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$L^p(L^q)$-Maximal Regularity for Damped Equations in a Cylindrical Domain

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Abstract: We show maximal regularity estimates for the damped hyperbolic and strongly damped wave equations with periodic initial conditions in a cylindrical domain. We prove that this property strongly depends on a critical combination on the parameters of the equation. Noteworthy, our results are still valid for fractional powers of the negative Laplacian operator. We base our methods on the theory of operator-valued Fourier multipliers on vector-valued Lebesgue spaces of periodic functions.

Keywords: Lebesgue maximal regularity; damped hyperbolic equation; strongly damped wave equation; cylindrical domain; fractional Laplacian operator; frictional damping

Citation: Alvarez, E.; Díaz, S.; Lizama, C. $L^p(L^q)$-Maximal Regularity for Damped Equations in a Cylindrical Domain. *Fractal Fract.* **2024**, *8*, 516. https://doi.org/10.3390/fractalfract8090516

Academic Editor: Carlo Cattani

Received: 12 July 2024
Revised: 23 August 2024
Accepted: 28 August 2024
Published: 30 August 2024

1. Introduction

In this paper, we study $L^p(L^q)$-maximal regularity for the damped equation

$$\begin{cases} \rho\partial_t^2 u(t,\mathbf{x}) + \kappa(-\Delta)^\beta u(t,\mathbf{x}) + \delta\partial_t u(t,\mathbf{x}) + \eta(-\Delta)^\beta\partial_t u(t,\mathbf{x}) = g(t,\mathbf{x}), \ t\in\mathbb{T},\, \mathbf{x}\in\Omega, \\ u(0,\mathbf{x}) = u(2\pi,\mathbf{x}), \quad \partial_t u(0,\mathbf{x}) = \partial_t u(2\pi,\mathbf{x}), \end{cases} \tag{1}$$

where $\mathbb{T} := [0,2\pi]$ and $(-\Delta)^\beta$ is the fractional Laplacian operator of order $\beta > 0$ in the sense of Balakrishnan (see (10) below), and where Δ represents a cylindrical decomposition of the Dirichlet Laplacian on $L^p(\Omega)$ with $\Omega \subset \mathbb{R}^{n\times d}$. We will assume that ρ, κ, δ and η are positive real numbers and that the forcing function f is regular enough. In general, the term $\delta\partial_t u$ describes frictional damping that we will assume is different from zero. It is noteworthy that damped differential equations have gained significant interest from various researchers across different time scales [1,2]. We observe that the choice of $\delta > 0$ is consistent with the physical interpretations presented in [3] for the case $\beta = 2$ and in the references [4–6] for the case $\beta = 1$.

The nonlinear equation that contains (1) in case $\beta = 2$ was introduced by Ball [3,7], who modified a previous model proposed by Woinowsky and Krieger [8,9] and Dickey [10] by introducing damping terms. The model introduced by Ball has served until now as motivation for a great amount of work. We cite, for example, the references [11–14]. In the literature, it is referenced as the *damped linear hyperbolic type equation*. We notice that, recently, mathematical models that contain (1) have drawn a lot of attention in the limiting case of $\delta = 0$ [15–19].

Equation (1) in case $\beta = 1$ appears, for example, as the linearized perturbed sine-Gordon equation, where $u(t,\mathbf{x})$ describes the evolution of the current [18]. In such cases, the parameters η, δ correspond to loss effects. This equation is also used in quantum mechanics, see [18]. In the literature, the case $\beta = 1$ is commonly known as the *strongly damped linear wave equation.*

The first study on regularity for (1) for $\beta = 2$ appeared in the work of Triggiani [20] and considers the case $\delta = 0$ in Hilbert spaces. In that case, the Equation (1) describes the vibration of a damped membrane [21]. After that, Chill and Srivastava proved a characterization for an abstract model of the damped wave equation that included (1), but for $t \in \mathbb{R}_+$, and which is valid whenever certain functions are L^p-Fourier multipliers (see Theorem 3.1 in [22]). Although very appealing, the application of such a characterization in each particular case requires a number of additional efforts due to its very nature, see Section 5 in [22]. A further generalization for the nonautonomous case, in terms of Hilbert spaces, has been studied by Achache [23]. Studies related to regularity in the context of two species of chemotaxis models can be found in [24]. Recently, the authors Bu and Cai [8] characterized the maximal regularity of an abstract model in $L^p(\mathbb{T}, X)$ spaces, which includes our fractional model (1). They show that the problem exhibits L^p-maximal regularity in Banach spaces X with unconditional martingale difference (UMD for short) properties, i.e., for some (or all) $1 < p < \infty$, the difference sequences (df_n) of martingales in $L^p(X)$ form an unconditional basic sequence, if and only if the associated resolvent set contain $\mathbb{Z}$ and certain sets of operator-valued symbols are Rademacher bounded (or R-bounded), see [25]. Maximal regularity has also been studied in a discrete setting, see, e.g., [26–28] and the references therein.

Since $X = L^q(\Omega), 1 < q < \infty$, is a typical example of UMD space, it is natural to ask whether it is possible that the remaining hypothesis of R-boundedness can be satisfied for the specific model (1), if not in all cases, eventually under some appropriate combination of the parameters of the equation. Our main objective is to provide an answer to this question.

