(k+1))^{\alpha+2}] + k[(ik)^2 - (i(k+1))^2]$$

$$
\begin{aligned}
&\quad - \gamma_\alpha k[(ik)^\alpha - (i(k+1))^\alpha]A\Big]S(ik) \\
&= \tau^\alpha k \frac{[(ik)^{\alpha+2} - (i(k+1))^{\alpha+2}]}{(ik)^{\alpha+2}}(ik)^{\alpha+2}S(ik) \\
&\quad + k\frac{[(ik)^2 - (i(k+1))^2]}{(ik)^2}\frac{1}{(ik)^\alpha}(ik)^{\alpha+2}S(ik) \\
&\quad - \gamma_\alpha k \frac{[(ik)^\alpha - (i(k+1))^\alpha]}{(ik)^\alpha}(ik)^\alpha AS(ik) \\
&= \tau^\alpha k \frac{[(ik)^{\alpha+2} - (i(k+1))^{\alpha+2}]}{(ik)^{\alpha+2}}M_k + k\frac{[(ik)^2 - (i(k+1))^2]}{(ik)^2}\frac{1}{(ik)^\alpha}M_k \\
&\quad - \gamma_\alpha k \frac{[(ik)^\alpha - (i(k+1))^\alpha]}{(ik)^\alpha}(ik)^\alpha AS(ik).
\end{aligned}
\tag{3.19}
$$

Inserting (3.18) in (3.19), and applying repeatedly (3.17) with $\xi \in \{\alpha+2, 2, \alpha\}$ we conclude that the set $\{kL_k\}_{k\in\mathbb{Z}}$ is R-bounded since it is the sum of R-bounded sets [3, Proposition 2.2.5]. Since all hypotheses of Theorem 3.7 in [19] are satisfied, the result follows. □

Remark 3.16. If X is a Hilbert space, R- boundedness is equivalent to uniform boundedness [3, Proposition 2.2.5 (d)] and then condition (ii) in Theorem 3.15 can be replaced by:

$$
\sup_{k\in\mathbb{Z}} \left\| \frac{-k^2(\tau^\alpha(ik)^\alpha + 1)}{c^2 + \gamma_\alpha(ik)^\alpha}\left(\frac{-k^2(\tau^\alpha(ik)^\alpha + 1)}{c^2 + \gamma_\alpha(ik)^\alpha} - A\right)^{-1}\right\| < \infty. \tag{3.20}
$$

4. Characterization of well-posedness for the afMGT equations of Type I and III

Let $0 < \alpha \le 1$ be given. In this section we first analyze well-posedness for the Type I afMGT equation given by:

$$
\tau^\alpha D^\alpha u''(t) + u''(t) - c^2 Au(t) - \tau^\alpha c^2 D^\alpha Au(t) - \delta D^{1-\alpha}Au'(t) = f(t), \quad t \in [0, 2\pi], \tag{4.1}
$$

in the Lebesgue spaces $L^p(0, 2\pi; X)$. Here, A is a closed linear operator defined on a Banach space X and $\tau, c, \delta > 0$.

We first introduce the correspondent notion of a strong L^p-solution for the abstract model (4.1).

Definition 4.17. Let $1 \le p < \infty$ and $0 < \alpha \le 1$. A function u is called a strong L^p-solution of (4.1) if

$$
u \in H^{\alpha+2,p}(0, 2\pi; X) \cap H^{2-\alpha,p}(0, 2\pi; D(A))
$$

and equation (4.1) holds for almost all $t \in [0, 2\pi]$.

Remark 4.18. Since $u \in H^{\alpha+2,p}(0, 2\pi; X)$ if u is a strong L^p-solution of (4.1), then, as shown in Remark 3.12, we must have $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $D^{\alpha+1}u(0) = D^{\alpha+1}u(2\pi)$. This means that (4.1) is in fact a problem with prescribed initial conditions.

Analogously to the previous section, we introduce the following definition.

Definition 4.19. Let $1 \le p < \infty$. We say that the problem (4.1) is strongly L^p well-posed if for every $f \in L^p(0, 2\pi; X)$ there exists a unique strong L^p-solution of (4.1).

The next result provides a characterization of L^p well-posedness for the problem (4.1) in terms of (L^p, L^p)-multipliers.

Theorem 4.20. *Let X be a UMD space, $\tau, c, \delta > 0$, $0 < \alpha \le 1$ and let $A : D(A) \subset X \to X$ be a closed linear operator. The following assertions are equivalent for $p \in [1, \infty)$:*

(a) The problem (4.1) is strongly L^p well-posed;
(b) $R(ik) := (\tau^\alpha (ik)^{\alpha+2} + (ik)^2 - (c^2 + c^2(ik)^\alpha + \delta(ik)^{2-\alpha})A)^{-1}$ exists in $\mathcal{B}(X)$ for all $k \in \mathbb{Z}$ and the set

$$\{(ik)^{\alpha+2} R(ik)\}_{k\in\mathbb{Z}}$$

is an (L^p, L^p)-multiplier.

Also, if (a) or (b) hold then we have the following a priori estimate for the solution:

$$\|D^\alpha u''\|_{H^{\alpha+2,p}(0,2\pi;X)} + ||u''||_{H^{2,p}(0,2\pi;X)} + ||Au||_{L^p(0,2\pi;X)} + ||D^\alpha Au||_{H^{\alpha,p}(0,2\pi;X)} + ||D^{1-\alpha} Au'||_{H^{2-\alpha,p}(0,2\pi;X)} \le C||f||_{L^p(0,2\pi;X)}.$$

Proof. (a) implies (b) can be proved similarly as in the first part of Theorem 3.14.

Next, we see that (b) implies (a). Let $f \in L^p(0, 2\pi; X)$ be fixed. By hypothesis, there exists $v \in L^p(0, 2\pi; X)$ such that

$$\widehat{v}(k) = (ik)^{\alpha+2} R(ik)\widehat{f}(k), \quad k \in \mathbb{Z}. \tag{4.2}$$

By Lemma 2.5 we deduce that $\{R(ik)\}_{k\in\mathbb{Z}}$ is an $(L^p, H^{\alpha+2,p})$-multiplier. Then, there exists $u \in H^{\alpha+2,p}(0, 2\pi; X)$ such that

$$\widehat{u}(k) = R(ik)\widehat{f}(k), \quad k \in \mathbb{Z}. \tag{4.3}$$

This implies that $\widehat{u}(k) \in D(A)$. Inserting (4.3) in (4.2) we get $\widehat{v}(k) = (ik)^{\alpha+2}\widehat{u}(k), \quad k \in \mathbb{Z}$. Therefore

$$u \in H^{\alpha+2,p}(0, 2\pi, X) \text{ and } D^{\alpha+2}u = v. \tag{4.4}$$

Moreover, by Proposition 2.2 we have that

$$u'' \in H^{\alpha,p}(0, 2\pi; X) \quad \text{ and } D^\alpha u'' = v. \tag{4.5}$$

Next, since the sequence $\{(ik)^{-\alpha}\}_{k\in\mathbb{Z}}$ is 1-regular and by hypothesis $\{(ik)^{\alpha+2} R(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier, then Lemma 2.7 and the identity

$$(ik)^2 R(ik) = \frac{1}{(ik)^2}(ik)^{\alpha+2} R(ik), \quad k \in \mathbb{Z},$$

imply that $\{(ik)^2 R(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier. It follows that there exists $v_1 \in L^p(0, 2\pi; X)$ such that

$$\widehat{v_1}(k) = (ik)^2 R(ik)\widehat{f}(k) = (ik)^2\widehat{u}(k), \quad k \in \mathbb{Z}. \tag{4.6}$$

Hence

$$u \in H^{2,p}(0, \pi; X) \text{ and } v_1 = u''. \tag{4.7}$$

From the identity

$$I = \tau^\alpha (ik)^{\alpha+2} R(ik) + (ik)^2 R(ik) - (c^2 + c^2\tau^\alpha (ik)^\alpha + \delta(ik)^{2-\alpha})AR(ik), \quad k \in \mathbb{Z},$$

we get

$$\begin{aligned}\widehat{f}(k) = \tau^\alpha (ik)^{\alpha+2} R(ik)\widehat{f}(k) + (ik)^2 R(ik)\widehat{f}(k) - (c^2 \\ + c^2\tau^\alpha (ik)^\alpha + \delta(ik)^{2-\alpha})AR(ik)\widehat{f}(k).\end{aligned} \tag{4.8}$$

Since $\{(ik)^{\alpha+2} R(ik)\}_{k\in\mathbb{Z}}$ and $\{(ik)^2 R(ik)\}_{k\in\mathbb{Z}}$ are (L^p, L^p)-multipliers, together with (4.8), imply that $\{(c^2 + c^2\tau^\alpha (ik)^\alpha + \delta(ik)^{1-\alpha}(ik))AR(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

Next, let

$$c_k := \frac{(ik)^\alpha}{c^2 + c^2\tau^\alpha (ik)^\alpha + \delta(ik)^{2-\alpha}}, \quad k \in \mathbb{Z}.$$

It is clear that the sequence $\{c_k\}_{k\in\mathbb{Z}}$ is bounded. Also using (3.10) with $\xi = \alpha$ and $\xi = 2 - \alpha$ we obtain that for all $k \in \mathbb{Z}$

$$\begin{aligned}|k(c_{k+1} - c_k)| \le \frac{c^2|k||(i(k+1))^\alpha - (ik)^\alpha|}{|q_k|} + \frac{\delta|k||(ik)^{2-\alpha}((i(k+1))^\alpha - (ik)^\alpha))|}{|q_k|} \\ + \frac{\delta|k||(ik)^\alpha(i(k+1))^{2-\alpha} - (ik)^{2-\alpha})|}{|q_k|} = \frac{O(|k|^\alpha) + O(k^2) + O(k^2)}{O(|k|^{4-2\alpha})},\end{aligned} \tag{4.9}$$

as $|k| \to \infty$, because $|q_k| := |(c^2 + c^2\tau^\alpha(i(k+1))^\alpha + \delta(i(k+1))^{2-\alpha})(c^2 + c^2\tau^\alpha(ik)^\alpha + \delta(ik)^{2-\alpha}| = O(k^{4-2\alpha})$ as $|k| \to \infty$. We conclude that

$$\sup_{k\in\mathbb{Z}} |c_k| + \sup_{k\in\mathbb{Z}} |k(c_{k+1} - c_k)| < \infty.$$

It follows from Lemma 2.7 that the sequence

$$\{c_k(c^2 + c^2\tau^\alpha(ik)^\alpha + \delta(ik)^{1-\alpha}(ik))AR(ik)\}_{k\in\mathbb{Z}} = \{(ik)^\alpha AR(ik)\}_{k\in\mathbb{Z}} \tag{4.10}$$

is an (L^p, L^p)-multiplier. Therefore there exists $v_3 \in L^p(0, 2\pi; X)$ such that

$$\hat{v}_3(k) = (ik)^\alpha AR(ik)\hat{f}(k) = (ik)^\alpha A\hat{u}(k), \quad k \in \mathbb{Z}, \tag{4.11}$$

where we have used (4.3). Define $w_3 := A^{-1}v_3$. Since $\cdot A^{-1}$ exists as a bounded operator, we have that $w_3 \in L^p(0, 2\pi, D(A))$ and $A\hat{w}_3(k) = \hat{v}_3(k) = (ik)^\alpha A\hat{u}(k)$. We conclude that $\hat{w}_3(k) = (ik)^\alpha \hat{u}(k)$, and therefore,

$$u \in H^{\alpha,p}(0, 2\pi, X), \quad D^\alpha u = w_3 \in D(A) \text{ and } AD^\alpha u = Aw_3 = v_3. \tag{4.12}$$

On the other hand, the identity

$$AR(ik) = \frac{1}{(ik)^\alpha}(ik)^\alpha AR(ik), \quad k \in \mathbb{Z},$$

and (4.10) show that the sequence $\{AR(ik)\}_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier, too. This means that there exists $v_4 \in L^p(0, 2\pi; X)$ such that

$$\widehat{v_4}(k) = AR(ik)\widehat{f}(k) = A\widehat{u}(k), \quad k \in \mathbb{Z}. \tag{4.13}$$

Since $0 \in \rho(A)$, we define $w_4 := A^{-1}v_4$. Then by (4.13), we have

$$u = w_2 \in L^p(0, 2\pi; D(A)), \quad v_4 = Au = Aw_4. \tag{4.14}$$

Since the sequences $\{AR(ik)\}_{k\in\mathbb{Z}}$, $\{(ik)^\alpha AR(ik)\}_{k\in\mathbb{Z}}$, $\{(ik)^{\alpha+2}R(ik)\}_{k\in\mathbb{Z}}$ and $\{(ik)^2R(ik)\}_{k\in\mathbb{Z}}$ are (L^p, L^p)-multipliers, and as a consequence of the identity (4.8), we obtain that the sequence $\{(ik)^{2-\alpha}AR(ik)\}_{k\in\mathbb{Z}}$ is also an (L^p, L^p)-multiplier. Therefore there exists $v_5 \in L^p(0, 2\pi; X)$ such that for all $k \in \mathbb{Z}$

$$\widehat{v_5}(k) = (ik)^{2-\alpha}AR(ik)\widehat{f}(k) = (ik)^{2-\alpha}A\hat{u}(k). \tag{4.15}$$