In this article, we succeed in proving the maximal regularity property for the Equation (1) in a *cylindrical domain* $\Omega = U \times V$, which is valid as long as the parameters of the equation satisfy the critical condition

$$\eta\delta - \kappa\rho > 0, \tag{2}$$

and thus provides new insights into how this property is highly dependent on the presence of frictional damping and its relationship to the other terms of the model. We note that, to the best of our knowledge, the critical condition (2) has not been carefully thought of in the literature.

The use of operator-valued Fourier multipliers (or symbols) to address boundary value problems in a cylindrical domain was first explored in [29] within a Besov-space setting. In that article, the author obtained semiclassical fundamental solutions for a wide variety of elliptic operators in infinite cylindrical domains $\mathbb{R}^n \times V$. As a result, they succeeded in solving related elliptic, parabolic as well as hyperbolic problems. Operators defined within a cylindrical domain with the same splitting property as in the present article were, in the case of an infinite cylinder, also examined by Nau et al. in [30–34].

In this article, we combine the general results of [8,31,35] and apply them to our model (1), obtaining results on the existence and uniqueness of the initial/boundary value problem with $L^p(L^q)$-maximal regularity estimates. As we observed in [8], to achieve our goal we have to verify the R-boundedness property in certain sets of operators, for which we use the criteria established by Denk, Hieber and Prüss in the reference [35], which reduces the problem to the study of the uniform boundedness of certain complex variable function outside the spectrum of the Laplacian operator subject to Dirichlet boundary conditions. We note that our method is general enough to admit a class of operators wider than the Laplacian, also allowing for the possibility of the fractional Laplacian of order β with $\beta > 0$.

Thus, initially, we establish our main result in an abstract setting that outlines how, in general terms, given an operator A defined on a complex Banach space X that meets specific sectoriality conditions, the abstract equation

$$\begin{cases} \rho u''(t) + (\delta + \eta A^\beta)u'(t) + \kappa A^\beta u(t) = g(t), \quad t \in \mathbb{T}, \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases}$$

has L^p–maximal regularity. Then, using the results of [31], we will establish that for each $f \in L^p(\mathbb{T}, L^q(\Omega))$ and $\eta\delta - \kappa\rho > 0$, the solution u to the problem (1) exists, is unique and belongs to the space $L^p(\mathbb{T}; D((-\Delta)^\beta)) \cap W^{1,p}_{per}(\mathbb{T}; L^q(\Omega))$. Moreover, the following inequality holds:

$$\|u\|_{L^p(\mathbb{T},L^q(\Omega))} + ||u'||_{L^p(\mathbb{T},L^q(\Omega))} + ||u''||_{L^p(\mathbb{T},L^q(\Omega))} + ||(-\Delta)^\beta u||_{L^p(\mathbb{T},L^q(\Omega))} + \\ + \|(-\Delta)^\beta u'\|_{L^p(\mathbb{T},L^q(\Omega))} \le C||g||_{L^p(\mathbb{T},L^q(\Omega))},$$

where the constant C is independent of g. Finally, using the implicit function theorem, an application to a semilinear problem, more precisely when g is replaced by $G(u) = (u')^2 + \xi g$ $(\xi \in \mathbb{R})$, is given.

2. Preliminaries

Let $p \in [1, \infty)$, $\mathbb{T} := [0, 2\pi]$ and X be a complex Banach space. By $L^p(\mathbb{T}, X)$ we denote the space of all equivalent classes of measurable functions $g : \mathbb{T} \to X$ such that

$$\|g\|^p_{L^p} := \frac{1}{2\pi} \int_0^{2\pi} \|g(t)\|^p dt < \infty.$$

In what follows, we recall some results included in [8], where the authors obtained a characterization of L^p-maximal regularity for an abstract model that includes the following second-order problem:

$$\begin{cases} v''(t) + \mathcal{B}v'(t) + \mathcal{A}v(t) = g(t), \quad t \in \mathbb{T}, \\ v(0) = v(2\pi), \quad v'(0) = v'(2\pi), \end{cases} \tag{3}$$

where $\mathcal{A}$ and $\mathcal{B}$ are closed linear operators with domains $D(\mathcal{A})$ and $D(\mathcal{B})$ defined on a Banach space X such that $D(\mathcal{A}) \cap D(\mathcal{B}) \neq \emptyset$.

Maximal regularity of evolution equations on the scale of Lebesgue spaces $L^p(\mathbb{T}, X)$ is an important topic that has received much attention in recent years [8,36–41].

We recall from [8] the notion of the resolvent set for the pair $(\mathcal{A}, \mathcal{B})$ as follows:

$$\rho(\mathcal{A}, \mathcal{B}) := \{s \in \mathbb{R} : -s^2 I + is\mathcal{B} + \mathcal{A} : [D(\mathcal{A}) \cap D(\mathcal{B})] \to X \\ \text{is invertible and} \quad [-s^2 I + is\mathcal{B} + \mathcal{A}]^{-1} \in B(X)\}. \tag{4}$$

Here, $[D(\mathcal{A}) \cap D(\mathcal{B})]$ is a Banach space endowed with the norm $\|x\|_{[D(\mathcal{A})\cap D(\mathcal{B})]} := \|x\| + \|\mathcal{A}x\| + \|\mathcal{B}x\|$.