Since $0 \in \rho(A)$, we define $w_5 := A^{-1}v_5$ which implies that $w_5 \in L^p(0, 2\pi, D(A))$ and $A\hat{w}_5(k) = \hat{v}_5(k) = (ik)^{2-\alpha}A\widehat{u}(k)$, that is, $A^{-1}\hat{v}_5(k) = (ik)^{2-\alpha}\widehat{u}(k)$. Consequently,

$$u \in H^{2-\alpha,p}(0, 2\pi; D(A)), \quad \text{and } v_5 = D^{2-\alpha}Au. \tag{4.16}$$

From Proposition 2.2 we get that

$$u' \in H^{1-\alpha,p}(0, 2\pi; X) \text{ and } v_5 = D^{1-\alpha}Au'. \tag{4.17}$$

Thus, inserting (4.2), (4.6), (4.13), (4.11) and (4.15) in (4.8) we obtain that

$$\widehat{f}(k) = \tau^\alpha\widehat{v}(k) + \widehat{v}_1(k) - c^2\widehat{v}_4(k) - c^2\tau^\alpha\widehat{v}_3(k) - \delta\widehat{v}_5(k), \quad k \in \mathbb{Z}.$$

Now, using (4.5), (4.7), (4.14), (4.12) and (4.17) we get

$$\tau^\alpha D^\alpha u''(t) + u''(t) - c^2Au(t) - c^2\tau^\alpha D^\alpha Au(t) - \delta D^{1-\alpha}Au'(t) = f(t) \tag{4.18}$$

for almost all $t \in [0, 2\pi]$. From (4.18) together with (4.4) and (4.16) we conclude that u is a strong L^p-solution of (4.1). The uniqueness can be proven analogously as in the proof of Theorem 3.14. The conclusion follows. □

The next result deals with a characterization of L^p well-posedness for the equation (4.1) in terms of the R-boundedness of the correspondent operator-valued symbol.

Theorem 4.21. *Let X be a UMD space and $\tau, c, \alpha > 0$. The following conditions are equivalent*

(i) Equation (4.1) is strongly L^p well-posed;
(ii) $\left\{\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha}\right\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and the set*

$$\left\{\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha}\left(\frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha} - A\right)^{-1}\right\}_{k\in\mathbb{Z}} \tag{4.19}$$

is R-bounded.

Proof. First we observe that since $\frac{(ik)^{2+\alpha}}{\tau^\alpha (ik)^{\alpha+2} + (ik)^2} = O(1)$ as $|k| \to \infty$, then R-boundedness of (4.19) is equivalent to R-boundedness of the set $\{(ik)^{\alpha+2} R(ik)\}_{k\in\mathbb{Z}}$. Therefore, the fact that (i) implies (ii) follows immediately from Theorem 4.20 and [8, Proposition 1.11]. We now assume that (ii) holds and let $M_k := (ik)^{\alpha+2} R(ik)$ with

$$R(ik) = (\tau^\alpha (ik)^{\alpha+2} + (ik)^2 - (c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha)A)^{-1}.$$

Following the same argument as in the proof of Theorem 3.15 it is sufficient to show that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded.

As it was shown in (3.17) the sequence $\{(ik)^{\alpha+2}\}_{k\in\mathbb{Z}}$ is 1-regular. We define

$$L_k := (R(ik)^{-1} - R(i(k+1))^{-1})R(ik).$$

From the identity $[\tau^\alpha (ik)^{\alpha+2} - k^2 - (c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha)A]R(ik) = I$, we obtain

$$AR(ik) = \frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha} R(ik) - \frac{1}{c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha} I.$$

Then we get

$$(ik)^\alpha AR(ik) = \frac{\tau^\alpha (ik)^\alpha + 1}{c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha} M_k - \frac{(ik)^\alpha}{c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha} I, \tag{4.20}$$

and we clearly have that $\{(ik)^\alpha AR(ik)\}_{k\in\mathbb{Z}}$ is R-bounded. On the other hand,

$$(ik)^{2-\alpha} AR(ik) = \frac{\tau^\alpha (ik)^2 + (ik)^{2-\alpha}}{[c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha](ik)^\alpha} M_k - \frac{(ik)^{2-\alpha}}{c^2 + \delta(ik)^{2-\alpha} + \tau^\alpha c^2 (ik)^\alpha} I \tag{4.21}$$

which shows the R-boundedness of $\{(ik)^{2-\alpha}AR(ik)\}_{k\in\mathbb{Z}}$. Finally, we verify that the set $\{kL_k\}_{k\in\mathbb{Z}}$ is R-bounded. Indeed, for any $k\in\mathbb{Z}$ we get

$$
\begin{aligned}
kL_k = k\Big[&\tau^\alpha(ik)^{\alpha+2}+(ik)^2-(c^2+\delta(ik)^{2-\alpha}\\
&+\tau^\alpha c^2(ik)^\alpha)A-\tau^\alpha(i(k+1))^{\alpha+2}-(i(k+1))^2\\
&+(c^2+\delta(i(k+1))^{2-\alpha}+\tau^\alpha c^2(i(k+1))^\alpha)A\Big]R(ik)\\
=\Big[&\tau^\alpha k[(ik)^{\alpha+2}-(i(k+1))^{\alpha+2}]+k[(ik)^2-(i(k+1))^2]\\
&-\delta k[(ik)^{2-\alpha}-(i(k+1))^{2-\alpha}]A-\tau^\alpha c^2 k[(ik)^\alpha-(i(k+1))^\alpha]A\Big]R(ik)\\
=\;&\tau^\alpha k\frac{[(ik)^{\alpha+2}-(i(k+1))^{\alpha+2}]}{(ik)^{\alpha+2}}(ik)^{\alpha+2}R(ik)\\
&+k\frac{[(ik)^2-(i(k+1))^2]}{(ik)^2}\frac{1}{(ik)^\alpha}(ik)^{\alpha+2}R(ik)\\
&-\frac{k[(ik)^{2-\alpha}-(i(k+1))^{2-\alpha}]}{(ik)^{2-\alpha}}\delta(ik)^{2-\alpha}AR(ik)\\
&-\frac{k[(ik)^\alpha-(i(k+1))^\alpha]}{(ik)^\alpha}\tau^\alpha c^2(ik)^\alpha AR(ik)\\
=\;&\tau^\alpha k\frac{[(ik)^{\alpha+2}-(i(k+1))^{\alpha+2}]}{(ik)^{\alpha+2}}M_k+k\frac{[(ik)^2-(i(k+1))^2]}{(ik)^2}\frac{1}{(ik)^\alpha}M_k\\
&-\frac{k[(ik)^{2-\alpha}-(i(k+1))^{2-\alpha}]}{(ik)^{2-\alpha}}\delta(ik)^{2-\alpha}AR(ik)\\
&-\frac{k[(ik)^\alpha-(i(k+1))^\alpha]}{(ik)^\alpha}\tau^\alpha c^2(ik)^\alpha AR(ik).
\end{aligned}
\tag{4.22}
$$

Inserting (4.20) and (4.21) in (4.22), and applying repeatedly (3.17) for $\xi\in\{2-\alpha,\alpha+2,2,\alpha\}$ we conclude that the set $\{kL_k\}_{k\in\mathbb{Z}}$ is R-bounded since it is sum of R-bounded sets. The conclusion holds from Theorem 3.7 in [19]. □

Secondly, we analyze the correspondent results about well-posedness for the Type III afMGT equation given by:

$$\tau u'''(t)+u''(t)-c^2Au(t)-\tau c^2Au'(t)-\delta D^{1-\beta}Au'(t)=f(t),\quad t\in[0,2\pi], \tag{4.23}$$

in periodic Lebesgue spaces $L^p(0,2\pi;X)$.

We first introduce the definition of strongly L^p well-posedness as follows.

Definition 4.22. Let $1\le p<\infty$ and $0<\beta\le 1$ be fixed. We say that the problem (4.23) is strongly L^p well-posed if for every $f\in L^p(0,2\pi;X)$ there exists a unique function $u\in H^{3,p}(0,2\pi;X)\cap H^{2-\beta,p}(0,2\pi;D(A))$ that satisfies equation (4.23) for almost all $t\in[0,2\pi]$.

Remark 4.23. If u is a strong L^p-solution of (4.23), then $u \in H^{3,p}(0, 2\pi; X)$ and therefore, from Remark 2.1, we have that the equation (4.23) must satisfy the initial conditions $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and $u''(0) = u''(2\pi)$.

The following theorem is the main result for (4.23). The proof is omitted since it follows similarly to the one of Theorem 4.20 without significant differences.

Theorem 4.24. *Let X be a UMD space, $\tau, c, \delta > 0$ and let A be a closed linear operator. The following assertions are equivalent for $p \in [1, \infty)$:*

(a) The problem (4.23) is strongly L^p well-posed;
(b) $\left\{\dfrac{\tau(ik)^3 + (ik)^2}{c^2 + \tau c^2 ik + \delta(ik)^{2-\beta}}\right\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and the set*

$$\left\{\frac{\tau(ik)^3 + (ik)^2}{c^2 + \tau c^2 ik + \delta(ik)^{2-\beta}}\left(\frac{\tau(ik)^3 + (ik)^2}{c^2 + \tau c^2 ik + \delta(ik)^{2-\beta}} - A\right)^{-1}\right\}_{k\in\mathbb{Z}} \tag{4.24}$$

is R-bounded.

Also, if (a) or (b) hold then we have the following estimate:

$$\|u'''\|_{H^{3,p}(0,2\pi;X)} + ||u''||_{H^{2,p}(0,2\pi;X)} + ||Au||_{L^p(0,2\pi;X)} + ||Au'||_{H^{1,p}(0,2\pi;X)} + ||D^{1-\beta}Au'||_{H^{2-\beta,p}(0,2\pi;X)} \le C||f||_{L^p(0,2\pi;X)}.$$

Remark 4.25. Although the method used in this article has been limited to periodic functions, other cases can be treated similarly. In fact, for functions defined on the positive real axis the Laplace transform can be used. In the case of functions defined on the whole real axis, the Fourier transform could be used. However, in such cases, characterizations of well-posedness as set out in this paper, might be more difficult to prove, because we need additional tools or stronger conditions on the abstract operator A. For example, in the case of the positive real axis, a similar theoretical approach is originally due to L. Weis [40] and requires A to be the generator of an analytic semigroup. In the case of the whole real axis, an approach similar to the one in this article could be performed on vector-valued Hölder spaces using a result due to Arendt, Batty and Bu [6]. In such a case, R-boundedness is no longer necessary, but uniqueness turns out to be a difficult problem, and tools like the Carleman transform need to be implemented. Our results are independent of these approaches, and particularly for spaces of 2π-periodic functions much simpler, because it uses the finite Fourier transform method in its simplest and most general form and a general result on operator-valued Fourier multipliers due to Arendt and Bu [6]. Also, no additional conditions on the operator A are needed.

5. Examples and consequences

In this section, we analyze and present some important consequences of our main abstract results which allows us to conclude some differences in the structure of each of the exposed models.

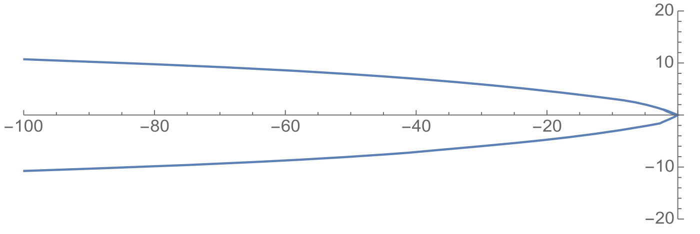

Fig. 1. $\alpha = 1, \tau = 0.5, c = 0.5, \delta = 0.5$.

In order to transfer to some extent the hyperbolic nature of the MGT equation to its fractional abstract version, it is necessary to study the location of the real and imaginary parts of the sequences $d^j(ik)$, $k \in \mathbb{Z}$, $j \in \{I, II, II\}$ in the complex plane.

5.1. Type II

Let $\tau, \delta, c > 0$ and $0 < \alpha \leq 1$. According to Theorem 3.15 the model (3.1) is structurally characterized by the sequence

$$d_k := \frac{-|k|^2(\tau^\alpha (ik)^\alpha + 1)}{c^2 + (\delta + \tau^\alpha c^2)(ik)^\alpha}, \quad k \in \mathbb{Z}.$$

We recall that $\gamma_\alpha := \delta + \tau^\alpha c^2$.

A computation shows that

$$Re(d_k) = \frac{-|k|^2 \left(\tau^\alpha |k|^{2\alpha} + \frac{c^2}{\gamma_\alpha} + |k|^\alpha (\tau^\alpha \frac{c^2}{\gamma_\alpha} + 1)\cos(\alpha\pi/2)\right)}{\gamma_\alpha \left(\left(\frac{c^2}{\gamma_\alpha} + |k|^\alpha \cos(\alpha\pi/2)\right)^2 + |k|^{2\alpha}\sin^2(\alpha\,\mathrm{sgn(k)}\pi/2)\right)} \tag{5.1}$$

and

$$Im(d_k) = \frac{\delta |k|^{2+\alpha}\sin(\alpha\,\mathrm{sgn(k)}\pi/2)}{\left(\frac{c^2}{\gamma_\alpha} + |k|^\alpha \cos(\alpha\pi/2)\right)^2 + |k|^{2\alpha}\sin^2(\alpha\,\mathrm{sgn(k)}\pi/2)}. \tag{5.2}$$

It is clear that $Re(d_k) < 0$ and $Im(d_k) \neq 0$ for each $k \neq 0$. Therefore, $\arg(d_k) < \pi$ for all $k \neq 0$. From (5.1) and (5.2) we get

$$\frac{Im(d_k)}{Re(d_k)} = \frac{-\delta |k|^\alpha \sin(\alpha\,\mathrm{sgn(k)}\pi/2)}{\tau^\alpha |k|^{2\alpha} + \frac{c^2}{\gamma_\alpha} + |k|^\alpha(\tau^\alpha \frac{c^2}{\gamma_\alpha} + 1)\cos(\alpha\pi/2)} = O(1/k^\alpha) \text{ as } |k| \to \infty. \tag{5.3}$$

For $\alpha = 1$ the above conclusions are exactly the same. See Fig. 1 that shows the generic location of the points d_k (over the continuous line drawn). Note that (5.3) implies that we cannot expect to admit a sectorial operator of a fixed angle $\theta < \pi$ in our Type II model. As a consequence, this case could be considered as prototypical.