Let $g \in L^1(\mathbb{T}; X)$. We denote

$$\hat{g}(k) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} g(t) dt, \quad k \in \mathbb{Z},$$

using the k-th Fourier coefficient g.

For $p \in [1, \infty)$ and $n \in \mathbb{N}$, the periodic Sobolev space of order n (see [8]) is defined by

$$W^{n,p}_{per}(\mathbb{T}, X) := \{v \in L^p(\mathbb{T}, X) : \exists z \in L^p(\mathbb{T}, X) \text{ such that } \hat{z}(k) = (ik)^n \hat{v}(k) \text{ for all } k \in \mathbb{Z}\}.$$

Remark 1. *We recall the following important properties related to the spaces* $W^{n,p}_{per}(\mathbb{T}, X)$*:*

(i) *Let $m, n \in \mathbb{N}$. If $n \leq m$, then $W^{m,p}_{per}(\mathbb{T}, X) \subseteq W^{n,p}_{per}(\mathbb{T}, X)$.*
(ii) *If $u \in W^{n,p}_{per}(\mathbb{T}, X)$, then for any $0 \leq k \leq n-1$, we obtain $u^{(k)}(0) = u^{(k)}(2\pi)$.*
(iii) *Let $u \in L^p(\mathbb{T}, X)$, then $u \in W^{1,p}_{per}(\mathbb{T}, X)$ if and only if u is differentiable, i.e., on $\mathbb{T}$ and $u' \in L^p(\mathbb{T}, X)$, in this case u is actually continuous and $u(0) = u(2\pi)$, see Lemma 2.1 in* [25].

Let

$$S_p(\mathcal{A}, \mathcal{B}) := \{v \in L^p(\mathbb{T}; D(\mathcal{A})) \cap W^{1,p}_{per}(\mathbb{T}; X) : \ v' \in L^p(\mathbb{T}; D(\mathcal{B})) \cap W^{1,p}_{per}(\mathbb{T}; X)\}.$$

$S_p(\mathcal{A}, \mathcal{B})$ is known as the maximal regularity space of the (3). It can be stated that the space $S_p(\mathcal{A}, \mathcal{B})$ equipped with the norm

$$\|v\|_{S_p(\mathcal{A},\mathcal{B})} := \|v\|_{L^p} + \|v'\|_{L^p} + \|v''\|_{L^p} + \|\mathcal{A}v\|_{L^p} + \|\mathcal{B}v'\|_{L^p},$$

is a Banach space, see [8]. The L^p–maximal regularity for the (3) is defined as follows:

Definition 1. *For a given function $g \in L^p(\mathbb{T}, X)$ where $1 \leq p < \infty$, we say that the Equation (3) has L^p-maximal regularity if there exists a unique solution $v \in S_p(\mathcal{A}, \mathcal{B})$ that satisfies the (3) almost everywhere on $\mathbb{T}$.*

Remark 2. *If the (3) has L^p–maximal regularity and $v \in S_p(\mathcal{A}, \mathcal{B})$ is the unique solution of the (3), then there exists a constant $C > 0$ such that for each $g \in L^p(\mathbb{T}, X)$ the estimate*

$$\|v\|_{S_p(\mathcal{A},\mathcal{B})} \leq C\|g\|_{L^p} \tag{5}$$

holds. See Section 2 in [8].

We recall the definition of Rademacher bounded (R-bounded) for certain sets of operators.

Definition 2. *Suppose X and Y are Banach spaces. A subset $\mathcal{T}$ of the bounded linear operators from X to Y, $\mathcal{T} \subset \mathcal{B}(X, Y)$, is said to be R-bounded if there is a constant $c \geq 0$ such that for all $T_1, ..., T_n \in \mathcal{T}$ and $x_1, ..., x_n \in X$, $n \in \mathbb{N}$, the following inequality holds:*

$$\|(T_1x_1, ..., T_nx_n)\|_R \leq c\|(x_1, ..., x_n)\|_R, \tag{6}$$

where

$$\|(x_1, ..., x_n)\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \Big\| \sum_{j=1}^n \epsilon_j x_j \Big\|.$$

The R-bound of $\mathcal{T}$, denoted by $R(\mathcal{T})$, refers to the smallest constant $c \geq 0$ for which the inequality (6) holds true.

Let $p \in (1, \infty)$. The class of Banach spaces X for which the Hilbert transform, defined as

$$(\mathcal{H}u)(t) := \lim_{\epsilon,R\to\infty} \frac{1}{\pi} \int_{\epsilon\leq|\tau|\leq R} u(t-\tau)\frac{1}{\tau}d\tau, \quad t \in \mathbb{R},$$

is bounded in $L^p(\mathbb{R}; X)$ is referred to as UMD. Now, let us recall the following result proved in Theorem 2.6 in [8].