With the above considerations in mind, we prove the following result:

Theorem 5.26. *Let A be a selfadjoint operator defined on a Hilbert space H with $0 \in \rho(A)$. Then equation (3.1) is strongly L^p well-posed.*

Proof. Since A is selfadjoint and $0 \in \rho(A)$ we have $\sigma(A) \subseteq \mathbb{R} \setminus \{0\}$. Since $Im(d_k) \neq 0$ for all $k \neq 0$, we have $d_k \in \rho(A)$ for all $k \in \mathbb{Z}$. From [28, Chapter 5, Section 3.5], we have that $\|(d_k - A)^{-1}\| = \frac{1}{dist(d_k,\sigma(A))}$, $k \in \mathbb{Z} \setminus \{0\}$. Since $dist(d_k, \sigma(A))$ has order $|k|^2$ and the same happens with $|d_k|$, we have $\sup_k(\frac{|d_k|}{dist(d_k,\sigma(A))}) < \infty$. Therefore, we can conclude that

$$\sup_{k\in\mathbb{Z}} \left\| d_k\, (d_k - A)^{-1} \right\| < \infty.$$

As a consequence of Theorem 3.15 and Remark 3.16, we conclude that equation (3.1) is L^p well-posed. □

We note that the above theorem applies in case $A = \Delta$ the Laplacian or $A = \Delta^2$ the bilaplacian operator in $L^2(\Omega)$ with Dirichlet boundary conditions where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

Remark 5.27. Concerning the problem of singular limit for the type II model, i.e. when $\tau \to 0$, we observe that, formally, the limit should be the time-fractional model $u'' - c^2 Au - \delta D_t^\alpha Au = f$.

5.2. *Type I*

Let $\tau, \delta, c > 0$ and $0 < \alpha \leq 1$. In this case, the relevant structural sequence is defined by:

$$d_k = \frac{\tau^\alpha (ik)^{\alpha+2} + (ik)^2}{c^2 + c^2\tau^\alpha (ik)^\alpha + \delta (ik)^{2-\alpha}}.$$

A computation shows that

$$\begin{aligned} Re(d_k) = &\frac{[\tau^\alpha |k|^{\alpha+2}\cos(\alpha\pi/2) + |k|^2][(\delta|k|^{2-\alpha} - \tau^\alpha c^2|k|^\alpha)\cos(\alpha\pi/2) - c^2)]}{p_k} \\ &- \frac{\tau^\alpha|k|^{\alpha+2}(\delta|k|^{2-\alpha} + \tau^\alpha c^2|k|^\alpha)\sin^2(\operatorname{sgn}(k)\alpha\pi/2)}{p_k} \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} Im(d_k) = &\frac{[\tau^\alpha|k|^{\alpha+2}\sin(\operatorname{sgn}(k)\alpha\pi/2)][(\delta|k|^{2-\alpha} - \tau^\alpha c^2|k|^\alpha)\cos(\alpha\pi/2) - c^2]}{p_k} \\ &+ \frac{[\tau^\alpha|k|^{\alpha+2}\cos(\alpha\pi/2) + |k|^2][(\delta|k|^{2-\alpha} + \tau^\alpha c^2)\sin(\operatorname{sgn}(k)\alpha\pi/2)]}{p_k} \end{aligned} \tag{5.5}$$

where $p_k := [(\delta|k|^{2-\alpha} - \tau^\alpha c^2|k|^\alpha)\cos(\alpha\pi/2) - c^2]^2 + (\delta|k|^{2-\alpha} + \tau^\alpha c^2|k|^\alpha)^2 \sin(\operatorname{sgn}(k)\alpha\pi/2)^2$.

After some computations, we obtain for each $k \in \mathbb{Z}$,

$$p_k Im(d_k) = \operatorname{sgn}(k)\big[\tau^\alpha[\delta|k|^4 \sin(\alpha\pi) - c^2|k|^{\alpha+2} \sin(\alpha\pi/2)] + \delta|k|^{4-\alpha} \sin(\alpha\pi/2) + \\ + \tau^\alpha c^2|k|^2 \sin(\alpha\pi/2)\big]$$

and hence $Im(d_k) \neq 0$ for each $k \in \mathbb{Z} \setminus \{0\}$.

The above property allows to argue as in Theorem 5.26 and obtain the following result in the Hilbert space setting.

Theorem 5.28. *Let A be a selfadjoint operator defined on a Hilbert space H with $0 \in \rho(A)$. Then equation (4.1) is strongly L^p well-posed.*

Proof. Since $Im(d_k) \neq 0$ for all $k \neq 0$, we have $d_k \in \rho(A)$ for all $k \in \mathbb{Z}$. Since $dist(d_k, \sigma(A))$ has order $|k|^{2\alpha}$ we can conclude that

$$\sup_{k \in \mathbb{Z}} \left\| d_k\,(d_k - A)^{-1} \right\| = M < \infty.$$

As a consequence of Theorem 4.21, equation (4.1) is L^p well-posed. □

A computation, taking into account that $Im(d_k) = -Im(d_{-k})$, so that it is enough to calculate for $k \in \mathbb{N}$, shows that

$$\frac{Im(d_k)}{Re(d_k)} \to \tan(\alpha\pi) \quad \text{as } k \to +\infty, \tag{5.6}$$

which coincides with (5.3) only in case $\alpha = 1$ for the range $0 < \alpha \leq 1$. Otherwise, the behavior is clearly different. Indeed, sectorial operators of a fixed angle $\theta < \pi$ can be included in our results depending on the values of the given parameters. This fact makes a difference with those operators that fit into the Type II model. See also the figures below.

After some calculus it can be observed that,

$$p_k Re(d_k) = \delta\big[|k|^{4-\alpha}\cos(\alpha\pi/2) + \tau^\alpha|k|^4\cos(\alpha\pi)\big] \\ - 2\tau^\alpha c^2|k|^{\alpha+2}\cos(\alpha\pi/2) - \tau^{2\alpha}c^2|k|^{2\alpha+2} - |k|^2c^2$$

whence we deduce that $Re(d_k) < 0$ for $k \neq 0$ if the condition

$$1/2 < \alpha \leq 1 \quad \text{and} \quad \ell(\alpha) := \cos(\alpha\pi/2) + \tau^\alpha\cos(\alpha\pi) < 0, \tag{5.7}$$

holds.

A picture of the typical structure of the sequence $\{d_k\}_{k\in\mathbb{Z}}$ is illustrated in the following figures where the points d_k are located over the continuous line drawn.

Figs. 2 and 3 show that the condition (5.7) is sufficient but not necessary to have $Re(d_k) < 0$ for $k \neq 0$.

In Fig. 4, we show that the condition (5.7) may fail to have $Re(d_k) < 0$ for $k \neq 0$.

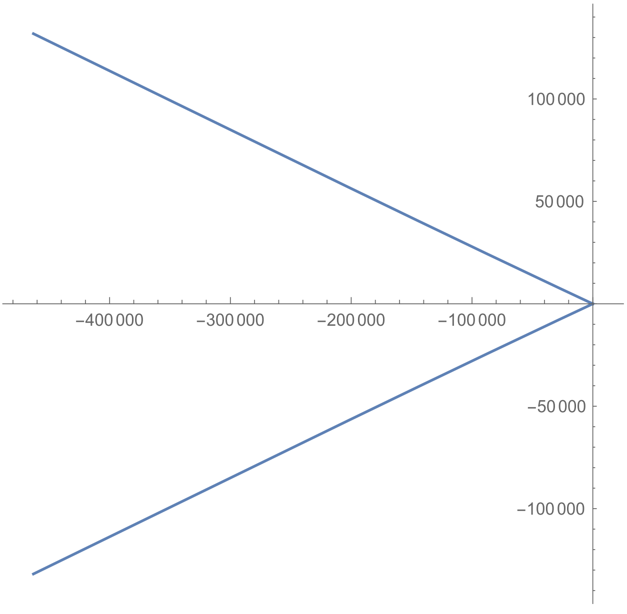

Fig. 2. $\alpha = 0.9$, $\tau = 2$, $c = 1$, $\delta = 3$ and $\ell(\alpha) = -1,6183 < 0$.

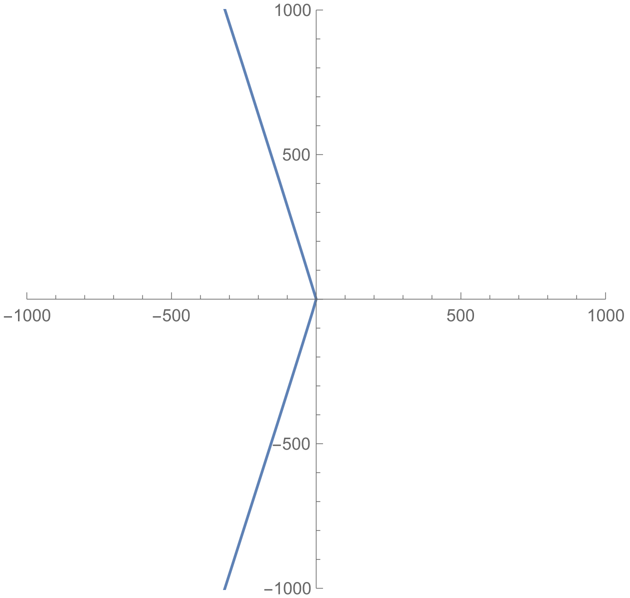

Fig. 3. $\alpha = 0.6$, $\tau = 2$, $c = 1$, $\delta = 3$ and $\ell(\alpha) = 0.119403 > 0$.

Remark 5.29. We observe that the first condition in (5.7), i.e. $\alpha > 1/2$, is consistent with the physical behavior of this type of model, as suggested in [25, Section 2].

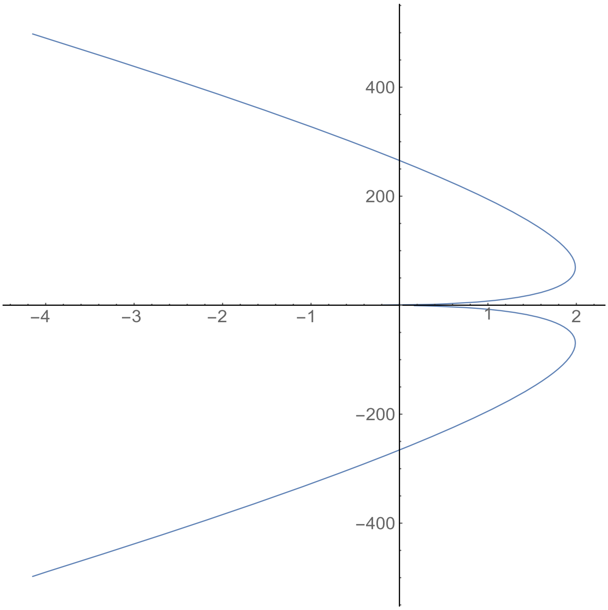

Fig. 4. $\alpha = 0.51$, $\tau = 2$, $c = 1$, $\delta = 3$ and $\ell(\alpha) = 0.707 > 0$.

Note that the second condition in (5.7), or directly from the formula for the real part of the sequence $\{d_k\}$, gives us information about a surprising and interesting dependence of the parameter τ and the fractional order α. Namely, if $\tau \to 0$ then $\alpha \to 1$.

It should be observed that the property that τ could be near to 0 is justified because, in practice, the parameter τ represents a positive constant accounting for relaxation, and it has been shown in a number of experiments that this parameter is small in several mediums as shown in [9, p.150] and references therein. In particular, the singular limit of the type I model assuming condition (5.7) should be the equation $u_{tt} - c^2 Au - \delta Au' = f$.

For our next result we need some preliminaries.

Let consider the space of functions: $\mathcal{H}^\infty(\Sigma_\psi) = \{g : \Sigma_\psi \to \mathbb{C}$ holomorphic and bounded$\}$ which is endowed with the norm

$$||g||_\infty^\psi = \sup_{|\arg z|<\psi} |g(z)|.$$

Le A be a sectorial operator that admits a bounded $\mathcal{H}^\infty-$ calculus [21], i.e. $A \in \mathcal{H}^\infty(X)$. If, moreover, the set $\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\psi), ||h||_\infty^\psi \leq 1\}$ is R-bounded for some $\psi > 0$ then we say A *admits an R-bounded $\mathcal{H}^\infty$-calculus* and that A belongs to the class $\mathcal{RH}^\infty(X)$. The correspondent angle will be noted as $\psi_T^{R_\infty}$.