Theorem 1. *Let X be a UMD space. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are closed linear operators defined on X with $D(\mathcal{A}) \cap D(\mathcal{B}) \neq \emptyset$. The following statements are equivalent:*
(i) *Equation (3) has L^p–maximal regularity;*

(ii) $\mathbb{Z} \subset \rho(\mathcal{A}, \mathcal{B})$ *and the sets* $\{k^2 N_k : k \in \mathbb{Z}\}$ *and* $\{k\mathcal{B}N_k : k \in \mathbb{Z}\}$ *are R-bounded where*

$$N_k := -[k^2 I - ik\mathcal{B} - \mathcal{A}]^{-1}, \quad k \in \mathbb{Z}. \tag{7}$$

Below, we briefly review the fundamentals of sectorial operators.

Let $\Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\}$. We consider the following function sets

$$\mathcal{H}(\Sigma_\phi) := \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\},$$
$$\mathcal{H}^\infty(\Sigma_\phi) := \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\},$$

where the last is endowed with the norm $||f||_\infty^\phi = \sup_{|\arg \lambda| < \phi} |f(\lambda)|$. Moreover, we define

$$\mathcal{H}_0(\Sigma_\phi) := \bigcup_{\alpha,\beta>0} \left\{ f \in \mathcal{H}(\Sigma_\phi) : ||f||_{\alpha,\beta}^\phi := \sup_{|\lambda|\le 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\ge 1} |\lambda^{-\beta} f(\lambda)| < \infty \right\}.$$

From now on, we use the following notation

$$\xi(t) = t(t + \mathcal{A})^{-1}.$$

Definition 3 ([42])**.** *A closed linear operator* $\mathcal{A}$ *on* X *is referred to as sectorial if it satisfies the following conditions:*

(i) $\overline{D(\mathcal{A})} = X$, $\overline{R(\mathcal{A})} = X$, *and* $(-\infty, 0) \subset \rho(\mathcal{A})$

(ii) *There exists* $K > 0$ *such that* $||\xi(t)|| \le K$ *for all* $t > 0$.

Note that this definition of sectorial operators includes those given in Definition 1.1.1 in [43] for non-negative operators.

On the other hand, we recall that a closed linear operator $\mathcal{A}$ in X is R-sectorial if the set $\{\xi(t)\}_{t>0}$ is R-bounded. Further, if $\mathcal{A}$ is sectorial then $\Sigma_\phi \subset \rho(-\mathcal{A})$ for some $\phi > 0$ and

$$\sup_{|\arg \lambda| < \phi} ||\xi(\lambda)|| < \infty.$$

We denote the *spectral angle* of a sectorial operator $\mathcal{A}$ by

$$\phi_{\mathcal{A}} = \inf\{\phi : \Sigma_{\pi-\phi} \subset \rho(-\mathcal{A}), \quad \sup_{\lambda \in \Sigma_{\pi-\phi}} ||\xi(\lambda)|| < \infty\}.$$

Definition 4 ([42])**.** *Given a sectorial operator* $\mathcal{A}$*, we say that it admits a bounded* $\mathcal{H}^\infty$*–calculus if there exist* $\phi > \phi_{\mathcal{A}}$ *and a constant* $K_\phi > 0$ *such that*

$$||f(\mathcal{A})|| \le K_\phi ||f||_\infty^\phi \text{ for all } f \in \mathcal{H}_0(\Sigma_\phi). \tag{8}$$

The class of sectorial operators $\mathcal{A}$ which admit a bounded $\mathcal{H}^\infty$–calculus is denoted by $\mathcal{H}^\infty(X)$. Moreover, the $\mathcal{H}^\infty-$angle is defined by $\phi_{\mathcal{A}}^\infty = \inf\{\phi > \phi_{\mathcal{A}} : (8) \text{ holds}\}$. When $\mathcal{A} \in \mathcal{H}^\infty(X)$ we say that $\mathcal{A}$ *admits an R-bounded* $\mathcal{H}^\infty$*–calculus* if the set

$$\{h(\mathcal{A}) : h \in \mathcal{H}^\infty(\Sigma_\theta), ||h||_\infty^\theta \le 1\}$$

is R-bounded for some $\theta > 0$. We denote the class of such operators by $\mathcal{RH}^\infty(X)$. The corresponding angle is defined in an obvious way and denoted by $\theta_{\mathcal{A}}^{R_\infty}$.

It is important to emphasize that if $\mathcal{A}$ is a sectorial operator in a Hilbert space, $L^p(\Omega)$, $1 < p < \infty$, $W^{s,p}(\Omega)$, $1 < p < \infty$, $s \in \mathbb{R}$ or Besov spaces $B_{p,q}^s(\Omega)$, $1 < p, q < \infty$, $s \in \mathbb{R}$ and A admits a bounded $\mathcal{H}^\infty$–calculus with angle β, then $\mathcal{A}$ admits a $\mathcal{RH}^\infty$–calculus on the same angle β on each of the above described spaces (see Kalton and Weis [44]). Moreover, in cases where X is a UMD space, then the previous claim holds.

The following result was proved in Proposition 4.10 in [42].