The following proposition provides sufficient conditions to ensure when $\{h_z(A)\}_{z\in\Lambda}$ is R-bounded.

Proposition 5.30. *[21] Let $A \in \mathcal{RH}^{\infty}(X)$ be given and assume that the set $\{h_z\}_{z\in\Lambda} \subset \mathcal{H}^{\infty}(\Sigma_{\psi})$ is uniformly bounded for some $\psi > \psi_T^{R_\infty}$, where Λ is an arbitrary index set. Then the set $\{h_z(A)\}_{z\in\Lambda}$ is R-bounded.*

Define

$$\theta^* := \sup_{k\in\mathbb{Z}} \arg(d_k),$$

where θ^* could depend on the values of α, τ, c, δ. From the above analysis and figures provided, it is clear that there exist $\tau, c, \delta > 0$ such that $\theta^* < \pi$ for some values of α.

Theorem 5.31. *Let $\tau, c, \delta > 0$ and $0 < \alpha \leq 1$ be given and assume that the condition $\theta^* < \pi$ holds. Let A be a closed linear operator defined on a UMD-space X. Suppose that $-A \in \mathcal{RH}^{\infty}(X)$ with angle $\theta_{-A}^{R_\infty} \in (0, \pi - \theta^*)$ and $0 \in \rho(A)$. Then equation (4.1) is strongly L^p well-posed for any $1 < p < \infty$.*

Proof. From Theorem 4.21 it suffices to show that the set

$$\left\{d_k\,(d_k - A)^{-1}\right\}_{k\in\mathbb{Z}} \tag{5.8}$$

is R-bounded. Due to the fact that $0 < \theta_{-A}^{R_\infty} < \pi - \theta^*$ there exists $s > \theta_{-A}^{R_\infty}$ such that $s < \pi - \theta^*$. For each $z \in \Sigma_s$ and $k \in \mathbb{Z}, k \neq 0$, define

$$h_k(z) := d_k(d_k + z)^{-1}.$$

Since $\delta > 0$, we note that $\frac{z}{d_k}$ belongs to the sector $\Sigma_{s+\theta^*}$ where $s + \theta^* < \pi$ and then the distance from the sector $\Sigma_{s+\theta^*}$ to -1 is always positive. As a result, there exists $M > 0$ independent of $k \in \mathbb{Z} \setminus \{0\}$ and $z \in \Sigma_\tau$ that satisfies the following:

$$|h_k(z)| = \left|\frac{1}{1 + \frac{z}{d_k}}\right| \leq M.$$

Since $-A \in \mathcal{RH}^{\infty}(X)$, then we can conclude from Proposition 5.30 that the set $\{h_k(-A)\}_{k\in\mathbb{Z}\setminus\{0\}}$ is R-bounded. Moreover, due to the fact that A is invertible, the operators $H(k) := (d_k - A)^{-1}$ exist for all $k \in \mathbb{Z}$. As a consequence, $H(k)$ belongs to $\mathcal{B}(X)$ for all $k \in \mathbb{Z}$ and the sequence $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded. □

As a consequence of the fact that any sectorial operator defined on a UMD space that admits a bounded $\mathcal{H}^{\infty}$-calculus of angle β also admits a $\mathcal{RH}^{\infty}$ calculus of the same angle β on the correspondent space (see [27] and [29]) we get the following corollary.

Corollary 5.32. *Let $1 < p < \infty$. Let X be a UMD-space and $-A$ a sectorial operator that admits a bounded $\mathcal{H}^{\infty}$ calculus with angle $\theta_A^{\infty} \in (0, \pi - \theta^*)$ and $0 \in \rho(A)$. Then the equation (4.1) is strongly L^p well-posed.*

As a concrete example, we consider the time-fractional Moore-Gibson-Thompson equation (4.1) of Type I on a cylindrical domain $\Omega = U \times V \subset \mathbb{R}^{n+d}$ where $U = \mathbb{R}^n_+$, $n \in \mathbb{N}$ and $V \subset \mathbb{R}^d$, $d \in \mathbb{N}_0$ is bounded, open and connected. Let $A = \Delta$ be a cylindrical decomposition of the Dirichlet Laplacian operator on $L^q(\Omega)$ with respect to the two cross-sections i.e. $\Delta = \Delta_1 + \Delta_2$ where Δ_i acts on the correspondent component of Ω. Following [36] we introduce L^q-realizations $\Delta_{q,i} = \Delta_i$ as follows:

$$D(\Delta_{q,1}) := \{u \in W^{2,q}(\mathbb{R}^n_+, L^q(V)) : \mathcal{D}_U = 0\};$$
$$D(\Delta_{q,2}) := W^{2,q}(V) \cap W^{1,q}_0(V),$$

with Dirichlet boundary conditions $\mathcal{D}_U$. We consider the Laplacian Δ_q in $L^q(\Omega)$ to be

$$D(\Delta_q) := D(\Delta_{q,1}) \cap D(\Delta_{q,2})$$
$$\Delta_q u := \Delta_{q,1} u + \Delta_{q,1} u = \Delta u, \quad u \in D(\Delta_q).$$

Let V be a C^2-standard domain ([36, Definition 3.1]). From [36, Theorem 4.2] we get $-\Delta_q \in \mathcal{RH}^\infty(L^q(\Omega))$ and $0 \in \rho(\Delta_q)$. Moreover, by [36, Proposition 5.1 (i)] we have $\theta^{\mathcal{R}_\infty}_{-\Delta_q} < \frac{\pi}{2}$. Since $X = L^q(\Omega)$, $1 < q < \infty$ is a UMD-space, as a consequence of Theorem 5.31 with $A = \Delta_q$ we obtain the next theorem.

Theorem 5.33. *Let $1 < p, q < \infty$. Suppose that $\theta^* < \theta^{\mathcal{R}_\infty}_{-\Delta_q}$. Then, for any given $f \in L^p(0, 2\pi; L^q(\Omega))$ there exists a unique $u \in H^{\alpha+2,p}(0, 2\pi; X) \cap H^{2-\alpha,p}(0, 2\pi; D(\Delta_q))$ such that equation (4.1) with $A = \Delta_q$ holds for almost all $t \in [0, 2\pi]$. Moreover, the estimate*

$$\begin{aligned}\|D^\alpha u''\|_{H^{\alpha+2,p}(0,2\pi;L^q(\Omega))} &+ ||u''||_{H^{2,p}(0,2\pi;L^q(\Omega))} + ||\Delta_q u||_{L^p(0,2\pi;L^q(\Omega))}\\ &+ ||D^{1-\alpha}\Delta_q u'||_{H^{2-\alpha,p}(0,2\pi;L^q(\Omega))} + \|D^\alpha_t \Delta_q u\|_{H^{\alpha,p}(0,2\pi;L^q(\Omega))}\\ &\leq C||f||_{L^p(0,2\pi;L^q(\Omega))},\end{aligned}$$

holds.