Proposition 1. *Let $\mathcal{A} \in \mathcal{RH}^\infty(X)$ and suppose that $\{h_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded for some $\theta > \theta_{\mathcal{A}}^{R_\infty}$, where Λ is an arbitrary index set. Then, the set $\{h_\lambda(\mathcal{A})\}_{\lambda\in\Lambda}$ is R-bounded.*

3. Main Results

Let $1 \le p < \infty$, $\rho, \eta, \kappa, \delta, \beta > 0$ and X be a Banach space. Initially, we will study necessary conditions for L^p–maximal regularity of the damped linear equation, given in abstract form as:

$$\begin{cases} \rho u''(t) + (\delta + \eta A^\beta)u'(t) + \kappa A^\beta u(t) = g(t), \quad t \in \mathbb{T}; \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases} \tag{9}$$

where A is a sectorial operator with domain $D(A)$ defined on a Banach space X (compare Definition 3 and Definition 1.1.1 in [43]), and for $0 < \beta < 1$

$$A^\beta f := \frac{\sin \beta\pi}{\pi} \int_0^\infty s^{\beta-1} A(s+A)^{-1} f ds, \quad f \in X, \tag{10}$$

where the integral is understood in Bochner's sense. This definition is due to Balakrishnan and can be naturally extended for $\beta \ge 1$, see, e.g., Definition 3.1.1 in [43].

We define the critical parameter:

$$\gamma := \eta\delta - \kappa\rho.$$

Our main abstract result in this article is the following.

Theorem 2. *Let X be a UMD-space, $1 < p < \infty$, $0 < \beta < 1$ and assume that $A \in \mathcal{RH}^\infty(X)$ with angle $\theta_A^{R_\infty} \in (0, \frac{\pi}{2\beta})$ and $0 \in \rho(A)$. If $\gamma > 0$, then the Equation (9) admits L^p–maximal regularity and the estimate (5) holds.*

Proof. Observe that our Equation (9) fits well into the (3) for $B := \rho^{-1}(\delta + \eta A^\beta)$ and $A := \rho^{-1}\kappa A^\beta$. In order to show maximal regularity for (9) we just need to prove that the sets $\{k^2 N_k : k \in \mathbb{Z}\}$ and $\{kBN_k : k \in \mathbb{Z}\}$ are R-bounded (see Theorem 1).

Indeed, we have

$$N_k = (\rho(ik)^2 + ik\delta + (ik\eta + \kappa)A^\beta)^{-1} = \frac{1}{ik\eta + \kappa}\Big(d_k + A^\beta\Big)^{-1},$$

where $d_k := \dfrac{\rho(ik)^2 + ik\delta}{ik\eta + \kappa}$. A computation shows that

$$\Re(d_k) = \frac{k^2(\eta\delta - \kappa\rho)}{(k\eta)^2 + \kappa^2} \text{ and } \Im(d_k) = \frac{k(\eta\rho k^2 + \kappa\delta)}{(k\eta)^2 + \kappa^2}.$$

Since $\kappa\rho < \eta\delta$, according to our hypothesis, we have $\Re(d_k) > 0$ and hence $\theta^* := \sup_{k\in\mathbb{Z}} |\arg(d_k)| < \pi/2$.

Now, since $0 < \theta_A^{R_\infty} < \frac{\pi}{2\beta}$, we get that there exists $s \in \mathbb{R}$ such that $\theta_A^{R_\infty} < s < \frac{\pi}{2\beta}$. Let us define

$$\mathscr{F}_k(z) := d_k\Big(d_k + z^\beta\Big)^{-1} = \left(1 + \frac{z^\beta}{d_k}\right)^{-1}$$

for all $z \in \Sigma_s$ and for all $k \in \mathbb{Z} \setminus \{0\}$.

One can see easily that $\frac{z^\beta}{d_k} \in \Sigma_{s\beta+\pi/2}$. From here we deduce that

$$\text{dist}(-1, \Sigma_{s\eta+\pi/2}) > 0.$$

Hence, there is a constant $M_0 > 0$ independent of $k \in \mathbb{Z}$ and $z \in \Sigma_s$ such that

$$|\mathscr{F}_k(z)| \le M_0.$$

Thus, Proposition 1 guarantees that $\{\mathscr{F}_k(A)\}_{k\in\mathbb{Z}\setminus\{0\}}$ is R-bounded. In particular, we have that the family of operators

$$\{d_k(d_k + A^\beta)^{-1}\}_{k\in\mathbb{Z}}$$

is R-bounded due to the invertibility of A and the fact that $\{d_k(d_k + A^\beta)^{-1}\}_{k\in\mathbb{Z}}$ exist for all $k \in \mathbb{Z}$.

In addition, the boundedness of $\left\{\frac{ik\eta+\kappa}{\rho(ik)^2+ik\delta}\right\}_{k\in\mathbb{Z}\setminus\{0\}}$ and the identity

$$(d_k + A^\beta)^{-1} = \frac{ik\eta + \kappa}{\rho(ik)^2 + ik\delta} d_k(d_k + A^\beta)^{-1}$$

guarantee that $\{(d_k + A^\beta)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded. Now, note that

$$k^2 N_k = \frac{k^2}{\rho(ik)^2 + ik\rho} d_k\left(d_k + A^\beta\right)^{-1}.$$

This implies that $\{k^2 N_k\}_{k\in\mathbb{Z}}$ is R-bounded, too. Finally, from the fact that

$$A^\beta(d_k + A^\beta)^{-1} = I - d_k(d_k + A^\beta)^{-1},$$

we get,

$$kBN_k = \frac{k\delta}{\rho(ik\eta + \kappa)}(d_k + A^\beta)^{-1} + \frac{k\eta}{\rho(ik\eta + \kappa)} I - \frac{k\eta}{\rho(ik\eta + \kappa)} d_k(d_k + A^\beta)^{-1}. \quad (11)$$

Next, according to the R-boundedness of the sets $\{(d_k + A^\beta)^{-1}\}_{k\in\mathbb{Z}}$ and $\{d_k(d_k + A^\beta)^{-1}\}_{k\in\mathbb{Z}}$, we conclude that the set $\{kBN_k \,:\, k \in \mathbb{Z}\}$ is also R-bounded.

Theorem 1 assures that Equation (9) admits L^p-maximal regularity. □

Remark 3. *As a new idea behind the proof of our main result, we note that only under the requirement that the operator A admit a $\mathcal{RH}^\infty$-calculus, we can derive the maximal regularity property for the Equation (9) involving fractional powers β of operator A. For that, we need to restrict the angle to the range $(0, \frac{\pi}{2\beta})$.*

The next corollary is a direct consequence of Theorem 2.

Corollary 1. *Let $p, q \in (1, \infty)$ be given. Assume that $\rho, \eta, \kappa, \delta > 0$ are such that $\gamma > 0$ and that $A \in \mathcal{RH}^\infty(L^q(\Omega))$ with angle $0 < \theta_A^{\mathcal{R}_\infty} < \frac{\pi}{2\beta}$ and 0 in the resolvent set of A. Then, for all $0 < \beta < 1$ Equation (9) admits L^p–maximal regularity.*

Finally, we consider the damped linear hyperbolic type equation in the cylindrical domain $\Omega = U \times V \subset \mathbb{R}^{n+d}$ where $U = \mathbb{R}^n_+, n \in \mathbb{N}$ and $V \subset \mathbb{R}^d, d \in \mathbb{N}_0$ is bounded, open and connected:

$$\begin{cases} \rho\partial_t^2 u(t,x,y) + \kappa(-\Delta)^\beta u(t,x,y) + \delta\partial_t u(t,x,y) + \eta(-\Delta)^\beta \partial_t u(t,x,y) = g(t,x,y), \\ u(0,x,y) = u(2\pi,x,y),\ (x,y) \in U \times V, \\ \partial_t u(0,x,y) = \partial_t u(2\pi,x,y),\ (x,y) \in U \times V, \\ u(t,x,y) = 0,\ (t,x,y) \in (0,2\pi) \times U \times \partial V, \\ u(t,x,y) = 0,\ (t,x,y) \in (0,2\pi) \times \partial U \times V, \end{cases} \quad (12)$$

where $(t, x, y) \in \mathbb{T} \times U \times V$, $\rho, \eta, \kappa, \delta > 0$, $0 < \beta < 1$ and Δ denotes a cylindrical decomposition of the Dirichlet Laplacian operator on $L^q(\Omega)$ with respect to the two cross-sections i.e., $\Delta = \Delta_1 + \Delta_2$ where Δ_i acts on the according component of Ω.

Following Section 3 in [31], we introduce L^q-realizations $\Delta_{q,i} = \Delta_i$ as follows:

$$D(\Delta_{q,1}) := \{\varphi \in W^{2,q}(U, L^q(V)) \, : \, \varphi(x,y) = 0, \; (x,y) \in \partial U \times V\};$$
$$D(\Delta_{q,2}) := W^{2,q}(V) \cap W_0^{1,q}(V).$$

We define the Laplacian Δ_q in $L^q(\Omega)$ subject to the Dirichlet boundary conditions on U and V to be

$$D(\Delta_q) := D(\Delta_{q,1}) \cap D(\Delta_{q,2}),$$
$$\Delta_q u := \Delta_{q,1} u + \Delta_{q,2} u = \Delta u, \quad u \in D(\Delta_q).$$

We recall from Definition 3.1 in [31] that a domain $V \subset \mathbb{R}^d$ is called a Lipschitz domain if there exists a $M > 0$ so that every point $x = (x_1, ..., x_n) \in \partial V$ has a neighborhood W such that, eventually after an affine change of coordinates, $\partial V \cap W$ is described by the equation $x_n = \varphi(x_1, ..., x_{n-1})$, where φ is a Lipschitz continuous function with a Lipschitz constant bounded by M and where $V \cap W$ equals the set $\{x \in W \; : \; x_n > \varphi(x_1, ..., x_{n-1})\}$.

If for some $m \in \mathbb{N}_0$ and all $x \in \partial V$ the function φ is a C^m-function, then V is called a C^m domain. If the boundary of V is also compact, that is, if V is a bounded or an exterior domain, then V is called a C^m standard domain.

We can apply Corollary 1 with any $0 < \beta < 1$ and $A = -\Delta_q$, obtaining the following result.