5.3. Type III

Let $\tau, \delta, c > 0$ and $0 < \beta \leq 1$. In this case, the sequence defined by

$$d_k = \frac{\tau(ik)^3 + (ik)^2}{c^2 + c^2\tau ik + \delta(ik)^{2-\beta}}$$

gives structural information of the model (4.23).

A computation shows that

$$Re(d_k) = -k^2 \frac{1 + \tau^2 k^2 + \delta c^{-2}\Big[|k|^{3-\beta}\tau \sin(\frac{\beta\pi}{2}) - |k|^{2-\beta}\cos(\frac{\beta\pi}{2})\Big]}{q_k}, \tag{5.9}$$

and

$$Im(d_k) = k^3 \frac{\delta c^{-2}\tau|k|^{2-\beta}\cos(\frac{\beta\pi}{2}) + \delta c^{-2}|k|^{1-\beta}\sin(\frac{\beta\pi}{2})}{q_k},$$

where $q_k = c^2(1+\delta c^{-2}|k|^{2-\beta}\cos(\frac{\beta\pi}{2}))^2 + c^2(\tau k + \delta c^{-2}|k|^{2-\beta}\sin^2(\text{sgn(k)}\frac{\beta\pi}{2}))$, for all $k \in \mathbb{Z}$.

We note that $Im(d_k) \neq 0$ for $k \neq 0$ and hence, analogously as in Theorem 5.28, we can prove the following result.

Theorem 5.34. *Let A be a selfadjoint operator defined on a Hilbert space H with $0 \in \rho(A)$. Then equation (4.23) is strongly L^p well-posed.*

For the proof, it is enough to observe that $|d_k| = O(|k|^{1+\beta})$ as $|k| \to \infty$ and hence the set

$$\left\{d_k\,(d_k - A)^{-1}\right\}_{k\in\mathbb{Z}}$$

is uniformly bounded.

Imposing that $|k|^{3-\beta}\tau\sin(\frac{\beta\pi}{2}) - |k|^{2-\beta}\cos(\frac{\beta\pi}{2}) \geq 0$ for all $k \in \mathbb{Z}$ in (5.9), it is not difficult to see that $Re(d_k) \leq 0$ if

$$\ell(\beta) = \cos\left(\frac{\beta\pi}{2}\right) - \tau\sin\left(\frac{\beta\pi}{2}\right) < 0, \qquad 0 < \beta \leq 1. \tag{5.10}$$

Remark 5.35. Since $\cot\left(\frac{\beta\pi}{2}\right) \to \infty$ as $\beta \to 0$ and $\cot\left(\frac{\beta\pi}{2}\right) \to 0$ as $\beta \to 1$, and in view that τ admits small values, we conclude that under condition (5.10), β should be near to 1, presenting a similar behavior as the Type I model. However, in contrast, no restriction on the values of β is necessary. If we consider the singular limit of the type III model assuming condition (5.10) then the limit is $u_{tt} - c^2Au - \delta Au' = f$ which coincides with the singular limit of the type I model. Compare it with Remark 5.29.

From the example presented in Fig. 5, it is clear that there exist $\tau, c, \delta > 0$ and $0 < \beta \leq 1$ such that $\theta^* < \pi$. More precisely, and taking into account that $Im(d_k) = -Im(d_{-k})$, $k \in \mathbb{Z}$, we obtain

$$\frac{Im(d_k)}{Re(d_k)} \to -\frac{\cos(\beta\pi/2)}{\sin(\beta\pi/2)} = -\cot(\beta\pi/2) \quad \text{as } k \to +\infty. \tag{5.11}$$

Note that if $\beta = 1$ the above condition naturally coincides with (5.3). As for Type I, sectorial operators could be admitted in the abstract model, in contrast with the Type II model.

Let

$$\theta^* := \sup_{k\in\mathbb{Z}} \arg(d_k).$$

θ^* could depend on the values of β, τ, c, δ. We obtain the following result.

Theorem 5.36. *Let $1 < p < \infty$, $\delta, c, \tau > 0$ and $0 < \beta \leq 1$ be given. Let X be a UMD-space and let A be an R-sectorial operator on X of angle $\theta < \theta^* < \pi$ with $0 \in \rho(A)$. Then the equation (4.23) is strongly L^p well-posed.*

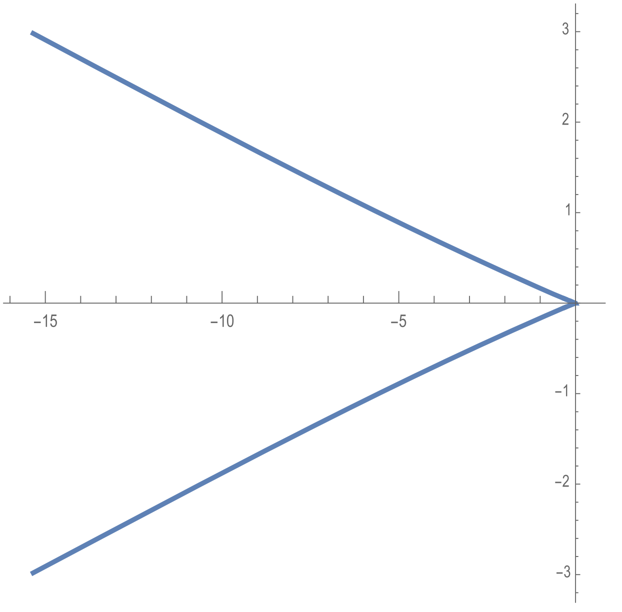

Fig. 5. $\beta = 0.7$, $\tau = 1$, $c = 100$, $\delta = 20$ and $\ell(\beta) = -0.437016 < 0$.

Proof. By hypothesis $d_k \in \Sigma_{\theta^*}$ for all $k \in \mathbb{Z}$. Therefore, the definition of R-sectoriality implies that $\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and that the sequence $\{d_k(d_k - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded. The conclusion holds from Theorem 4.24. □

We finish this article with the following example.

Example 5.37. Let us consider the L^q-realization of Δ_q in $X = L^q(\Omega)$ of Δ, where $1 < q < \infty$. It has been proved in [24, Appendix] that Δ_q is an R-sectorial operator in X for any angle $\theta \in (0, \pi)$. Assuming that $0 \in \rho(\Delta_q)$ (for instance if $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary) then all the hypotheses of Theorem 5.36 are satisfied. Also, for the proof of Proposition 2.2 in [4] we note that the operator $-(-\Delta_q)^{1/2}$ is R-sectorial in X with angle $\theta \in (0, \pi/2)$. Moreover, by [4, Proof of Proposition 2.3], the same happens for the operator $-\Delta_q^2$, with equal angle.

Data availability

No data was used for the research described in the article.

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