Theorem 3. *Let $1 < p, q < \infty$ and $0 < \beta < 1$. Assume that $\rho, \eta, \kappa, \delta > 0$ are such that $\gamma > 0$. Suppose that V is a C^2 standard domain. Then, for any given $g \in L^p(\mathbb{T}, L^q(\Omega))$ the problem (12) has a unique solution u that belongs to the maximal regularity space:*

$$MR_p(L^q(\Omega)) := \{v \in L^p(\mathbb{T}; D((-\Delta_q)^\beta) \cap W^{1,p}_{per}(\mathbb{T}; L^q(\Omega)) : \; v' \in L^p(\mathbb{T}; D((-\Delta_q)^\beta) \cap W^{1,p}_{per}(\mathbb{T}; L^q(\Omega))\}.$$

Moreover, there exists a constant $C > 0$, independent of p, q such that the following estimate

$$\|v\|_{MR_p(L^q(\Omega))} := \|v\|_{L^p(\mathbb{T},L^q(\Omega))} + \|v'\|_{L^p(\mathbb{T},L^q(\Omega))} + \|v''\|_{L^p(\mathbb{T},L^q(\Omega))} + \|(-\Delta_q)^\beta)v\|_{L^p(\mathbb{T},L^q(\Omega))} + \|(-\Delta_q)^\beta)v'\|_{L^p(\mathbb{T},L^q(\Omega))} \leq C\|g\|_{L^p(\mathbb{T},L^q(\Omega))} \tag{13}$$

holds.

Proof. Applying Based on Theorem 4.2 in [31], we conclude under the given hypothesis that $-\Delta_q \in \mathcal{RH}^\infty(L^q(\Omega))$ and $0 \in \rho(\Delta_q)$. Moreover, since V is bounded, according to Remark 4.7 in [31], we have $\theta^{\mathcal{R}_\infty}_{-\Delta_q} = 0$. We conclude from Corollary 1 that the problem (12) has L^p maximal regularity. According to Definition 1 we conclude that for $g \in L^p(\mathbb{T}, L^q(\Omega))$ there exists a unique solution $u \in S_p((-\Delta_q)^\beta, \delta I + \eta(-\Delta_q)^\beta)$ that satisfies (12). According to the definition of the maximal regularity space we have

$$\begin{aligned} &S_p((-\Delta_q)^\beta, \delta I + \eta(-\Delta_q)^\beta) \\ &\quad = \{v \in L^p(\mathbb{T}; (-\Delta_q)^\beta) \cap W^{1,p}_{per}(\mathbb{T}; L^q(\Omega)) : \; v' \in L^p(\mathbb{T}; \delta I + \eta(-\Delta_q)^\beta) \cap W^{1,p}_{per}(\mathbb{T}; L^q(\Omega))\} \\ &\quad = MR_p(L^q(\Omega)). \end{aligned}$$

Moreover, it follows from Remark 2 that the estimate (5) holds and that there exists a constant $C > 0$ such that

$$\|v\|_{MR_p(L^q(\Omega))} \leq C\|g\|_{L^p(\mathbb{T},L^q(\Omega))}$$

holds. Replacing the left-hand side according to the definition, we obtain the estimate given in the conclusion of the theorem. □

We now analyze a semilinear case for problem (12), more precisely, when the forcing term g is replaced by the following nonlinearity

$$G(u)(t,x,y) := (\partial_t u)^2(t,x,y) + \xi g(t,x,y), \quad \xi \in \mathbb{R}. \tag{14}$$

Our next result shows the existence of solutions for the semilinear problem based on a general version of the implicit function theorem and Theorem 3.

Theorem 4. *Let $1 < p,q < \infty$ and $0 < \beta < 1$. Assume that $\rho, \eta, \kappa, \delta > 0$ are such that $\gamma > 0$. Then, for any given $g \in L^p(\mathbb{T}, L^q(\Omega))$ there exists $\xi^* > 0$ such that for each $0 < |\xi| < \xi^*$ the problem (12) with g, replaced by G in (14), has a nontrivial solution $u_\xi \in L^p(\mathbb{T}; D((-\Delta_q)^\beta) \cap W^{1,p}_{per}(\mathbb{T}; L^q(\Omega))$ with $u'_\xi \in L^p(\mathbb{T}; D((-\Delta_q)^\beta) \cap W^{1,p}_{per}(\mathbb{T}; L^q(\Omega))$.*

Proof. Recall that $(MR_p(L^q(\Omega)), \|\cdot\|_{MR_p(L^q(\Omega))})$ is a Banach space. We define a linear operator $L : MR_p(L^q(\Omega)) \to L^p(\mathbb{T}; L^q(\Omega))$ by

$$L(u) := \rho u'' + \kappa(-\Delta)^\beta u + \delta u' + \eta(-\Delta)^\beta u'.$$

From the definition of the operator L we deduce that there exists a constant $M > 0$ such that $\|L(u)\|_{L^p(\mathbb{T}, L^q(\Omega))} \leq M\|u\|_{MR_p(L^q(\Omega))}$.

As a consequence of Theorem 3, we obtain the existence of a constant $C > 0$ such that $\|u\|_{MR_p(L^q(\Omega))} \leq C\|L(u)\|_{L^p(\mathbb{T}, L^q(\Omega))}$. Therefore, L is an isomorphism. Moreover, as a consequence of Theorem 3, it follows that L is onto.

Let us now define mapping F as follows:

$$F(u) := (u')^2, \quad u \in MR_p(L^q(\Omega)).$$

Note that $u \in MR_p(L^q(\Omega))$ implies $u' \in W^{1,p}(\mathbb{T}; L^q(\Omega))$ and hence we conclude that $u'' \in L^p(\mathbb{T}; L^q(\Omega))$ and u' is continuous, see Remark 1. Therefore, we obtain

$$\int_0^{2\pi} \|u'(t)u'(t)\|_q^p dt \leq \int_0^{2\pi} \|u'(t)\|_q^p \|u'(t)\|_q^p dt \leq 2\pi(\sup_{t \in [0,2\pi]} \|u'(t)\|_q^p)^2.$$

Thus, $(u')^2 \in L^p(\mathbb{T}; L^q(\Omega))$. This implies that $F : MR_p(L^q(\Omega)) \to L^p(\mathbb{T}; L^q(\Omega))$ is well defined.

We claim that map F is continuous. Indeed, given a sequence $(u_n) \subset MR_p(L^q(\Omega))$ such that $\|u_n - u\|_{MR_p(L^q(\Omega))} \to 0$ we conclude that the sequence (u_n) is bounded, that is, there exists a constant $C > 0$ such that $\sup_n \|u_n\|_{MR_p(L^q(\Omega))} < C$ and, in particular, $\sup_n \|u'_n\|_{L^p(\mathbb{T}; L^q(\Omega))} < C$. Hence,

$$\begin{aligned}
\|F(u_n) - F(u)\|_{L^p(\mathbb{T}; L^q(\Omega))} &\leq \|(u'_n)^2 - (u')^2\|_{L^p(\mathbb{T}; L^q(\Omega))} \\
&= \|[(u'_n) + (u')][(u'_n) - (u')]\|_{L^p(\mathbb{T}; L^q(\Omega))} \\
&\leq \Big(\|u'_n\|_{L^p(\mathbb{T}; L^q(\Omega))} + \|u'\|_{L^p(\mathbb{T}; L^q(\Omega))} \Big) \|(u'_n) - (u')\|_{L^p(\mathbb{T}; L^q(\Omega))} \\
&\leq (C + \|u'\|_{L^p(\mathbb{T}; L^q(\Omega))}) \|u_n - u\|_{MR_p(L^q(\Omega))},
\end{aligned}$$

proving the claim.

Now, we consider the uniparametric family $H : \mathbb{R} \times MR_p(L^q(\Omega)) \to L^p(\mathbb{T}; L^q(\Omega))$ defined by

$$H(\tau, u) := -L(u) + F(u) + \tau g.$$

From its definition, it is clear that $H(0,0) = 0$. On the other hand, we note that

$$DF(u) = 2(\partial_t u)\partial_t$$

where D denotes the Fréchet derivative of F and, therefore,

$$DF(u)(h) = 2(\partial_t u)\partial_t h.$$

In particular, this implies that $DF(0) = 0$. Since $D_u H(\tau, u) = -L + DF(u)$ we then obtain $D_u H(0, 0) = -L$ which is linear, bounded and invertible. Using Theorem 3, we conclude according to the implicit function theorem that there exists a neighborhood $I \subset \mathbb{R}$ of 0 and a unique $\psi : I \to MR_p(L^q(\Omega))$ such that $H(\tau, \psi(\tau)) = 0$ for all $\tau \in I$. Since $u \equiv 0$ is a trivial solution in case $\tau = 0$, we conclude that for all $\tau \in I \setminus \{0\}$ there exists $u_\tau := \psi(\tau) \in MR_p(L^q(\Omega))$ such that u_τ is a nontrivial solution for the Equation (12), and the proof is finished. □

4. Conclusions

The research conducted in this paper demonstrates that the maximal $L_p(L_q)$ regularity property for damped wave equations and damped hyperbolic equations in a cylindrical domain critically depends on the relationship between the parameters of the equation, particularly expressed by the condition $\eta\delta - \kappa\rho > 0$. This finding underlines the importance of the damping parameters and the domain geometry in the dynamic behavior of the solutions. The proof of this dependence provides a solid foundation for understanding the regularity in these systems, offering a valuable tool for predicting and controlling solutions in practical applications.

Moreover, this paper explicitly addresses different models, including wave equations with multiple damping terms, which extends the applicability of the results to a wider range of physical and mathematical problems. These models serve as a basis for future research, in which the influence of more complex domain geometries, variations in boundary conditions or even the inclusion of nonlinear terms could be explored. Extending these results to other types of partial differential equations on times scale could provide new insights into the analysis of regularity and stability in damped dynamic systems.

Author Contributions: C.L. conceived the presented idea. C.L., S.D. and E.A. developed the theory and performed the computations. C.L., S.D. and E.A. verified the analytical methods. Writing—original draft, E.A. and C.L.; Writing—review & editing, S.D. All authors have read and agreed to the published version of the manuscript.

Funding: C.L. is partially supported by ANID, FONDECYT Grant 1220036.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

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